BONN LECTURES

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Contents

All references are to the first arXiv version of [EG19a], and I do not intend to update the numbering to match any future versions.

1. Definitions related to étale φ-modules and étale (φ, Γ)-modules

1.1. **Rings.** We begin with some material from [EG19a, §2.1]. We fix a finite extension K of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_K$ and residue field k, and regard K a subfield of some fixed algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, and let $\mathbf{C}$ denote the completion of $\overline{\mathbf{Q}}_p$. It is a perfectoid field, whose tilt $\mathbf{C}^\flat$ is a complete non-archimedean valued perfect field of characteristic p. If F is a perfectoid closed subfield of $\mathbf{C}$, then its tilt $F^\flat$ is a closed, and perfect, subfield of $\mathbf{C}^\flat$. We will only need to consider two examples of F (other than $\mathbf{C}$ itself), which arise from the theories of (φ,Γ)-modules and Breuil–Kisin modules.

1.1.1. *Example* (The Kummer case). If we choose a uniformizer π of K, as well as a compatible system of p-power roots π^{1/p^n} of π (here, "compatible" has the obvious meaning, namely that $(\pi^{1/p^{n+1}})^p = \pi^{1/p^n}$), then we define $K_\infty = K(\pi^{1/p^\infty}) := \bigcup_n K(\pi^{1/p^n})$. If we let $\widehat{K}_\infty$ denote the closure of K_∞ in $\mathbf{C}$, then $\widehat{K}_\infty$ is again a perfectoid subfield of $\mathbf{C}$.

1.1.2. *Example* (The cyclotomic case). We write $K(\zeta_{p^\infty})$ to denote the extension of K obtained by adjoining all p-power roots of unity. It is an infinite degree Galois extension of K, whose Galois group is naturally identified with an open subgroup of $\mathbf{Z}_p^\times$. We let K_{cyc} denote the unique subextension of $K(\zeta_{p^\infty})$ whose Galois group over K is isomorphic to $\mathbf{Z}_p$ (so K_{cyc} is the "cyclotomic $\mathbf{Z}_p$-extension" of K). If we let $\widehat{K}_{\mathrm{cyc}}$ denote the closure of K_{cyc} in $\mathbf{C}$, then $\widehat{K}_{\mathrm{cyc}}$ is a perfectoid subfield of $\mathbf{C}$.

1.1.3. *Remark.* Let $L = K_{\mathrm{cyc}}$ or K_∞. Then $\widehat{L} \otimes_L \overline{\mathbf{Q}}_p$ is an algebraic closure of $\widehat{L}$, so that the absolute Galois groups of L and $\widehat{L}$ are canonically identified. The action of G_L on $\overline{\mathbf{Q}}_p$ extends to an action on $\mathbf{C}$, and by a theorem of Ax–Tate–Sen [Ax70], we have $\mathbf{C}^{G_L} = \widehat{L}$.

If $\mathcal{O}_F$ denotes the ring of integers in F (with $F = \mathbf{C}$ or one of the two possibilities just discussed), then $\mathcal{O}_F^\flat$ is the ring of integers in $F^\flat$. Since $F^\flat$ and $\mathcal{O}_F^\flat$ are perfect, we may form their rings of (truncated) Witt vectors $W(\mathcal{O}_F^\flat)$, $W(F^\flat)$, $W_a(F^\flat)$, $W_a(\mathcal{O}_F^\flat)$ (where $a \ge 1$ is an integer). We write $\mathbf{A}_{\mathrm{inf}} := W(\mathcal{O}_{\mathbf{C}}^\flat)$.

We always consider these rings of Witt vectors as topological rings with the so-called *weak topology*, which admits the following description: if x is any element of $\mathcal{O}_F^\flat$ of positive valuation, and if $[x]$ denotes the Teichmüller lift of x, then we endow $W_a(\mathcal{O}_F^\flat)$ with the $[x]$-adic topology, so that $W(\mathcal{O}_F^\flat)$ is then endowed with the $(p,[x])$-adic topology. The topology on $W_a(F^\flat)$ is then characterized by the fact that $W_a(\mathcal{O}_F^\flat)$ is an open subring (and the topology on $W(F^\flat)$ is the inverse limit topology).

While we could formulate all of our results in terms of φ-modules over these rings of Witt vectors, for the purposes of proving our structural results about the stacks (and for connecting to the usual theories of (φ,Γ)-modules and Breuil–Kisin modules), we need to consider various smaller (in particular Noetherian) subrings. These come from the Fontaine–Wintenberger theory of the field of norms, but we will skip over this and go straight to the definitions.

Firstly, in the Kummer case, we set $\mathfrak{S} = W(k)[[u]]$, with a φ-semi-linear endomorphism φ determined by $\varphi(u) = u^p$, and let $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[1/u]$. The choice of compatible system of p-power roots of π gives an element $\pi^{1/p^\infty} \in \mathcal{O}^\flat_{\widehat{K_\infty}}$, and there is a continuous φ-equivariant embedding

$$\mathfrak{S} \hookrightarrow W(\mathcal{O}^\flat_{\widehat{K_\infty}})$$

sending $u \mapsto [\pi^{1/p^\infty}]$. This embedding extends to a continuous φ-equivariant embedding

$$\mathcal{O}_{\mathcal{E}} \hookrightarrow W((\widehat{K_\infty})^\flat).$$

Now consider the cyclotomic case, where we write $\widetilde{\Gamma}_K := \mathrm{Gal}(K(\zeta_{p^\infty})/K)$, and the cyclotomic character induces an embedding $\chi : \widetilde{\Gamma}_K \hookrightarrow \mathbf{Z}_p^\times$. Consequently, there is an isomorphism $\widetilde{\Gamma}_K \cong \Gamma_K \times \Delta$, where $\Gamma_K \cong \mathbf{Z}_p$ and Δ is finite. We have $K_{\mathrm{cyc}} = (K(\zeta_{p^\infty}))^\Delta$. We will usually write Γ for Γ_K from now on. **We now make a simplifying assumption for the purposes of exposition: assume that $K/\mathbf{Q}_p$ is unramified.** (The general case is of course important, but there is a subtlety: not only is it hard to write down the action of φ explicitly, but there is no φ-stable analogue of the ring $\mathbf{A}_K^+$. This means that in many arguments in [EG19a] we reduce unramified case, but we want to avoid the ensuing technicalities in this course.) If we choose a compatible system of p^nth roots of 1, then these give rise in the usual way to an element $\varepsilon \in (\widehat{K(\zeta_{p^\infty})})^\flat$. There is then a continuous embedding

$$W(k)[[T]] \hookrightarrow W(\mathcal{O}^\flat_{\widehat{K(\zeta_{p^\infty})}})$$

(the source being endowed with its (p,T)-adic topology, and the target with its weak topology), defined via $T \mapsto [\varepsilon] - 1$. We denote the image of this embedding by $(\mathbf{A}'_K)^+$. This embedding extends to an embedding

$$\widehat{W(k)((T))} \hookrightarrow W\big((\widehat{K(\zeta_{p^\infty})})^\flat\big)$$

(here the source is the p-adic completion of the Laurent series ring $W(k)((T))$), whose image we denote by $\mathbf{A}'_K$. Write $T'_K \in \mathbf{A}'_K$ for the image of T. The actions of φ and $\gamma \in \widetilde{\Gamma}_K$ on $T'_K \in \mathbf{A}'_K$ are given by the explicit formulae

$$\varphi(T'_K) = (1 + T'_K)^p - 1, \tag{1.1.4}$$

$$\gamma(1 + T'_K) = (1 + T'_K)^{\chi(\gamma)}, \tag{1.1.5}$$

where $\chi : \widetilde{\Gamma}_K \to \mathbf{Z}_p^\times$ again denotes the cyclotomic character. We set $\mathbf{A}_K := (\mathbf{A}'_K)^\Delta$, $T_K = \mathrm{tr}_{\mathbf{A}'_K/\mathbf{A}_K}(T'_K)$, and $\mathbf{A}_K^+ = W(k)[[T_K]]$; then we have $\mathbf{A}_K = W(k)((T_K))^\wedge$, and $\mathbf{A}_K^+$ is (φ, Γ_K)-stable. We have $\varphi(T_K) \in T_K\mathbf{A}_K^+$, and $g(T_K) \in T_K\mathbf{A}_K^+$ for all $g \in \Gamma_K$. From now on we will often write T for T_K.

1.1.6. *Remark.* We could work with the rings $\mathbf{A}'_K$ and the theory of $(\varphi, \widetilde{\Gamma})$-modules, but it is often convenient to work with the procyclic group Γ and not have to carry around the finite group Δ.

1.1.7. *Remark.* Note that in both the Kummer and cyclotomic settings, we are considering rings which are abstractly a power series ring over the Witt vectors, equipped with a lift of Frobenius. This will mean that various foundational parts of the theory can be developed in parallel for the two cases.

1.2. **Coefficients.** We now consider coefficients. Recall that if A is a p-adically complete $\mathbf{Z}_p$-algebra, then A is said to be *topologically of finite type over* $\mathbf{Z}_p$ if it can be written as a quotient of a restricted formal power series ring in finitely many variables $\mathbf{Z}_p\langle\langle X_1, \dots, X_n\rangle\rangle$; equivalently, if and only if A/p is a finite type $\mathbf{F}_p$-algebra ([FK18, §0, Prop. 8.4.2]). In particular, if A is a $\mathbf{Z}/p^a$-algebra for some $a \geq 1$, then A is topologically of finite type over $\mathbf{Z}_p$ if and only if it is of finite type over $\mathbf{Z}/p^a$. Our coefficient rings will always be assumed to be topologically of finite type over $\mathbf{Z}_p$, and they will usually also be (finite type) $\mathbf{Z}/p^a$-algebras for some $a \geq 1$.

1.2.1. *Remark.* Our stacks are all limit preserving, so that values on A-valued points (for any A) are determined by their values on A which are (topologically) of finite type. It therefore does not cause us any problems to restrict to coefficients A of this kind. In addition many of our arguments with φ-modules require this assumption on A, and we do not know if the results hold without it.

Our various coefficient rings are all defined by taking completed tensor products. More precisely, if $a \geq 1$, let v denote an element of the maximal ideal of $W_a(\mathcal{O}_{\mathbf{C}}^\flat)$ whose image in $\mathcal{O}_{\mathbf{C}}^\flat$ is non-zero. We then set

$$W_a(\mathcal{O}_{\mathbf{C}}^\flat)_A = W_a(\mathcal{O}_{\mathbf{C}}^\flat) \widehat{\otimes}_{\mathbf{Z}_p} A := \varprojlim_i \big(W_a(\mathcal{O}_{\mathbf{C}}^\flat) \otimes_{\mathbf{Z}_p} A\big)/v^i$$

(so that the indicated completion is the v-adic completion). Note that any two choices of v induce the same topology on $W_a(\mathcal{O}_{\mathbf{C}}^\flat) \otimes_{\mathbf{Z}_p} A$, so that $W_a(\mathcal{O}_{\mathbf{C}}^\flat) \widehat{\otimes}_{\mathbf{Z}_p} A$ is well-defined independent of the choice of v. We then define

$$W_a(\mathbf{C}^\flat)_A = W_a(\mathbf{C}^\flat) \widehat{\otimes}_{\mathbf{Z}_p} A := W_a(\mathcal{O}_{\mathbf{C}}^\flat)_A[1/v];$$

this ring is again well-defined independently of the choice of v. We set

$$W(\mathcal{O}_{\mathbf{C}}^\flat)_A = W(\mathcal{O}_{\mathbf{C}}^\flat) \widehat{\otimes}_{\mathbf{Z}_p} A := \varprojlim_a W_a(\mathcal{O}_{\mathbf{C}}^\flat)_A,$$

and similarly

$$W(\mathbf{C}^\flat)_A = W(\mathbf{C}^\flat) \widehat{\otimes}_{\mathbf{Z}_p} A := \varprojlim_a W_a(\mathbf{C}^\flat)_A.$$

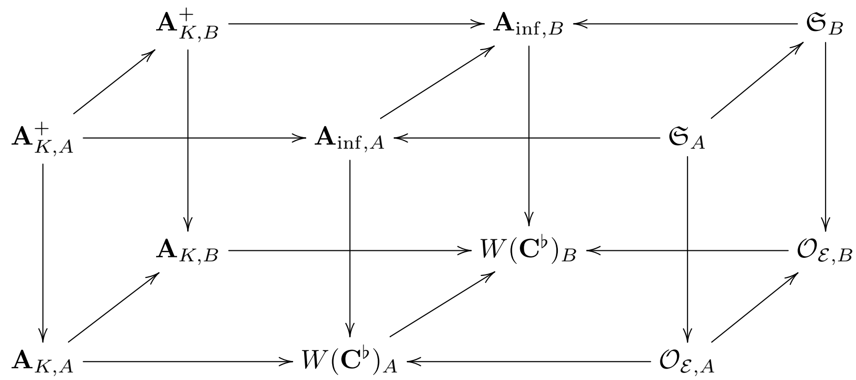
In keeping with our notation above, we will usually write

$$\mathbf{A}_{\mathrm{inf},A} := W(\mathcal{O}_{\mathbf{C}}^\flat)_A.$$

We define the imperfect coefficient rings in the Kummer and cyclotomic cases in the same way, but we can be more explicit: we have $\mathfrak{S}_A = (W(k) \otimes_{\mathbf{Z}_p} A)[[u]]$, and $\mathcal{O}_{\mathcal{E},A}$ is the p-adic completion of $\mathfrak{S}_A[1/u]$, and similarly for $\mathbf{A}_{K,A}^+$ and $\mathbf{A}_{K,A}$.

1.2.2. *A digression on flatness and completion.* Since $\mathbf{A}_{\mathrm{inf}}$ is not Noetherian, we have to be a bit careful when taking completions, and for example it does not seem to be obvious that the natural map $\mathfrak{S}_A \to \mathbf{A}_{\mathrm{inf},A}$ is injective. The following is [EG19a, Prop. 2.2.11], and is proved using [FGK11, §5] and [BMS19, Rem. 4.31].

1.2.3. **Proposition.** *Suppose that $A \to B$ is a flat homomorphism of finite type $\mathbf{Z}/p^a\mathbf{Z}$-algebras for some $a \geq 1$. Then all the maps in the following diagram are flat. Furthermore the vertical arrows are all injections, while the horizontal arrows are all faithfully flat (and so in particular also injections). If $A \to B$ is furthermore faithfully flat, then the same is true of the diagonal arrows.*



It is straightforward to show that φ extends to a continuous endomorphism of $\mathbf{A}_{\mathrm{inf},A}$, $W(\mathcal{O}_C^\flat)_A$, and so on; and the natural action of G_K on these rings is continuous, as is the action of Γ on $\mathbf{A}^+_{K,A}$ and $\mathbf{A}_{K,A}$.

1.3. **φ-modules, Breuil–Kisin(–Fargues) modules, (φ,Γ)-modules, and Galois representations.** Let R be a $\mathbf{Z}_p$-algebra, equipped with a ring endomorphism φ, which is congruent to the (p-power) Frobenius modulo p. If M is an R-module, we write

$$\varphi^*M := R \otimes_{R,\varphi} M.$$

1.3.1. **Definition.** An *étale φ-module over R* is a finite R-module M, equipped with a φ-semi-linear endomorphism $\varphi_M : M \to M$, which has the property that the induced R-linear morphism

$$\Phi_M : \varphi^*M \xrightarrow{1\otimes\varphi_M} M$$

is an isomorphism. A morphism of étale φ-modules is a morphism of the underlying R-modules which commutes with the morphisms Φ_M. We say that M is *projective* (resp. *free*) if it is projective of constant rank (resp. free of constant rank) as an R-module.

We apply this in particular with $R = \mathbf{A}_{K,A}$ or $\mathcal{O}_{\mathcal{E},A}$. In the former case, an *étale (φ,Γ)-module* is an étale φ-module over $\mathbf{A}_{K,A}$, equipped with a commuting continuous semilinear action of Γ. In both cases there is a relationship with Galois representations as follows; the case without coefficients is due to Fontaine (and was the motivation for introducing (φ,Γ)-modules in the first place), and the version with coefficients is due to Dee [Dee01].

Let $\widehat{\mathbf{A}}^{\mathrm{ur}}_K$ denote the p-adic completion of the ring of integers of the maximal unramified extension of $\mathbf{A}_K[1/p]$ in $W(\mathbf{C}^\flat)[1/p]$; this is preserved by the natural actions of φ and G_K on $W(\mathbf{C}^\flat)[1/p]$. Define $\widehat{\mathbf{A}}^{\mathrm{ur}}_{K,A}$ as usual. Then if A is an Artinian $\mathbf{Z}_p$-algebra, we have an equivalence of categories between the category of finite projective étale (φ,Γ)-modules M with A-coefficients, and the category of finite free A-modules V with a continuous action of G_K, given by the functors

$$V \mapsto (\widehat{\mathbf{A}}^{\mathrm{ur}}_{K,A} \otimes_A V)^{G_{K_{\mathrm{cyc}}}},$$

$$M \mapsto (\widehat{\mathbf{A}}^{\mathrm{ur}}_{K,A} \otimes_{\mathbf{A}_{K,A}} M)^{\varphi=1}.$$

Taking limits (and forming $\widehat{\mathbf{A}}^{\mathrm{ur}}_{K,A}$ with respect to the $\mathfrak{m}_A$-adic topology), we can extend this to the case that A is complete local Noetherian, or $A = \overline{\mathbf{F}}_p$ or $\overline{\mathbf{Z}}_p$; this is important later on for identifying the versal rings of our moduli stacks with Galois deformation rings. Note though that for non-formal A it is not the case that there is an equivalence between (φ, Γ)-modules and Galois representations (*cf.* the example of a Fontaine–Laffaille family from Matt's first lecture).

There is also a version of this in the Kummer setting: if $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ is the p-adic completion of the ring of integers in the maximal unramified extension of $\mathrm{Frac}(\mathcal{O}_{\mathcal{E}})$ in $W(\mathbf{C}^\flat)[1/p]$, then for A as above, we have an equivalence of categories between étale φ-modules over $\mathcal{O}_{\mathcal{E},A}$ and G_{K_∞}-representations, given by

$$V \mapsto (\widehat{\mathcal{O}}_{\widehat{\mathcal{E}^{\mathrm{ur}}},A} \otimes_A V)^{G_{K_\infty}},$$

$$M \mapsto (\widehat{\mathcal{O}}_{\widehat{\mathcal{E}^{\mathrm{ur}}},A} \otimes_{\mathcal{O}_{\mathcal{E},A}} M)^{\varphi=1}.$$

In the next few lectures we will define moduli stacks of étale φ-modules and étale (φ, Γ)-modules. One of the key tools in proving the basic properties of these stacks is to study the corresponding stacks of *finite height* φ-modules over $\mathbf{A}^+_{K,A}$ or $\mathfrak{S}_A$. We'll do this more generally in the next lecture, but for now we'll just have the following definition, which is important in the crystalline and semistable theory. (Actually, because $K/\mathbf{Q}_p$ is assumed unramified, the finite height modules over $\mathbf{A}^+_{K,A}$ have an intrinsic utility for describing crystalline representations in terms of *Wach modules*. However, this doesn't extend to general $K/\mathbf{Q}_p$, or to the semistable case, so we will not make any use of this property of Wach modules in these lectures.)

1.3.2. **Definition.** Let $E(u)$ be the minimal polynomial over $W(k)$ of our fixed uniformiser π of K. We define a (projective) Breuil–Kisin module (resp. a Breuil–Kisin–Fargues module) of height at most h with A-coefficients to be a finitely generated projective $\mathfrak{S}_A$-module (resp. $\mathbf{A}_{\mathrm{inf},A}$-module) $\mathfrak{M}$, equipped with a φ-semi-linear morphism $\varphi : \mathfrak{M} \to \mathfrak{M}$, with the property that the corresponding morphism $\Phi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is injective, with cokernel killed by $E(u)^h$. If $\mathfrak{M}$ is a Breuil–Kisin module then $\mathbf{A}_{\mathrm{inf},A} \otimes_{\mathfrak{S}_A} \mathfrak{M}$ is a Breuil–Kisin–Fargues module.

Note that these are "effective", i.e. φ-stable. This means that we will end up only considering non-negative Hodge–Tate weights; the general case follows by twisting by a sufficiently large power of $E(u)$, so this is harmless.

The connection between Breuil–Kisin modules and crystalline/semistable Galois representations will be reviewed this afternoon, and we will also come back to it later in the week.

1.4. **Almost Galois descent.** Let F be a closed perfectoid subfield of $\mathbf{C}$. The following is [EG19a, Thm. 2.4.1], which is proved by a somewhat elaborate argument making use of almost Galois descent.

1.4.1. **Theorem.** *Let A be a finite type $\mathbf{Z}/p^a$-algebra, for some $a \geq 1$. The inclusion $W(F^\flat)_A \to W(\mathbf{C}^\flat)_A$ is a faithfully flat morphism of Noetherian rings, and the functor $M \mapsto W(\mathbf{C}^\flat)_A \otimes_{W(F^\flat)_A} M$ induces an equivalence between the category of finitely generated projective $W(F^\flat)_A$-modules and the category of finitely generated projective $W(\mathbf{C}^\flat)_A$-modules endowed with a continuous semi-linear G_F-action. A quasi-inverse functor is given by $N \mapsto N^{G_F}$.*

We will find this result very useful later on, because it lets us combine the Kummer and cyclotomic settings. The key extra ingredient (which also takes some work) is that when we have the additional structure of a φ-module, we can descend from the perfect coefficient ring $W(F^\flat)_A$ (with $F = \widehat{K}_{\mathrm{cyc}}$ or $F = \widehat{K}_\infty$) to the imperfect coefficient rings $\mathbf{A}_{K,A}$ and $\mathcal{O}_{\mathcal{E},A}$ respectively.

1.4.2. **Definition.** Let A be a p-adically complete $\mathbf{Z}_p$-algebra. An *étale* (G_K, φ)*-module with* A*-coefficients* (resp. an *étale* $(G_{K_{\mathrm{cyc}}}, \varphi)$*-module with* A*-coefficients*, resp. an *étale* (G_{K_∞}, φ)*-module with* A*-coefficients*) is by definition a finitely generated $W(\mathbf{C}^\flat)_A$-module M equipped with an isomorphism of $W(\mathbf{C}^\flat)_A$-modules

$$\Phi_M : \varphi^* M \xrightarrow{\sim} M,$$

and a $W(\mathbf{C}^\flat)_A$-semi-linear action of G_K (resp. $G_{K_{\mathrm{cyc}}}$, resp. G_{K_∞}), which is continuous and commutes with Φ_M. We say that M is projective if it is projective of constant rank as a $W(\mathbf{C}^\flat)_A$-module.

The following is [EG19a, Prop. 2.7.4].

1.4.3. **Proposition.** *Let A be a finite type $\mathbf{Z}/p^a$-algebra for some $a \geq 1$.*

(1) The functor $M \mapsto W(\mathbf{C}^\flat)_A \otimes_{\mathbf{A}_{K,A}} M$ is an equivalence between the category of finite projective étale φ-modules over $\mathbf{A}_{K,A}$ and the category of finite projective étale $(G_{K_{\mathrm{cyc}}}, \varphi)$-modules with A-coefficients.
   It induces an equivalence of categories between the category of finite projective étale (φ, Γ_K)-modules with A-coefficients and the category of finite projective étale (G_K, φ)-modules with A-coefficients.

(2) The functor $M \mapsto W(\mathbf{C}^\flat)_A \otimes_{\mathcal{O}_{\mathcal{E},A}} M$ is an equivalence of categories between the category of finite projective étale φ-modules over $\mathcal{O}_{\mathcal{E},A}$ and the category of finite projective étale (G_{K_∞}, φ)-modules with A-coefficients.

2. Moduli stacks of φ-modules

2.1. **Results of Pappas–Rapoport.** We begin our discussion of stacks with φ-modules; the more complicated case of (φ, Γ)-modules will be built on this. These stacks were first studied by Pappas–Rapoport [PR09] in the context of Breuil–Kisin modules; they showed (among other things) that there are algebraic stacks of Breuil–Kisin modules, and that the morphism to the stacks of étale φ-modules are well-behaved. These results were built upon in [EG19b, §5], which showed that the stack of étale φ-modules is itself well-behaved, and in particular is Ind-algebraic.

We begin by recalling the results of Pappas–Rapoport (in a slightly more general context, where φ is not necessarily given by $\varphi(u) = u^p$; this makes some arguments messier but doesn't change any of the key points).

2.1.1. **Situation.** Fix a finite extension $k/\mathbf{F}_p$ and write $\mathbf{A}^+ := W(k)[[T]]$. Write $\mathbf{A}$ for the p-adic completion of $\mathbf{A}^+[1/T]$.

If A is a p-adically complete $\mathbf{Z}_p$-algebra, we write $\mathbf{A}^+_A := (W(k) \otimes_{\mathbf{Z}_p} A)[[T]]$; we equip $\mathbf{A}^+_A$ with its (p, T)-adic topology, so that it is a topological A-algebra (where A has the p-adic topology). Let $\mathbf{A}_A$ be the p-adic completion of $\mathbf{A}^+_A[1/T]$, which we regard as a topological A-algebra by declaring $\mathbf{A}^+_A$ to be an open subalgebra. Note that the formation of $\mathfrak{S}_A$, $\mathcal{O}_{\mathcal{E},A}$, $\mathbf{A}^+_{K,A}$ and $\mathbf{A}_{K,A}$ above are particular instances of this construction.

Let φ be a ring endomorphism of $\mathbf{A}$ which is congruent to the (p-power) Frobenius endomorphism modulo p, and satisfies $\varphi(\mathbf{A}^+) \subseteq \mathbf{A}^+$.

By [EG19b, Lem. 5.2.2 and 5.2.5] and [EG19a, Lem. 3.2.5, 3.2.6], φ is faithfully flat, and induces the usual Frobenius on $W(k)$, and it extends uniquely to an A-linear continuous endomorphism of $\mathbf{A}_A^+$ and $\mathbf{A}_A$.

Fix a polynomial $F \in W(k)[T]$ which is congruent to a positive power of T modulo p. (For example, if we are working in the Breuil–Kisin setting, we will take $T = u$, $\varphi(u) = u^p$, and $F = E(u)$.)

In order to know that the various categories in groupoids that we will define are actually stacks, we use the following result of Drinfeld [Dri06, Thm. 3.11] (see [EG19b, Thm. 5.1.18] for the precise statements given here).

2.1.2. **Theorem.** *The following notions are local for the fpqc topology on* $\operatorname{Spec} A$*.*

(1) A finitely generated projective $\mathbf{A}_A$*-module.*
(2) A projective $\mathbf{A}_A$*-module of rank* d*.*
(3) A finitely generated projective $\mathbf{A}_A$*-module which is fpqc locally free of rank* d*.*
(4) A finitely generated projective $\mathbf{A}_A^+$*-module.*
(5) A projective $\mathbf{A}_A^+$*-module of rank* d*.*
(6) A finitely generated projective $\mathbf{A}_A^+$*-module which is fpqc locally free of rank* d*.*

2.1.3. *Remark.* More precisely, saying that the notion of a finitely generated projective $\mathbf{A}_A$-module is local for the *fpqc* topology on $\operatorname{Spec} A$ means the following (and the meanings of the other statements in Theorem 2.1.2 are entirely analogous):

If A' is any faithfully flat A-algebra, set $A'' := A' \otimes_A A'$. Then the category of finitely generated projective $\mathbf{A}_A$-modules is canonically equivalent to the category of finitely generated projective $\mathbf{A}_{A'}$-modules M' which are equipped with an isomorphism

$$M' \otimes_{\mathbf{A}_{A'}, a \mapsto 1 \otimes a} \mathbf{A}_{A''} \xrightarrow{\sim} M' \otimes_{\mathbf{A}_{A'}, a \mapsto a \otimes 1} \mathbf{A}_{A''}$$

which satisfies the usual cocycle condition.

If we fix integers $a, d \geq 1$, then by Theorem 2.1.2 we may define an *fpqc* stack in groupoids $\mathcal{R}_d^a$ over $\operatorname{Spec} \mathbf{Z}/p^a$ as follows: For any $\mathbf{Z}/p^a$-algebra A, we define $\mathcal{R}_d^a(A)$ to be the groupoid of étale φ-modules M over $\mathbf{A}_A$ which are projective of rank d. There is a closely related version of this considered in [PR09], namely $\mathcal{R}_{d,\mathrm{free}}^a$, where we demand that M is furthermore *fpqc*-locally free. (In fact, we don't know whether or not this is a consequence of M being projective, so we don't know whether $\mathcal{R}_{d,\mathrm{free}}^a = \mathcal{R}_d^a$.) In either case, if $A \to B$ is a morphism of $\mathbf{Z}_p$-algebras, and M is an object of $\mathcal{R}_d^a(A)$, then the pull-back of M to $\mathcal{R}_d^a(B)$ is defined to be the tensor product $\mathbf{A}_B \otimes_{\mathbf{A}_A} M$.

From now on we regard $\mathcal{R}_d^a$ as an *fppf* stack over $\mathbf{Z}/p^a$. By [Sta, Tag 04WV], we may also regard the stack $\mathcal{R}_d^a$ as an *fppf* stack over $\mathbf{Z}_p$, and as a varies, we may form the 2-colimit $\mathcal{R} := \varinjlim_a \mathcal{R}_d^a$, which is again an *fppf* stack over $\mathbf{Z}_p$.

The following definition generalises that of a Breuil–Kisin module.

2.1.4. **Definition.** Let h be a non-negative integer. A *φ-module of F-height at most h over* $\mathbf{A}_A^+$ is a pair $(\mathfrak{M}, \varphi_M)$ consisting of a finitely generated T-torsion free $\mathbf{A}_A^+$-module $\mathfrak{M}$, and a φ-semi-linear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$, with the further properties that if we write

$$\Phi_{\mathfrak{M}} := 1 \otimes \varphi_{\mathfrak{M}} : \varphi^* \mathfrak{M} \to \mathfrak{M},$$

then $\Phi_{\mathfrak{M}}$ is injective, and the cokernel of $\Phi_{\mathfrak{M}}$ is killed by F^h.

A *φ-module of finite F-height over $\mathbf{A}_A^+$* is a φ-module of F-height at most h for some $h \geq 0$. A morphism of φ-modules is a morphism of the underlying $\mathbf{A}_A^+$-modules which commutes with the morphisms $\Phi_{\mathfrak{M}}$.

We say that a φ-module of finite F-height is projective of rank d if it is a finitely generated projective $\mathbf{A}_A^+$-module of constant rank d.

If we fix integers $a, d \geq 1$ and an integer $h \geq 0$, then we may define an *fpqc* stack in groupoids $\mathcal{C}_{d,h}^a$ over $\operatorname{Spec} \mathbf{Z}/p^a$ as follows: For any $\mathbf{Z}/p^a$-algebra A, we define $\mathcal{C}_{d,h}^a(A)$ to be the groupoid of φ-modules of F-height at most h over $\mathbf{A}_A^+$ which are projective of rank d.

Just as for the stack $\mathcal{R}_d^a$, we may and do also regard the stack $\mathcal{C}_{d,h}^a$ as an *fppf* stack over $\mathbf{Z}_p$, and we then, allowing a to vary, we define $\mathcal{C}_{d,h} := \varinjlim_a \mathcal{C}_{d,h}^a$, obtaining an *fppf* stack over $\operatorname{Spf} \mathbf{Z}_p$. There are canonical morphisms $\mathcal{C}_{d,h}^a \to \mathcal{R}_d^a$ and $\mathcal{C}_{d,h} \to \mathcal{R}_d$ given by tensoring with $\mathbf{A}_A$ over $\mathbf{A}_A^+$. One can show that any projective $\mathbf{A}_A^+$-module is even Zariski locally free, so these morphisms factor through the Pappas–Rapoport versions $\mathcal{R}_{d,\mathrm{free}}^a$ (resp. $\mathcal{R}_{d,\mathrm{free}}$). Reassuringly, we have the following lemma [EG19a, Lem. 3.1.3].

2.1.5. **Lemma.** *If A is a p-adically complete $\mathbf{Z}_p$-algebra, then there is a canonical equivalence between the groupoid of morphisms $\operatorname{Spf} A \to \mathcal{R}_d$ and the groupoid of rank d étale φ-modules over $\mathbf{A}_A$; and there is a canonical equivalence between the groupoid of morphisms $\operatorname{Spf} A \to \mathcal{C}_{d,h}$ and the groupoid of φ-modules of rank d and F-height at most h over $\mathbf{A}_A^+$.*

The following is essentially [PR09, Thm. 2.1 (a), Cor. 2.6].

2.1.6. **Theorem.**

(1) *The stack $\mathcal{C}_{d,h}^a$ is an algebraic stack of finite presentation over $\operatorname{Spec} \mathbf{Z}/p^a$, with affine diagonal.*
(2) *The morphism $\mathcal{C}_{d,h}^a \to \mathcal{R}_{d,\mathrm{free}}^a$ is representable by algebraic spaces, proper, and of finite presentation.*
(3) *The diagonal morphism $\Delta : \mathcal{R}_{d,\mathrm{free}}^a \to \mathcal{R}_{d,\mathrm{free}}^a \times_{\mathbf{Z}/p^a} \mathcal{R}_{d,\mathrm{free}}^a$ is representable by algebraic spaces, affine, and of finite presentation.*

We very briefly indicate the main ideas of the proof. One of the key points is the following [PR09, Prop. 2.2] (see [EG19b, Lem. 5.2.9] for this version). Assume that A is a $\mathbf{Z}/p^a\mathbf{Z}$-algebra. For $n \geq 0$, write

$$U_n = 1 + T^n M_d(\mathbf{A}_A^+),$$
$$V_n = \{A \in \mathrm{GL}_d(\mathbf{A}_A) \mid A, A^{-1} \in T^{-n} M_d(\mathbf{A}_A^+)\}.$$

2.1.7. **Lemma.** *For each $m \geq 0$ there is an $n(m) \geq 0$ (implicitly depending also on a) such that if $n \geq n(m)$, then:*

(1) *For each $g \in U_n$, $B \in V_m$, there is a unique $h \in U_n$ such that $g^{-1}B\varphi(g) = h^{-1}B$.*
(2) *For each $h \in U_n$, $B \in V_m$ there is a unique $g \in U_n$ such that $g^{-1}B\varphi(g) = h^{-1}B$.*

In each case the uniqueness statement is easy, and the existence is proved by a (slightly tricky) successive approximation argument. Given the lemma, the first part of Theorem 2.1.6 follows from standard results about the affine Grassmannian:

the point is that working locally, the data of a finite height $\mathbf{A}_A^+$-module is the data of the matrix in $M_d(\mathbf{A}_A^+)$ of the corresponding linearized φ, and the finite height condition implies that this is in V_n for some n depending on a and h. Changes of basis correspond to φ-conjugacy, and Lemma 2.1.7 lets us replace φ-conjugacy by left multiplication provided we work in a sufficiently deep congruence subgroup, and this is enough to get the result.

The rest of Theorem 2.1.6 is also proved by reducing to explicit statements about matrices over $\mathbf{A}_A$. The key point in each case is to obtain a bound on the T-adic poles in some matrix entries, and in the bound on the height h gives a bound on how different the denominators of g and $\varphi(g)$ can be (for some matrix g), which can be played off against the fact that the poles of $\varphi(g)$ have approximately p times the order of the poles of g.

We then deduce that Theorem 2.1.6 holds with $\mathcal{R}^a_{d,\mathrm{free}}$ replaced by $\mathcal{R}^a_d$; roughly speaking the idea is to regard a projective $\mathbf{A}_A$-module as a pair consisting of an *fpqc*-locally free projective $\mathbf{A}_A$-module of higher rank, together with an idempotent. We also make use of the following technical statement [EG19b, Lem. 5.1.23]: if M is a finitely generated projective $A((u))$-module, then there exists an $n_0 \geq 1$ such that for all $n \geq n_0$, M is Nisnevich (and in particular *fpqc*) locally free as an $A((u^n))$-module. In particular we find it useful to regard a projective $A((u))$-module as a locally free $A((u^n))$-module together with an action of u.

2.2. **Scheme-theoretic images.** The main result of [EG19b] is the following [EG19b, Thm. 5.4.20].

2.2.1. **Theorem.** *$\mathcal{R}^a_d$ is a limit preserving Ind-algebraic stack, whose diagonal is representable by algebraic spaces, affine, and of finite presentation.*

The key point is of course the Ind-algebraicity. In fact we show that $\mathcal{R}^a_d = \varinjlim_h \mathcal{R}^a_{d,h}$, where $\mathcal{R}^a_{d,h}$ is the scheme-theoretic image (in the sense explained in Matt's 2nd lecture) of the morphism $\mathcal{C}^a_{d,h} \to \mathcal{R}^a_d$. It is at least intuitively reasonable that we have $\mathcal{R}^a_d = \varinjlim_h \mathcal{R}^a_{d,h}$ (this says that étale locally, every étale φ-module comes from a finite height φ-module, which in the free case is immediate by just choosing a basis and scaling it by a sufficiently large power of F), and the hard part is to show that $\mathcal{R}^a_{d,h}$ is actually an algebraic stack.

We do this using the criterion presented in Matt's lectures. The key point is to show that $\mathcal{R}^a_d$ admits versal rings at all finite type points, and that $\mathcal{R}^a_{d,h}$ admits *effective* Noetherian versal rings. Apart from the effectivity, this is at least morally quite straightforward: the versal rings for $\mathcal{R}^a_d$ correspond to unrestricted Galois deformation rings, and those for $\mathcal{R}^a_{d,h}$ are closely related to their finite height analogues. In particular, if we are in the Kummer setting and we take $h = 1$, then $\mathcal{R}^a_d$ really admits the unrestricted deformation rings for G_{K_∞} as versal rings, and $\mathcal{R}^a_{d,1}$ admits a version of the height 1 deformation rings, which can be identified with finite flat deformation rings for G_K (which are in particular Noetherian).

Making this precise involves some work, in part because we don't have a simple description of the A-valued points of $\mathcal{R}^a_{d,h}$ unless A is a field. The real work is in proving the effectivity, i.e. that the versal morphism $\operatorname{Spf} R \to \mathcal{R}^a_{d,h}$ can be upgraded to a morphism $\operatorname{Spec} R \to \mathcal{R}^a_{d,h}$. The key point here is to check that the composite morphism $\operatorname{Spf} R \to \mathcal{R}^a_d$ comes from a morphism $\operatorname{Spec} R \to \mathcal{R}^a_d$; it is then relatively straightforward to check that this morphism factors through $\mathcal{R}^a_{d,h}$, using arguments which are very similar to those that we will give in Section 3.3. Slightly more explicitly, having a morphism $\mathrm{Spf}\, R \to \mathcal{R}_d^a$ means that we have an étale φ-module over the $\mathfrak{m}_R$-adic completion of $\mathbf{A}_R$ (i.e. where we are allowed to have infinite Laurent tails, as long as they tend to zero $\mathfrak{m}_R$-adically), and we need to show that this arises from an étale φ-module over $\mathbf{A}_R$ itself.

To show this effectivity, we show that the versal ring R to $\mathcal{R}_{d,h}^a$ is a quotient of a ring R^h, which in turn is the quotient of the versal ring to $\mathcal{R}_{d,h}^a$ which is universal for the property that for each Artinian quotient A of R^h, with corresponding étale φ-module M_A, there is a (not necessarily projective) $\mathbf{A}_A^+$-module $\mathfrak{M}_A$ of F-height $\le h$ with $M_A = \mathfrak{M}_A[1/T]$. This is hopefully at least plausible; the actual argument is a little involved, as the versal ring $\mathrm{Spf}\, R$ is defined as the scheme-theoretic image of the pullback of $\mathcal{C}_{d,h}^a \to \mathcal{R}_d$ to $\mathrm{Spf}\, R^{\mathrm{univ}}$, where R^{univ} is a versal ring to $\mathcal{R}_d$, which is a deformation ring for an étale φ-module.

If we let A run over the Artinian quotients of R^h, then by a Mittag–Leffler argument we can choose the $\mathfrak{M}_A$ compatibly, and obtain a (not necessarily projective) $\mathbf{A}_{R^h}^+$-module $\mathfrak{M}_{R^h}$ of F-height $\le h$. We then set $M_{R^h} = \mathfrak{M}_{R^h}[1/T]$, and apply the following theorem ([EG19b, Thm. 5.3.22]) with $R = R^h$ and $M = M_{R^h}$.

2.2.2. **Theorem.** *Let R be a complete Noetherian local $\mathbf{Z}/p^a$-algebra with maximal ideal $\mathfrak{m}$, let M be an étale φ-module over $\mathbf{A}_R$, and suppose that the $\mathfrak{m}$-adic completion $\widehat{M}$ is projective, or equivalently, free (over the $\mathfrak{m}$-adic completion of $\mathbf{A}_R$). Then M itself is projective (over $\mathbf{A}_R$).*

The projective module M_{R^h} gives (by definition) a morphism $\mathrm{Spec}\, R^h \to \mathcal{R}_d^a$, and thus a composite morphism $\mathrm{Spec}\, R \to \mathrm{Spec}\, R^h \to \mathcal{R}_d^a$, as required.

Finally, passing to the limit over $a \ge 1$ and using some results of [Eme], we obtain the following.

2.2.3. **Corollary.**

(1) *$\mathcal{C}_{d,h}$ is a p-adic formal algebraic stack of finite presentation over* $\mathrm{Spf}\, \mathbf{Z}_p$, *with affine diagonal.*
(2) *$\mathcal{R}_d$ is a limit preserving Ind-algebraic stack, whose diagonal is representable by algebraic spaces, affine, and of finite presentation.*
(3) *The morphism $\mathcal{C}_{d,h} \to \mathcal{R}_d$ is representable by algebraic spaces, proper, and of finite presentation.*
(4) *The diagonal morphism $\Delta : \mathcal{R}_d \to \mathcal{R}_d \times_{\mathrm{Spf}\, \mathbf{Z}_p} \mathcal{R}_d$ is representable by algebraic spaces, affine, and of finite presentation.*

3. Stacks of (φ, Γ)-modules

3.1. **Moduli stacks of (φ, Γ)-modules.** Recall that for simplicity we are always assuming that $K/\mathbf{Q}_p$ is unramified when we discuss (φ, Γ)-modules.

3.1.1. **Definition.** We let $\mathcal{X}_{K,d}$ denote the moduli stack of projective étale (φ, Γ_K)-modules of rank d. More precisely, if A is a p-adically complete $\mathbf{Z}_p$-algebra, then we define $\mathcal{X}_{K,d}(A)$ (i.e., the groupoid of morphisms $\mathrm{Spf}\, A \to \mathcal{X}_{K,d}$) to be the groupoid of projective étale (φ, Γ_K)-modules of rank d with A-coefficients, with morphisms given by isomorphisms. If $A \to B$ is a morphism of complete $\mathbf{Z}_p$-algebras, and M is an object of $\mathcal{X}_{K,d}(A)$, then the pull-back of M to $\mathcal{X}_{K,d}(B)$ is defined to be the tensor product $\mathbf{A}_{K,B} \otimes_{\mathbf{A}_{K,A}} M$. It again follows from Theorem 1.4.1 that this is indeed a stack.

Note that by the equivalence between (φ, Γ_K)-modules and G_K-representations over Artinian rings, the $\overline{\mathbf{F}}_p$-points of $\mathcal{X}_{K,d}$ are in bijection with representations $\overline{\rho} : G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$. More generally, fix a point $\operatorname{Spec} \mathbf{F} \to \mathcal{X}_{K,d}$ for some finite field $\mathbf{F}$, giving rise to a continuous representation $\overline{\rho} : G_K \to \mathrm{GL}_d(\mathbf{F})$, and let $R^{\square}_{\overline{\rho}}$ denote the universal framed deformation $W(\mathbf{F})$-algebra for lifts of $\overline{\rho}$. Then it is easy to check that the natural morphism $\operatorname{Spf} R^{\square}_{\overline{\rho}} \to \mathcal{X}_{K,d}$ is versal.

One of the main results that we will prove is that $\mathcal{X}_{K,d}$ is a Noetherian formal algebraic stack. However, the proof of this is quite involved, and in this lecture we establish the preliminary result that $\mathcal{X}_{K,d}$ is an Ind-algebraic stack, which we deduce from the results that we have proved for stacks of étale φ-modules.

3.1.2. **Definition.** We let $\mathcal{R}_{K,d}$ (frequently abbreviated to $\mathcal{R}_d$) denote the moduli stack of rank d projective étale φ-modules, taking $\mathbf{A}$ to be $\mathbf{A}_K$.

We now turn to studying Γ-actions on our φ-modules. We choose a topological generator γ of Γ, and let $\Gamma_{\mathrm{disc}} := \langle \gamma \rangle$; so $\Gamma_{\mathrm{disc}} \cong \mathbf{Z}$. Note that since Γ_{disc} is dense in Γ, in order to endow M with the structure of an étale (φ, Γ)-module, it suffices to equip M with a continuous action of Γ_{disc} (where we equip Γ_{disc} with the topology induced on it by Γ).

There is a canonical action of Γ_{disc} on $\mathcal{R}_d$ (that is, a canonical morphism $\gamma : \mathcal{R}_d \to \mathcal{R}_d$): if M is an object of $\mathcal{R}_d(A)$, then $\gamma(M)$ is given by $\gamma^* M := \mathbf{A}_{K,A} \otimes_{\gamma, \mathbf{A}_{K,A}} M$. Then we set

$$\mathcal{R}_d^{\Gamma_{\mathrm{disc}}} := \mathcal{R}_d \underset{\Delta, \mathcal{R}_d \times \mathcal{R}_d, \Gamma_\gamma}{\times} \mathcal{R}_d,$$

where Δ is the diagonal of $\mathcal{R}_d$ and Γ_γ is the graph of γ, so that $\Gamma_\gamma(x) = (x, \gamma(x))$.

We claim that $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ is nothing other than the moduli stack of projective étale φ-modules of rank d equipped with a semi-linear action of Γ_{disc}. This is an exercise in unwinding the usual construction of the 2-fibre product: $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ consists of tuples (x, y, α, β), with x, y being objects of $\mathcal{R}_d$, and $\alpha : x \xrightarrow{\sim} y$ and $\beta : \gamma(x) \xrightarrow{\sim} y$ being isomorphisms. This is equivalent to the category fibred in groupoids given by pairs (x, ι) consisting of an object x of $\mathcal{R}_d$ and an isomorphism $\iota : \gamma(x) \xrightarrow{\sim} x$. Thus an object of $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}(A)$ is a projective étale φ-module of rank d with A-coefficients M, together with an isomorphism of φ-modules $\iota : \gamma^* M \xrightarrow{\sim} M$; and this isomorphism is precisely the data of a semi-linear action of $\Gamma_{\mathrm{disc}} = \langle \gamma \rangle$ on M, as required.

Since $\mathcal{R}_d$ is an Ind-algebraic stack, so is $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$; indeed it follows from Corollary 2.2.3 that $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ is a limit preserving Ind-algebraic stack, whose diagonal is representable by algebraic spaces, affine, and of finite presentation.

Restricting the Γ-action on an étale (φ, Γ)-module to Γ_{disc}, we obtain a morphism $\mathcal{X}_{K,d} \to \mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$, which is fully faithful. Thus $\mathcal{X}_{K,d}$ may be regarded as a substack of $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$; in particular, we deduce that its diagonal is representable by algebraic spaces, affine, and of finite presentation. Although $\mathcal{X}_{K,d}$ is a substack of the Ind-algebraic stack $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$, it is not a closed substack, but should rather be thought of as a certain formal completion (as in the discussion of the rank one case in Matt's first lecture); in particular, since it is not a closed substack, we cannot immediately deduce the Ind-algebraicity of $\mathcal{X}_{K,d}$ from the Ind-algebraicity of $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$. Instead, we will argue as in the proof that $\mathcal{R}_d$ is Ind-algebraic, and exhibit $\mathcal{X}_{K,d}$ as the scheme-theoretic image of an Ind-algebraic stack.

From now on we will typically drop K from the notation, simply writing $\mathcal{X}_d$, $\mathcal{R}_d$ and so on. To go further we need to understand the continuity condition in the definition of $\mathcal{X}_d$ more carefully. It is easy to check that we have $\gamma(T) - T \in (p, T)T\mathbf{A}_A^+$, and using this one can check the following special case of [EG19a, Lem. D.24]:

3.1.3. **Lemma.** *Suppose that A is a $\mathbf{Z}/p^a$-algebra for some $a \geq 1$. Let M be a finite projective $\mathbf{A}_A$-module, equipped with a semi-linear action of Γ_{disc}. Then the following are equivalent:*

1. *The action of Γ_{disc} extends to a continuous action of Γ.*
2. *For any lattice $\mathfrak{M} \subseteq M$, there exists $s \geq 0$ such that $(\gamma^{p^s} - 1)(\mathfrak{M}) \subseteq T\mathfrak{M}$. (Here a* lattice *is a finitely generated $\mathbf{A}_{K,A}^+$-submodule $\mathfrak{M} \subseteq M$ whose $\mathbf{A}_{K,A}$-span is M.)*
3. *For some lattice $\mathfrak{M} \subseteq M$ and some $s \geq 0$, we have $(\gamma^{p^s} - 1)(\mathfrak{M}) \subseteq T\mathfrak{M}$.*
4. *The action of $\gamma - 1$ on $M \otimes_{\mathbf{Z}/p^a} \mathbf{F}_p$ is topologically nilpotent.*

It is easy to use this criterion, together with the fact that $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ is limit preserving, to show that $\mathcal{X}_d$ is limit preserving; see [EG19a, Lem. 3.2.18].

3.2. **Weak Wach modules.** In this section we introduce the notion of a weak Wach module of height at most h and level at most s. These will play a purely technical auxiliary role for us, and will be used only in order to show that $\mathcal{X}_d$ is an Ind-algebraic stack; we won't use their relation to crystalline representations.

By Lemma 3.1.3, if A is a $\mathbf{Z}/p^a$-algebra for some $a \geq 1$, and $\mathfrak{M}$ is a rank d projective φ-module of T-height $\leq h$ over A, such that $\mathfrak{M}[1/T]$ is equipped with a semi-linear action of Γ_{disc}, then this action extends to a continuous action of Γ if and only if for some $s \geq 0$ we have $(\gamma^{p^s} - 1)(\mathfrak{M}) \subseteq T\mathfrak{M}$. This motivates the following definition.

3.2.1. **Definition.** A *rank d projective weak Wach module of T-height $\leq h$ and level $\leq s$* is a rank d projective φ-module $\mathfrak{M}$ over $\mathbf{A}_{K,A}^+$, which is of T-height $\leq h$, such that $\mathfrak{M}[1/T]$ is equipped with a semilinear action of Γ_{disc} which satisfies $(\gamma^{p^s} - 1)(\mathfrak{M}) \subseteq T\mathfrak{M}$.

3.2.2. **Definition.** We let $\mathcal{W}_{d,h}$ denote the moduli stack of rank d projective weak Wach modules of T-height $\leq h$. We let $\mathcal{W}_{d,h,s}$ denote the substack of rank d projective weak Wach modules of T-height $\leq h$ and level $\leq s$.

We will next show that the stacks $\mathcal{W}_{d,h,s}$ are p-adic formal algebraic stacks of finite presentation over Spf $\mathbf{Z}_p$. Since the canonical morphism $\varinjlim_s \mathcal{W}_{d,h,s} \to \mathcal{W}_{d,h}$ is an isomorphism (by definition), this will show in particular that $\mathcal{W}_{d,h}$ is an Ind-algebraic stack; we will also see that the transition maps in this injective limit are closed immersions.

Recall that we have the p-adic formal algebraic stack $\mathcal{C}_{d,h}$ classifying rank d projective φ-modules over $\mathbf{A}_{K,A}^+$ of T-height at most h. We consider the fibre product $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}} \times_{\mathcal{R}_d} \mathcal{C}_{d,h}$, where the map $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}} \to \mathcal{R}_d$ is the canonical morphism given by forgetting the Γ_{disc} action; this is the moduli stack of rank d projective φ-modules $\mathfrak{M}$ over $\mathbf{A}_{K,A}^+$ of T-height at most h, equipped with a semi-linear action of Γ_{disc} on $\mathfrak{M}[1/T]$. It follows from Corollary 2.2.3 that $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}} \times_{\mathcal{R}_d} \mathcal{C}_{d,h}$ is a p-adic formal algebraic stack of finite presentation over Spf $\mathbf{Z}_p$.

Restricting the Γ-action on a weak Wach module to Γ_{disc}, we may regard $\mathcal{W}_{d,h}$ as a substack of $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}} \times_{\mathcal{R}_d} \mathcal{C}_{d,h}$. The following is [EG19a, Prop. 3.3.5].

3.2.3. **Proposition.** *For $s \geq 1$, the morphism*

$$\mathcal{W}_{d,h,s} \longrightarrow \mathcal{R}_d^{\Gamma_{\mathrm{disc}}} \times_{\mathcal{R}_d} \mathcal{C}_{d,h}$$

is a closed immersion of finite presentation. In particular, each of the stacks $\mathcal{W}_{d,h,s}$ is a p-adic formal algebraic stack of finite presentation over $\operatorname{Spf} \mathbf{Z}_p$*; and for each $s' \geq s$, the canonical monomorphism $\mathcal{W}_{d,h,s} \hookrightarrow \mathcal{W}_{d,h,s'}$ is a closed immersion of finite presentation.*

The proof of this is fairly straightforward: by definition, we need to show that the condition that $(\gamma^{p^s} - 1)(\mathfrak{M}) \subseteq T\mathfrak{M}$ is a closed condition, and is determined by finitely many equations. We do this by reducing to the free case and considering the equations on the level of matrices.

3.3. $\mathcal{X}_d$ **is an Ind-algebraic stack.** By definition, we have a 2-Cartesian diagram

$$\begin{array}{ccc} \mathcal{W}_{d,h} & \longrightarrow & \mathcal{R}_d^{\Gamma_{\mathrm{disc}}} \times_{\mathcal{R}_d} \mathcal{C}_{d,h} \\ \downarrow & & \downarrow \\ \mathcal{X}_d & \longrightarrow & \mathcal{R}_d^{\Gamma_{\mathrm{disc}}} \end{array} \tag{3.3.1}$$

If $h' \geq h$ then the closed immersion $\mathcal{C}_{d,h} \hookrightarrow \mathcal{C}_{d,h'}$ is compatible with the morphisms from each of its source and target to $\mathcal{R}_d$, and so we obtain a closed immersion

$$\mathcal{W}_{d,h} \hookrightarrow \mathcal{W}_{d,h'}. \tag{3.3.2}$$

By construction, the morphisms $\mathcal{W}_{d,h} \to \mathcal{X}_d$ are compatible, as h varies, with the closed immersions (3.3.2). Thus we also obtain a morphism

$$\varinjlim_h \mathcal{W}_{d,h} \to \mathcal{X}_d. \tag{3.3.3}$$

Roughly speaking, we will prove that $\mathcal{X}_d$ is an Ind-algebraic stack by showing that it is the "scheme-theoretic image" of the morphism $\varinjlim_h \mathcal{W}_{d,h} \to \mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ induced by (3.3.3). More precisely, choose $s \geq 0$, and consider the composite

$$\mathcal{W}_{d,h,s} \to \mathcal{W}_{d,h} \to \mathcal{X}_d \to \mathcal{R}_d^{\Gamma_{\mathrm{disc}}}. \tag{3.3.4}$$

This admits the alternative factorization

$$\mathcal{W}_{d,h,s} \to \mathcal{W}_{d,h} \to \mathcal{R}_d^{\Gamma_{\mathrm{disc}}} \times_{\mathcal{R}_d} \mathcal{C}_{d,h} \to \mathcal{R}_d^{\Gamma_{\mathrm{disc}}}.$$

Proposition 3.2.3 shows that the composite of the first two arrows is a closed embedding of finite presentation, while Corollary 2.2.3 shows that the third arrow is representable by algebraic spaces, proper, and of finite presentation. Thus (3.3.4) is representable by algebraic spaces, proper, and of finite presentation.

Fix an integer $a \geq 1$, and write $\mathcal{W}_{d,h,s}^a := \mathcal{W}_{d,h,s} \times_{\operatorname{Spf} \mathbf{Z}_p} \operatorname{Spec} \mathbf{Z}/p^a$. Proposition 3.2.3 shows that $\mathcal{W}_{d,h,s}$ is a p-adic formal algebraic stack of finite presentation over $\operatorname{Spf} \mathbf{Z}_p$, and so $\mathcal{W}_{d,h,s}^a$ is an algebraic stack, and a closed substack of $\mathcal{W}_{d,h,s}$.

Note that since at this point we don't know that $\mathcal{X}_d$ is Ind-algebraic, we can't directly define a scheme-theoretic image of $\mathcal{W}_{d,h,s}^a$ in $\mathcal{X}_d$. It might be possible to do this using the formalism of [EG19b]; we take a slightly different approach.

3.3.5. **Definition.** We let $\mathcal{X}^a_{d,h,s}$ denote the scheme-theoretic image of the composite

$$\mathcal{W}^a_{d,h,s} \hookrightarrow \mathcal{W}_{d,h,s} \xrightarrow{(3.3.4)} \mathcal{R}_d^{\Gamma_{\mathrm{disc}}}. \tag{3.3.6}$$

This is a morphism of Ind-algebraic stacks, and the scheme-theoretic image has the obvious meaning: since $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ is an Ind-algebraic stack, constructed as the 2-colimit of a directed system of algebraic stacks whose transition morphisms are closed immersions, the morphism (3.3.6), which is representable by algebraic spaces, proper, and of finite presentation, factors through a closed algebraic substack $\mathcal{Z}$ of $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$. We then define $\mathcal{X}^a_{d,h,s}$ to be the scheme-theoretic image of $\mathcal{W}^a_{d,h,s}$ in $\mathcal{Z}$. Then $\mathcal{X}^a_{d,h,s}$ is a closed algebraic substack of $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$, and is independent of the choice of $\mathcal{Z}$.

Our next goal is to prove that $\mathcal{X}^a_{d,h,s}$ is a (necessarily closed) substack of $\mathcal{X}^a_d$. Our argument for this is a little indirect. By definition, it is enough to check that if A is a finite type $\mathbf{Z}/p^a$-algebra, then for any morphism $\operatorname{Spec} A \to \mathcal{X}^a_{d,h,s}$, the composite morphism $\operatorname{Spec} A \to \mathcal{X}^a_{d,h,s} \to \mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ factors through $\mathcal{X}_d$. More concretely, if M denotes the étale φ-module over A, endowed with a Γ_{disc}-action, associated to the given point $\operatorname{Spec} A \to \mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$, then we must show that the Γ_{disc}-action on M is continuous. By Lemma 3.1.3, we need to show that M contains a (not necessarily projective) lattice $\mathfrak{M}$ such that for some $s \geq 0$, we have $(\gamma^{p^s} - 1)(\mathfrak{M}) \subseteq T\mathfrak{M}$.

To do this, we note that the natural map $A \to B := \varprojlim_i A_i$ is injective, where A_i runs over the Artinian quotients of A. It is enough to find such a lattice for M_B, and by a Mittag-Leffler argument, it is enough to find a lattice of some fixed height for each M_{A_i}; that is, we can reduce to the following lemma [EG19a, Lem. 3.4.8].

3.3.7. **Lemma.** *Suppose that M is a projective étale φ-module of rank d over a finite type Artinian $\mathbf{Z}/p^a$-algebra A, and that M is endowed with an action of Γ_{disc}, such that the corresponding morphism $\operatorname{Spec} A \to \mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ factors through $\mathcal{X}^a_{d,h,s}$. Then M contains a φ-invariant lattice $\mathfrak{M}$ of T-height $\leq h$, such that $(\gamma^{p^s} - 1)(\mathfrak{M}) \subseteq T\mathfrak{M}$.*

Our proof of this lemma is a little involved. We can immediately reduce to the Artin local case, and then to a problem about the universal framed deformation rings of a fixed φ-module equipped with an action of Γ_{disc}. Writing R for this universal deformation ring, we can consider the subfunctor of deformations which admit a lattice $\mathfrak{M}$ of the required type, and it is straightforward to check that this is representable by a quotient S of R. The statement of the lemma reduces to showing that $\operatorname{Spf} S$ contains the scheme-theoretic image $\operatorname{Spf} T$ of the morphism

$$X := \mathcal{W}_{d,h,s} \times_{\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}} \operatorname{Spf} R \to \operatorname{Spf} R.$$

We prove this using the following criterion [EG19a, Lem. A.30].

3.3.8. **Lemma.** *Let $R \to S$ be a continuous surjection of pro-Artinian local rings, and let $X \to \operatorname{Spf} R$ be a finite type morphism of formal algebraic spaces.*

Make the following assumption: if A is any finite-type Artinian local R-algebra for which the canonical morphism $R \to A$ factors through a discrete quotient of R, and for which the canonical morphism $X_A \to \operatorname{Spec} A$ admits a section, then the canonical morphism $R \to A$ furthermore factors through S.

Then the scheme-theoretic image of $X \to \operatorname{Spf} R$ is a closed formal subscheme of $\operatorname{Spf} S$.

The result then follows by unwinding the definitions: the point is that admitting a section to X_A in particular gives a morphism $\operatorname{Spec} A \to \mathcal{W}_{d,h,s}$, and the corresponding weak Wach module is a lattice of the kind being considered, which gives the required factorisation through S.

Finally, we can prove that $\mathcal{X}_d$ is Ind-algebraic.

3.3.9. **Proposition.** *The canonical morphism $\varinjlim \mathcal{X}^a_{d,h,s} \to \mathcal{X}_d$ is an isomorphism. Thus $\mathcal{X}_d$ is an Ind-algebraic stack, and may in fact be written as the inductive limit of algebraic stacks of finite presentation, with the transition maps being closed immersions.*

Proof. We just need to show that if $T = \operatorname{Spec} A$ for a Noetherian $\mathbf{Z}/p^a$-algebra A, then any morphism $T \to \mathcal{X}_d$ factors through some $\mathcal{X}^a_{d,h,s}$, or equivalently, that the closed immersion

$$\mathcal{X}^a_{d,h,s} \times_{\mathcal{X}_d} T \to T \tag{3.3.10}$$

is an isomorphism, for some choice of h and s.

If M denotes the étale (φ, Γ)-module corresponding to the morphism $\operatorname{Spec} A \to \mathcal{X}_d$, then [EG19b, Prop. 5.4.7] shows that we may find a scheme-theoretically dominant morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ such that M_B is free of rank d. (In outline, the proof of that result is to note that $\mathfrak{M}$ is free if and only if $\mathfrak{M}/T\mathfrak{M}$ is free, so that $\mathfrak{M}$ is automatically free over a dense open subset of $\operatorname{Spec} A$, and use Noetherian induction.) If we show that the composite $\operatorname{Spec} B \to \operatorname{Spec} A \to \mathcal{X}_d$ factors through $\mathcal{X}^a_{d,h,s}$ for some h and s, then we see that the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ factors through the closed subscheme $\mathcal{X}^a_{d,h,s} \times_{\mathcal{X}_d} \operatorname{Spec} A$ of $\operatorname{Spec} A$. Since $\operatorname{Spec} B \to \operatorname{Spec} A$ is scheme-theoretically dominant, this implies that (3.3.10) is indeed an isomorphism, as required.

Since M_B is free, we may choose a φ-invariant free lattice $\mathfrak{M} \subseteq M_B$, of height $\leq h$ for some sufficiently large value of h. Since the Γ_{disc}-action on M, and hence on M_B, is continuous by assumption, Lemma 3.1.3 then shows that $(\gamma^{p^s} - 1)(\mathfrak{M}) \subseteq T\mathfrak{M}$ for some sufficiently large value of s. Then $\mathfrak{M}$ gives rise to a B-valued point of $\mathcal{W}^a_{d,h,s}$, whose image in $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ is equal to the étale φ-module M_B. Thus the morphism $\operatorname{Spec} B \to \mathcal{X}_d$ corresponding to M_B does indeed factor through $\mathcal{X}^a_{d,h,s}$. ∎

4. Crystalline and semistable moduli stacks

From now on we allow $K/\mathbf{Q}_p$ to be arbitrary.

4.1. Breuil–Kisin–Fargues modules admitting all descents.

Our approach to constructing crystalline and semistable moduli stacks relies on the idea that while most Breuil–Kisin modules do not give crystalline or semistable representations of G_K, this is "mostly" for the simple reason that the corresponding G_{K_∞}-representations do not even extend to G_K. However, if ρ is a G_K-representation such that $\rho|_{G_{K_\infty}}$ admits a Breuil–Kisin module, then ρ should be "almost" semistable: in particular, ρ is potentially semistable, and becomes semistable over $K(\pi^{1/p^m})$ for some m depending only on K.

Such a statement was first claimed by [Car13]. Using this result it is straightforward to deduce that ρ is semistable if and only if $\rho|_{G_{K_\infty}}$ admits a Breuil–Kisin module *for all possible choices of* $(\pi$ *and*$)$ $(\pi^{1/p^n})_{n \geq 0}$. This is the main idea behind our construction of the stacks. (While I was led to this statement by [Car13], I am sure in retrospect that I was influenced by some remarks of Peter Scholze's about prismatic crystals that I didn't understand at the time!) Unfortunately, there is a mistake in [Car13]. Very recently Hui Gao [Gao19] has given a completely different proof of the results of [Car13], but in the meantime Tong Liu and I proved a slightly weaker statement that suffices for our purposes, and in fact it is useful for us to build in to the constructions of our stacks the additional structures involved in this weaker statement.

For each choice of uniformiser π of K, and each choice $\pi^\flat \in \mathcal{O}_{\mathbf{C}}^\flat$ of p-power roots of π, we write $\mathfrak{S}_{\pi^\flat}$ for $\mathfrak{S} = W(k)[[u]]$, regarded as a subring of $\mathbf{A}_{\mathrm{inf}}$ via $u \mapsto [\pi^\flat]$. For each choice of $\pi^\flat$ and each $s \geq 0$ we write $K_{\pi^\flat,s}$ for $K(\pi^{1/p^s})$, and $K_{\pi^\flat,\infty}$ for $\cup_s K_{\pi^\flat,s}$. Write $E_\pi(u)$ for the Eisenstein polynomial for π, and $E_{\pi^\flat}$ for its image in $\mathbf{A}_{\mathrm{inf}}$. Then $E_{\pi^\flat}$ is a generator of the kernel of the natural ring homomorphism $\theta : \mathbf{A}_{\mathrm{inf}} \to \mathcal{O}_{\mathbf{C}}$ which for any $x \in \mathcal{O}_{\mathbf{C}}^\flat$ satisfies $\theta([x]) = x^\sharp$.

4.1.1. **Definition.** A Breuil–Kisin–Fargues G_K-module of height at most h is a Breuil–Kisin–Fargues module of height at most h which is equipped with a semilinear G_K-action which commutes with φ.

4.1.2. *Remark.* Note that if $\mathfrak{M}^{\mathrm{inf}}$ is a Breuil–Kisin–Fargues G_K-module, then $W(\mathbf{C}^\flat)\otimes_{\mathbf{A}_{\mathrm{inf}}} \mathfrak{M}^{\mathrm{inf}}$ is naturally a (G_K, φ)-module in the sense of Definition 1.4.2.

4.1.3. **Definition.** Let $\mathfrak{M}^{\mathrm{inf}}$ be a Breuil–Kisin–Fargues G_K-module of height at most h with a semilinear G_K-action. Then we say that $\mathfrak{M}^{\mathrm{inf}}$ *admits all descents* if the following conditions hold.

(1) For every choice of π and $\pi^\flat$, there is a Breuil–Kisin module $\mathfrak{M}_{\pi^\flat}$ of height at most h with $\mathfrak{M}_{\pi^\flat} \subset (\mathfrak{M}^{\mathrm{inf}})^{G_{K_{\pi^\flat,\infty}}}$ for which the induced morphism $\mathbf{A}_{\mathrm{inf}} \otimes_{\mathfrak{S}_{\pi^\flat}} \mathfrak{M}_{\pi^\flat} \to \mathfrak{M}^{\mathrm{inf}}$ is an isomorphism.
(2) The $W(k)$-submodule $\mathfrak{M}_{\pi^\flat}/[\pi^\flat]\mathfrak{M}_{\pi^\flat}$ of $W(\overline{k}) \otimes_{\mathbf{A}_{\mathrm{inf}}} \mathfrak{M}^{\mathrm{inf}}$ is independent of the choice of π and $\pi^\flat$.
(3) The $\mathcal{O}_K$-submodule $\varphi^*\mathfrak{M}_{\pi^\flat}/E_{\pi^\flat}\varphi^*\mathfrak{M}_{\pi^\flat}$ of $\mathcal{O}_{\mathbf{C}}\otimes_{\theta,\mathbf{A}_{\mathrm{inf}}}\varphi^*\mathfrak{M}^{\mathrm{inf}}$ is independent of the choice of π and $\pi^\flat$.

4.1.4. **Definition.** Let $\mathfrak{M}^{\mathrm{inf}}$ be a Breuil–Kisin–Fargues G_K-module which admits all descents. We say that $\mathfrak{M}^{\mathrm{inf}}$ is furthermore *crystalline* if for each choice of π and $\pi^\flat$, and each $g \in G_K$, we have

$$(g-1)(\mathfrak{M}_{\pi^\flat}) \subset \varphi^{-1}([\varepsilon]-1)[\pi^\flat]\mathfrak{M}^{\mathrm{inf}}.$$

There is an equivalence of categories between the category of (G_K, φ)-modules M of rank d and the category of free $\mathbf{Z}_p$-modules T of rank d which are equipped with a continuous action of G_K, with the Galois representation corresponding to M being given by $T(M) = M^{\varphi=1}$. Write $V(M) := T(M) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

We deduce the following theorem [EG19a, Thm. F.11] from the results of Tong Liu's paper [Liu18] and Fargues' correspondence between Breuil–Kisin–Fargues modules and $\mathrm{B}_{\mathrm{dR}}^+$-lattices [BMS19, Thm. 4.28].

4.1.5. **Theorem** (T.G. and Tong Liu)**.** *Let M be a (G_K, φ)-module. Then $V(M)$ is semistable with Hodge–Tate weights in $[0,h]$ if and only if there is a (necessarily unique) Breuil–Kisin–Fargues G_K-module $\mathfrak{M}^{\mathrm{inf}}$ which is of height at most h, which admits all descents, and which satisfies $M = W(\mathbf{C}^\flat) \otimes_{\mathbf{A}_{\mathrm{inf}}} \mathfrak{M}^{\mathrm{inf}}$.*

Furthermore, $V(M)$ is crystalline if and only if $\mathfrak{M}^{\mathrm{inf}}$ is crystalline.

4.2. **Definition of the crystalline and semistable stacks.** For simplicity of exposition, we will only mention the crystalline case from now on; the proofs in the semistable case are identical (in fact slightly simpler, because the crystalline case has the extra condition on the Breuil–Kisin–Fargues modules). The actual definition of these stacks, and the proof of their basic properties, is quite involved, but the basic idea is quite simple: we will take the scheme-theoretic image in $\mathcal{X}_d$ of the moduli stack of Breuil–Kisin–Fargues G_K-modules which admit all descents.

Firstly, note that Definition 4.1.4 carries over in an obvious fashion to the case with coefficients in a topologically of finite type p-adically complete $\mathbf{Z}_p$-algebra A (see [EG19a, Defn. 4.2.4]). Using the faithful flatness results of Proposition 1.2.3, one can check that the existence of a descent to a Breuil–Kisin module depends only on π, and not on $\pi^\flat$, and that any such descent is unique ([EG19a, Lem. 4.2.7, 4.2.8]).

4.2.1. **Definition.** For any $h \geq 0$ we let $\mathcal{C}^a_{d,\mathrm{crys},h}$ denote the limit preserving category of groupoids over $\operatorname{Spec} \mathbf{Z}/p^a$ determined by decreeing, for any finite type $\mathbf{Z}/p^a$-algebra A, that $\mathcal{C}^a_{d,\mathrm{crys},h}(A)$ is the groupoid of Breuil–Kisin–Fargues G_K-modules with A-coefficients, which are of height at most h, which admit all descents, and which are crystalline. We let $\mathcal{C}_{d,\mathrm{crys},h} := \varinjlim_a \mathcal{C}^a_{d,\mathrm{crys},h}$.

It is easy to check that if A be a p-adically complete $\mathbf{Z}_p$-algebra which is topologically of finite type, then $\mathcal{C}_{d,\mathrm{crys},h}(A)$ is the groupoid of Breuil–Kisin–Fargues G_K-modules with A-coefficients, which are of height at most h, which admit all descents, and which are crystalline. There is a natural morphism

$$\mathcal{C}_{d,\mathrm{crys},h} \to \mathcal{X}_d,$$

which is defined, for finite type $\mathbf{Z}/p^a$-algebras, via $\mathfrak{M}^{\mathrm{inf}} \mapsto W(\mathbf{C}^\flat)_A \otimes_{\mathbf{A}_{\mathrm{inf},A}} \mathfrak{M}^{\mathrm{inf}}$, with the target object being regarded as an A-valued point of $\mathcal{X}_d$ via the equivalence of Proposition 1.4.3.

For now we will admit the following theorem [EG19a, Thm. 4.5.18].

4.2.2. **Theorem.** *$\mathcal{C}_{d,\mathrm{crys},h}$ is a p-adic formal algebraic stack of finite presentation and affine diagonal. The morphism $\mathcal{C}_{d,\mathrm{crys},h} \to \mathcal{X}_d$ is representable by algebraic spaces, proper, and of finite presentation.*

We will deduce this from the corresponding statements for the stacks $\mathcal{C}_{d,h}$ of Breuil–Kisin modules; the key point is (roughly) to show that the additional structure of a G_K-action admitting all descents is only a finite amount of data.

Let $\mathcal{C}^{\mathrm{fl}}_{d,\mathrm{crys},h}$ denote the flat part of $\mathcal{C}_{d,\mathrm{crys},h}$ (i.e. the maximal substack which is flat over $\operatorname{Spf} \mathbf{Z}_p$; see [Eme, Ex. 9.11]). Then we define $\mathcal{X}_d^{\mathrm{crys},h}$ to be the scheme-theoretic image of the morphism $\mathcal{C}^{\mathrm{fl}}_{d,\mathrm{crys},h} \to \mathcal{X}_d$. The following result is essentially [EG19a, Thm. 4.8.12] (for the purposes of exposition, we are ignoring inertial types and Hodge–Tate weights for now).

4.2.3. **Theorem.** *The closed substack $\mathcal{X}_d^{\mathrm{crys},h}$ of $\mathcal{X}_d$ is a p-adic formal algebraic stack, which is of finite type and flat over $\operatorname{Spf} \mathbf{Z}_p$, and is uniquely determined as a $\mathbf{Z}_p$-flat closed substack of $\mathcal{X}_d$ by the following property: if A° is a finite flat $\mathbf{Z}_p$-algebra, then $\mathcal{X}_d^{\mathrm{crys},h}(A^\circ)$ is precisely the subgroupoid of $\mathcal{X}_d(A^\circ)$ consisting of G_K-representations which are crystalline with Hodge–Tate weights contained in $[0,h]$.*

That $\mathcal{X}_d^{\mathrm{crys},h}$ is a p-adic formal algebraic stack and is of finite type and flat over $\operatorname{Spf} \mathbf{Z}_p$ follows from its construction and [EG19a, Prop. A.21] (that is, "the scheme-theoretic image of a p-adic formal algebraic stack is a p-adic formal algebraic stack"). The characterisation of its A°-points can be reduced to the local case, and thus to the following important statement about versal rings: fix a point $\operatorname{Spec} \mathbf{F} \to \mathcal{X}_d(\mathbf{F})$ for some finite field $\mathbf{F}$, giving rise to a continuous representation $\overline{\rho} : G_K \to \mathrm{GL}_d(\mathbf{F})$. Let $R_{\overline{\rho}}^{\mathrm{crys},h}$ denote the universal framed deformation $W(\mathbf{F})$-algebra for lifts of $\overline{\rho}$ which are crystalline with Hodge–Tate weights in $[0, h]$ (which exists by [Kis08]). Then there is an induced morphism $\operatorname{Spf} R_{\overline{\rho}}^{\mathrm{crys},h} \to \mathcal{X}_d$, and we have [EG19a, Prop. 4.8.10]:

4.2.4. **Proposition.** *The morphism* $\operatorname{Spf} R_{\overline{\rho}}^{\mathrm{crys},h} \to \mathcal{X}_d$ *factors through a versal morphism* $\operatorname{Spf} R_{\overline{\rho}}^{\mathrm{crys},h} \to \mathcal{X}_d^{\mathrm{crys},h}$.

Proof. This is proved using Lemma 3.3.8, in a similar way to the way we used it in the sketch proof of Lemma 3.3.7; the actual argument is slightly more complicated as we prove an algebraization statement and then work with p inverted. □

4.3. **Potentially semistable and potentially crystalline moduli stacks with fixed Hodge–Tate weights.** Let L/K be a fixed finite Galois extension; then there is an obvious notion of a Breuil–Kisin–Fargues G_K-module which admits all descents over L, and an obvious extension of the above results to give stacks of G_K-representations which become crystalline over L with Hodge–Tate weights contained in $[0, h]$. We write $\mathcal{C}_{d,\mathrm{crys},h}^{L/K,\mathrm{fl}}$ for the corresponding stack of Breuil–Kisin–Fargues modules.

Since the inertia types and Hodge–Tate weights are discrete invariants, we at least morally expect these stacks to decompose as a disjoint union of stacks of potentially crystalline representations with fixed inertial and Hodge types; so the key point should be to see how to read this data off from the Breuil–Kisin–Fargues modules.

Fix a finite extension $E/\mathbf{Q}_p$ with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field $\mathbf{F}$, and assume that E is large enough to contain the images of all embeddings $K \hookrightarrow \overline{\mathbf{Q}}_p$. We will abusively write $\mathcal{X}_d$ for the base change of $\mathcal{X}_d$ to $\operatorname{Spf} \mathcal{O}$, without further comment. From now on A° will denote a p-adically complete flat $\mathcal{O}$-algebra which is topologically of finite type over $\mathcal{O}$, and we write $A = A^\circ[1/p]$.

Let $\mathfrak{M}_{A^\circ}$ be a Breuil–Kisin–Fargues G_K-module with A°-coefficients which admits all descents over L, and write $\overline{\mathfrak{M}}_{A^\circ} := \mathfrak{M}_{A^\circ,\pi^\flat}/[\pi^\flat]\mathfrak{M}_{A^\circ,\pi^\flat}$ (for some choice of $\pi^\flat$, with π now denoting a uniformiser of L). Then $\overline{\mathfrak{M}}_{A^\circ}$ has a natural $W(l)\otimes_{\mathbf{Z}_p} A$-semilinear action of $\mathrm{Gal}(L/K)$, which is defined as follows: if $g \in \mathrm{Gal}(L/K)$, then $g(\mathfrak{M}_{A^\circ,\pi^\flat}) = \mathfrak{M}_{A^\circ,g(\pi^\flat)}$, so the morphism $g : \mathfrak{M}_{A^\circ,\pi^\flat} \to g(\mathfrak{M}_{A^\circ,\pi^\flat}) = \mathfrak{M}_{A^\circ,g(\pi^\flat)}$ induces a morphism

$$g : \mathfrak{M}_{A^\circ,\pi^\flat}/[\pi^\flat]\mathfrak{M}_{A^\circ,\pi^\flat} \to \mathfrak{M}_{A^\circ,g(\pi^\flat)}/[g(\pi^\flat)]\mathfrak{M}_{A^\circ,g(\pi^\flat)},$$

and the source and target are both canonically identified with $\overline{\mathfrak{M}}_{A^\circ}$.

This action of $\mathrm{Gal}(L/K)$ on $\overline{\mathfrak{M}}_{A^\circ}$ induces an $L_0 \otimes_{\mathbf{Q}_p} A$-linear action of $I_{L/K}$ on the projective $L_0 \otimes_{\mathbf{Q}_p} A$-module $\overline{\mathfrak{M}}_{A^\circ} \otimes_{A^\circ} A$. Fix a choice of embedding $\sigma : L_0 \hookrightarrow E$, and let $e_\sigma \in L_0 \otimes_{\mathbf{Q}_p} E$ be the corresponding idempotent. Then $e_\sigma(\overline{\mathfrak{M}}_{A^\circ} \otimes_{A^\circ} A)$ is a projective A-module of rank d, with an A-linear action of $I_{L/K}$. Up to canonical isomorphism, this module does not depend on the choice of σ. We write

$$\mathrm{WD}(\mathfrak{M}_{A^\circ}^{\mathrm{inf}}) := e_\sigma(\overline{\mathfrak{M}}_{A^\circ} \otimes_{A^\circ} A),$$

a projective A-module of rank d with an A-linear action of $I_{L/K}$. It is easy to check that this is compatible with base change (of p-adically complete flat $\mathcal{O}$-algebras which are topologically of finite type over $\mathcal{O}$).

We now turn to Hodge types. Fix some choice of $\pi^\flat$ a uniformiser of L, write $\mathfrak{M}_{A^\circ}$ for $\mathfrak{M}_{\pi^\flat,A^\circ}$, and u for $[\pi^\flat]$. For each $0 \le i \le h$ we define $\operatorname{Fil}^i \varphi^*\mathfrak{M}_{A^\circ} = \Phi_{\mathfrak{M}_{A^\circ}}^{-1}(E(u)^i\mathfrak{M}_{A^\circ})$, and we set $\operatorname{Fil}^i \varphi^*\mathfrak{M}_{A^\circ} = \varphi^*\mathfrak{M}_{A^\circ}$ for $i < 0$. Then for each $0 \le i \le h$,

$$(\operatorname{Fil}^i \varphi^*\mathfrak{M}_{A^\circ}/E(u)\operatorname{Fil}^{i-1}\varphi^*\mathfrak{M}_{A^\circ}) \otimes_{A^\circ} A$$

is a finite projective $L \otimes_{\mathbf{Q}_p} A$-module, whose formation is compatible with base change (see [EG19a, Prop. 4.7.1]). There is a natural action of $\operatorname{Gal}(L/K)$, which is semi-linear with respect to the action of $\operatorname{Gal}(L/K)$ on $L \otimes_{\mathbf{Q}_p} A$ induced by its action on the first factor. Since L/K is a Galois extension, the tensor product $L \otimes_{\mathbf{Q}_p} A$ is an étale $\operatorname{Gal}(L/K)$-extension of $K \otimes_{\mathbf{Q}_p} A$, and so étale descent allows us to descend $(\varphi^*\mathfrak{M}_{A^\circ}/E(u)\varphi^*\mathfrak{M}_{A^\circ}) \otimes_{A^\circ} A$ to a filtered module over $K \otimes_{\mathbf{Q}_p} A$; concretely, this descent is achieved by taking $\operatorname{Gal}(L/K)$-invariants. This leads to the following definition.

4.3.1. **Definition.** In the preceding situation, we write

$$D_{\mathrm{dR}}(\mathfrak{M}^{\mathrm{inf}}_{A^\circ}) := \big((\varphi^*\mathfrak{M}_{A^\circ}/E(u)\varphi^*\mathfrak{M}_{A^\circ}) \otimes_{A^\circ} A\big)^{\operatorname{Gal}(L/K)},$$

and more generally, for each $i \ge 0$, we write

$$\operatorname{Fil}^i D_{\mathrm{dR}}(\mathfrak{M}^{\mathrm{inf}}_{A^\circ}) := \big((\operatorname{Fil}^i \varphi^*\mathfrak{M}_{A^\circ}/E(u)\operatorname{Fil}^{i-1}\varphi^*\mathfrak{M}_{A^\circ}) \otimes_{A^\circ} A\big)^{\operatorname{Gal}(L/K)}$$

(and for $i < 0$, we write $\operatorname{Fil}^i D_{\mathrm{dR}}(\mathfrak{M}^{\mathrm{inf}}_{A^\circ}) := D_{\mathrm{dR}}(\mathfrak{M}^{\mathrm{inf}}_{A^\circ})$). The property of being a finite rank projective module is preserved under étale descent, and so we find that $D_{\mathrm{dR}}(\mathfrak{M}^{\mathrm{inf}}_{A^\circ})$ is a rank d projective $K \otimes_{\mathbf{Q}_p} A$-module, filtered by projective submodules.

Since A is an E-algebra, we have the product decomposition $K \otimes_{\mathbf{Q}_p} A \xrightarrow{\sim} \prod_{\sigma:K\hookrightarrow E} A$, and so, if we write e_σ for the idempotent corresponding to the factor labeled by σ in this decomposition, we find that

$$D_{\mathrm{dR}}(\mathfrak{M}^{\mathrm{inf}}_{A^\circ}) = \prod_{\sigma:K\hookrightarrow E} e_\sigma D_{\mathrm{dR}}(\mathfrak{M}^{\mathrm{inf}}_{A^\circ}),$$

where each $e_\sigma D_{\mathrm{dR}}(\mathfrak{M}^{\mathrm{inf}}_{A^\circ})$ is a projective A-module of rank d. For each i, we write

$$\operatorname{Fil}^i e_\sigma D_{\mathrm{dR}}(\mathfrak{M}^{\mathrm{inf}}_{A^\circ}) = e_\sigma \operatorname{Fil}^i D_{\mathrm{dR}}(\mathfrak{M}^{\mathrm{inf}}_{A^\circ}).$$

Each $\operatorname{Fil}^i e_\sigma D_{\mathrm{dR}}(\mathfrak{M}^{\mathrm{inf}}_{A^\circ})$ is again a projective A-module.

4.3.2. **Definition.** A Hodge type $\underline{\lambda}$ of rank d is by definition a set of tuples of integers $\{\lambda_{\sigma,i}\}_{\sigma:K\hookrightarrow\overline{\mathbf{Q}}_p,1\le i\le d}$ with $\lambda_{\sigma,i} \ge \lambda_{\sigma,i+1}$ for all σ and all $1 \le i \le d-1$.

If $\underline{D} := (D_\sigma)_{\sigma:K\hookrightarrow E}$ is a collection of rank d vector bundles over $\operatorname{Spec} A$, labeled (as indicated) by the embeddings $\sigma : K \hookrightarrow E$, then we say that $\underline{D}$ has Hodge type $\underline{\lambda}$ if $\operatorname{Fil}^i D_\sigma$ has constant rank equal to $\#\{j \mid \lambda_{\sigma|K,j} \ge i\}$.

As the notation suggests, the Hodge type of $V(\mathfrak{M}^{\mathrm{inf}})$ agrees with the Hodge type of $D_{\mathrm{dR}}(\mathfrak{M}^{\mathrm{inf}})$. Putting this all together, we have [EG19a, Prop. 4.8.2]:

4.3.3. **Proposition.** *Let L/K be a finite Galois extension. Then the stack $\mathcal{C}^{L/K,\mathrm{fl}}_{d,\mathrm{crys},h}$ is a scheme-theoretic union of closed substacks $\mathcal{C}^{L/K,\mathrm{fl},\underline{\lambda},\tau}_{d,\mathrm{crys},h}$, where $\underline{\lambda}$ runs over all* *effective Hodge types that are bounded by h, and τ runs over all d-dimensional E-representations of $I_{L/K}$. These latter closed substacks are uniquely characterised by the following property: if A° is a finite flat $\mathcal{O}$-algebra, then an A°-point of $\mathcal{C}^{L/K,\mathrm{fl}}_{d,\mathrm{crys},h}$) is a point of $\mathcal{C}^{L/K,\mathrm{fl},\underline{\lambda},\tau}_{d,\mathrm{crys},h}$ if and only if the corresponding Breuil–Kisin–Fargues module $\mathfrak{M}^{\mathrm{inf}}_{A^\circ}$ has Hodge type $\underline{\lambda}$ and inertial type τ.*

4.3.4. *Remark.* It is not obvious (at least to us) from the definition that the Hodge filtration on $D_{\mathrm{dR}}(\mathfrak{M}^{\mathrm{inf}})$ is independent of the choice of $\pi^\flat$; rather, we deduce this independence from its compatibility with the Hodge filtration on $D_{\mathrm{dR}}(V(\mathfrak{M}^{\mathrm{inf}}))$.

We then define the corresponding stacks $\mathcal{X}^{\mathrm{crys},\underline{\lambda},\tau}_{d,}$; we can extend the definition to possibly negative Hodge–Tate weights by twisting by an appropriate power of the cyclotomic character. Finally we can deduce [EG19a, Thm. 4.8.14]:

4.3.5. **Theorem.** *The algebraic stacks $\mathcal{X}^{\mathrm{crys},\underline{\lambda},\tau}_{d} \times_{\mathrm{Spf}\,\mathcal{O}} \mathrm{Spec}\,\mathbf{F}$ and $\mathcal{X}^{\mathrm{ss},\underline{\lambda},\tau}_{d} \times_{\mathrm{Spf}\,\mathcal{O}} \mathrm{Spec}\,\mathbf{F}$ are equidimensional of dimension*

$$\sum_\sigma \#\{1 \le i < j \le d | \lambda_{\sigma,i} > \lambda_{\sigma,j}\}.$$

In particular, if $\underline{\lambda}$ is regular, then the algebraic stacks $\mathcal{X}^{\mathrm{crys},\underline{\lambda},\tau}_{d} \times_{\mathrm{Spf}\,\mathcal{O}} \mathrm{Spec}\,\mathbf{F}$ and $\mathcal{X}^{\mathrm{ss},\underline{\lambda},\tau}_{d} \times_{\mathrm{Spf}\,\mathcal{O}} \mathrm{Spec}\,\mathbf{F}$ are equidimensional of dimension $[K : \mathbf{Q}_p]d(d-1)/2$.

This follows from Proposition 4.2.4, from which it follows that the versal rings to $\mathcal{X}^{\mathrm{crys},\underline{\lambda},\tau}_{d}$ are given by the corresponding crystalline deformation rings $R^{\mathrm{crys},\underline{\lambda},\tau}_{\overline{\rho}}$, and the computation of the dimensions of these deformation rings in [Kis08] (which comes down to a computation on the generic fibre, using weakly admissible modules).

4.4. **Canonical actions.** It remains to prove Theorem 4.2.2. The key finiteness property comes from the following reinterpretation of some of the arguments of [CL11, §2] (see [EG19a, §4.3, 4.4]):

4.4.1. **Proposition.** *For any fixed a, h, and any sufficiently large N, there is a positive integer $s(a, h, N)$ with the property that for any finite type $\mathbf{Z}/p^a$-algebra A, any projective Breuil–Kisin module $\mathfrak{M}$ of height at most h, and any $s \ge s(a, h, N)$, there is a unique continuous action of G_{K_s} on $\mathfrak{M}^{\mathrm{inf}} := \mathbf{A}_{\mathrm{inf},A} \otimes_{\mathfrak{S}_A} \mathfrak{M}$ which commutes with φ and is semi-linear with respect to the natural action of G_{K_s} on $\mathbf{A}_{\mathrm{inf},A}$, with the additional property that for all $g \in G_{K_s}$ we have $(g-1)(\mathfrak{M}) \subset u^N \mathfrak{M}^{\mathrm{inf}}$.*

Conversely, if $\mathfrak{M}^{\mathrm{inf}}$ be a Breuil–Kisin–Fargues G_K-module of height at most h with A-coefficients, which admits all descents, then for each choice of $\pi^\flat$, and each $s \ge s(a, h, N)$, the action of $G_{K_{\pi^\flat,s}}$ agrees with the canonical action obtained from $\mathfrak{M}_{\pi^\flat}$ as above.

The proof of the first part is quite straightforward; the point is that by a standard argument with the φ-structure, one can uniquely upgrade approximate homomorphisms (i.e. maps which are homomorphisms modulo u^N) between Breuil–Kisin(–Fargues) modules to actual homomomorphisms (see [EG19a, Lem. 4.3.2]). We can write down an “approximate action” of G_{K_s} on $\mathfrak{M}^{\mathrm{inf}}$, by just letting it act trivially on $\mathfrak{M}$ (after fixing some basis - since the action is semilinear, it doesn’t literally make sense to ask that it be trivial).

For the second part, it's easy to check that this holds for $s \gg 0$ just by continuity of the action of G_K on $\mathfrak{M}^{\mathrm{inf}}$, and to deduce it for the optimal value of s, we consider the subgroup of G_K generated by the $G_{K_{\pi^\flat,s}}$ as we run over all choices of $\pi^\flat$.

For any $h \geq 0$ and any choice of $\pi^\flat$, we write $\mathcal{C}_{\pi^\flat,d,h}$ for the moduli stack of rank d Breuil–Kisin modules for $\mathfrak{S}_{\pi^\flat,A}$ of height at most h, and $\mathcal{R}_{\pi^\flat,d}$ for the corresponding stack of étale φ-modules. We have a natural morphism $\mathcal{X}_{K,d} \to \mathcal{R}_{\pi^\flat,d}$ given by Proposition 1.4.3 and restriction from G_K to G_{K_∞}; for each $s \geq 1$, we can factor this as

$$\mathcal{X}_{K,d} \to \mathcal{X}_{K_{\pi^\flat,s},d} \to \mathcal{R}_{\pi^\flat,d}.$$

Then for any fixed a, h, and any N and $s \geq s(a,h,N)$ as in Proposition 4.4.1, there is a canonical morphism $\mathcal{C}^a_{\pi^\flat,d,h} \to \mathcal{X}^a_{K_{\pi^\flat,s},d}$ obtained from the canonical action of Proposition 4.4.1, which fits into a commutative triangle

$$\begin{array}{ccc} & & \mathcal{C}^a_{\pi^\flat,d,h} \\ & \swarrow & \downarrow \\ \mathcal{X}^a_{K_{\pi^\flat,s},d} & \longrightarrow & \mathcal{R}^a_{\pi^\flat,d} \end{array}$$

We then define a stack $\mathcal{C}^a_{\pi^\flat,s,d,h}$ by the requirement that it fits into a 2-Cartesian diagram

$$\begin{array}{ccc} \mathcal{C}^a_{\pi^\flat,s,d,h} & \longrightarrow & \mathcal{C}^a_{\pi^\flat,d,h} \\ \downarrow & & \downarrow \\ \mathcal{X}^a_{K,d} & \longrightarrow & \mathcal{X}^a_{K_{\pi^\flat,s},d} \end{array} \tag{4.4.2}$$

The lower horizontal arrow in this diagram is again defined via Proposition 1.4.3, and it is not so hard to show (using the properties that we have proved about $\mathcal{X}_{K,d}$ and its diagonal) that it is representable by algebraic spaces and of finite presentation. Since $\mathcal{C}^a_{\pi^\flat,d,h}$ is an algebraic stack of finite presentation over $\mathbf{Z}/p^a$, it follows that $\mathcal{C}^a_{\pi^\flat,s,d,h}$ is also an algebraic stack of finite presentation over $\mathbf{Z}/p^a$. It is also straightforward to check that the right hand vertical arrow in this diagram is representable by algebraic spaces, proper, and of finite presentation, so that the same is true of the left hand vertical arrow.

Given all this, the key statement remaining to be proved is the following [EG19a, Prop. 4.4.8].

4.4.3. **Proposition.** *For each $\pi^\flat$ and each s as above, there is a natural closed immersion $\mathcal{C}^a_{d,\mathrm{crys},h} \to \mathcal{C}^a_{\pi^\flat,s,d,h}$. In particular, $\mathcal{C}^a_{d,\mathrm{crys},h}$ is an algebraic stack of finite presentation over $\mathbf{Z}/p^a$, and its diagonal is affine.*

Proof. We have a morphism $\mathcal{C}^a_{d,\mathrm{crys},h} \to \mathcal{X}^a_{K,d}$ (given by extending scalars to $W(\mathbf{C}^\flat)$), and it follows that there is a natural morphism $\mathcal{C}^a_{d,\mathrm{crys},h} \to \mathcal{C}^a_{\pi^\flat,d,h}$, defined via $\mathfrak{M}^{\mathrm{inf}} \mapsto \mathfrak{M}_{\pi^\flat}$. The composite morphisms $\mathcal{C}^a_{d,\mathrm{crys},h} \to \mathcal{C}^a_{\pi^\flat,d,h} \to \mathcal{X}^a_{K_{\pi^\flat,s},d}$ and $\mathcal{C}^a_{d,\mathrm{crys},h} \to \mathcal{X}^a_{K,d} \to \mathcal{X}^a_{K_{\pi^\flat,s},d}$ coincide by construction, so we have an induced morphism

$$\mathcal{C}^a_{d,\mathrm{crys},h} \to \mathcal{C}^a_{\pi^\flat,s,d,h} \tag{4.4.4}$$

which we need to show is a closed immersion.

To see that (4.4.4) is at least a monomorphism, it is enough to note that if A is a finite type $\mathbf{Z}/p^a$-algebra, and $\mathfrak{M}^{\mathrm{inf}}$ is a Breuil–Kisin–Fargues module over A which admits all descents, then $\mathfrak{M}^{\mathrm{inf}} = \mathbf{A}_{\mathrm{inf},A} \otimes_{\mathfrak{S}_{\pi^\flat,A}} \mathfrak{M}_{\pi^\flat}$ is determined by $\mathfrak{M}_{\pi^\flat}$, and the G_K-action on $\mathfrak{M}^{\mathrm{inf}}$ is determined by the G_K-action on $W(\mathbf{C}^\flat)_A \otimes_{\mathbf{A}_{\mathrm{inf},A}} \mathfrak{M}^{\mathrm{inf}}$.

Finally, the proof (4.4.4) is a closed immersion is a bit more involved: we need to show that the conditions that $\mathfrak{M}^{\mathrm{inf}} = \mathbf{A}_{\mathrm{inf},A} \otimes_{\mathfrak{S}_{\pi^\flat,A}} \mathfrak{M}_{\pi^\flat}$ is G_K-stable and admits all descents are closed conditions. We make repeated use of the results of [EG19a, App. B], particularly [EG19a, Lem. B.28] to show that the vanishing loci of various morphisms of Breuil–Kisin–Fargues modules are closed. □

5. Geometric Breuil–Mézard

As in the previous lectures, we will concentrate on the potentially crystalline version of the Breuil–Mézard conjecture; most of what we say goes over unchanged to the potentially semistable version.

5.1. **The irreducible components of $\mathcal{X}_{d,\mathrm{red}}$.** We can now complete the analysis of the irreducible components of $\mathcal{X}_{d,\mathrm{red}}$. Recall that $\mathcal{X}_{d,\mathrm{red}}$ is an algebraic stack of finite presentation over $\mathbf{F}$, and has dimension $[K:\mathbf{Q}_p]d(d-1)/2$. Furthermore, for each Serre weight $\underline{k}$, there is a corresponding irreducible component $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ of $(\mathcal{X}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$, and the components for different weights $\underline{k}$ are distinct. Using our results on crystalline lifts, we can now show that these are the only irreducible components. The following is [EG19a, Thm. 6.5.1].

5.1.1. **Theorem.** *$\mathcal{X}_{d,\mathrm{red}}$ is equidimensional of dimension $[K:\mathbf{Q}_p]d(d-1)/2$, and the irreducible components of $(\mathcal{X}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$ are precisely the various closed substacks $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$; in particular, $(\mathcal{X}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$ is maximally nonsplit of niveau 1. Furthermore each $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ can be defined over $\mathbf{F}$, i.e. is the base change of an irreducible component $\mathcal{X}^{\underline{k}}_{d,\mathrm{red}}$ of $\mathcal{X}_{d,\mathrm{red}}$.*

Proof. To see that the $\mathcal{X}^{\underline{k}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ may all be defined over $\mathbf{F}$, we need to show that the action of $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$ on the irreducible components of $(\mathcal{X}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$ is trivial. This follows immediately by considering its action on the maximally nonsplit representations of niveau 1 (since the action of $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$ preserves the property of being maximally nonsplit of niveau 1 and weight $\underline{k}$). It is therefore enough to prove that each irreducible component of $(\mathcal{X}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$ is of dimension of at least $[K:\mathbf{Q}_p]d(d-1)/2$; this follows by choosing a closed point not contained in any other irreducible component, and noting that (by our result on the existence of crystalline lifts) it is contained in the special fibre of some $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda}}$ with $\underline{\lambda}$ regular. □

We can also prove [EG19a, Prop. 6.5.3]:

5.1.2. **Proposition.** *$\mathcal{X}_d$ is not a p-adic formal algebraic stack.*

Proof. Assume that $\mathcal{X}_d$ is a p-adic formal algebraic stack, so that its special fibre $\overline{\mathcal{X}}_d := \mathcal{X}_d \times_{\mathcal{O}} \mathbf{F}$ is an algebraic stack, which is furthermore of finite type over $\mathbf{F}$. Since the underlying reduced substack of $\overline{\mathcal{X}}_d$ is $\mathcal{X}_{d,\mathrm{red}}$, which is equidimensional of dimension $[K:\mathbf{Q}_p]d(d-1)/2$, we see that $\overline{\mathcal{X}}_d$ also has dimension $[K:\mathbf{Q}_p]d(d-1)/2$.

Computing with versal rings, we see that for every $\overline{\rho} : G_K \to \mathrm{GL}_d(\mathbf{F})$ the unrestricted framed deformation ring $R^{\square}_{\overline{\rho}}/\varpi$ must have dimension $d^2+[K:\mathbf{Q}_p]d(d-1)/2$. However, it is known that there are representations $\overline{\rho}$ for which $R^{\square}_{\overline{\rho}}/\varpi$ is formally smooth of dimension $d^2+[K:\mathbf{Q}_p]d^2$ (see for example [All19, Lem. 3.3.1]). Thus we must have $d^2 = d(d-1)/2$, a contradiction. □

5.2. **The qualitative geometric Breuil–Mézard conjecture.** If $\underline{\lambda}$ is a regular Hodge type, and τ is any inertial type, then the stack $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\tau}$ is a finite type p-adic formal algebraic stack over $\mathcal{O}$, which is $\mathcal{O}$-flat and equidimensional of dimension $1+[K:\mathbf{Q}_p]d(d-1)/2$. It follows that its special fibre $\overline{\mathcal{X}}_d^{\mathrm{crys},\underline{\lambda},\tau}$ is an algebraic stack over $\mathbf{F}$ which is equidimensional of dimension $[K:\mathbf{Q}_p]d(d-1)/2$. Since $\mathcal{X}_d^{\mathrm{crys},\underline{\lambda},\tau}$ is a closed substack of $\mathcal{X}_d$, $\overline{\mathcal{X}}_d^{\mathrm{crys},\underline{\lambda},\tau}$ is a closed substack of the special fibre $\overline{\mathcal{X}}_d$, and its irreducible components (with the induced reduced substack structure) are therefore closed substacks of the algebraic stack $\overline{\mathcal{X}}_{d,\mathrm{red}}$.

Since $\overline{\mathcal{X}}_{d,\mathrm{red}}$ is equidimensional of dimension $[K:\mathbf{Q}_p]d(d-1)/2$, it follows that the irreducible components of $\overline{\mathcal{X}}_d^{\mathrm{crys},\lambda,\tau}$ are irreducible components of $\overline{\mathcal{X}}_{d,\mathrm{red}}$, and are therefore of the form $\overline{\mathcal{X}}_{d,\mathrm{red}}^{\underline{k}}$ for some Serre weight $\underline{k}$.

For each $\underline{k}$, we write $\mu_{\underline{k}}(\overline{\mathcal{X}}_d^{\mathrm{crys},\underline{\lambda},\tau})$ for the multiplicity of $\overline{\mathcal{X}}_{d,\mathrm{red}}^{\underline{k}}$ as a component of $\overline{\mathcal{X}}_d^{\mathrm{crys},\underline{\lambda},\tau}$. We write $Z_{\mathrm{crys},\underline{\lambda},\tau} = Z(\overline{\mathcal{X}}_d^{\mathrm{crys},\underline{\lambda},\tau})$ for the corresponding cycle, i.e. for the formal sum

$$Z_{\mathrm{crys},\underline{\lambda},\tau} = \sum_{\underline{k}} \mu_{\underline{k}}(\overline{\mathcal{X}}_d^{\mathrm{crys},\underline{\lambda},\tau}) \cdot \overline{\mathcal{X}}_d^{\underline{k}}, \tag{5.2.1}$$

which we regard as an element of the finitely generated free abelian group $\mathbf{Z}[\mathcal{X}_{d,\mathrm{red}}]$ whose generators are the irreducible components $\overline{\mathcal{X}}_d^{\underline{k}}$.

Fix some representation $\overline{\rho} : G_K \to \mathrm{GL}_d(\mathbf{F})$, corresponding to a point $x : \operatorname{Spec}\mathbf{F} \to \mathcal{X}_d$. For each regular Hodge type $\underline{\lambda}$ and inertial type τ, we have an effective versal morphism $\operatorname{Spec} R_{\overline{\rho}}^{\mathrm{crys},\underline{\lambda},\tau}/\varpi \to \overline{\mathcal{X}}^{\mathrm{crys},\underline{\lambda},\tau}$. For each $\underline{k}$ we set

$$\mathcal{C}_{\underline{k}}(\overline{\rho}) := \operatorname{Spf} R^{\square}_{\overline{\rho}} \times_{\mathcal{X}_d} \mathcal{X}_d^{\underline{k}},$$

which we regard as a cycle of dimension $d^2+[K:\mathbf{Q}_p]d(d-1)/2$ in $\operatorname{Spec} R_{\overline{\rho}}/\varpi$ (note that since $\mathcal{X}_d^{\underline{k}}$ is algebraic, it has effective versal rings, so we really get a subscheme of $\operatorname{Spec} R_{\overline{\rho}}/\varpi$, rather than of $\operatorname{Spf} R_{\overline{\rho}}/\varpi$). The following theorem gives a qualitative version of the refined Breuil–Mézard conjecture [EG14, Conj. 4.2.1]. While its statement is purely local, we do not know how to prove it without making use of the stack $\mathcal{X}_d$.

5.2.2. **Theorem.** *Let $\overline{\rho} : G_K \to \mathrm{GL}_d(\mathbf{F})$ be a continuous representation. Then there are finitely many cycles of dimension $d^2+[K:\mathbf{Q}_p]d(d-1)/2$ in $\operatorname{Spec} R^{\square}_{\overline{\rho}}/\varpi$ such that for any regular Hodge type $\underline{\lambda}$ and any inertial type τ, each of the special fibres $\operatorname{Spec} R_{\overline{\rho}}^{\mathrm{crys},\underline{\lambda},\tau}/\varpi$ is set-theoretically supported on some union of these cycles.*

Proof. We have $\operatorname{Spec} R_{\overline{\rho}}^{\mathrm{crys},\underline{\lambda},\tau}/\varpi = \operatorname{Spf} R^{\square}_{\overline{\rho}} \times_{\mathcal{X}_d} \overline{\mathcal{X}}^{\mathrm{crys},\underline{\lambda},\tau}$. It follows from (5.2.1), together with the definition of $\mathcal{C}_{\underline{k}}(\overline{\rho})$, that we may write the underlying cycle as

$$Z(\operatorname{Spec} R_{\overline{\rho}}^{\mathrm{crys},\underline{\lambda},\tau}/\varpi) = \sum_{\underline{k}} \mu_{\underline{k}}(\overline{\mathcal{X}}_d^{\mathrm{crys},\underline{\lambda},\tau}) \cdot \mathcal{C}_{\underline{k}}(\overline{\rho}). \tag{5.2.3}$$

The theorem follows immediately (taking our finite set of cycles to be the $\mathcal{C}_{\underline{k}}(\overline{\rho})$). ☐

We can regard this theorem as isolating the "refined" part of [EG14, Conj. 4.2.1]; that is, we have taken the original numerical Breuil–Mézard conjecture, formulated a geometric refinement of it, and then removed the numerical part of the conjecture. The numerical part of the conjecture (in the optic of this paper) consists of relating the multiplicities $\mu_{\underline{k}}(\overline{\mathcal{X}}_d^{\mathrm{crys},\underline{\lambda},\tau})$ to the representation theory of $\mathrm{GL}_n(k)$, as we now recall.

The following is essentially due to Schneider–Zink [SZ99].

5.2.4. **Theorem.** *Let $\tau : I_K \to \mathrm{GL}_d(\overline{\mathbf{Q}}_p)$ be an inertial type. Then there is a finite-dimensional smooth irreducible $\overline{\mathbf{Q}}_p$-representation $\sigma^{\mathrm{crys}}(\tau)$ with the property that if π is an irreducible smooth $\overline{\mathbf{Q}}_p$-representation of $\mathrm{GL}_d(K)$, then the $\overline{\mathbf{Q}}_p$-vector space $\mathrm{Hom}_{\mathrm{GL}_d(\mathcal{O}_K)}(\sigma^{\mathrm{crys}}(\tau),\pi)$ has dimension at most 1, and is nonzero precisely if $\mathrm{rec}_p(\pi)|_{I_F} \cong \tau$, and $N = 0$ on $\mathrm{rec}_p(\pi)$.*

For each regular Hodge type $\underline{\lambda}$ we let $W(\underline{\lambda})$ be the corresponding representation of $\mathrm{GL}_d(\mathcal{O}_K)$, defined as follows: For each $\sigma : K \hookrightarrow \overline{\mathbf{Q}}_p$, we write $\xi_{\sigma,i} = \lambda_{\sigma,i} - (d-i)$, so that $\xi_{\sigma,1} \geq \cdots \geq \xi_{\sigma,d}$. We view each $\xi_\sigma := (\xi_{\sigma,1},\ldots,\xi_{\sigma,d})$ as a dominant weight of the algebraic group GL_d (with respect to the upper triangular Borel subgroup), and we write M_{ξ_σ} for the algebraic $\mathcal{O}_K$-representation of $\mathrm{GL}_d(\mathcal{O}_K)$ of highest weight ξ_σ. Then we define $L_{\underline{\lambda}} := \otimes_\sigma M_{\xi_\sigma} \otimes_{\mathcal{O}_K,\sigma} \mathcal{O}$.

For each τ we let $\sigma^{\mathrm{crys},\circ}(\tau)$ denote a choice of $\mathrm{GL}_d(\mathcal{O}_K)$-stable $\mathcal{O}$-lattice in $\sigma^{\mathrm{crys}}(\tau)$, and write $\overline{\sigma}^{\mathrm{crys}}(\lambda,\tau)$ for the semisimplification of the $\mathbf{F}$-representation of $\mathrm{GL}_d(k)$ given by $L_{\underline{\lambda}} \otimes_{\mathcal{O}} \sigma^{\mathrm{crys},\circ}(\tau) \otimes_{\mathcal{O}} \mathbf{F}$. For each Serre weight $\underline{k}$, we write $F_{\underline{k}}$ for the corresponding irreducible $\mathbf{F}$-representation of $\mathrm{GL}_d(k)$. Then there are unique integers $n_{\underline{k}}^{\mathrm{crys}}(\lambda,\tau)$ such that

$$\overline{\sigma}^{\mathrm{crys}}(\lambda,\tau) \cong \oplus_{\underline{k}} F_{\underline{k}}^{\oplus n_{\underline{k}}^{\mathrm{ss}}(\lambda,\tau)}.$$

Then our "universal" geometric Breuil–Mézard conjecture is as follows.

5.2.5. **Conjecture.** *There are cycles $Z_{\underline{k}}$ with the property that for each regular Hodge type $\underline{\lambda}$ and each inertial type τ, we have $Z_{\mathrm{crys},\underline{\lambda},\tau} = \sum_{\underline{k}} n_{\underline{k}}^{\mathrm{crys}}(\lambda,\tau)\cdot Z_{\underline{k}}$, $Z_{\mathrm{ss},\underline{\lambda},\tau} = \sum_{\underline{k}} n_{\underline{k}}^{\mathrm{ss}}(\lambda,\tau)\cdot Z_{\underline{k}}$.*

5.3. **The relationship between the numerical, refined and geometric Breuil–Mézard conjectures.** In brief (see [EG19a, §5.3] for the details), the relationship is as follows: Conjecture 5.2.5 implies (by pulling back to versal rings) the geometric conjecture of [EG14], which in turn implies the numerical conjecture.

Conversely, the numerical conjecture implies Conjecture 5.2.5; in fact, it is enough to know the numerical conjecture for a single sufficiently generic $\overline{\rho}$ on each irreducible component of $\mathcal{X}_{d,\mathrm{red}}$. To see this, recall that the numerical conjecture for $\overline{\rho}$ is that there are integers $\mu_{\underline{k}}(\overline{\rho})$ such that for all $\underline{k}$, we have

$$e(\mathrm{Spec}\, R_{\overline{\rho}}^{\mathrm{crys},\underline{\lambda},\tau}/\varpi) = \sum_{\underline{k}} n_{\underline{k}}^{\mathrm{crys}}(\lambda,\tau)\mu_{\underline{k}}(\overline{\rho}). \tag{5.3.1}$$

For each $\underline{k}$ we choose a point $x_{\underline{k}} : \mathrm{Spec}\,\mathbf{F} \to \overline{\mathcal{X}}_{d,\mathrm{red}}$ which is contained in $\overline{\mathcal{X}}^{\underline{k}}$ and not in any $\overline{\mathcal{X}}^{\underline{k}'}$ for $\underline{k}' \neq \underline{k}$. We furthermore demand that $x_{\underline{k}}$ is a smooth point of $\overline{\mathcal{X}}_{d,\mathrm{red}}$. (Since $\overline{\mathcal{X}}_{d,\mathrm{red}}$ is reduced and of finite type over $\mathbf{F}$, there is a dense set of points of $\overline{\mathcal{X}}^{\underline{k}}$ satisfying these conditions.) Write $\overline{\rho}_{\underline{k}} : G_K \to \mathrm{GL}_d(\mathbf{F})$ for the representation corresponding to $X_{\underline{k}}$, and assume that the numerical conjecture holds for each $\overline{\rho}_{\underline{k}}$. Then if we set

$$Z_{\underline{k}} := \sum_{\underline{k}'} \mu_{\underline{k}}(\overline{\rho}_{\underline{k}'}) \cdot \overline{\mathcal{X}}^{\underline{k}'}, \tag{5.3.2}$$

it is easy to check that Conjecture 5.2.5 holds.

In particular, by the results discussed in Vytas' and James' lectures, Conjecture 5.2.5 holds if $K = \mathbf{Q}_p$ and $d = 2$. It also holds if $d = 2$, $p > 2$, $\underline{\lambda} = (\underline{0}, \underline{1})$, and τ is arbitrary. We can make the cycles $Z_{\underline{k}}$ completely explicit: we say that a Serre weight $\underline{k}$ for GL_2 is "Steinberg" if for each $\overline{\sigma}$ we have $k_{\overline{\sigma},1} - k_{\overline{\sigma},2} = p - 1$. If $\underline{k}$ is Steinberg then we define $\underline{\tilde{k}}$ by $\tilde{k}_{\overline{\sigma},1} = \tilde{k}_{\overline{\sigma},2} = k_{\overline{\sigma},2}$. Then if $\underline{k}$ is not Steinberg, we have $Z_{\underline{k}} = \mathcal{X}_2^{\underline{k}}$, while if $\underline{k}$ is Steinberg, $Z_{\underline{k}} = \mathcal{X}_2^{\underline{k}} + \mathcal{X}_2^{\underline{\tilde{k}}}$.

This explicit description follows from the results of [CEGS19]; for the details, see [EG19a, Thm. 8.6.2]. (Roughly speaking, the point is that it's easy to compute the tamely Barsotti–Tate deformation rings for generic extensions of characters, and they're either zero or smooth.)

5.4. **The weight part of Serre's conjecture.** Finally, we briefly explain the relationship between Conjecture 5.2.5 and the weight part of Serre's conjecture. For more details, see [GHS18] (particularly Section 6).

We expect that the cycles $Z_{\underline{k}}$ will be effective, in the sense that they are combinations of the $\mathcal{X}_d^{\underline{k}}$ with non-negative coefficients. This expectation is borne out in all known examples, and in any case would be a consequence of standard conjectures about the Taylor–Wiles method.

Assume Conjecture 5.2.5, and assume that the cycles $Z_{\underline{k}}$ are effective. As explained in Section 5.3, it follows that the numerical Breuil–Mézard holds for every $\overline{\rho}$, with $\mu_{\underline{k}}(\overline{\rho})$ being the Hilbert–Samuel multiplicity of the cycle $Z_{\underline{k}}(\overline{\rho})$, which (since $Z_{\underline{k}}(\overline{\rho})$ is effective) is positive if and only if $Z_{\underline{k}}(\overline{\rho})$ is nonzero, i.e. if and only if $Z_{\underline{k}}$ is supported at $\overline{\rho}$. Thus we can rephrase the Breuil–Mézard version of the weight part of Serre's conjecture (i.e. the version suggested by [GK14]) as saying that $W(\overline{\rho})$ is the set of $\underline{k}$ such that $Z_{\underline{k}}$ is supported at $\overline{\rho}$.

Alternatively, we can rephrase this conjecture in the following way: to each irreducible component of $\mathcal{X}_{d,\mathrm{red}}$, we assign the set of weights $\underline{k}$ with the property that $Z_{\underline{k}}$ is supported on this component. Then $W(\overline{\rho})$ is simply the union of the sets of weights for the irreducible components of $\mathcal{X}_{d,\mathrm{red}}$ which contain $\overline{\rho}$.

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