

# The boundary of chaos for interval mappings

Trevor Clark

Sofía Trejo

## Abstract

A goal in the study of dynamics on the interval is to understand the transition to positive topological entropy. There is a conjecture from the 1980's that the only route to positive topological entropy is through a cascade of period doubling bifurcations. We prove this conjecture in natural families of smooth interval maps, and use it to study the structure of the boundary of mappings with positive entropy. In particular, we show that in families of mappings with a fixed number of critical points the boundary is locally connected, and for analytic mappings that it is a cellular set.

**Keywords.** transition to positive entropy, multimodal mappings, renormalization

## 1 Introduction

This paper is motivated by the following conjectures in one-dimensional dynamics about the boundary of mappings with positive topological entropy:

Given a map  $f$  of an interval,  $I$ , let

$$\mathcal{P}er(f) = \{n \in \mathbb{N} : f^n(p) = p \text{ for some } p \in I\}.$$

We refer to  $\mathcal{P}er(f)$  as the *set of periods of  $f$* .

**Boundary of Chaos Conjecture I.** *All endomorphisms of the interval,  $f \in \mathcal{C}^k(I)$ ,  $k = 0, 1, 2, \dots, \infty, \omega$ , with  $\mathcal{P}er(f) = \{2^n : n \in \mathbb{N}\}$ , are on the boundary of mappings with positive topological entropy and on the boundary of the set of mappings with finitely many periods.*

Interest in this conjecture is strongly motivated by its implications on the *routes to chaos*, that is, on the transition from zero to positive entropy, for mappings of the circle or

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T. Clark: Imperial College London; e-mail: t.clark@imperial.ac.uk (corresponding author)

S. Trejo: e-mail: sofia.trejo.a@gmail.com

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the interval, see [MaT1, MaT2]. This conjecture implies that the transition to positive entropy for mappings on the interval occurs through successive period doubling bifurcations. Conjecture I leads to a better understanding of the parameter spaces of smooth dynamical systems. In [OT], the following conjecture was made about the internal structure of the boundary of mappings with positive topological entropy:

**Boundary of Chaos Conjecture II.** *An open and dense subset of the boundary of mappings with positive topological entropy splits into disjoint cells such that each cell is contained in the basin of the quadratic-like fixed point of renormalization. See [OT] for a more precise statement.*

Conjecture I was first made for the space  $\mathcal{C}^1(I)$  in [BHa] and later for each  $k = 0, 1, 2, \dots, \infty, \omega$ , see [MaT1, MaT2, HT, OT]. It is known in  $\mathcal{C}^0(I)$  and  $\mathcal{C}^1(I)$ . In [K1], it was proved that any  $f \in \mathcal{C}^0(I)$  with  $\mathcal{P}er(f) = \{2^n : n \in \mathbb{N}\}$  is on the boundary of mappings with positive topological entropy. In fact, for any compact manifold  $M$ , infinite topological entropy is a generic property amongst endomorphisms of  $M$  in the  $\mathcal{C}^0$  topology [Y]. In [JS], it was proved that  $f \in \mathcal{C}^0(I)$  with  $\mathcal{P}er(f) = \{2^n : n \in \mathbb{N}\}$  can be approximated by mappings with finitely many periods. In [J], Conjecture I was proved for  $\mathcal{C}^1(I)$ . The results in lower regularity are perturbative, and this approach does not seem to work in higher regularity.

## 1.1 Main results

Let  $\mathcal{A}_{\underline{b}}(I)$ , where  $\underline{b} = (\ell_1, \dots, \ell_b)$  is a vector of even integers greater than one, denote the space of analytic mappings of the interval, with critical points  $c_1 < c_2 < \dots < c_b$ , such that the order of  $c_i$  is  $\ell_i$ . We prove the following result for analytic mappings:

**Theorem A.** *All analytic endomorphisms  $f \in \mathcal{A}_{\underline{b}}(I)$  with  $\mathcal{P}er(f) = \{2^n : n \in \mathbb{N}\}$  and all critical points of even order are on the boundary of mappings with positive topological entropy and on the boundary of mappings with finitely many periods in  $\mathcal{A}_{\underline{b}}(I)$ .*

Recall that by Sharkovskii's Theorem, a mapping  $f$  has finitely many periods if and only if for some  $N \in \mathbb{N}$ ,  $\mathcal{P}er(f) = \{2^n : 0 \leq n \leq N\}$ . Let us point out that mappings with finitely many periods are in the interior of mappings with zero topological entropy in  $\mathcal{C}^1(I)$ , [Mi2], so one may replace “boundary of the mappings with finitely many periods” with “boundary of the interior of mappings with zero entropy” in the statement of the theorem. Let us also recall that a mapping with  $\mathcal{P}er(f) = \{2^n : n \in \mathbb{N}\}$  has zero entropy [Mi1].

Theorem A is closely related to the *Density of hyperbolicity*, [KSvS2], which tells us that every mapping can be approximated by mappings where every critical point converges to a periodic attractor, but it does not specify the combinatorics of the mapping used to carry out the approximation. Conjecture I implies that for mappings  $f$  with

$\mathcal{P}er(f) = \{2^n : n \in \mathbb{N}\}$ , this approximation can be done in two combinatorially different ways, and it specifies the combinatorics of the approximating mappings with zero entropy precisely.

Our method used to prove Theorem A, leads us to the following:

**Theorem B.** *The boundary of mappings with positive entropy,  $\Gamma \subset \mathcal{A}_{\underline{b}}(I)$ , where  $\underline{b}$  has only even entries, admits a cellular decomposition. Moreover, there exists an open and dense subset of  $\Gamma$  of disjoint cells, each contained in the basin of a unimodal, polynomial-like fixed point of renormalization.*

A *cell* is a connected set of codimension- $k$  whose boundary contains a (relatively) open and dense set of codimension- $(k+1)$ . A set  $X$  admits a *cellular decomposition* if it can be expressed as a disjoint union of cells. By *basin* we mean the set of all analytic mappings, whose renormalizations converge to the unimodal fixed point of renormalization with the appropriate degree. Note that the approximation of a mapping by mappings in such a basin is not unique.

Theorem B implies that the symbolic dynamics on the Cantor set is the same as the symbolic dynamics on the Cantor set for the unimodal Feigenbaum mapping for an open and dense set of mappings in  $\Gamma$ .

Using the complex bounds of [CvST], we are able to extend Theorem A to spaces of smooth mappings with critical points of even order.

**Theorem C.** *Let  $k \geq 3$  and  $b \in \mathbb{N}$ . If  $\underline{b}$  is a  $b$ -tuple with only even entries, then each  $f \in \mathcal{A}_{\underline{b}}^k(I)$  with  $\mathcal{P}er(f) = \{2^n : n \in \mathbb{N}\}$  is on the boundary of mappings with positive topological entropy and on the boundary of the set of mappings with finitely many periods in  $\mathcal{A}_{\underline{b}}^k(I)$ .*

See page 10 for the definition of the space  $\mathcal{A}_{\underline{b}}^k(I)$ .

To prove Conjecture II, for smooth mappings, we make use of the hyperbolicity of renormalization of  $C^{2+\alpha}$ -unimodal mappings with period-doubling combinatorics, [Da]. See [dFdMP] for the generalization of this result to all bounded combinatorics. The hyperbolic structure at the quadratic-like fixed point of renormalization, gives us a means to understand the structure of the set of mappings on the boundary of positive entropy in spaces of mappings with several critical points. We let  $\mathcal{A}_{even,b}^r(I)$  denote the space of mappings with  $b$  critical points all of even order, see page 10.

**Theorem D.** *An open and dense set of mappings on the boundary of positive entropy in  $\mathcal{A}_{even,b}^r(I)$ ,  $r > 3$ , is a union of disjoint codimension-one manifolds, each of which is contained in the basin of a unimodal, quadratic-like fixed point of renormalization.*

Specifically, the dense set of mappings which can be decomposed into codimension-one manifolds consists of mappings with all critical points non-degenerate and with exactly one solenoidal attractor. The boundaries of these manifolds contain mappings where the solenoidal attractor contains more than one critical point. Since we do not know that sets of such mappings are manifolds, we are unable to obtain the cellular decomposition of the boundary of positive entropy.

The following is an interesting consequence of Theorem [D](#).

**Theorem E.** *Let  $r > 3$ , and let  $\underline{b}$  be a  $b$ -tuple of even integers. The connected components of the boundary of mappings with positive topological entropy in  $\mathcal{A}_{\underline{b}}^r(I)$  are locally connected.*

This result should be contrasted with the Theorem of [\[FrT\]](#) that the boundary of mappings with positive entropy in the family of bimodal mappings of the circle is not locally connected, and the result of [\[BruvS2\]](#), which shows that many isentropes in families of polynomials are not locally connected. Let us point out that the mechanisms used to produce non-local connectivity in these cases are not present in our setting. The result of [\[FrT\]](#) relies on there being an accumulation of pieces of Arnold tongues in the boundaries of phase locking regions with definite “height” above the critical line in the boundary of mappings with positive entropy. This phenomenon creates a comb-like structure in the boundary. The families considered in [\[BruvS2\]](#) do not have a constant number of critical points, and the proof there that certain isentropes are non-locally connected requires that the entropy of the isentrope is positive.

Mappings with  $\mathcal{P}er(f) = \{2^n : n \in \mathbb{N}\}$  are infinitely renormalizable [\[HT, Theorem 3\]](#), see Section [3](#), and such mappings have been the subject of intense study over the past thirty years. Previous results in the direction of those in this paper have been obtained via proofs of the Hyperbolicity of Renormalization Conjectures, [\[TC, CT, F1, F2\]](#) (or at least convergence of renormalization together with certain rigidity results, [\[Sm2\]](#)). For unimodal mappings with critical points of even order, the solution of the renormalization conjectures imply, roughly, that the stable manifold of the (period-doubling) renormalization operator consists of mappings which are topologically conjugate to the fixed point,  $f_*$ , of renormalization, and the family  $\{f_* + \lambda v\}$ , where  $v$  is the expanding direction for renormalization and  $\lambda \in (-\varepsilon, \varepsilon)$ , is transverse to the topological conjugacy class of  $f_*$ . Moreover,  $f_*$  is a polynomial-like mapping, which is hybrid conjugate to the Feigenbaum polynomial, and the family  $z \mapsto z^2 + c$  is transverse to the topological conjugacy class. Thus one obtains Conjecture I for such unimodal mappings from the solution of the renormalization conjectures together with the solution of Conjecture I for unicritical, real, polynomials. Theorem [A](#) has been proved for analytic unimodal mappings [\[La, S, L3\]](#). Renormalization results for smooth unimodal mappings with quadratic critical points were ob-

tained in [Da] and [dFdMP]. In [dFdMP] for  $\gamma \in (0, 1)$ , sufficiently close to one, the authors of that paper proved hyperbolicity of renormalization (with bounded combinatorics) for  $C^{2+\gamma}$  mappings, and proved that the stable manifold of the renormalization operator is a  $C^1$  codimension-one submanifold of the space of  $C^{3+\gamma}$  mappings. Thus proving Theorem A for  $C^{3+\gamma}$  unimodal mappings with non-degenerate critical points. In [Sm2], using convergence of renormalization and rigidity, Smania proved Conjecture I for multimodal mappings with all critical points non-degenerate and with the same  $\omega$ -limit set (indeed, in [Sm5] he goes beyond this to prove hyperbolicity of renormalization for these mappings). In this paper, we remove these two conditions to prove Theorem A. We remove the condition that each critical point is non-degenerate by using the complex bounds of [CvST], see Theorem 2.4. The condition on the number of solenoidal attractors is removed through a technical perturbation argument, Lemma 5.5.

While we do not focus on renormalization in this paper, let us point out that by now it is not difficult to remove the condition that all critical points are non-degenerate from [Sm2]. McMullen, [McM2], proved exponential convergence of renormalization acting on quadratic-like mappings, which are infinitely renormalizable of bounded type. This was extended to multimodal mappings with quadratic critical points by Smania, [Sm2]. From the complex bounds of [CvST] and the quasiconformal rigidity of analytic mappings, [CvS], it is possible to extend this proof to infinitely renormalizable mappings of bounded type in  $\mathcal{A}_b(I)$ . Let us mention that using the *decomposition of a renormalization*, exponential convergence of renormalization for  $C^k$ ,  $k \geq 3$ , symmetric unimodal mappings, in the  $C^k$ -topology, was proved in [AMdM]. Renormalization ideas figure heavily in our proof; however, we leave the investigation of the rate of convergence of renormalization (of, in particular, smooth mappings) to future work.

We believe that the methods used in this paper can be improved on to extend Theorem C to  $C^2$  mappings with critical points of integer power; however, developing these tools would take us far from the goal of this paper. Since our proof of Theorem D depends on hyperbolicity of the quadratic fixed point of renormalization, extending this result to mappings with lower regularity would require a different approach. Let us also remark that our methods depend heavily on complex tools, so we do not obtain results for mappings with flat critical points or with critical points of non-integer order.

## 1.2 Outline of the paper

In Section 2, we state some basic definitions which will be used throughout this paper, and give the necessary background in real dynamics. In Section 3, to make this paper more self-contained, we reduce Theorem A to an equivalent statement about infinitely renormalizable mappings with zero entropy, Theorem F. In Section 4, we introduce the different spaces of mappings in which we will work.

Of particular importance to us is the space of stunted sawtooth mappings,  $\mathcal{S}$ , see Section 4.1. Stunted sawtooth mappings were introduced in [MT]. From a combinatorial point of view, they model mappings of the interval with finitely many critical points well. Moreover, the space of stunted sawtooth mappings is a convenient space of mappings to work in since in this space, entropy is monotone in each of the parameters (the “signed heights” of the plateaus). Indeed the analogue of Theorem A is known in this space:

**Theorem 1.1.** [HT] *Let  $T_\xi \in \mathcal{S}$  be so that  $\text{Per}(T_\xi) = \{2^n : n \in \mathbb{N}\}$ . Given  $\nu > 0$  there exists  $\alpha, \beta \in [-e, e]^m$  so that  $|\xi - i| < \nu$  for  $i = \alpha, \beta$  where  $h(T_\alpha) > 0$  and  $T_\beta$  has only finitely many periods.*

This result is the starting point for the results of this paper. In Section 5, we will transfer it successively to the space of polynomials using ideas from [BruvS], then via the Douady-Hubbard Straighening Theorem to polynomial-like mappings, and finally to analytic mappings with even critical points via renormalization and specifically the complex bounds of [CvST]. Similar tools to those used to prove Theorem A together with the transversal non-singularity of the derivative of the renormalization operator acting from the space of analytic mappings to the space of polynomial-like germs are used to prove Theorem B. We obtain Theorem C from Theorem A via an approximation argument, which is similar to one used in [GMdM]. Once we have proved Theorem C, we use it together with results of [Da] on the hyperbolicity of the period-doubling renormalization operator acting on smooth unimodal mappings to prove Theorem D. Finally we deduce Theorem E.

## 2 Preliminaries

### 2.1 Notation and terminology

Given a topological space  $X$  and  $A \subset X$  we denote the *boundary* of  $A$  by  $\partial A$  and its *closure* by  $\text{cl}(A)$ . If  $X$  is a metric space we denote the *open ball of radius  $\varepsilon$*  centred around  $x \in X$  by  $B_\varepsilon(x) = \{y \in X : \text{dist}(x, y) < \varepsilon\}$ .

As usual,  $\mathbb{R}$  and  $\mathbb{C}$  denote the real line and the complex plane, respectively, and  $I$  will always denote a compact interval in  $\mathbb{R}$ . It will be convenient to assume that  $I = [-1, 1]$ . We denote the circle  $\mathbb{R} \bmod 1$  by  $\mathbb{T}$ . If  $X$  is a set and  $x \in X$ , we let  $\text{Comp}_x(X)$  be the connected component of  $X$  containing  $x$ .

Given a (continuous) piecewise monotone map  $f : I \rightarrow I$  we call its local extrema *turning points*. If  $f$  has finitely many turning points and  $f(\partial I) \subset \partial I$ , then  $f$  is called a *multimodal map*. The images of the turning points of a multimodal mapping are called *critical values*.

## 2.2 Background in dynamics

Given a function  $f : X \rightarrow X$  acting on a topological space  $X$  the *orbit* of a point  $x \in X$  is defined as the set  $\mathcal{O}_f(x) = \{f^n(x) : n \in \mathbb{N}\}$ . The set of accumulation points of  $\mathcal{O}_f(x)$  is known as the  $\omega$ -*limit set* of  $x$  and is denoted by  $\omega(x)$ . A point  $x \in X$  is called *non-wandering* if given any open set  $U \ni x$  there exists  $n \in \mathbb{N}$  such that  $f^n(U) \cap U \neq \emptyset$ . The set of non-wandering points of a map  $f$  will be denoted by  $\Omega(f)$ . In particular, if  $x \in \omega(x)$ , then we say that  $x$  is *recurrent*.

**Definition 2.1.** In a space of mappings  $\mathcal{X}$ , we let  $\Gamma_{\mathcal{X}}$  denote the subset of  $\mathcal{X}$  consisting of mappings  $f$  with  $\mathcal{P}er(f) = \{2^n : n \in \mathbb{N}\}$ . When it will not cause confusion, we omit  $\mathcal{X}$  from the notation.

Given a piecewise monotone map  $f : I \rightarrow I$  with  $m$ -turning points  $-1 < c_1 < \dots < c_m < 1$  we denote by  $i_f(x)$  the *itinerary of  $x$*  and the *kneading sequence* of  $c_j$  by

$$\nu_j := \lim_{x \downarrow c_j} i_f(x),$$

where the sequence  $\nu_j$  consists of the symbols  $I_0, \dots, I_m$ , where the  $I_i$ 's are the intervals from  $I \setminus \{c_1, \dots, c_m\}$ . Finally, we denote by

$$\nu(f) = (\nu_1, \dots, \nu_m)$$

the *kneading invariant of  $f$* . See [dMvS] for the definition of the itinerary of a point.

**Definition 2.2.** Let  $f : I \rightarrow I$  be a piecewise monotone map with turning points  $-1 < c_1 < c_2 < \dots < c_r < 1$  and critical values  $v_1 < v_2 < \dots < v_s$ . Then, we define its *shape* as the set:

$$\tau = \{(i, j_i) : 1 \leq i \leq r\},$$

where  $j_i \in \{1, \dots, s\}$  is so that  $f(c_i) = v_{j_i}$ .

For example the map in Figure 1 has shape

$$\tau = \{(1, 3), (2, 2), (3, 3), (4, 1), (5, 3), (6, 2), (7, 3)\}.$$

The shape keeps track of the order of critical values, which critical points have which critical values and, in particular, which critical points have the same critical values. This notion of shape is useful in the study of mappings arising as renormalizations. Since such mappings are compositions of unimodal mappings, they have more ‘‘symmetries’’ than general polynomials.

We say that a set of ordered pairs

$$\tau = \{(i, j_i) : i \in \{1, 2, \dots, r\} \text{ and } -1 \leq j_i \leq 1\}$$

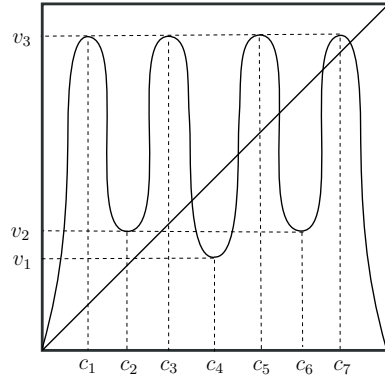


Figure 1

is a *shape*. Observe that given a shape we can define a kneading invariant.

Given a multimodal map  $f$  and a forward invariant set  $A \subset I$  we say that  $A$  is a *topological attractor* if its basin  $\mathcal{B}(A) = \{x : \omega(x) \subset A\}$  satisfies the following properties:

- the closure of  $\mathcal{B}(A)$  contains intervals;
- every closed forward invariant set  $A' \subsetneq A$  has a smaller basin of attraction, *i.e.*  $\text{cl}(\mathcal{B}(A)) \setminus \text{cl}(\mathcal{B}(A'))$  contains intervals.

### 2.2.1 Renormalization

**Definition 2.3.** Let  $f : I \rightarrow I$  be an interval map and let  $n \in \mathbb{N}$ . A proper subinterval  $J \subset I$  is called a *restrictive interval* of period  $n$  if:

- the interiors of  $J, \dots, f^{n-1}(J)$  are pairwise disjoint;
- $f^n(J) \subset J$  and  $f^n(\partial J) \subset \partial J$ ;
- at least one of the intervals  $J, \dots, f^{n-1}(J)$  contains a turning point of  $f$ ;
- $J$  is maximal with respect to these properties.

If  $f$  has a restrictive interval,  $J$ , we say that  $f$  is *renormalizable*. Furthermore, if  $\Phi : J \rightarrow I$  is an affine surjection, the *renormalization operator*,  $f \rightarrow \mathcal{R}(f)$ , is defined by

$$\mathcal{R}(f) = \Phi \circ f^n \circ \Phi^{-1} : I \rightarrow I,$$

and  $\mathcal{R}(f)$  is known as a *renormalization of  $f$* .

Let  $c$  be a turning point of  $f$ . Assume  $f$  possesses infinitely many restrictive intervals  $J_n \ni c$  of period  $q_n$ . If  $q_n \rightarrow \infty$  we say that  $f$  is *infinitely renormalizable* at  $c$ . Under these circumstances the set  $\omega(c)$  is a solenoidal attractor  $L$  with

$$L = \bigcap_{n=0}^{\infty} L_n \quad \text{where} \quad L_n = \bigcup_{n=0}^{q_n-1} J_n.$$



For a proof see Theorem 4.1 in [dMvS].

Suppose that  $f$  is infinitely renormalizable, and the  $\{q_n\}_{n=1}^{\infty}$  is the strictly increasing sequence of  $q_n \in \mathbb{N}$  so that  $f$  has a restrictive interval of period  $q_n$  and no other periods. We say that  $f$  has *bounded combinatorics* if there exists  $M \in \mathbb{N}$  so that  $q_{n+1}/q_n \leq M$  for all  $n$ . A mapping is infinitely renormalizable with period-doubling combinatorics if and only if  $q_{n+1}/q_n = 2$  for all  $n$ .

**Definition 2.4.** Let  $M \in \mathbb{N} \cup \{\infty\}$ . We say that two  $M$ -times renormalizable mappings  $f$  and  $g$  each with exactly one solenoidal attractor *have the same combinatorics up to level  $N \leq M$* , if there are critical points  $c_f$  and  $c_g$ , of  $f$  and  $g$  respectively, such that the following hold: Let  $J_0(f) \supset J_1(f) \supset J_2(f) \supset \cdots \supset J_N(f)$ , be the sequence of all maximal periodic intervals about  $c_f$  for  $f$  and let  $J_0(g) \supset J_1(g) \supset J_2(g) \supset \cdots \supset J_N(g)$ , be the sequence of all maximal restrictive intervals about  $c_g$  for  $g$  (of course if  $N$  is infinite it does not terminate). Assume that the period of  $J_n(f)$ ,  $0 \leq n \leq N$ , is  $p_n^f$ , and for  $0 \leq i < p_n^f$ , let  $J_n^i(f) = \text{Comp}_{f^i(c_f)} f^{-i}(J_n(f))$ , and similarly for  $g$ . We require that for all  $n$ :

1.  $p_n(f) = p_n(g) =: p_n$  and
2. the order of the intervals  $\{J_n^i(f), 0 \leq i < p_n\}$ ,  $\{J_n^i(g), 0 \leq i < p_n\}$  in  $I$  are the same, and
  - $J_n^i(f)$  contains a critical point if and only if  $J_n^i(g)$  contains a critical point and the critical points have the same orders.

If  $N = \infty$  we simply say that  $f$  and  $g$  *have the same combinatorics*.

Let us point out that our definition of “same combinatorics” where we keep track of the orders of critical points is useful for when we consider analytic mappings.

The following result makes use of restrictive intervals to decompose  $\Omega(f)$ , the *non-wandering set* of  $f$ .

**Theorem 2.1.** [dMvS, Theorem III.4.2] *Given a multimodal mapping  $f$ , there exists  $N \in \mathbb{N} \cup \{\infty\}$  such that the following holds.*

1.  $\Omega(f)$  can be decomposed into closed forward invariant subsets  $\Omega_n$ :

$$\Omega(f) = \bigcup_{n \leq N} \Omega_n,$$

where the set  $\Omega_n$  is defined as follows. Let  $K_0 = I$  and let  $K_{n+1}$  be the union of all maximal restrictive intervals of  $f|_{K_n}$ . Then  $K_n$  is a decreasing sequence of nested sets, each consisting of a finite union of intervals for each finite  $n \leq N$ . Then

$$\Omega_n := \Omega(f) \cap \text{cl}(K_n \setminus K_{n+1})$$

for  $n < N$  and  $\Omega_N = \Omega(f) \cap K_N$ . If  $N = \infty$  we define  $K_\infty = \bigcap_{n=0}^{\infty} K_n$ .

2. For each finite  $n \leq N$ , the set  $\Omega_n$  is a union of transitive sets. If  $N = \infty$  we have that  $\Omega_\infty = K_\infty$  is a union of solenoidal attractors.
3. The map  $f$  has zero entropy if and only if  $\Omega_n$  consists of periodic orbits of period  $2^n$  for every finite  $n \leq N$ .

Theorem 2.1 implies that the attractors of maps in  $\Gamma$  can only be periodic or solenoidal. In the latter case the attractor is equal to  $\omega(c)$ , where  $c$  is some turning point at which  $f$  is infinitely renormalizable.

### 2.3 Analytic and smooth mappings

Given  $a > 0$ , let  $\Omega_a = \{z \in \mathbb{C} : \text{dist}(z, I) < a\}$ . We let  $\mathcal{B}_{\Omega_a}$  denote the space of complex analytic mappings on  $\Omega_a$  which are continuous on  $\text{cl}(\Omega_a)$ . We endow  $\mathcal{B}_{\Omega_a}$  with the sup-norm. We let  $\mathcal{B}_{\Omega_a}^{\mathbb{R}}$  denote the set of mappings in  $\mathcal{B}_{\Omega_a}$  that commute with complex conjugation, and call such mappings *real*.

Given  $k \in \mathbb{N}$  we let  $\mathcal{C}^k(I)$  denote the space of  $\mathcal{C}^k$  multimodal maps of the compact interval  $I$ ; *i.e.* maps which are  $k$ -times differentiable with continuous  $k$ -th derivative on some small (real) neighbourhood of  $I$ . We endow  $\mathcal{C}^k(I)$  with the usual norm:

$$\|f\|_{\mathcal{C}^k(I)} = \max_{0 \leq i \leq k} \sup_{x \in I} |f^{(i)}(x)|,$$

where  $f^{(i)}$  denotes the  $i$ -th derivative of  $f$ .

We let  $\mathcal{C}^\omega(I)$  denote the space of real-analytic functions on  $I$ . We endow  $\mathcal{C}^\omega(I)$  with a topology defined as follows: We say that a net  $\{f_\alpha\}$  converges to  $f$  if all the  $f_\alpha$  are analytic on some fixed neighbourhood  $\Omega$  of  $I$  and  $f_\alpha$  converges pointwise to  $f$  on every compact subset of  $\Omega$ .

Given  $\underline{b} = (\ell_1, \dots, \ell_b)$  a vector of positive integers we say that  $f \in \mathcal{C}^k(I)$  belongs to  $\mathcal{A}_{\underline{b}}^k(I)$  if the following holds. The map  $f$  has finitely many parabolic cycles and  $b$  critical points  $c_i$ ,  $1 \leq i \leq b$ , labeled so that  $c_1 < c_2 < \dots < c_b$ , and each  $c_i$  has a neighbourhood on which we can express  $f$  as

$$f(x) = \pm \phi_i(x - c_i)^{\ell_i} + f(c_i),$$

where  $\phi_i$  is a local  $\mathcal{C}^k$  diffeomorphism with  $\phi_i(0) = 0$  and  $\ell_i \in \mathbb{N}$  is at least two. We say that  $\ell_i$  is the *degree* or *order* of  $c_i$ . If  $\ell_i$  is even we say that the corresponding critical point  $c_i$  has even order. Let  $\text{Crit}(f)$  denote the set of critical points of  $f$ . We will denote by  $\mathcal{A}_{\underline{b}}(I)$  the set of analytic maps in  $\mathcal{A}_{\underline{b}}^k(I)$ . When it will not cause confusion we will drop the subscript  $\underline{b}$  from the notation. For  $b \in \mathbb{N}$ , let  $\mathcal{A}_{\text{even}, b}^k(I) = \cup_{\underline{b}} \mathcal{A}_{\underline{b}}^k(I)$ , where the union is taken over  $b$ -tuples,  $\underline{b}$ , with all entries even.

For many of the results in real dynamics that we recall later, the condition that the critical points have integer order is unnecessary. The results which use complex analysis require the condition on the order of the critical points.

We say that a mapping  $f$  is critically finite when its post-critical set

$$\{f^i(c) : c \in \text{Crit}(f), i \in \mathbb{N}\}$$

is a finite set. A mapping is critically finite if and only if all of its critical points are periodic or pre-periodic.

### 2.3.1 Real bounds

Real *a priori* bounds, were first proved for unimodal infinitely renormalizable mappings with bounded combinatorics by Sullivan, [S, dMvS]. For multimodal mappings with all critical points even, real bounds were obtained in [Sm1] for infinitely renormalizable with bounded combinatorics. These were generalized in [Sh2]. We have the following real bounds for infinitely renormalizable mappings.

**Theorem 2.2** (Real Bounds, [CvS, CvST]). *There exists  $\delta > 0$  so that the following holds. Suppose that  $f \in \mathcal{A}^3(I)$  is infinitely renormalizable at a critical point  $c$ , suppose that*

$$J_1 \supset J_2 \supset J_3 \supset \dots$$

*is a sequence of restrictive intervals about  $c$  where  $p_i$  is the period of  $J_i$  and  $p_i \rightarrow \infty$  as  $i \rightarrow \infty$ . Then for  $n$  sufficiently large, if  $f^s : J \rightarrow J_n$  is a diffeomorphism, we have that there exists an interval  $\hat{J} \supset J$  so that  $f^s : \hat{J} \rightarrow (1 + \delta)J_n$  is a diffeomorphism. Moreover,  $(1 + \delta)J_{n+1} \subset J_n$ .*

### 2.3.2 Asymptotically holomorphic mappings

Asymptotically holomorphic mappings have proved to be vital in extending known results for analytic mappings to the case of smooth mappings. One of their first uses in dynamical systems was in a proof of rigidity of quadratic Fibonacci mappings, [L1]. We will make use of a particularly effective asymptotically holomorphic extension given by [GSS]. These extensions have been used to study smooth mappings of the interval, [CvST, CdFvS, CvS], and on the circle, [GdM, GMdM].

Suppose that  $J \subset \mathbb{R}$ , and that  $U$  is an open subset of  $\mathbb{C}$  containing  $J$ . We say that a mapping  $F : U \rightarrow \mathbb{C}$  is *asymptotically holomorphic of order  $k$*  on  $J$  if

$$\frac{\partial F}{\partial \bar{z}}(x) = 0 \text{ for } x \in J, \quad \text{and} \quad \frac{\partial F}{\partial \bar{z}}(x + iy) \rightarrow 0 \text{ as } y \rightarrow 0.$$

Let  $\kappa \geq 1$  and let  $U \in \mathbb{C}$  be an open set. We say that a mapping  $F : U \rightarrow \mathbb{C}$  is  $\kappa$ -*quasiregular* if it is orientation preserving, with local square integrable derivatives,  $f_z$  and  $f_{\bar{z}}$ , which satisfy

$$\max_{\alpha} |\partial_{\alpha} f(z)| \leq \kappa \min_{\alpha} |\partial_{\alpha} f(z)|,$$

for almost every  $z \in U$ , where

$$\partial_{\alpha} f(z) = \cos(\alpha) f_x(z) + \sin(\alpha) f_y(z), \text{ for } \alpha \in [0, 2\pi).$$

We say that  $F$  is *quasiregular* if it is  $\kappa$ -quasiregular for some  $\kappa \geq 1$ .

**Theorem 2.3.** [GSS] *Suppose that  $f$  is a  $\mathcal{C}^k(I)$ , then there is an asymptotically holomorphic extension of  $f$  to a neighbourhood of the interval in the complex plane.*

### 2.3.3 Smooth polynomial-like mappings

A *polynomial-like mapping* is a proper holomorphic branched covering map  $F : U \rightarrow V$ , where  $U \Subset V \neq \mathbb{C}$  are two simply connected complex domains. We will consider polynomial-like mappings up to affine conjugacy. We define the *filled Julia set* for a polynomial-like map  $F$  as:

$$K(F) = \bigcap_{n \in \mathbb{N}} F^{-n}(V).$$

The *Julia set* of  $F$ , denoted by  $J(F)$ , is the boundary of  $K(F)$ . We say that  $F : U \rightarrow V$  is a *real polynomial-like mapping* if it is a polynomial-like mapping,  $U$  and  $V$  are real-symmetric and  $F$  commutes with complex conjugation.

We say that two polynomial-like mappings  $F : U_F \rightarrow V_F$  and  $G : U_G \rightarrow V_G$  are *quasiconformally equivalent* if there exists a qc-mapping  $H$  defined on a neighbourhood  $W$  of  $K(F)$  to a neighbourhood of  $K(G)$  such that  $H \circ F(z) = G \circ H(z)$ ,  $z \in W$ . If additionally we have that  $\bar{\partial}H = 0$  on  $K(F)$ , then we say that  $F$  and  $G$  are *hybrid equivalent*.

An *asymptotically holomorphic polynomial-like mapping*, abbreviated *AHPL-mapping*, of order  $k$  is a proper  $\mathcal{C}^k$  branched covering map  $F : U \rightarrow V$ , where  $U \Subset V \neq \mathbb{C}$  are two simply connected complex domains, which is asymptotically holomorphic of order  $k$  on  $U \cap \mathbb{R}$ . Every such asymptotically holomorphic polynomial-like mapping in this paper has the properties that  $U$  and  $V$  are real-symmetric and  $F$  commutes with complex conjugation.

### 2.3.4 Complex bounds

Complex bounds for real mappings have a long history, see the introduction of [CvST]. In the classes of mappings most relevant to us, they were first proved for real-analytic, infinitely renormalizable unimodal mappings with bounded combinatorics by Sullivan,

[S]. This result was extended to analytic multimodal mappings with all critical points even by Smania [Sm1]. The authors together with van Strien proved the following, which built on work of [Sh2, KSvS1].

**Theorem 2.4.** [CvST] *Suppose that  $f \in \mathcal{A}^k(I)$ ,  $k \geq 3$ , is infinitely renormalizable at an even critical point  $c_0$ . Let  $J_1 \supset J_2 \supset \dots$  denote the sequence of restrictive intervals for  $f$  about  $c_0$ , where the period of  $J_i$  is  $q_i$ . Then for all  $i$  sufficiently large, there exists an asymptotically holomorphic polynomial-like mapping of order  $k$ ,  $F : U \rightarrow V$  with  $F = f^{q_i}|_U$ ,  $U \supset J_i$  and  $\text{mod}(V \setminus U)$  bounded away from zero. Moreover, the dilatation of  $F$  tends to zero as  $i$  tends to infinity.*

*If  $f$  is analytic, then  $F$  is a polynomial-like mapping.*

When  $f$  is analytic, the polynomial-like mapping  $F$  is constructed using the holomorphic extension of  $f$  to a neighbourhood of the interval. If  $f \in \mathcal{A}^k(I)$ , the extension is constructed for any  $C^k$  asymptotically holomorphic extension of order  $k$  of  $f$  to a neighbourhood of  $I$ . By Theorem 2.3 at least one such extension exists.

The following lemma is useful for working with asymptotically holomorphic mappings.

**Lemma 2.5** (Stoilow Factorization). *If  $F : U \rightarrow V$  is a quasiregular mapping, then we can factor  $F$  as  $F = h \circ \phi$  where  $\phi : U \rightarrow U$  is quasiconformal and  $h : U \rightarrow V$  is analytic.*

Indeed Stoilow Factorization together with compactness of the spaces of (holomorphic) polynomial-like mappings [McM1, Theorem 5.8] and  $K$ -qc mappings implies:

**Lemma 2.6.** *Let  $K \geq 1$ ,  $\delta \geq 0$  and  $b \in \mathbb{N}$ . Then the space of  $K$ -quasiregular asymptotically holomorphic polynomial-like mappings  $F : U \rightarrow V$  of degree  $b$  with  $\text{mod}(V \setminus U) \geq \delta$  is compact.*

### 2.3.5 Quasisymmetric rigidity

Quasisymmetric (quasiconformal) techniques were introduced into one-dimensional dynamics by Dennis Sullivan, who observed that quasisymmetric rigidity of unimodal mappings could be used to prove density of hyperbolicity. Quasiconformal rigidity was first proved in [L2, GS] for quadratic polynomials. It was later proved for all real polynomials in [KSvS1]. The first author together with van Strien proved the following:

**Theorem 2.7.** [CvS] *For mappings of the interval we have the following:*

1. Rigidity for analytic mappings. *Suppose that  $f, \tilde{f} \in \mathcal{A}_b(I)$ , are topologically conjugate mappings by a conjugacy which is a bijection on*

- the sets of parabolic points,
- the sets of critical points and corresponding critical points have the same order.

Then  $f$  and  $\tilde{f}$  are quasimetrically conjugate.

2. Rigidity for smooth mappings. Suppose that  $f, \tilde{f} \in \mathcal{A}_b^k(I)$ ,  $k \geq 3$ , do not have parabolic cycles, and that they are topologically conjugate mappings by a conjugacy which is a bijection on the sets of critical points and corresponding critical points have the same order. Then  $f$  and  $\tilde{f}$  are quasimetrically conjugate.

*Remark.* If  $f$  and  $\tilde{f}$  are  $C^k$  we can allow for parabolic points as in the theorem for analytic mappings under some additional regularity assumptions, see [CvS]. We will only apply rigidity of smooth mappings to deep renormalizations, which do not have parabolic cycles by [dMvS, Theorem IV.B].

For real polynomials we have the following:

**Theorem 2.8.** [KSvSI, CvST] Suppose that  $f$  and  $\tilde{f}$  are two real polynomials, with real critical points. Assume that  $f$  and  $\tilde{f}$  are topologically conjugate as dynamical systems on the real line, that corresponding critical points for  $f$  and  $\tilde{f}$  have the same order and that parabolic points correspond to parabolic points, then  $f$  and  $\tilde{f}$  are quasiconformally conjugate as dynamical systems on the complex plane.

This result was proved for mappings with all critical points of even order in [KSvS1], and this restriction on the degrees of the critical points was removed in [CvST].

### 2.3.6 Absence of invariant line fields

A *line field* on a subset  $E$  of  $\mathbb{C}$  is a choice of a line through the origin in the tangent space  $T_e X$  at each point  $e \in E$ . For a polynomial, absence of invariant line fields on the Julia set is an ergodic property of the dynamics, which is closely related to rigidity [McM1]. Complex bounds are a key tool in the proof of quasimetric rigidity, and they play a crucial role in establishing the absence of invariant line fields for polynomials. Absence of invariant line fields were first proved in [McM1]. Building on this, they were proved for real infinitely renormalizable polynomial-like mappings in [Sm2], and for real rational maps with all critical points real and with even degrees in [Sh1].

*Remark [McM1].* A line field may be identified with a Beltrami differential  $\mu = \mu(z) \frac{d\bar{z}}{dz}$  with  $|\mu| = 1$ : The real line through  $v = a(z) \frac{\partial}{\partial z}$  corresponds to the Beltrami differential  $\frac{a}{\bar{a}} \frac{d\bar{z}}{dz}$ . Conversely a Beltrami differential determines a function  $\mu(v) = \mu(z) \frac{\bar{a}(z)}{a(z)}$ , where

$v = a(z) \frac{\partial}{\partial z}$  is a tangent vector; the line field consists of those tangent vectors  $v$  for which  $\mu(v) = 1$ .

We will make use of the following theorems about polynomials:

**Theorem 2.9.** [McMI, ShI, CvST] *Suppose that  $f$  is a real polynomial with real critical points. Then  $f$  supports no measurable invariant line field on its Julia set.*

Theorems 2.9 and 2.8 together with the Böttcher Theorem imply the following:

**Corollary 2.10.** *Suppose that  $f$  and  $\tilde{f}$  are topologically conjugate mappings as in the statement of the Theorem 2.8 and with all periodic points repelling. Then  $f$  and  $\tilde{f}$  are affinely conjugate.*

### 3 Entropy and renormalization

In this section we study maps  $f$  with  $\mathcal{P}er(f) = \{2^n : n \in \mathbb{N}\}$  our goal is to show that to prove Theorem A it is enough to prove:

**Theorem F.** Every map  $f \in \mathcal{A}_b^3(I)$ , which is infinitely renormalizable with entropy zero can be approximated by mappings with positive topological entropy and by mappings with finitely many periods.

The equivalence of Theorems A and F is not new, but we include it to help make the paper more self contained.

There are many equivalent definitions of topological entropy. For simplicity of exposition, we use the one introduced in [MiSz1]. Given a continuous piecewise monotone map  $f : I \rightarrow I$  we define the *lap number* of  $f$ , denoted by  $\ell(f)$ , as the number of maximal intervals on which  $f$  is monotone. The topological entropy is defined as the rate of exponential growth of  $\ell(f^n)$ .

**Definition 3.1.** Given a continuous piecewise monotone map  $f : I \rightarrow I$  we define its *topological entropy* as

$$h(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log(\ell(f^n)).$$

For simplicity, we will refer to topological entropy as *entropy*.

The following classical result, see [LiMi], relates the entropy of a map with the periods of its periodic orbits.

**Proposition 3.1.** *A map  $f \in \mathcal{C}^0(I)$  has positive entropy if and only if  $f$  has a periodic orbit of a period which is not a power of two.*



To get a characterization of the boundary of chaos we will take a closer look at the level sets of the entropy map.

**Proposition 3.2.** *We have the following:*

1. *the set of maps with positive entropy is open in the space  $\mathcal{C}^k(I)$  for  $k \geq 2$ , and*
2.  *$\Gamma$  is closed in  $\mathcal{C}^k(I)$ , for  $k \geq 1$ .*

*Proof.* The first statement follows from the fact that the topological entropy is continuous on  $\mathcal{A}^k(I)$ , for  $k \geq 2$ , see Theorem 6 in [MiSz1]. The second statement corresponds to Proposition 2.1 in [Mi2]. □

**Remark:** To show that  $\Gamma$  is a closed set the author in [Mi2] proves the following: The set of maps for which the set  $\mathcal{P}er(f)$  is bounded is  $\mathcal{C}^k$ -open for  $k \geq 1$ .

This remark and Proposition 3.1 imply the following result.

**Corollary 3.3.** *If  $f \in \mathcal{C}^k(I)$  for  $k \geq 2$  is on the boundary of the set of maps with positive entropy, then  $f \in \Gamma$ . The same holds for maps which lie on the boundary of the interior of the set of maps with zero entropy.*

This corollary, also proved in [BHa], provides a characterization of maps on the boundary of chaos in  $\mathcal{C}^k(I)$  for  $k > 1$  which remains true for maps in  $\mathcal{A}(I)$ .

The next result will help us determine the combinatorics of renormalizable maps with zero entropy.

**Proposition 3.4.** [dMvS, Proposition III.4.2] *If  $f \in \mathcal{A}^k(I)$  and  $h(f) = 0$ , then each restrictive interval is contained in a restrictive interval of period 2. Furthermore, every point in  $I$  is either eventually mapped into a restrictive interval of period 2, or is asymptotic to a fixed point.*

**Lemma 3.5.** [HT, Theorem 2] *If  $f \in \Gamma$ , then  $f$  is infinitely renormalizable. Furthermore, if  $J_n$  and  $J_{n+1}$  are consecutive restrictive intervals (meaning that  $J_{n+1}$  is a maximal, with respect to containment, restrictive interval in  $J_n$ ), then the period of  $J_{n+1}$  inside of  $J_n$  is two.*

*Proof.* Consider  $\Delta_j(f)$ , the set of accumulation points of periodic orbits of periods greater or equal to  $2^j$ , and let

$$\Delta(f) = \bigcap_{j \in \mathbb{N}} \Delta_j(f).$$

It is clear from the definition that  $\Delta(f)$  is closed and  $f$ -invariant. In addition, by Lemma 1 in [HT] we know that no point in  $\Delta(f)$  is periodic. Proposition 3.1 and Proposition 3.4,



imply that every point which is not eventually mapped into a restrictive interval of period two is asymptotic to a fixed point. Given  $p \in \Delta(f)$  we have that the orbit of  $p$  enters  $J_0$ , a restrictive interval of period two. By definition, there exists a turning point  $c$  contained either  $J_0$  or in  $f(J_0)$ . Repeating the argument, substituting  $f$  by  $f^{2^n}$  for  $n \in \mathbb{N}$  we can find a nested sequence of restrictive intervals  $J_n$ , with  $J_{n+1}$  of period two under  $f^{2^n}$  inside  $J_n$  and such that  $c \in J_n$ .  $\square$

**Corollary 3.6.** *If  $f \in \mathcal{A}^k$  finitely renormalizable and  $h(f) = 0$  then the period of its periodic orbits is bounded.*

*Proof that Theorem F implies Theorem A.* From Corollary 3.3 we know that maps which can be approximated by mappings with finitely many periods and by mappings with positive entropy belong to  $\Gamma$ , i.e. that the set of periods of their periodic orbits corresponds to the set  $\{2^n : n \in \mathbb{N}\}$ . By Proposition 3.1 and Lemma 3.5 we know that  $\Gamma$  corresponds to the set of infinitely renormalizable maps with entropy zero. Hence Theorem F implies Theorem A.  $\square$

## 4 Spaces of mappings

### 4.1 Stunted sawtooth mappings

In this section, we recall the definition of stunted sawtooth mappings and collect some useful facts about these mappings. In particular, we prove: Any stunted sawtooth mapping  $T$  with  $\mathcal{P}er(T) = \{2^n : n \in \mathbb{N}\}$  can be approximated by

- a stunted sawtooth mapping with all plateaus periodic and entropy zero, and
- by a stunted sawtooth mapping with all plateaus periodic and with positive entropy.

see Proposition 4.2 below.

#### 4.1.1 The definition of the space of stunted sawtooth mappings

We start by defining an auxiliary piecewise linear mapping  $S_0$ , which will be used in the definition of stunted sawtooth mappings. The *basic shape* of a piecewise linear mapping  $S$  is defined as

$$\epsilon(S) = \begin{cases} 1 & \text{if } S \text{ is increasing at the left endpoint of } I, \\ -1 & \text{otherwise.} \end{cases}$$

Let  $\epsilon \in \{-1, 1\}$ . Fix a constant  $m \in \mathbb{N}$ , to be the number of turning points, and set  $\lambda = m + 2$ . The slopes of the piecewise monotone mapping are either  $\lambda$  or  $-\lambda$ . Let  $e = m\lambda/(\lambda - 1)$ , and set  $A = [-e, e]$ . One easily sees that there exists a unique

$m$ -modal piecewise linear mapping  $S_0$  with  $\epsilon(S_0) = \epsilon$ ,  $m$  turning points,  $c_1, \dots, c_m$  at  $-m+1, -m+3, \dots, m-3, m-1$  with the following properties:

- $m+1$  intervals of monotonicity  $I_0 = [-e, c_1], I_1 = [c_1, c_2], \dots, I_m = [c_m, e]$ .
- slopes  $\pm\lambda$ , extremal values  $\pm\lambda$ ; and
- $S_0(\{-e, e\}) \subset \{-e, e\}$ .

See Figure 2.

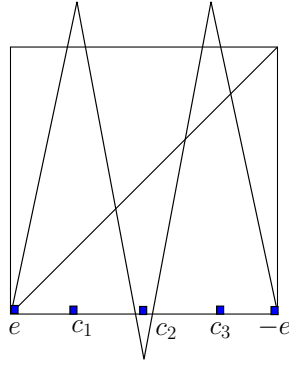


Figure 2: The map  $S_0$

The space of  $\mathcal{S} = \mathcal{S}_{\epsilon, m}$  of *stunted sawtooth maps* with  $m$  turning points consists of continuous maps  $T$  with plateaus  $Z_{i,T}$  with  $i \in \{1, \dots, m\}$  which are obtained from the map  $S_0$  (see Figure 4) and satisfy the following:

- $Z_{i,T}$  is a closed symmetric interval around  $c_i$ ;
- $T$  and  $S_0$  agree outside  $\cup_i Z_{i,T}$ ;
- $T|_{Z_{i,T}}$  is constant and  $T(Z_{i,T}) \in [-e, e]$ ;
- $Z_{i,T}$  have pairwise disjoint interiors.

It is important to remark that a map  $T \in \mathcal{S}$  could have touching plateaus. In other words, two of its plateaus could have one endpoint in common. In this case, we say that  $T$  is  *$m$ -modal in the degenerate sense*. We use the  $m$ -signed extremal values  $\xi \in [-e, e]^m$  to parametrize  $\mathcal{S}$  in the following way.

$$\xi_i = \begin{cases} T(Z_{i,T}) & \text{if } c_i \text{ is a maximum of } S_0, \\ -T(Z_{i,T}) & \text{if } c_i \text{ is a minimum of } S_0. \end{cases}$$

Figure 3 illustrates the parametrization. We denote by  $T_\xi$  the map in  $\mathcal{S}$  with parameters  $\xi = (\xi_1, \dots, \xi_m)$ .

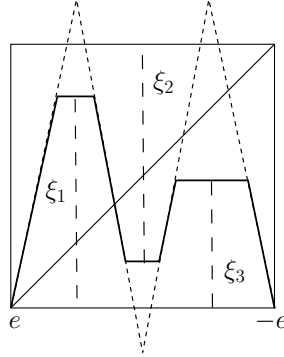


Figure 3: The stunted sawtooth mapping parameterized by  $\xi = (\xi_1, \xi_2, \xi_3)$ .

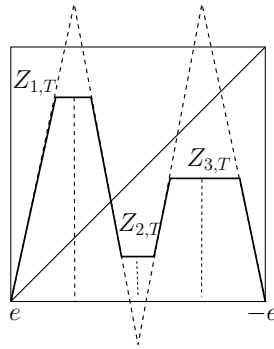


Figure 4: The stunted sawtooth mapping with shape  $\tau = \{(1, 3), (2, 1), (3, 2)\}$  obtained from  $S_0$ .

We can identify the space  $\mathcal{S}$  with the set

$$\{\xi = (\xi_1, \dots, \xi_m) : \xi_i \in [-e, e] \text{ and } \xi_i \geq -\xi_{i+1}\}.$$

We define  $T_\xi < T_{\xi'}$  if for the corresponding parameters  $\xi_i \leq \xi'_i$  for all  $1 \leq i \leq m$  with at least one strict inequality.

The definition of the shape of a stunted sawtooth mapping is the same as for piecewise monotone mappings if we replace turning points by plateaus.

**Definition 4.1.** Given a map  $T \in \mathcal{S}$  we will define its *shape* in the following way. Let  $\ell \leq m$  be the number of distinct values of  $T$  on the plateaus  $Z_{i,T}$ ,  $1 \leq i \leq m$ , and label these values by  $v_j$ ,  $1 \leq j \leq \ell$ , so that  $v_1 < \dots < v_\ell$ . The shape of  $T$  is defined as the set of ordered pairs:

$$\tau(T) = \{(i, j_i) : 1 \leq i \leq m\}$$

where  $j_i$  is so that  $T(Z_{i,T}) = v_{j_i}$ . For example, the shape of the map in Figure 4 is  $\tau(T) = \{(1, 3), (2, 1), (3, 2)\}$ .

Given a map  $T \in \mathcal{S}$  with shape  $\tau$  we define

$$\mathcal{S}(\tau) = \{T' \in \mathcal{S} : \text{the shape of } T' \text{ is equal to } \tau\}.$$

Let us recall two useful facts related to the entropy of stunted sawtooth mappings.

**Proposition 4.1.** [*BruvS, Proposition 4.1*] *The map  $\xi = (\xi_1, \dots, \xi_m) \rightarrow h(T_\xi)$  is non-decreasing in each coordinate.*

The following is a slight variation of the Theorem of Hu-Tresser.

**Proposition 4.2.** [*HT, c.f. Theorem 1*] *Suppose that  $T_\xi$  is a stunted sawtooth mapping with  $\mathcal{P}er(T_\xi) = \{2^n : n \in \mathbb{N}\}$ . Let  $\tau = \tau(T_\xi)$ . Then for any  $\varepsilon > 0$ , there exist,  $\xi', \xi''$  with  $|\xi - \xi'| < \varepsilon$ ,  $|\xi - \xi''| < \varepsilon$ , such that*

1.  $\tau(T_\xi) = \tau(T_{\xi'}) = \tau(T_{\xi''})$ ,
2. both  $T_{\xi'}$  and  $T_{\xi''}$  have all plateaus periodic or pre-periodic,
3.  $T_{\xi'}$  has positive entropy, and
4.  $T_{\xi''}$  has finitely many periods.

Proposition 4.2 is given by a slight variation of the proof of Theorem 1 of [HT] to additionally obtain conclusions (1) and (2) in the statement.

*Proof.* We can apply Theorem 1.1 to find  $\alpha$  and  $\beta$  with  $T_\alpha, T_\beta \in \mathcal{S}_{\varepsilon, m}$ ,  $h(T_\alpha) > 0$  and so that  $T_\beta$  has only finitely many periods. Since the entropy of  $T_\xi$  is zero, either the right endpoint of each plateau of  $T_\xi$  is periodic or it can be approximated by periodic orbits, and since  $\mathcal{P}er(S) = \{2^n : n \in \mathbb{N}\}$ , at least one right endpoint of a plateau is not periodic. Hence, there is at least one plateau whose right end point is in  $\Delta(T_\xi)$ , where  $\Delta(T_\xi)$  is as in Lemma 3.5. Using this information we can use the proof of Theorem 1.1 to ensure two extra properties: First, that  $T_{\xi'}$  and  $T_{\xi''}$  have the same shape as  $T_\xi$ . Second, that the right end points of all plateaus of  $T_{\xi'}$  and  $T_{\xi''}$  are periodic.  $\square$

## 4.2 Multimodal mappings of type $\underline{b}$

In the next two subsections, we introduce two types of mappings which arise naturally when one studies renormalization of multimodal mappings: multimodal mappings of type  $\underline{b}$  and polynomials of type  $\underline{b}$ . These sorts of mappings were considered by Smania in [Sm2] under the additional assumption that all critical points of the mappings have order two, which simplifies the description of the spaces a little.

**Definition 4.2.** Given a vector  $\underline{b} = (\ell_1, \dots, \ell_b)$  of positive even integers, we say that  $f$  is a *multimodal map of type  $\underline{b}$*  if it can be written as a decomposition of  $b$  maps  $f_i \in \mathcal{A}(I)$  (or more generally in  $\mathcal{A}^k(I)$ ,  $k = 0, 1, 2, \dots, \infty, \omega$ ) as follows:

- $f = f_b \circ \dots \circ f_1$ ;

- $f_i$  has a unique critical point  $c_i$ , which is a maximum and has order  $\ell_i$ .
- $f_i(\partial I) \subset \partial I$ ;
- for  $1 \leq i \leq b-1$ ,  $f(c_i) \geq c_{i+1}$  and  $f(c_b) \geq c_1$ .

The vector  $(f_1, \dots, f_b)$  gives a *decomposition* of  $f$ . To each decomposition of  $f$  we associated an *extended map*  $F : I \times \{1, 2, \dots, b\} \rightarrow I \times \{1, 2, \dots, b\}$ , defined as

$$F(x, i) = (f_i(x), (i \bmod b) + 1).$$

We define the *critical set* of  $F$  as  $\text{Crit}(F) = \{(c_i, i) : 1 \leq i \leq b\}$ , where  $c_i$  is the critical point of  $f_i$ . We will always assume that the critical point  $c_1$  has even order.

Multimodal mappings of type  $\underline{b}$  arise naturally as renormalizations of multimodal mappings in  $\mathcal{A}^k(I)$ .

As for multimodal maps, we define renormalizable maps on this space using restrictive intervals. More formally, let  $f$  be a multimodal map of type  $\underline{b}$  and consider an extended map  $F$  induced by a decomposition  $(f_1, \dots, f_b)$ . We say that  $J$  is a *k-periodic interval* of  $F$  if:

- $c_1 \in J$ , where  $c_1$  is the critical point of  $f_1$ .
- The interval  $J, F(J) \dots F^{k-1}(J)$  are pairwise disjoint.
- $F^k(J) \subset J$
- The orbit of  $J$  contains  $\text{Crit}(F)$ .

Under these circumstances we call  $k$  the period of  $J$ . Furthermore, if  $J$  is the maximal periodic interval of period  $k$ , then  $F^k(\partial J) \subset \partial J$  and we say that  $J$  is a *restrictive interval* of  $F$  of period  $k$ . If a map  $F$  possesses a restrictive interval we will say that it is *renormalizable*. It is easy to see that the renormalization of a multimodal mapping of type  $\underline{b}$  is a multimodal mapping of type  $\underline{b}'$ , where  $\underline{b}'$  depends on  $\underline{b}$  and the combinatorics of  $F$ . If  $F$  possesses an infinite sequence of restrictive intervals with periods tending to infinity, we say that the mapping  $F$  is *infinitely renormalizable*.

### 4.3 Polynomials of type $\underline{b}$

**Definition 4.3.** Given a vector  $\underline{b} = (\ell_1, \dots, \ell_b)$  of positive even integers we define the space  $\mathcal{P}_{\underline{b}}$  of *polynomials of type  $\underline{b}$*  as follows. A polynomial  $p : I \rightarrow I$  belongs to  $\mathcal{P}_{\underline{b}}$  if

$$p = q_b \circ \dots \circ q_1,$$

where  $q_i : I \rightarrow I$  has the following properties for  $i \in \{1, \dots, b\}$ :  $q_i(-1) = q_i(1) = -1$ ,  $q_i(0) > 0$  for  $i \neq b$ , and  $q_i = A_i^{-1} \circ p_i \circ A_i$ , where  $p_i : \mathbb{R} \rightarrow \mathbb{R}$  is a polynomial of the form  $z^{\ell_i} + a_i$  which has an invariant interval  $J_i$  and  $A_i : I \rightarrow J_i$  is an affine bijection. The vector given by  $(q_1, \dots, q_b)$  will be called a *decomposition of  $p$* . We identify affinely conjugate polynomials.

Observe that, if in addition we have that  $q_b(0) > 0$  then  $p \in \mathcal{P}_{\underline{b}}$  is a multimodal map of type  $\underline{b}$ .

**Standing assumption:** from now on we will work with a fixed vector  $\underline{b} = (\ell_1, \dots, \ell_b)$  of positive even integers. We will denote the family  $\mathcal{P}_{\underline{b}}$  simply as  $\mathcal{P}$ .

It is important to note that the number of turning points of maps in  $\mathcal{P}$  is not constant, but it is uniformly bounded by a constant depending only on the  $b$  (the length of  $\underline{b}$ ).

Given a polynomial  $p \in \mathcal{P}$  with shape  $\tau$  we let

$$\mathcal{P}(\tau) = \{q \in \mathcal{P} : \text{the shape of } q \text{ is equal to } \tau\}.$$

We say that a shape  $\tau$  is *admissible for polynomials of type  $\underline{b}$*  if  $\mathcal{P}(\tau) \neq \emptyset$ .

**Lemma 4.3.** *Each map  $p \in \mathcal{P}$  has a unique decomposition.*

*Proof.* Let  $p = q_n \circ \dots \circ q_1$  and let  $q_i, p_i, A_i$  and  $J_i$  be as in the definition of  $\mathcal{P}$ . Since  $p_i = z^{\ell_i} + a_i$  the interval  $J_i$  is symmetric with respect to the origin and its right end point is the point  $b_i > 0$  so that  $p_i(b_i) = b_i$ . Since the  $\ell_i$  is fixed, the value of  $b_i$  depends only on  $a_i$ . There exist only two affine maps which map  $[-1, 1]$  to  $[-b_i, b_i]$  bijectively, which are  $z \rightarrow b_i z$  and  $z \rightarrow -b_i z$ . Since  $q_i(-1) = q_i(1) = 1$  we get that  $A_i(z) = -b_i z$ . Hence,  $q_i$  depends only on  $a_i$ . The uniqueness of the decomposition of  $p$  can be proved by induction on  $b$ , the length of  $\underline{b}$ . If  $b = 1$  then  $q_i = q'_i$  if

$$\frac{1}{b_i}[(b_i z)^d + a_i] = \frac{1}{b'_i}[(b'_i z)^d + a'_i]$$

Hence  $a_i = a'_i$ . By definition of  $b_i$  and  $b'_i$  this implies that  $b_i = b'_i$ . So the decomposition is unique. To prove the result for  $b + 1$  we observe the following. We have  $p = q_{b+1} \circ \dots \circ q_1$ . The polynomial  $q_{b+1}$  has exactly one turning point, and the critical value of  $p$  corresponding to the critical point of  $q_{b+1}$  is determined by the critical value of  $q_{b+1}$ , hence it depends only on  $a_{b+1}$ . Since the map  $p' = q_b \circ \dots \circ q_1$  has a unique decomposition the result follows.  $\square$

**Lemma 4.4.** *Suppose that  $\tau$  is admissible for polynomials of type  $\underline{b}$ , and that  $g : I \rightarrow I$  is a piecewise monotone map with  $\tau(g) = \tau$ , then there exists a unique map  $q \in \mathcal{P}(\tau)$  with the same critical values as  $g$ .*

*Proof.* This result can be shown by induction on the length of  $\underline{b}$ , in a similar fashion as the proof of Lemma 4.3.  $\square$

We say that two mappings are *essentially conjugate* if they are topologically conjugate outside of their basins of attraction.

**Proposition 4.5.** *Given a shape  $\tau$  that is admissible for polynomials of type  $\underline{b}$  and a piecewise monotone map  $g : I \rightarrow I$  with  $\tau(g) = \tau$ , there exists a map  $q \in \mathcal{P}(\tau)$ , which is essentially conjugate to  $g$ .*

*Proof.* This result follows from the previous lemma and the proof of Step 1 in Theorem 4.1 in [dMvS].  $\square$

The following two results of [Sm2] generalize immediately to polynomials with critical points not a power of two, so we have not included their proofs. Before we can state them, we must introduce some notation. Let  $\mathcal{P}^*$  denote the set of maps of the form  $p = p_b \circ \dots \circ p_1$ , where  $p_i = z^{\ell_i} + a_i$  and  $(\ell_1, \dots, \ell_b) = \underline{b}$  and let  $Pol_y(\hat{b})$  denote the set of monic polynomials of degree  $\hat{b} = \ell_1 \dots \ell_b$ .

**Proposition 4.6.** [Sm2, Proposition 3.1] *The space  $\mathcal{P}^*$  is a complex submanifold of  $Pol_y(\hat{b})$  with parametrization*

$$(a_1, \dots, a_b) \rightarrow P_{a_b} \circ \dots \circ P_{a_1}.$$

The *connectedness locus* of  $Pol_y(\hat{b})$  is the set of all mappings in  $Pol_y(\hat{b})$  with connected Julia set.

**Proposition 4.7.** [Sm2, Proposition 3.2] *The connectedness locus in  $Pol_y(\hat{b})$  is compact.*

### 4.3.1 Stunted sawtooth mappings and polynomials

In this section we present the results from [BruvS] which we will use in later sections. For a fixed vector  $\underline{m} = (\ell_1, \dots, \ell_m)$  of even integers let  $\mathcal{Q}$  denote the space of polynomials  $q : I \rightarrow I$  with  $q(-1) = q(1) = -1$ , with  $m$  turning points  $-1 < c_1 < \dots < c_m < 1$  where the order of  $c_i$  is  $\ell_i$ .

Given  $p \in \mathcal{Q}$  the following holds. Let  $\mathcal{S} = \mathcal{S}_m$  and  $S_0$  be defined as in Section 4.1. Let  $\nu(p) = (\nu_1, \dots, \nu_m)$  be the kneading invariant of  $p$  and let  $s_i$  be the unique point in the  $(i+1)$ -th lap of  $S_0$  such that

$$\lim_{y \downarrow s_i} i_{s_0}(y) = \nu_i := \lim_{x \downarrow c_i} i_f(x).$$

Let  $Z_i$  be the symmetric interval around the  $i$ -th turning point of  $S_0$  with right end point  $s_i$ . Then we can define a map

$$\Psi : \mathcal{Q} \rightarrow \mathcal{S} \quad \text{by} \quad p \rightarrow \Psi(p),$$

where  $\Psi(p)$  is the unique map in  $\mathcal{S}$  which agrees with  $S_0$  outside  $\cup_{i=1}^m Z_i$  and which is constant on  $Z_i$  with value  $S_0(s_i)$ .

The following result summarizes some of the key properties of  $\Psi$ .

**Lemma 4.8.** [*BruvS*, Lemma 5.1] *The map  $\Psi : \mathcal{Q} \rightarrow \mathcal{S}$*

- *is well-defined;*
- *the kneading invariant of  $p$  and of  $T = \Psi(p)$  are the same in the sense that  $\lim_{y \downarrow Z_i} i_T(y) = \nu_i$ ;*
- *$p$  and  $\Psi(p)$  have the same topological entropy;*
- *$\Psi(p)$  is non-degenerate (see the next paragraph).*

Recall that given a map  $T \in \mathcal{S}$  a pair of plateaus  $(Z_i, Z_j)$  is called *wandering* if there exists  $n \in \mathbb{N}$  such that  $T^n$  of the set  $[Z_i, Z_j]$  (the convex hull of  $Z_i$  and  $Z_j$ ) is a point. We say that a map  $T \in \mathcal{S}$  is *non-degenerate* if for every wandering pair  $(Z_i, Z_j)$  its convex hull belongs to the closure of a component of the basin of a periodic plateau. We will denote by  $\mathcal{S}^*$  the set of non-degenerate maps in  $\mathcal{S}$ . In particular, Lemma 4.16 in [*BruvS*] tells us the following.

**Lemma 4.9.** *If we take a map  $T \in \mathcal{S}^*$  with no periodic attractors, then there exists  $\nu_0 > 0$  so that  $B_{\nu_0}(T) \subset \mathcal{S}^*$ .*

## 4.4 Polynomial-like mappings and germs

### 4.4.1 Polynomial-like mappings of type $\underline{b}$

**Definition 4.4.** Given a vector  $\underline{b} = (\ell_1, \ell_2, \dots, \ell_b)$  of positive even integers we say that map  $F : U \rightarrow V$  is a *polynomial-like map of type  $\underline{b}$*  if there exist simply connected domains  $U = U_1, \dots, U_b, U_{b+1} = V$  and holomorphic maps  $F_i : U_i \rightarrow U_{i+1}$  with  $i \in \{1, \dots, b\}$  satisfying:

- For  $1 \leq i \leq b$ ,  $F_i : U_i \rightarrow U_{i+1}$  is a branched covering of degree  $\ell_i$  with exactly one ramification point.
- $F = F_b \circ \dots \circ F_1$ .



We denote the space of polynomial-like mappings of type  $\underline{b}$  by  $\mathcal{PL}_{\underline{b}}$ . We define the *type* of an AHPL-mapping in the same way.

The following result is an analogue of the Douady-Hubbard Straightening Theorem for polynomial-like mappings of type  $\underline{b}$ .

**Proposition 4.10.** [Sm2, Proposition 4.1] *Let  $\underline{b} = (\ell_1, \ell_2, \dots, \ell_b)$  be a vector of non-negative even integers. Assume  $F : U \rightarrow V$  is a polynomial-like map of type  $\underline{b}$  and that the critical values of  $F$  are contained in  $U$ . Then  $f$  is hybrid conjugate to a polynomial  $P = \chi(F)$  in  $\mathcal{P}^*$ .*

We call the polynomial  $P$  the *straightening of  $F$* , and we refer to the mapping  $\chi$  as the *straightening map*. See page 12 for the definition of hybrid conjugate.

Following [McM1], we endow the space of polynomial-like mappings with the *Carathéodory topology*. A *pointed disk* is a topological disk  $U \subset \mathbb{C}$  with a marked point  $u \in U$ . Let  $\mathcal{D}$  denote the set of pointed disks  $(U, u)$ . We first define the *Carathéodory topology* on  $\mathcal{D}$ . We say that  $(U_n, u_n) \rightarrow (U, u)$  in  $\mathcal{D}$  if

- $u_n \rightarrow u$ ;
- for any compact set  $K \subset U$ ,  $K \subset U_n$  for all  $n$  sufficiently large; and
- for any connected  $N \ni u$ , if  $N \subset U_n$  for infinitely many  $n$ , then  $N \subset U$ .

Now, we define the Carathéodory topology on the space of all holomorphic mappings  $f : (U, u) \rightarrow \mathbb{C}$ , where  $(U, u)$  is a pointed disk. We say that  $f_n : (U_n, u_n) \rightarrow \mathbb{C}$  *converges to  $f : (U, u) \rightarrow \mathbb{C}$*  if:

- $(U_n, u_n) \rightarrow (U, u)$  in  $\mathcal{D}$ , and
- for all  $n$  sufficiently large  $f_n$  converges to  $f$  uniformly on compact subsets of  $U$ .

We endow the space of polynomial-like mappings  $F : U \rightarrow V$  with the Carathéodory topology by choosing the marked point in the filled Julia set.

#### 4.4.2 Polynomial-like germs

We have the following equivalence relation on the space  $\mathcal{PL}_{\underline{b}}$ : Suppose that  $F : U \rightarrow V$  and  $F' : U' \rightarrow V'$  are polynomial-like mappings of type  $\underline{b}$ . We say that  $F \sim F'$  if  $F$  and  $F'$  have a common polynomial-like restriction. By [McM1], Theorem 5.11, we have that if  $F \sim F'$ , then  $K_F = K_{F'}$ , and for mappings with connected Julia set the relation is an equivalence relation. Classes of this equivalence relation are called *polynomial-like germs* and we denote the equivalence class of a polynomial-like mapping  $F$  by  $[F]$ . Let  $\mathcal{PG}$  represent the space of polynomial-like germs, up to affine conjugacy, and let  $\mathcal{PG}^{\mathbb{R}}$  be

the subset of real polynomial-like germs. The space of polynomials is naturally embedded in the space of polynomial-like germs with only real critical points (all of even order). We let  $\mathcal{C}$  denote the connectedness locus in  $\mathcal{PG}$ , and let  $\mathcal{C}^{\mathbb{R}} = \mathcal{C} \cap \mathcal{PG}^{\mathbb{R}}$ .

We say that a polynomial-like germ  $f : U \rightarrow \mathbb{C}$  is *renormalizable* at a point  $c \in \text{Crit}(f)$ , if there exists a neighbourhood  $U_1 \subset U$  of  $c$  and an  $s \in \mathbb{N}$  so that  $f^s : U_1 \rightarrow V$  is a polynomial-like mapping with connected Julia set.

The definitions of quasiconformal equivalence and hybrid equivalence for polynomial-like germs are the same as for polynomial-like mappings. We denote the hybrid class of a polynomial-like mapping or germ  $F$  by  $\mathcal{H}_F$ . Any two polynomial-like germs  $[F]$  and  $[G]$  in the same hybrid class  $\mathcal{H}$  can be included in a *Beltrami disk*: Let  $h$  be a hybrid conjugacy between representatives  $F$  and  $G$  and let  $\mu = \bar{\partial}h/\partial h$  be its Beltrami differential. Let  $\varepsilon > 0$  be so small that  $(1 + \varepsilon)\|\mu\|_{\infty} < 1$ . Define  $\mu_{\lambda} = \lambda\mu$ ,  $\lambda \in \mathbb{D}_{1+\varepsilon}$ . By the Measurable Riemann Mapping Theorem, we obtain a family  $h_{\lambda}$ ,  $\lambda \in \mathbb{D}_{1+\varepsilon}$ , of quasiconformal mappings, the solutions of the associated Beltrami equations. The Beltrami disk through  $F$  and  $G$  is the family of mappings  $\{F_{\lambda} = h_{\lambda} \circ F \circ h_{\lambda}^{-1} : \lambda \in \mathbb{D}_{1+\varepsilon}\}$ . The real one parameter family  $\{F_{\lambda} : |\lambda| < 1 + \varepsilon\}$  is called the *Beltrami path* through  $F$  and  $G$ .

To define the topology on  $\mathcal{PG}$ , we push down the Carathéodory topology which we defined on the space of polynomial-like mappings, see [L3]. We say that a sequence of polynomial-like germs  $[F_n] \rightarrow [F]$  if the sequence of  $[F_n]$  and be split into finitely many subsequences  $[F_m^i]$  which admit representatives  $F_m^i$  which converge to representatives  $F^i$  of  $F$ . In the case when  $J(F)$  is connected, which is the one that is important to us, we do not need to split the sequence  $[F_n]$  into subsequences.

#### 4.4.3 External mappings and matings

Let us fix  $d \in \mathbb{N}$ ,  $d \geq 2$ . Let  $g : \mathbb{T} \rightarrow \mathbb{T}$  be a degree  $d$  real-analytic endomorphism of the unit circle. We say that  $g$  is *expanding* if it admits an extension to a degree  $d$  covering map  $g : U \rightarrow V$  between annular neighbourhoods of  $\mathbb{T}$  with  $U \Subset V$ . We normalize  $g$  by the condition that  $g(0) = 0$ . Let  $\mathcal{E}$  denote this space of normalized expanding endomorphisms of the circle. Let  $\mathcal{E}^{\mathbb{R}}$  denote the space of real-symmetric expanding maps of the circle.

For  $g \in \mathcal{E}$ , let  $\text{mod}(g) = \sup \text{mod}(V \setminus (U \cup \mathbb{D}))$ , where the supremum is taken over all extensions  $g : U \rightarrow V$  as above.

To each  $f \in \mathcal{PG}$  of degree  $d$ , we associate its external mapping  $\pi(f) = g$ : we let  $f$  be a polynomial-like representative of  $f$ , and then use the construction of [DH], Section 2. See also, [L3], Section 3.2. We say that two polynomial-like germs,  $f$  and  $g$ , are externally equivalent if  $\pi(f) = \pi(g)$ .

**Lemma 4.11.** *We have the following:*

- *If  $F$  and  $G$  are externally equivalent polynomial-like germs with connected Julia*

sets, then there is a conformal mapping  $h : \mathbb{C} \setminus K_F \rightarrow \mathbb{C} \setminus K_G$  which conjugates  $F$  and  $G$  near their Julia sets.

- The external mapping  $\pi(F)(z) = z^d$  if and only if  $F$  is a polynomial of degree  $d$ .

**Theorem 4.12** (Mating Theorem). *If  $P$  is a real polynomial of degree  $d$  with connected Julia set and  $g \in \mathcal{E}$ , then there exists, up to affine conjugacy, a unique germ  $F = i_P(g) \in \mathcal{P}\mathcal{G}$  such that  $\chi(F) = P$  and  $\pi(F) = g$ .*

The following theorem can be obtained in exactly the same way as for unicritical mappings, see [AL]. Let  $\mathcal{M} \subset \mathcal{P}_{\underline{b}}$  denote the subset of  $\mathcal{P}_{\underline{b}}$  of mappings  $f$  such that the Julia set,  $J(f)$ , is connected. We let  $\mathcal{M}^{\mathbb{R}}$  denote the real slice of  $\mathcal{M}$ . To simplify matters, we restrict to the real slices of these complex spaces.

**Theorem 4.13.** [AL, c.f. Theorem 2.2] *There is a canonical choice of the straightening  $\chi(f) \in \mathcal{M}^{\mathbb{R}}$ , and an external mapping  $\pi(f) \in \mathcal{E}^{\mathbb{R}}$  associated to each germ  $f \in \mathcal{C}^{\mathbb{R}}$ , which depends continuously on  $f$ . It has the following properties:*

1. For each  $P \in \mathcal{M}^{\mathbb{R}}$ , the hybrid leaf  $\mathcal{H}_P^{\mathbb{R}}$  is the fiber  $\chi^{-1}(P) \cap \mathcal{C}^{\mathbb{R}}$  and the external map  $\pi$  restricts to a homeomorphism  $\mathcal{H}_P^{\mathbb{R}} \rightarrow \mathcal{E}^{\mathbb{R}}$ , whose inverse is denoted by  $i_P$  and is called the (canonical mating).
2.  $(\pi, \chi) : \mathcal{C}^{\mathbb{R}} \rightarrow \mathcal{E}^{\mathbb{R}} \times \mathcal{M}^{\mathbb{R}}$  is a homeomorphism.
3. For  $P, P' \in \mathcal{M}^{\mathbb{R}}$ , if  $f_\lambda$  is a Beltrami path in  $\mathcal{H}_P$ , then  $i_{P'} \circ i_P^{-1}(f_\lambda)$  is a Beltrami path in  $\mathcal{H}_{P'}$ .
4. the external map, straightening and mating are equivariant with respect to complex conjugation.

*Proof.* We will not reproduce the proof given in [AL] here. Let us make a few remarks. First, for the proof that the straightening is a mapping from the space of polynomial-like mappings of type  $\underline{b}$  to the space of polynomials of type  $\underline{b}$ , we refer the reader to [Sm2, Proposition 4.1]. Second, it is easy to check that the mating,  $\mathcal{E} \times \mathcal{P}_{\underline{b}}^{\mathbb{R}} \rightarrow \mathcal{C}^{\mathbb{R}}$  defined by,  $(g, P) \mapsto i_P(g)$ , is continuous in  $g$ . Let us show that it is continuous in  $P$ .

Fix  $g \in \mathcal{E}$ , and choose a path  $g_t \in \mathcal{E}$  that connects  $g_0 : z \mapsto z^d$  with  $g_1 = g$ . Note that  $\mathcal{E}^{\mathbb{R}}$  is contractible by [AL, Lemma 2.1]. Consider the sequence of paths  $f_{t,n} = i_{P_n}(g_t)$ . Passing to a subsequence, we may assume that  $f_{t,n} \rightarrow f_t$ . By the compactness of  $K$ -qc mappings, we have that each  $f_t$  is qc conjugate with  $P$ . Suppose that  $f_t$  is not hybrid conjugate with  $P$ . But this implies that  $K(P)$  supports a measurable invariant line field, so by Theorem 2.9, we have that,  $J(P) \neq K(P)$ , and so  $P$  has a hyperbolic attracting cycle or a parabolic periodic point. If  $P$  has no hyperbolic cycles, then then  $P$  is qc-rigid by [KSvS1], and we have that  $f_t \in \mathcal{H}_P$ . Assume  $P$  has a hyperbolic attractor.

Then there exists a one-parameter family of mappings  $P_s$  through  $P$  so that  $J(P_s)$  moves holomorphically with  $s$ . Now we can conclude that  $f_t \in \mathcal{H}_P$ , and finish the proof of the continuity of the mating as on page 183 of [AL].

□

**Proposition 4.14.** *Let  $b \in \mathbb{N}$ , and let  $\underline{b}$  be a  $b$ -tuple of even integers. Assume that  $\mathcal{X}$  is a compact subset of  $\mathcal{PG}_{\underline{b}}$ . Suppose that  $\mathcal{V}_n \subset \mathcal{PG}_{\underline{b}}$ , is the set of mappings that are at least  $n$ -times renormalizable. Then if  $f_n \in \mathcal{V}_n \cap \mathcal{X}$  is any sequence,  $f_n \rightarrow \Gamma_{\mathcal{PG}_{\underline{b}}}$  in the Carathéodory topology.*

*Proof.* Since the  $f_n$  are contained in compact set, any subsequence of the  $f_n$  must have a subsequence which converges. By Theorem 2.7, this limit is infinitely renormalizable, so it is in  $\Gamma$ . □

#### 4.4.4 Convergence of renormalization for analytic mappings

McMullen proved exponential convergence of renormalization of quadratic-like mappings with bounded combinatorics, [McM2]. These results were generalized by Smania to multimodal mappings with all critical points of degree 2 [Sm2], [Sm3]. By Theorem 2.4 and the quasiconformal rigidity of analytic mappings, Theorem 2.7, we have exponential convergence of renormalization for infinitely renormalizable analytic mappings with bounded combinatorics.

**Theorem 4.15.** *For any  $\underline{b}$ , there exists  $\lambda \in (0, 1)$  so that the following holds. Suppose that  $f, g \in \mathcal{A}_{\underline{b}}(I)$  are infinitely renormalizable mappings of type  $\underline{b}$  (so that  $f$  and  $g$  each has exactly one solenoidal attractor which contains all of its critical points). There exists  $C > 0$ , depending also on the combinatorics of  $f$ , so that*

$$\|\mathcal{R}^n(f) - \mathcal{R}^n(g)\| \leq C\lambda^n,$$

*in the Carathéodory topology. Moreover the limit set of  $\mathcal{R}^n(f)$  is contained in a Cantor set  $\mathcal{K}$ .*

*Remark.* The attractor of the period doubling renormalization operator for multimodal mappings, with more than one critical point, is a horseshoe, [Sm2], [OT]. This is in contrast with the unimodal case where the period-doubling renormalization operator has stationary combinatorics and the attractor is a fixed point.

## 5 The boundary of chaos

### 5.1 Boundary of chaos for polynomials of type $\underline{b}$

Given a map  $p \in \mathcal{P}$  with decomposition  $(q_1, \dots, q_b)$  and with  $q_b(0) > 0$ , let

$$\hat{p}: I \times \{1, \dots, b\} \rightarrow I \times \{1, \dots, b\}$$

denote the extension of  $p$  given in Definition 4.2.

**Theorem 5.1.** *Let  $p \in \Gamma = \Gamma_{\mathcal{P}}$  with  $b$ -turning points and shape  $\tau$ . Given  $\nu > 0$  there exists  $q, r \in \mathcal{P}(\tau) \cap B_\nu(p)$  both will all turning points periodic or pre-periodic, with:*

- $Per(q)$  is finite,
- $h(r) > 0$ .

*Proof.* Apply Lemma 4.8 to obtain a map  $T = \Psi(p) \in \mathcal{S}$  with the same kneading invariant as  $p$  and with the same entropy,  $h(T) = 0$ . Since none of the plateaus of  $T$  are periodic or pre-periodic, all points with periodic itineraries for  $T$  are contained in  $I \setminus (\text{int}(\cup_{i=1}^m Z_{i,T}))$ . Hence, each periodic itinerary corresponds to a unique periodic orbit of  $T$  and  $Per(T) = \{2^n : n \in \mathbb{N}\}$ .

By Proposition 4.2, we can find maps  $\{T_k\}_{k \in \mathbb{N}}$  with all plateaus periodic or pre-periodic and with finitely many periodic orbits. Furthermore, we can guarantee that  $T_k \in B_{1/k}(T)$  for each  $k \in \mathbb{N}$ . In addition, by Lemma 4.9 we may assume that  $T_k \in \mathcal{S}^*$ , see page 24. Now we follow a procedure used in Proposition 5.9 in [BruvS]. For each  $k \in \mathbb{N}$  define  $x \sim_k y$  if there exists  $i > 0$  so that  $T_k^i(x)$  maps the convex hull  $[x, y]$  into one of the plateaus of  $T_k$ . Then collapse each of these intervals  $[x, y]$  to a point and let  $\hat{T}_k$  be the corresponding map. By definition we have that  $\hat{T}_k$  is continuous and since  $T \in \mathcal{S}^*$  has  $m$ -turning points so does  $\hat{T}_k$ . Furthermore, by construction each  $\hat{T}_k \in \mathcal{S}^*(\tau)$ , has no wandering intervals, no inessential attractors and its kneading invariant corresponds to the one of  $T_k$ . By Proposition 4.5 there exists  $p_k \in \mathcal{P}(\tau)$ , which is essentially conjugate to  $\hat{T}_k$ . So  $p_k$  and  $\hat{T}_k$  are conjugate. Hence  $p_k$  has finitely many periodic orbits and entropy zero. Since the connectedness locus of  $\mathcal{P}^*$  is compact, there exists a subsequence  $\{p_{k_j}\}_{j \in \mathbb{N}}$  which converges to a map  $p'$ . Without loss of generality assume  $p_k \rightarrow p'$ . Corollary 3.3 and Lemma 3.5 imply that  $p'$  is an infinitely renormalizable map with entropy zero. Finally, since the kneading invariant of the maps  $T_k$  converges to the kneading invariant of  $T$ , we have that the kneading invariants of the maps  $p_k$  converge to the kneading invariant of  $p'$ . Hence,  $p'$  has the same kneading invariant as  $p$ . By Theorem 2.8,  $p$  and  $p'$  are conjugate by an affine map.

By an analogous argument as the one used to construct the maps  $p_k$ , we can find a sequence of maps  $q_k \in \mathcal{P}(\tau)$ , which have all critical points periodic or pre-periodic and positive entropy so that  $q_k \rightarrow q'$ .  $\square$

## 5.2 Boundary of chaos for polynomial-like germs

**Proposition 5.2.** *Suppose that  $F : U \rightarrow V$  is a real polynomial-like germ which is infinitely renormalizable at a critical point  $c$ . Assume that  $h(F|_{U \cap \mathbb{R}}) = 0$  and that all*

*critical points of  $F$  are even and have the same  $\omega$ -limit set. Then there exist polynomial-like germs  $G : U_G \rightarrow V_G$  and  $H : U_H \rightarrow V_H$  arbitrarily close to  $F$  in the Carathéodory topology such that*

- $G$  has finitely many periods, and
- $h(H|_{U_H \cap \mathbb{R}}) > 0$ .

*Remark.* By infinitely renormalizable, we mean that the restriction of  $F$  to its real trace is infinitely renormalizable about  $c$ .

*Proof.* Let  $P = \chi(F)$  denote the straightening of  $F$ . By Theorem 5.1, there exists a sequence of polynomials  $P_k$  converging to  $P$  such that each  $P_k$  is critically finite and  $h(P_k) = 0$ . By Theorem 4.13, the hybrid classes of the  $P_k$  are connected submanifolds in the space of polynomial-like germs and laminate  $\mathcal{PG}$ . Hence for any neighbourhood  $B \subset \mathcal{PG}$  of  $F$ , there exists  $k$  so that  $\mathcal{H}_{P_k} \cap B \neq \emptyset$ . Hence there exists a sequence of critically finite polynomial-like germs  $G_k$  with  $h(G_k) = 0$  converging to  $F$ .

Similarly, there exists a sequence of polynomials  $P_k$  converging to  $P$  such that each  $P_k$  is critically finite and  $h(P_k) > 0$ , and the same argument implies that there is a sequence of critically finite polynomial-like germs  $H_k$  with positive entropy converging to  $F$ .  $\square$

### 5.3 Boundary of chaos for analytic mappings

Suppose that  $\Lambda$  is a  $\mathcal{C}^k$ -manifold with base point  $\lambda_0 \in \Lambda$ . Suppose that  $X \subset \mathbb{C}$ , we say that  $h_\lambda : X \rightarrow \mathbb{C}, \lambda \in \Lambda$ , is a  $\mathcal{C}^k$ -motion if

- $h_{\lambda_0} = \text{id}$ ,
- $h_\lambda$  is an injection for each  $\lambda \in \Lambda$ , and
- $z \mapsto h_\lambda(z)$  is  $\mathcal{C}^k$  in  $\lambda$ .

We say that  $h_\lambda : X \rightarrow \mathbb{C}, \lambda \in \Lambda$  is a *holomorphic motion* if additionally we require  $\Lambda$  to be a complex Banach manifold, and the mapping  $z \mapsto h_\lambda(z)$  to be holomorphic.

Suppose that  $f \in \mathcal{B}_{\Omega_a}^{\mathbb{R}}$ . Let  $\mathcal{W}$  be a neighbourhood of  $f$  in  $\mathcal{B}_{\Omega_a}^{\mathbb{R}}$ . We say that a periodic interval  $K = K^f \subset [-1, 1]$ , of period  $s$  *persists* in  $\mathcal{W}$  if for each  $g \in \mathcal{W}$ , there is a  $\mathcal{C}^k$ -motion  $h^g : K^f \rightarrow K^g$ , and  $K^g$  is a restrictive interval of period  $s$  and  $h^g \circ f^s(z) = g^s \circ h^g(z)$  for  $z \in \partial K$ . We call  $K^g$  the *continuation of  $K^f$  to  $g$* . Similarly, if  $f^s|_U = F : U \rightarrow V$  is a polynomial-like mapping, we say that  $F : U \rightarrow V$  *persists over  $\mathcal{W}$*  if for each  $g$  in  $\mathcal{W}$  there is a holomorphic motion  $h^g : (U, V) \rightarrow (U_G, V_G)$ , a polynomial-like mapping,  $g^s|_{U_G} = G : U_G \rightarrow V_G$ , and  $h^g \circ F(z) = G \circ h^g(z)$  for  $z \in \partial U$ .

**Lemma 5.3.** *Let  $f \in \Gamma_{\mathcal{A}^k(I)}$  and let  $c$  be a turning point at which  $f$  is infinitely renormalizable. Let  $\{J_n\}_{n \in \mathbb{N}}$  be a sequence of restrictive intervals containing  $c$ . For  $n$  large enough there exists  $\epsilon_n > 0$  so that  $J_n$  persists on  $B_{\epsilon_n}(f)$ .*

*Proof.* Let  $J_n$  be a sequence of restrictive intervals containing a turning point  $c$ . By definition, the boundary points of  $J_n$  are: a periodic point  $p_n$  of period  $2^n$  and a preimage of  $p_n$  under  $f^{2^n}$ . By Theorem IV.B in [dMvS] there exists  $M \in \mathbb{N}$  so that all periodic orbits of prime period greater than  $M$  are repelling. Since  $p_n$  is hyperbolic for all  $n > M$  we have that the interval  $J_n$  persists on a  $C^1$ -neighbourhood of  $f$ . In other words, there exist a neighbourhood  $U_n \ni f$  so that the interval  $J_n$  has a continuation on  $U_n$ . Given a map  $g \in U_n$ , we will denote by  $J_n^g$  its corresponding continuation. Let us show that  $J_n^g$  is a restrictive interval for  $g$ . For all  $n$  sufficiently large, we can guarantee that the results from [CvST] hold for  $f$ . Since  $f \in \Gamma$  we get that  $I_n = J_n$ , where  $I_n$  is an interval from the generalized enhanced nest. By Theorem 3.1 (a) in [CvST] there exists  $\delta > 0$  so that  $V_{n+1} = (1 + \delta)J_{n+1} \subset J_n$ . Make  $\epsilon_n > 0$  small enough so that  $\|f^{2^n} - g^{2^n}\| < \delta/4|J_n|$  and  $J_n$  persists on  $B_{\epsilon_n}(f)$ . Then, if  $g \in B_{\epsilon_n}(f)$  all turning points of  $g^{2^n}|_{J_n^g}$  are contained in  $V_{n+1} \subset J_n^g$ . Hence  $J_n^g$  is a restrictive interval for  $g$  and the result follows.  $\square$

**Lemma 5.4.** *Let  $f \in \Gamma_{\mathcal{A}^k(I)}$  and let  $K_n$  be as in Theorem 2.1. For  $n \in \mathbb{N}$  large enough, there exists  $\nu_n > 0$  so that  $K_n$  persists on  $B_{\nu_n}(f)$ .*

*Proof.* By Theorem 2.1, we know that  $f$  has a finite number of solenoidal attractors  $C_i$ . Furthermore,  $C_i = \omega(c_i)$  for a turning point  $c_i$  at which  $f$  is infinitely renormalizable. Each  $c_i$  has associated a sequence of restrictive intervals  $J_n^i \ni c_i$  of period  $2^n$ . If we let  $K_n$  be as in Theorem 2.1, then for  $n$  large

$$K_n = \bigcup_i \bigcup_{k=0}^{2^n-1} f^k(J_n^i)$$

The persistence of  $K_n$  follows directly from Lemma 5.3 by taking  $\nu_n > 0$  equal to the minimum of the constants  $\epsilon_n$  associated to the intervals  $J_n^i$ . In addition, if  $g \in B_{\epsilon_n}(f)$  then the continuation  $J_n^i(g)$  of  $J_n^i$  associated to  $g$ , is a restrictive interval of period  $2^n$  and

$$K_n(g) = \bigcup_i \bigcup_{k=0}^{2^n-1} f^k(J_n^i(g)).$$

$\square$

**Lemma 5.5.** *Let  $f \in \Gamma_{\mathcal{A}^k(I)}$  and let  $K_n$  be as in Theorem 2.1. Given  $n$  large enough, there exists  $\epsilon_n > 0$  so that  $K_n$  and  $K_{n+1}$  persist on  $B_{\epsilon_n}(f)$ . Furthermore, let  $K_i^g$  be the continuation of  $K_i$ ,  $i = n, n+1$ , associated to  $g \in B_{\epsilon_n}(f)$ . Then for  $0 \leq j \leq n$ , the any  $x \in \Omega_j^g := \Omega(g) \cap cl(K_j^g \setminus K_{j+1}^g)$  is a periodic point of period  $2^m$  for some  $0 \leq m \leq n$ .*



*Proof.* In Lemma 5.4 we proved that for  $n$  sufficiently large, there exists  $\nu_n > 0$  so that  $K_n$  persists on  $B_{\nu_n}(f)$ . Taking  $\nu_n$  smaller if necessary, we can assume that all hyperbolic attracting basins for  $f$  and all repelling periodic points with period less than  $2^n$  persist over  $B_{\nu_n}(f)$

*Claim 1.* Let  $K'_0, K'_1$  be the intervals associated to  $g$  by Theorem 2.1. There exists  $\epsilon_0 > 0$  so that for  $g \in B_{\epsilon_0}(f)$ ,  $\Omega_0^g = \Omega(g) \cap \text{cl}(K'_0 \setminus K'_1)$  consists of fixed points of  $g$ .

The lemma follows inductively from the claim: Let  $J$  be a component of  $K_n$  and consider  $f^{2^n}|_J$ . By the claim there exists  $\epsilon_n \leq \epsilon_{n-1}$  so that if  $g \in B_{\epsilon_n}(f)$ ,  $\Omega_n(g) = \Omega(g) \cap \text{cl}(K_n(g) \setminus K_{n+1}(g))$  consists of fixed points of  $g^{2^n}$ .

*Proof of Claim 1.* To conclude the proof of the lemma, we now prove Claim 1. Let us start by describing how parabolic fixed points bifurcate over small  $\mathcal{C}^3$  neighbourhoods of  $f$ . Suppose that  $p$  is a parabolic periodic point with multiplier 1. We say that  $p$  is of *crossing type*, if on one side of  $p$  the graph of  $f$  is above the diagonal and on the other it is below. Parabolic fixed points with multiplier -1 always cross the diagonal.

There exists a neighbourhood  $Q$  of the set of parabolic points of  $f$  such that if  $g$  is sufficiently close to  $f$ , every fixed point of  $g$  is either in  $Q$  or is a continuation of a hyperbolic fixed point of  $f$ , and each component of  $Q$  contains a parabolic fixed point of  $f$ . We denote the component of  $Q$  that contains  $p$  by  $Q_p$ . We will show that close to the boundary of  $Q_p$ , the behaviour of  $g$  is similar to the behavior of  $f$  and that in  $Q_p$  either there are no periodic cycles, a periodic cycle or an invariant interval for  $g$ .

**Case 1:  $p$  is a parabolic fixed point of  $f$  with multiplier 1, which is not of crossing type.** Assume that the graph of  $f$  is above the diagonal in a neighbourhood of  $p$ . Then  $p$  is attracting from the left and repelling from the right. If  $Q_p$  contains no fixed point of  $g$ , we say that a *gate* opens between the graph of  $f$  and the diagonal. In this case, locally, orbits under the perturbed mapping travel from the left of  $p$  to the right, and  $g$  has no fixed points in  $Q_p$ . So suppose that there is a fixed point of  $g$  in  $Q_p$ . If there is a non-parabolic fixed point, then since  $g$  is close to  $f$ , there are at least two fixed points for the perturbed map. Assume this is the case and let  $q$  denote the fixed point in  $Q_p$  furthest to the left and  $q'$  the fixed point in  $Q_p$  furthest to the right. We have that  $q$  is attracting from the left,  $q'$  is repelling from the right, and  $[q, q']$  is an invariant interval (if it was not invariant, it would contain a critical point, but then  $g$  would not be close to  $f$  in the  $\mathcal{C}^1$  topology). The dynamics in the invariant interval are simple, each orbit converges to a fixed point. Similar analysis holds when the graph of  $f$  is below the diagonal.

**Case 2:  $p$  is a parabolic fixed point of crossing type and multiplier 1.** Either  $p$  is attracting or repelling, and the periodic point persists under small perturbations. We have that if  $p$  is an attracting parabolic fixed point of crossing type for  $f$ , then either there is an attracting (not necessarily hyperbolic) fixed point for  $g$  close to  $p$ , or  $g$  has an invariant



interval containing no turning points near  $p$  that is attracting from the left and the right. A similar analysis holds when  $p$  is a repelling parabolic fixed point of  $f$  with multiplier 1, which is of crossing type: either there is a repelling (not necessarily hyperbolic) fixed point for  $g$  close to  $p$ , or  $g$  has an invariant interval containing no turning points near  $p$  that is repelling from the left and the right.

**Case 3:  $p$  is a parabolic fixed point with multiplier  $-1$ .** Then  $p$  is of crossing type and  $p$  is a parabolic fixed point with multiplier 1 of crossing type for  $f^2$ , and we can apply the above analysis to  $f^2$  in a small neighbourhood of  $p$ .

Suppose that there are parabolic fixed points  $p_0, p_1, \dots, p_{k-1}$  of  $f$  each with multiplier one and not of crossing type such that for each  $i \in \{0, \dots, k-1\}$  there is a point  $x_i$  such that the following holds:

- $f^j(x_i)$  converges to  $p_{i+1 \bmod k}$ ,
- $(f|_{\mathcal{Y}_p})^{-j}(x_i)$  converges to  $p_i$ , where  $\mathcal{Y}_p$  is the monotone branch of  $f$  containing  $p$ .

We will call such a sequence a *pseudo-cycle of orbits*.

*Claim 2.* If the entropy of  $f$  is zero, then no such pseudo-cycle of orbits exists.

*Proof of Claim 2.* Let us recall that if a return mapping to an interval has two full branches, then it has positive entropy [Mi1]. Suppose that  $p_j$  is the parabolic fixed point that is furthest to the right in  $I$ , and  $p_i$  is furthest to the left. If the graph of  $f$  is above the diagonal near  $p_j$ , then  $p_j$  must be repelling from the right. By assumption, there is a pseudo-cycle of orbits which enters  $(p_j - \lambda, p_j)$  for any  $\lambda > 0$ . So the closest turning point to the right of  $p_j$ ,  $c_1$ , is a local maximum. Furthermore, there are no fixed points between  $p_j$  and  $c_1$ .

Let  $\alpha$  be the orientation reversing fixed point closest to  $c_1$ . Let  $J$  be the interval in  $I \setminus \{f^{-1}(\alpha)\}$  that contains  $c_1$ , then since  $J$  is not invariant under  $f^2$  as there is pseudo-cycle, we have that the dynamics of  $f^2$  on  $J$  has positive entropy (the return map has two full branches). So we can assume that the graph of  $f$  is below the diagonal at  $p_j$ . But then  $p_j$  is attracting from the left, and we have that there is a turning point  $c_2$  contained in  $[p_i, p_j]$ , with  $f(c_2) > p_j$ . But now, the graph of  $f$  must cross the diagonal between  $c_2$  and  $p_j$ , and the point where it crosses cannot be attracting, since that would violate the condition on orbits near the parabolic point, so it must be repelling, but now we can argue as before to see that  $f$  must have positive entropy. So we can assume that there is a turning point between  $c_2$  and  $p_j$ , this point must correspond to a minimum of  $f$ , and it must be less than the parabolic point closest to, and on the left of  $c_2$ . Again we have that  $f$  has positive entropy. So no such parabolic fixed points exist.  $\checkmark$

For each repelling periodic point  $p$  of  $f$ , let  $\varepsilon > 0$  be chosen so small that  $B_\varepsilon(p)$  is contained in a neighbourhood of  $p$  where  $f$  is conjugate to  $x \mapsto f'(p)x$ . Let  $U$  be the

union of  $Q$  together with  $\cup B_\varepsilon(p)$ , where the last union is taken over all repelling fixed points of  $f$  in the complement of  $\text{int}(K_1)$ .

Let  $\mathcal{B}$  denote the union of  $K_1$ , basins of hyperbolic attractors, small neighbourhoods of attracting parabolic points of crossing type for  $f$ . From Proposition 3.4, any point  $x$  which is accumulated by  $f^{-n}(\mathcal{B})$ , but which is not in  $f^{-n}(\mathcal{B})$  for any  $n$ , is a (pre)fixed point of  $f$  or is contained in the basin of a one-sided parabolic attractor, so we have that for  $M$  large enough  $K_0 \setminus (\cup_{n=0}^M f^{-n}(\mathcal{B}))$  together with  $\cup_{n=0}^\infty f^{-n}(U)$  contains all but countably many points of  $I$ , each whose forward orbit is eventually fixed, and the complement of  $K_0 \setminus (\cup_{n=0}^M f^{-n}(\mathcal{B}))$  consists of points that are eventually mapped to small neighbourhoods (possibly one-sided) of repelling (not necessarily hyperbolic and possibly one-sided) points and points that converge to a one-sided parabolic attractor.

Suppose first that  $K_1$  persists. Then for any  $x \in \cup_{n=0}^M f^{-n}(\mathcal{B})$  under  $g$  one of the following holds:

- the orbit of  $x$  eventually lands in  $K_1(g)$ ;
- the orbit of  $x$  converges to a hyperbolic attractor;
- the orbit of  $x$  is eventually contained in some  $Q_p$  where  $p$  is a parabolic point of  $f$  and converges to a fixed point of  $g$  in  $Q_p$ .

If  $x \in \cup_{n=0}^\infty f^{-n}(U)$  then either the orbit of  $x$  eventually enters  $\cup_{n=0}^M f^{-n}(\mathcal{B})$ , in which case we know the possibilities for its forward orbit, or the orbit of  $x$  enters  $U$ . In this case, either

- the orbit of  $x$  is eventually contained in some  $Q_p$  where  $p$  is a parabolic point of  $f$  and converges to a fixed point or
- the orbit of  $x$  enters  $\cup_{n=0}^M f^{-n}(\mathcal{B})$ .

So let us assume now that  $\partial K_1$  contains a parabolic point  $p$  with multiplier 1 and that this point cannot be continued to all nearby mappings. Then for some nearby map  $g$  a gate opens up at the boundary of  $K_1$ . Let  $K'_1$  be the union of maximal restrictive intervals of  $g$  and let  $\mathcal{B}'$  denote the union of  $K'_1$  together with

- the corresponding basins of hyperbolic attractors and
- neighbourhoods of the corresponding attracting parabolic points of crossing type.

By the analysis above there are no pseudo-cycles of orbits outside  $K_1(g)$ , so we have that an orbit travels through a bounded number of gates, and eventually passes through one that has the property that any fundamental domain for the dynamics is covered (except for possibly finitely many points) by  $\cup_{n=0}^M g^{-n}(\mathcal{B}') \cup \cup_{n=0}^\infty f^{-n}(U)$ . In particular, every point eventually converges to a fixed point for  $g$  or enters  $K'_1$ . Thus Claim 1 follows.  $\square$

Now we prove Theorem **F** for analytic mappings, and thus obtain Theorem **A**, see the end of Section **3**.

**Theorem 5.6.** *Suppose that  $f : I \rightarrow I$ ,  $f \in \mathcal{A}_{\underline{b}}(I)$  is in  $\mathcal{B}_{\Omega_a}$ , for some  $a > 0$ , with all critical points of even order, which is infinitely renormalizable at some critical point  $c$  and that  $h(f) = 0$ . Then there exist mappings  $g : I \rightarrow I$  and  $\tilde{g} : I \rightarrow I$  arbitrarily close to  $f$ , both in  $\mathcal{A}_{\underline{b}}(I)$ , such that*

- $g$  has finitely many periods, and
- $h(\tilde{g}) > 0$ .

*Proof.* Let  $c$  be a critical point of  $f$  at which  $f$  is infinitely renormalizable. Let  $J_n \ni c$  be the sequence of restrictive intervals with periods  $2^n$  about  $c$ . By Theorem **2.4**, for all  $n$  sufficiently large there exists a polynomial-like mapping of type  $b_1$ ,  $F : U \rightarrow V$ ,  $U \ni c$ ,  $J_n \subset U$  and  $F = f^{2^n}|_U$ , where  $\underline{b}_1$  depends on  $\underline{b}$  and the combinatorics of the renormalization. Moreover, there exists a neighbourhood  $\mathcal{U} \subset \mathcal{B}_{\Omega_a} \cap \mathcal{A}_{\underline{b}}(I)$  of  $f$ , so that the polynomial-like mapping  $F : U \rightarrow V$  persists over  $\mathcal{U}$ . Observe that for each  $G \in \mathcal{U}$ ,  $G : U_G \rightarrow V$  where  $U_G = \text{Comp}_{c_1}(G^{-1}(V))$  is a polynomial-like mapping. Let  $\mathcal{R} : \mathcal{U} \rightarrow \mathcal{PG}_{\underline{b}_1}$  be the renormalization operator from  $\mathcal{U}$  to the space of polynomial-like germs of type  $\underline{b}_1$ , mapping  $g \mapsto G$  where  $G = g^{2^n}|_{U_g}$ .

Since  $\mathcal{R}$  is a composition of affine rescalings and composition of analytic mappings,  $\mathcal{R}$  is analytic. By Lemma **5.4**,  $\mathcal{R}(\mathcal{U})$  contains an open set  $\mathcal{U}'$  containing  $F$ . By Proposition **5.2**, there exists a real polynomial-like mapping  $G \in \mathcal{U}'$ , arbitrarily close to  $F$  so that  $G|_{U_G \cap \mathbb{R}}$  has positive topological entropy. Thus, by continuity of  $\mathcal{R}$ , there exists an analytic mapping  $g \in \mathcal{U}$ , which is a pre-image of  $G$  under  $\mathcal{R}$ . The mapping  $g$  has positive topological entropy, since its renormalization  $G|_{U_G \cap \mathbb{R}}$  has positive topological entropy.

Showing that there is a sequence with zero topological entropy is a little harder, we need to ensure that the preimage under  $\mathcal{R}$  still has zero entropy, and we need to consider all turning points at which  $f$  is infinitely renormalizable.

Let  $c_i$ ,  $1 \leq i \leq m$ , denote the critical points of  $f$  such that  $\omega(c_i)$  is a solenoidal attractor. Let  $m'$  be the number of distinct such solenoidal attractors. For each distinct  $\omega(c_i)$  choose a critical point  $c_{i,0}$ ,  $1 \leq i \leq m'$ , of even order so that  $\omega(c_{i,0}) = \omega(c_i)$  and  $f$  is infinitely renormalizable at  $c_{i,0}$ . By Theorem **2.4**, and since at each (of the at most  $|\underline{b}|$ ) critical points  $c_{i,0}$  of  $f$ ,  $f$  is infinitely renormalizable with periods  $2^n$ , there exists a neighbourhood  $\mathcal{U} \subset \mathcal{A}_{\underline{b}}(I) \cap \mathcal{B}_{\Omega_a}$  of  $f$  and  $N \in \mathbb{N}$ , so that the following holds: For each  $c_{i,0}$ , there exists a  $b_i$ -tuple,  $\underline{b}_i$ , depending on  $\underline{b}$  and the combinatorics of the renormalization, so that the mapping  $f^{2^N} : J_N^i \rightarrow J_N^i$  extends to a polynomial-like mapping of type  $\underline{b}_i$ ,  $F_i : U_{F_i} \rightarrow V_{F_i}$ , which persists over  $\mathcal{U}$ , where  $J_N^i \ni c_{i,0}$  is the restrictive interval of period  $2^N$  containing  $c_{i,0}$ . Let  $\mathcal{R}_i : \mathcal{U} \rightarrow \mathcal{PG}_{\underline{b}_i}$ , and let  $\hat{\mathcal{R}} : \mathcal{U} \rightarrow \mathcal{PG}_{\underline{b}_1} \times \cdots \times \mathcal{PG}_{\underline{b}_{m'}}$  be the

mapping defined by  $\hat{\mathcal{R}}(f) = (\mathcal{R}_1(f), \dots, \mathcal{R}_{m'}(f))$ . We have that  $\hat{\mathcal{R}}$  is continuous (it is a composition of iteration and rescaling in each coordinate), and  $\hat{\mathcal{R}}(\mathcal{U})$  is open. For each  $i$ , there exists an open neighbourhood  $\mathcal{V}_i$  of  $F_i$  in  $\mathcal{PG}_{b_i}$ , which intersects the interior of the set of mappings with zero entropy in  $\mathcal{PG}_{b_i}$ . Moreover, we can choose  $\mathcal{U}$  small enough that Lemma 5.5 holds.

Let  $\mathcal{U}'$  be the preimage of  $(\mathcal{V}_1, \dots, \mathcal{V}_{m'})$  under  $\hat{\mathcal{R}}$ . Let  $g \in \mathcal{U} \cap \mathcal{U}'$  be so that if  $\hat{\mathcal{R}}(g) = (G_1, \dots, G_{m'})$ , then each  $G_i$ ,  $1 \leq i \leq m'$ , has finitely many periods. Let  $K_N$  be the forward invariant set from Theorem 2.1. It is the union of restrictive intervals of period  $2^N$  for  $g$ . By Lemma 5.5, we have that the set of periodic points of  $g$  in  $I \setminus K_N$  has finitely many periods, and we have constructed  $g$  so that its set of periodic points in  $K_N$  also has finitely many periods. Thus, since  $K_N$  is forward invariant,  $g$  has zero entropy.  $\square$

## 5.4 Proof of Theorem B

Theorems 5.7 and 5.8 below imply Theorem B.

**Theorem 5.7.** *Let  $\underline{b}$  be a  $b$ -tuple of even integers. There exists an open and dense subset of  $\Gamma_{A_{\underline{b}}(I)}$  of disjoint cells, each contained in the basin of a unimodal, polynomial-like fixed point of renormalization.*

*Proof.* Let  $\Gamma_1$  denote the subset of  $\Gamma$  consisting of mappings with exactly one solenoidal attractor. Let us show that  $\Gamma_1$  is open and dense in  $\Gamma$ . Suppose that  $f \in \Gamma_1$ , then  $f$  has a single solenoidal attractor and the critical points not in the solenoidal attractor are asymptotic to periodic points of period  $2^n$ , where  $n$  is bounded from above. Thus in any sufficiently small neighbourhood of  $f$ , each mapping has at most one solenoidal attractor. Let us now show that  $\Gamma_1$  is dense in  $\Gamma$ . Suppose that  $f \in \Gamma \setminus \Gamma_1$ . We need to show that we can approximate  $f$  by mappings with a single solenoidal attractor.

We can argue as in the proof of Theorem 5.6. Let  $f$  be an analytic mapping with at least two solenoidal attractors. For ease of exposition, assume that  $F$  has exactly two solenoidal attractors. Then there exist vectors  $\underline{b}_1$  and  $\underline{b}_2$ , and a neighbourhood  $\mathcal{U}$  of  $f$  so that  $\hat{\mathcal{R}} : \mathcal{U} \rightarrow \mathcal{PG}_{b_1} \times \mathcal{PG}_{b_2}$ . Let  $\hat{\mathcal{R}}(f) = (G_1, G_2)$ . By Proposition 5.2, there exist mappings  $G$  arbitrarily close to  $G_2$  in the interior of zero entropy. Thus, since  $\hat{\mathcal{R}}$  is an open, continuous mapping, there exist analytic mappings  $g$  arbitrarily close to  $f$  with exactly one solenoidal attractor. Moreover, as in the proof of Theorem 5.6, by Lemma 5.5, we can find such a  $g$  with zero entropy.

Mappings in  $\Gamma_1$  could have several critical points in their solenoidal attractors. We will now show that there is an open and dense set  $\Gamma_2$  of  $\Gamma_1$  consisting of mappings such that there is only one critical point in the solenoidal attractor. The proof that  $\Gamma_2$  is open (that is, relatively open in  $\Gamma$ ) is the same as the proof that  $\Gamma_1$  is open, and so we omit it. To

prove that  $\Gamma_2$  is dense, we use the strategy used to prove Theorem 5.6. First let us show that in the space of stunted sawtooth mappings we can approximate any mapping in  $\Gamma_{S_b}$  by mappings  $T_k$  in  $\Gamma_{S_b}$  with all but one plateau periodic. If  $h(T) = 0$ , and  $T$  is at most finitely renormalizable, let  $T'$  be the last renormalization of  $T$ . Then, by [BruvS, Lemma 7.6] the  $\omega$ -limit set of each point under  $T'$  is a fixed point of  $T'$ . Moreover, since this fixed point is necessarily attracting, it is contained in a fixed plateau of  $T'$ . By [BruvS, Lemma 7.7], if  $T$  is a stunted sawtooth mapping in the interior of zero entropy, then each point under  $T$  is either (pre)periodic or in the basin of one of the periodic attractors (periodic plateaus) of  $T$ . By [HT], see Theorem 1.1,  $\Gamma_{S_m}$  is the limit of stunted sawtooth mappings with periodic plateaus of period  $2^n$  as  $n \rightarrow \infty$ .

*Claim.* For any  $\varepsilon_0 > 0$ , there exists  $n \in \mathbb{N}$ , so that if  $T = (t_1, t_2, \dots, t_b)$  has a periodic plateau of period  $2^n$  and zero entropy, for some  $t_i$  and some  $\varepsilon \in (0, \varepsilon_0)$ , either  $T_1 = (t_1, \dots, t_{i-1}, t_i + \varepsilon, t_{i+1}, \dots, t_b)$  or  $T_2 = (t_1, \dots, t_{i-1}, t_i - \varepsilon, t_{i+1}, \dots, t_b)$  is in  $\Gamma$ .

*Proof of Claim.* Observe that the space of stunted sawtooth mappings is compact and recall that period-doubling bifurcations occur in each parameter separately. Suppose the claim fails. Then there exists  $\varepsilon_0 > 0$ , so that for any  $n \in \mathbb{N}$ , there exists  $T = (t_1, t_2, \dots, t_b)$  with a periodic plateau of period  $2^n$ , zero entropy, so that for each  $i$ , we have that for each  $\varepsilon \in (0, \varepsilon_0)$ ,  $T = (t_1, \dots, t_{i-1}, t_i \pm \varepsilon, t_{i+1}, \dots, t_b)$  is not in  $\Gamma$ . Since there are at most finitely many plateaus, this implies that for some  $i \in \{1, \dots, b\}$ , there is a sequence of stunted sawtooth mappings  $T_n = (t_1^n, \dots, t_b^n)$  with the  $i$ -th plateau periodic with period  $2^{j_n}$ , and no plateau periodic with period greater than  $2^{j_n}$ , where  $j_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Since period-doubling bifurcations occur sequentially in the space of stunted sawtooth mappings, we can assume that the parameters  $t_i^n$  are monotone. Thus they converge to a limit  $t_*$ . This limiting parameter is accumulated by periodic points of period  $2^n$ . Since  $|t_* - t_i^n| \rightarrow 0$ , we arrive at a contradiction and the claim follows.  $\checkmark$

Now, by Theorem A, by taking  $N$  large we can approximate  $T$  arbitrarily well by mappings with  $\mathcal{P}er(T) = \{2^n : 0 \leq n \leq N\}$ . But now, by the claim, we can perturb such a mapping by moving just one plateau up or down to obtain a mapping in  $\Gamma$ , moreover, the size of this perturbation tends to zero as  $N \rightarrow \infty$ .

To conclude the proof, we can argue as in the proof of Theorem 5.6. Let  $f$  be an analytic mapping with exactly one solenoidal attractor, and let  $c$  be a critical point in the solenoidal attractor. Then for some  $b' \in \mathbb{N}$ , there is a  $b'$ -tuple,  $\underline{b}'$ , so that  $f$  has a polynomial-like renormalization of type  $\underline{b}'$ ,  $F : U \rightarrow V$ , about  $c$ . Let  $P = \chi(F)$  be its straightening. Then by Theorem 1.1,  $\Psi(P)$  is a stunted sawtooth mapping in the boundary of mappings with finitely many periods. Recall the definition of  $\Psi$  on page 24. By the claim, we can approximate  $\Psi(P)$  by stunted sawtooth mappings  $T_j \in \Gamma$  with exactly one plateau in a solenoidal attractor. Thus arguing as in the proof of Theorem 5.1, we can approximate  $P$  by polynomials  $P_j$  of type  $\underline{b}'$  with exactly one solenoidal attractor,

which contains exactly one critical point. So as in the proof of Proposition 5.2, there exist polynomial-like germs converging to  $F$ , which are hybrid conjugate to the  $P_j$ , and finally, as in the proof of Theorem 5.6, we can approximate  $f$ , by analytic mappings in  $\Gamma_2$ .  $\square$

**Theorem 5.8.** *Let  $\underline{b}$  be a  $b$ -tuple of even integers. Then  $\Gamma_{\mathcal{A}_{\underline{b}}(I)}$  admits a cellular decomposition.*

*Proof.* We have proved that there is an open and dense subset of  $\Gamma$  consisting of codimension-1 open sets. Let  $X$  denote one of these cells. We need to show that there is a relatively open and dense subset of  $\partial X$  consisting of codimension-2 cells. Let  $X_1$  denote the subset of  $\partial X$  consisting of mappings with a single solenoidal attractor containing exactly 2 critical points and let  $X_2$  denote the subset of  $\partial X$  consisting of mappings with exactly two solenoidal attractors each containing exactly one critical point.

*Claim 1.*  $X_1 \cup X_2$  is open and dense in  $\partial X$ .

*Proof of Claim 1.* First we show that  $X_1$  and  $X_2$  are open in  $\partial X$ . Suppose that  $f \in X_1$ . Then, relabeling the critical points of  $f$  if necessary, we can assume that  $f$  has a solenoidal attractor which contains  $c_1$  and  $c_2$ , but not  $c_3, \dots, c_b$ . Let  $\mathcal{J}_n$  denote the cycle of the restrictive interval of period  $2^n$ . For  $n$  sufficiently large,  $\mathcal{J}_n \cap \{c_3, \dots, c_b\} = \emptyset$ . Thus, by Lemma 5.3 for  $n$  sufficiently large, there is an open set of mappings  $\mathcal{U}$  containing  $f$ , such that for all  $g \in \mathcal{U}$ , each  $g$  has a restrictive interval of period  $2^n$  and the orbit of this interval contains exactly two critical points of  $g$ . For mappings  $g \in \mathcal{U} \cap \partial X$ , the number of critical points in the solenoidal attractor cannot drop to one, since the condition that  $f$  have a solenoidal attractor containing exactly one critical point is relatively open in  $\Gamma$ . Thus we have that  $X_1$  is relatively open in  $\partial X$ . The proof that  $X_2$  is relatively open is similar, just consider two disjoint cycles of restrictive intervals with sufficiently high period.

Let us now explain how to see that  $X_1 \cup X_2$  is dense in  $\partial X$ . Suppose that  $f \in \partial X$ . If  $f$  has exactly one solenoidal attractor (which must contain at least two critical points) then we show that we can approximate  $f$  by mappings with a single solenoidal attractor, which contains exactly two critical points. If  $f$  has more than one solenoidal attractor, then we show that we can approximate  $f$  by mappings with two solenoidal attractors, each containing exactly one critical point. The strategy for carrying out these approximations is no different than in the proof that codimension-one cells (consisting of mappings with a solenoidal attractor containing exactly one critical point) are dense in  $\Gamma$ , and so we omit the details. One first proves the corresponding statement in the space of stunted sawtooth mappings, and then transfers it successively to polynomials, polynomial-like germs and finally to analytic mappings.  $\checkmark$

*Claim 2.* Each of  $X_1$  and  $X_2$  have codimension-two in  $\mathcal{A}(I)$ .



*Proof of Claim 2.* We refer the reader to [L3, Section 4] for the definition of the tangent space to the space of polynomial-like germs, and to [Sm5, Section 2.4] for the generalization to multimodal mappings.

Suppose that  $f \in X_1$ . Then  $f$  has a renormalization  $R(f) = f^s|_U = F : U \rightarrow V$  that is contained in a space of polynomial-like germs  $\mathcal{PG}_{b'}$  with exactly two critical points, and indeed there is an open set  $\mathcal{U} \ni f$ , such that if  $g \in \mathcal{U}$ , then  $g$  has a polynomial-like renormalization  $g^s|_{U_g} = G : U_G \rightarrow V_G$  in  $\mathcal{PL}_{b'}$ . The codimension of the hybrid class of  $F$  in the space of polynomial-like germs is two. Thus we have that there are vectors  $v_1, v_2 \in T_F \mathcal{PG}_{b'}$ , which are transverse to the hybrid class of  $F$ , and since  $F \in \Gamma_{\mathcal{PG}_{b'}}$ , we have that we can choose these vectors so that for  $t > 0$  and small,  $F - tv_1$  is in the interior of zero entropy and  $F - tv_2$  in  $R(\mathcal{U} \cap X)$ .

Suppose that  $v \in T_{R(f)} \mathcal{PG}_b$ , which is transverse to the hybrid class of  $R(f)$ . Then there exists a sequence  $w_i \in T_f \mathcal{A}_b(I)$ , so that  $DR(f)w_i \rightarrow v$ , see [ALdM, Lemma 4.8] and [Sm5, Theorem 3] for a generalization to the multimodal case. So the renormalization operator is transversally non-singular, and we have that there exist vectors  $w_1, w_2 \in T_f \mathcal{A}(I)$ , so that  $w_1$  and  $w_2$  are transverse to  $\partial X$ . If  $f \in X_2$ , the proof is similar - consider the renormalizations about each of the critical points separately.  $\checkmark$

Proceeding inductively we see that the union of codimension- $j$  cells in  $\Gamma_{\mathcal{A}(I)}$ , where  $j$  runs from 1 though to  $b$  exhausts  $\Gamma$ .  $\square$

## 5.5 Boundary of chaos for smooth mappings.

In this section, we prove Theorems **C** and **D**, which extend Theorems **A** and **B** to smooth mappings.

Suppose that  $f \in \mathcal{A}_b^k(I)$  and let  $\mathcal{W}$  be a neighbourhood of  $f$  in  $\mathcal{A}_b^k(I)$ . If  $f^s|_U = F : U \rightarrow V$  is an asymptotically holomorphic polynomial-like mapping, we say that  $F : U \rightarrow V$  persists over  $\mathcal{W}$  if for each  $g$  in  $\mathcal{W}$  there is a  $\mathcal{C}^k$ -motion  $h^g : (U, V) \rightarrow (U_G, V_G)$ , an asymptotically holomorphic polynomial-like mapping,  $g^s|_{U_G} = G : U_G \rightarrow V_G$ , and  $h^g \circ F(z) = G \circ h^g(z)$  for  $z \in \partial U$ .

Before proving Theorem **C**, let us collect some general tools.

**Lemma 5.9.** [GdM, Proposition 5.5] *For any bounded domain  $U$  in the complex plane, there exists  $C = C(U) > 0$ , with  $C(U) \leq C(W)$  if  $U \subset W$ , such that the following holds: Let  $\{G_n : U \rightarrow G_n(U)\}_{n \in \mathbb{N}}$  be sequence of quasiconformal homeomorphisms such that*

- *the domain  $G_n(U)$  are uniformly bounded; that is, there exists  $R > 0$  so that  $G_n(U) \subset B_R(0)$  for all  $n$ ; and*
- *$\mu_n \rightarrow 0$  in  $L^\infty$ , where  $\mu_n$  is the Beltrami coefficient of  $G_n$  in  $U$ .*

Then given any domain  $U' \Subset U$ , there exists  $n_0 \in \mathbb{N}$  and a sequence  $\{H_n : U' \rightarrow H_n(U')\}_{n \geq n_0}$  of biholomorphisms such that

$$\|H_n - G_n\|_{C^0(U')} \leq C(U) \left( \frac{R}{d(\partial U, \partial U')} \right) \|\mu_n\|_\infty,$$

where  $d$  is the Euclidean distance between the disjoint sets  $\partial U$  and  $\partial U'$ .

**Lemma 5.10.** [*GMdM*, Proposition 11.2] Let  $I$  be a compact interval in the real line and let  $U$  be an open set in the complex plane containing  $I$ . Fix  $M > 0$  and consider the family

$$\mathcal{F} = \{f : U \rightarrow \mathbb{C}, \text{holomorphic} : \|f\|_{C^0} \leq M\}.$$

Then for any  $k \in \mathbb{N}$  and any  $\alpha \in (0, 1)$  there exists  $L = L(k, \alpha, M) > 0$  such that

$$\|f\|_{C^k(I)} \leq L(\|f\|_{C^0})^\alpha.$$

Combining Lemmas 5.9 and 5.10 we obtain a bound on a  $C^k$  norm by a bound on the dilatation of Beltrami differential.

We say that a diffeomorphism  $\phi : I \rightarrow I$  is in the *Epstein class*,  $\mathcal{E}_\beta$ , if there exists  $\beta > 0$ , so that  $\phi^{-1}$  extends to a holomorphic, univalent mapping from the slit complex plane

$$\mathbb{C}_{(-1-\beta, 1+\beta)} = \mathbb{C} \setminus ((-\infty, -1-\beta] \cup [1+\beta, \infty))$$

into  $\mathbb{C}$ . Given a set  $P = \{p_1, \dots, p_b\}$  of  $b$  real unimodal polynomials which preserve the interval  $[-1, 1]$ , we say that a (multimodal) mapping  $f \in \mathcal{A}(I)$  of the interval is in the *Epstein class*,  $\mathcal{E}_{\beta, P}$  if it can be expressed in the form

$$f = \phi_j \circ p_j \circ \phi_{j-1} \circ p_{j-1} \circ \dots \circ \phi_1 \circ p_1,$$

with  $j \leq b$ , where each  $\phi_j$  is in  $\mathcal{E}_\beta$ .

**Lemma 5.11.** [*ShT*, Theorem 2] Suppose that  $f \in \mathcal{A}^k(I)$ ,  $k \geq 2$ . let  $T$  be an open interval such that  $f^s : T \rightarrow f^s(T)$  is a diffeomorphism. Then for any  $S, \alpha, \varepsilon > 0$ , there exists  $\delta = \delta(S, \alpha, \varepsilon) > 0$  and  $\beta = \beta(\alpha) > 0$  satisfying the following. Suppose that  $\sum_{j=0}^{s-1} |f^j(T)| \leq S$  and that  $J$  is a closed subinterval of  $T$  such that

- $f^s(T)$  is a  $\alpha$ -scaled neighbourhood of  $f^s(J)$ , and
- $|f^j(J)| < \delta$  for  $0 \leq j < s$ .

Then letting  $\psi_0 : J \rightarrow I$  and  $\psi_s : f^s(J) \rightarrow I$  be affine diffeomorphisms, there exists a mapping  $G : I \rightarrow I$  in the Epstein class  $\mathcal{E}_\beta$  such that

$$\|\psi_s \circ f^s \circ \psi_0^{-1} - G\|_{C^k} < \varepsilon.$$



To simplify the statements of the following two results about Epstein mappings, let us fix  $b$  real unimodal polynomials,  $p_1, \dots, p_b$ , which preserve the interval  $[-1, 1]$ . Let  $\hat{\mathcal{E}}_\beta = \mathcal{E}_{\beta, P}$ .

By Lemma 5.11 and Theorem 2.2, we have

**Lemma 5.12.** *There exist  $\beta \in (0, 1)$  so that the following holds. Let  $\varepsilon > 0$ . Given any mapping  $f \in \Gamma_{\mathcal{A}^k(I)}$ , there exists  $j_0 \in \mathbb{N}$  and a sequence of Epstein mappings  $H_j$  in  $\hat{\mathcal{E}}_\beta$  with the same domain as  $R^j(f)$ , such that for  $j \geq j_0$ ,*

$$\|R^j(f) - H_j\|_{C^k(I)} \leq \varepsilon.$$

**Lemma 5.13.** *For any  $\beta \in (0, 1)$  and  $b \in \mathbb{N}$ , there exists a Jordan domain  $U_\beta$  containing  $I = [-1, 1]$  and a positive constant  $M_\beta$  so that for any Epstein mapping  $g \in \hat{\mathcal{E}}_\beta$  of  $I$  the holomorphic extension of  $I$  is well-defined in  $U_\beta$  and satisfies  $|g(z)| \leq M_\beta$  for all  $z \in U_\beta$ .*

*Proof.* Since each mapping in the Epstein class  $\hat{\mathcal{E}}_\beta$  can be expressed as a composition:

$$h_j \circ p_j \circ h_{j-1} \circ p_{j-1} \circ \dots \circ h_1 \circ p_1,$$

where  $p_i$  is a polynomial and  $h_i$  is a diffeomorphism in  $\mathcal{E}_\beta$  for  $1 \leq i \leq k$  and  $1 \leq k \leq n$ , the result follows from [GMdM, Proposition 11.5].  $\square$

**Proposition 5.14** (c.f. [GMdM], Theorem 11.1). *There exists a compact set  $\mathcal{K}$  of polynomial-like germs of type  $\underline{b}$  with the following property: Let  $k \geq 3$ . For any  $\varepsilon$  and  $f \in \Gamma_{\mathcal{A}^k(I)}$ , there exists a sequence  $\{f_n\} \subset \mathcal{K}$ , and  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ ,*

$$\|R^n(f) - f_n\|_{C^k} \leq \varepsilon,$$

and  $f_n$  is infinitely renormalizable with the same combinatorics as  $R^n(f)$ .

*Remark.* In Proposition 5.14, we have convergence of renormalization to a limit set in the  $C^k$  topology; whereas, [GMdM, Theorem 11.1] implies exponential convergence in the  $C^{k-1}$  topology.

*Proof.* We start with the following claim.

*Claim.* There exists a compact set of polynomial-like germs  $\mathcal{K}$  such that given  $f \in \Gamma_{\mathcal{A}^k(I)}$ , there exists a sequence  $g_n \in \mathcal{K}$  so that  $\|g_n - R^n(f)\|_{C^0} \rightarrow 0$  as  $n \rightarrow \infty$ , and  $g_n$  has the same combinatorics as  $R^n(f)$ .

*Proof of Claim.* By Theorem 2.4, for each  $n$  sufficiently big there exist a  $b$ -tuple  $\underline{b}$  and an asymptotically holomorphic polynomial-like renormalization  $F_n : U_n \rightarrow V_n$  of type  $\underline{b}$  of  $f$ . By Lemma 2.5, for each  $n$  we can express  $F_n$  as the composition  $h_n \circ \phi_n : U_n \rightarrow V_n$  where  $\phi_n : U_n \rightarrow U_n$  is quasiconformal with dilatation bounded by  $\text{diam}(U_n)$  and

$h_n : U_n \rightarrow V_n$  is a real polynomial-like mapping. By Lemma 5.9 and Theorem 2.4, the mappings  $\phi_n$  converge to the identity in  $C^0(U')$ , where  $U' = F_n^{-1}(U)$ . By Theorem 4.15, the  $h_n$  converge to a compact set of infinitely renormalizable polynomial-like germs  $\mathcal{K}$ . For each  $j$  let  $\mathcal{V}_j \subset \mathcal{P}\mathcal{G}_{\underline{b}}$  be the neighbourhood of  $\mathcal{K}$  consisting of mappings so that their  $j$ -th polynomial-like renormalization persists over  $\mathcal{V}_j$ . Since for any  $j \in \mathbb{N}$ ,  $h_n$  eventually enters  $\mathcal{V}_j$ , we have that  $h_n : U_n \rightarrow V_n$  is  $j_n$  times renormalizable where  $j_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

Each  $h_n$  with  $n$  large is an analytic polynomial-like mapping, which is  $j_n$  times renormalizable, with  $j_n$  very large. Moreover, by Theorem 2.4 and [McM1, Theorem 5.8] for  $n$  sufficiently large, the  $j_n$  are contained in a compact family of analytic mappings. Let  $\delta > 0$  be so that for all  $n$  sufficiently large,  $\text{mod}(V_n \setminus U_n) \geq \delta$ , and let  $\Gamma'$  be the intersection of  $\Gamma_{\mathcal{P}\mathcal{G}_{\underline{b}}}$  with the set of polynomial-like germs with moduli bounded from below by  $\delta$ . Let  $\mathcal{V}_n$  be the set of all polynomial-like germs with moduli bounded from below by  $\delta$  which are at least  $n$ -times renormalizable. Then  $\Gamma' = \bigcap_{n=0}^{\infty} \mathcal{V}_n$ . Moreover, by Theorem 4.13 for any  $\varepsilon > 0$ ,  $\mathcal{V}_n$  is eventually contained in a  $\varepsilon$ -neighbourhood of  $\Gamma'$ . Thus for any  $\delta > 0$ , if  $n$  is sufficiently large, there exists a polynomial-like mapping  $g_n$  in the topological conjugacy class of  $R^n(f)$  within distance  $\delta$  from  $h_n$  in the Carathéodory topology. ✓

Associated to each  $\underline{b}$ , there exist a family of polynomials  $P$  and  $\beta > 0$ , so that by Lemma 5.12, we have that there exists mapping  $H_n$  in the Epstein class,  $\mathcal{E}_{\beta, P}$ , which is arbitrarily close to  $R^n(f)$  in the  $C^k$ -topology. Thus we have that  $\|g_n - H_n\|_{C^0}$  is small. Hence, since  $g_n$  and  $H_n$  are both analytic, by Lemmas 5.10 and 5.13, we have that  $\|g_n - H_n\|_{C^k}$  is small. Thus we have that  $\|R^n(f) - g_n\|_{C^k}$  is small. □

*Proof of Theorem C.* Suppose that  $f \in \mathcal{A}_{\underline{b}}^k(I)$ ,  $k \geq 3$ , is a mapping with zero topological entropy, and which is infinitely renormalizable at a critical point  $c$ , where  $\underline{b}$  is  $b$ -tuple with all even entries. Throughout the proof,  $\mathcal{R}$  denotes the renormalization operator with period-doubling combinatorics determined by the combinatorics of restrictive intervals for  $f$  about  $c$ . For  $N \in \mathbb{N}$  sufficiently large, there exists  $\mathcal{W} \subset \mathcal{A}_{\underline{b}}^k(I)$ , a small open neighbourhood of  $f$  chosen so that each  $g \in \mathcal{W}$  has an asymptotically holomorphic polynomial-like renormalization  $\mathcal{R}^N g = G : U_G \rightarrow V_G$ . Such a neighbourhood  $\mathcal{W}$  exists by Theorem 2.4 and since the renormalization operator is open. Let  $\mathcal{W}' = \mathcal{R}^N(\mathcal{W})$ . Each  $G \in \mathcal{W}'$ , is asymptotically holomorphic. Let  $F = \mathcal{R}^N f$ .

By Proposition 4.14, to show that there are mappings with positive entropy and mappings with finitely many periods in  $\mathcal{W}'$ , it is enough to show that there exists an analytic polynomial-like mapping arbitrarily close to  $F : U \rightarrow V$  in the  $C^k$ -topology on the real line. As in the proof of Proposition 5.14, it is sufficient to prove that we can approximate  $F$  by a polynomial-like mapping in the  $C^0$  topology on a complex neighbourhood of the

interval.

Let  $n \in \mathbb{N}$ . There exists a  $b_1$ -tuple  $\underline{b}_1$  with all entries even so that  $\mathcal{R}^n(F) = F_n : U_n \rightarrow V_n$ , is an asymptotically holomorphic polynomial-like mapping of type  $\underline{b}_1$ . Associated to  $\underline{b}_1$ , there is a family of polynomials  $P$ , and  $\beta > 0$  so that by Lemma 5.12, for any  $\varepsilon_1 > 0$ , there exists a mapping  $G_n : I \rightarrow I$  in the Epstein class,  $\mathcal{E}_{\beta, P}$ , so that  $\|G_n - F_n\|_{C^k(I)} < \varepsilon_1$ . Moreover, by the claim in the proof of Proposition 5.14, as  $n \rightarrow \infty$ ,  $F_n \rightarrow \mathcal{K}$  in  $\mathcal{C}^0(U'_n)$ , where  $U'_n = F_n^{-1}(V_n)$ . The mappings in  $\mathcal{K}$  are analytic, so for  $n$  sufficiently large, if  $G_n$  is sufficiently close to  $F_n$  in  $\mathcal{C}^k(I)$ , then  $G_n$  is close to  $\mathcal{K}$  in  $\mathcal{C}^0(X)$ , where  $X$  is the open neighbourhood of the interval given by Lemma 5.13. Then, since the mappings in  $\mathcal{K}$  are polynomial-like mappings, for some  $M \in \mathbb{N} \cup \{0\}$ , uniformly bounded in  $\beta$ ,  $\mathcal{R}^M G_n : U_{\mathcal{R}^M G_n} \rightarrow V_{\mathcal{R}^M G_n}$  is a polynomial-like mapping. Moreover, since we can take  $\varepsilon_1$  as small as we like, by continuity of the renormalization operator in the  $\mathcal{C}^k$  topology, see the appendix of [AMdM], we can assume that  $\mathcal{R}^M G_n \in \mathcal{R}^{n+M}(\mathcal{W})$ , and now we can conclude the proof as Theorem 5.6.

□

### 5.5.1 Proof of Theorem D

In this section, we prove Theorem D. The key step is the construction of a codimension- $b$  manifold consisting of mappings that are infinitely renormalizable at one critical point, and whose remaining critical points are periodic.

As usual, we say that a critical point  $c$  of  $f$  is *non-degenerate* if  $D^2 f(c) \neq 0$ . Let  $b \in \mathbb{N}$ , and  $\mathcal{A}_{\text{even}, b}^r(I) = \cup_{\underline{b}} \mathcal{A}_{\underline{b}}^r(I)$ , where the union is taken over all  $b$ -tuples  $\underline{b}$  with all entries even.

**Lemma 5.15.** *Let  $r = 3 + \alpha$ , where  $\alpha > 0$ . The set of mappings with all critical points non-degenerate is open and dense in  $\Gamma_{\mathcal{A}_{\text{even}, b}^r}(I)$ .*

*Proof.* Let  $\underline{2}$  denote the  $b$ -tuple where every entry is a two. It is well-known that the set  $\mathcal{A}_{\underline{2}}^r(I)$  of mappings with all critical points non-degenerate is open and dense in the space  $\mathcal{A}_{\text{even}, b}^r(I)$ , [W]. Thus the set of mappings with all critical points non-degenerate is relatively open in  $\Gamma_{\mathcal{A}_{\text{even}, b}^r}(I)$ .

We will now prove density. Let us assume that  $f$  has exactly one solenoidal attractor, the case when it has more than one is similar. Let  $f \in \Gamma_{\mathcal{A}_{\text{even}, b}^r}(I)$ , and let  $\mathcal{U}$  be an open neighbourhood of  $f$ . Let  $c$  be a critical point at which  $f$  is infinitely renormalizable. Let  $J_n \ni c$  be a restrictive interval of period  $2^n$ . Then for  $n$  sufficiently large, we have that each interval  $J_n^i = \text{Comp}_{f^i(c)} f^{-(2^n-i)}(J_n)$ ,  $i \in \{0, 1, \dots, 2^n - 1\}$ , contains at most one critical point. Moreover, by Lemma 5.4, there exists a neighbourhood  $\mathcal{U} \subset \mathcal{A}_{\text{even}, b}^r(I)$  of  $f$  so that  $\cup_{i=0}^{2^n-1} J_n^i$  persists over  $\mathcal{U}$ . By Theorem C, and the fact that the set of mappings with a non-degenerate critical point is open and dense, we can approximate  $f$  by mappings

$f_0, f_1 \in \mathcal{U} \cap \mathcal{A}_{2,b}(I)$ , where  $f_0$  is in the interior of mappings with zero entropy and  $f_1$  has positive entropy. Let  $P(x) = x^2$ . Since  $f_0, f_1 \in \mathcal{U}$ , we can express  $\mathcal{R}^{2^n}(f_0) = h_{0,b} \circ P \circ h_{0,b-1} \circ P \circ \cdots \circ h_{0,1} \circ P$ , and  $\mathcal{R}^{2^n}(f_1) = h_{1,b} \circ P \circ h_{1,b-1} \circ P \circ \cdots \circ h_{1,1} \circ P$ , where each  $h_{i,j} : I \rightarrow I$ ,  $i \in \{0,1\}$  and  $j \in \{1, \dots, b\}$ , is a  $C^r$  diffeomorphism of the interval. Now, for each  $j \in \{1, 2, \dots, b\}$ , let  $h_j^\lambda : I \rightarrow I$ ,  $0 \leq \lambda \leq 1$  be a path of diffeomorphisms between  $h_{0,j}$  and  $h_{1,j}$ . Thus we obtain a path  $F_\lambda = h_b^\lambda \circ P \circ \cdots \circ h_1^\lambda \circ P$  of multimodal mappings from  $\mathcal{R}^{2^n}(f_0)$  to  $\mathcal{R}^{2^n}(f_1)$ . Moreover, we can assume that the diameter of the path is as small as we like by choosing  $f_0, f_1$  close enough to  $f$ . Taking the preimage of the path under  $\mathcal{R}^{2^n}$ , we obtain a path  $f_\lambda$  from  $f_0$  to  $f_1$ , which crosses  $\Gamma_{\mathcal{A}_{2,b}^r}$ . Thus there exists a mapping with all critical points non-degenerate arbitrarily close to  $f$ .

When  $f$  has more than one solenoidal attractor, we have to choose the mapping in the interior of zero entropy as we did at the end of Theorem 5.6. This family of mappings lies in the submanifold of mappings with at least one degenerate critical point.  $\square$

We will make use of the period-doubling renormalization operator acting on unimodal mappings with non-degenerate critical points, [Da]. Let  $\alpha > 0$ . We let  $\mathcal{A}_2^{2+\alpha}(I)$  be the space of unimodal  $C^{2+\alpha}$  mappings on the interval with a non-degenerate critical point. The period-doubling renormalization operator acting on the  $C^{2+\alpha}(I)$  has a unique fixed point,  $F_*$ . By Sullivan's complex bounds, [S], we can regard  $F_*$  as a quadratic like germ. Moreover, at  $F_*$  the renormalization operator is hyperbolic. Let  $\mathbf{u}_*$  denote the unstable vector at  $F_*$ . The next proposition describes the stable manifold.

**Proposition 5.16.** [Da] *Let  $\alpha > 0$ . The local stable set of  $F_*$ ,  $W_\varepsilon^{s,2+\alpha} \subset \mathcal{A}_2^{2+\alpha}(I)$  is a codimension-one,  $C^1$ -submanifold.*

Let us say that a multimodal mapping of type  $\underline{b}$ ,  $F$ , with critical points  $\{c_1, c_2, \dots, c_b\}$  has combinatorics  $\sigma_*^0$  if  $b-1$  of its critical points are contained in a periodic cycle and at the remaining critical point, say  $c_0$ ,  $F$  is infinitely renormalizable with period-doubling combinatorics.

Let  $b \in \mathbb{N}$  and let  $\underline{b}$  be a  $b$ -tuple of with all entries even. From now on, we fix  $\underline{b}$ .

**Lemma 5.17.** [dFdMP, Proposition 8.7] *For real numbers  $r > s+1 \geq 2$ , the composition operator from  $\mathcal{C}^r \times \mathcal{C}^s \rightarrow \mathcal{C}^s$  is a  $C^1$  mapping.*

**Proposition 5.18** (c.f. [dFdMP], Theorem 9.1). *For every  $r > 3$ , if  $F$  has combinatorics  $\sigma_*^0$ , then the connected component containing  $F$  of the topological conjugacy class of  $F$  is an embedded, codimension- $b$ ,  $C^1$ , Banach submanifold of the space of smooth multimodal mappings.*

*Proof.* Let  $\alpha > 0$  be so that  $r = 3 + \alpha$ , and choose  $0 < \alpha' < \alpha$ . By Proposition 5.16, the local stable manifold through  $F_*$  in the space  $C^{2+\alpha'}$  is a codimension-one  $C^1$ -submanifold. Let us denote it by  $W_\varepsilon^{s,2+\alpha'}$ . We may assume that  $\varepsilon > 0$  so small that the vector  $\mathbf{u}_* \in T_F \mathcal{A}_b^r(I)$  is transversal to the local stable set  $W_\varepsilon^{s,2+\alpha'}(F_*)$  at each  $F \in W_\varepsilon^{s,2+\alpha'}(F_*)$ .

Let  $G \in W_\varepsilon^{s,3+\alpha}(F_*)$ , the stable set of  $F_*$  in  $\mathcal{A}_b^{3+\alpha}(I)$ . There exists  $N = N(G) > 0$  so large that

$$\mathcal{R}^N(G) \in W_\varepsilon^{s,3+\alpha}(F_*) \subset W_\varepsilon^{s,2+\alpha'}(F_*).$$

Since  $v = \mathbf{u}_*$  is transversal at  $\mathcal{R}^N(G)$  to  $W_\varepsilon^{s,2+\alpha'}(F_*)$ , there exist a small open set  $\mathbb{O}_0 \subset \mathcal{A}_b^{2+\alpha'}(I)$  containing  $\mathcal{R}^N(G)$  and a  $C^1$  function  $\Phi : \mathbb{O}_0 \rightarrow \mathbb{R}$  such that  $\Phi^{-1}(0) = W_\varepsilon^{s,2+\alpha'}(F_*) \cap \mathbb{O}_0$  for which  $0 \in \mathbb{R}$  is a regular value and  $D\Phi(\mathcal{R}^N(G))v \neq 0$ .

By Lemma 5.17, the operator  $\mathcal{R}^N$  is a  $C^1$  map from  $\mathcal{A}_b^{3+\alpha}(I)$  into  $\mathcal{A}_b^{2+\alpha'}(I)$ . Let  $\mathbb{O}_1 \subset \mathcal{A}_b^{3+\alpha}(I)$  be an open set containing  $G$  such that  $\mathcal{R}^N(\mathbb{O}_1) \subset \mathbb{O}_0$ . We want to show that  $0 \in \mathbb{R}$  is a regular value for  $\Phi \circ \mathcal{R}^N : \mathbb{O}_1 \rightarrow \mathbb{R}$ . Defining  $g_t = \mathcal{R}^N(G) + tv$ , with  $|t|$  small, we get a  $C^1$  family  $\{g_t\}$  of mappings in  $\mathcal{A}_b^{3+\alpha}(I)$ , which is transversal to  $W_\varepsilon^{s,2+\alpha'}(F_*)$  at  $g_0 = \mathcal{R}^N(G)$ .

*Claim.* There exists a  $C^1$  family  $\{G_t\} \subset \mathcal{A}_b^{3+\alpha}(I)$  such that for all small  $t$  we have  $\mathcal{R}^N(G_t) = g_t$ . Moreover, for each of the  $b - 1$  critical points  $c_t$  of  $G_t$  which do not correspond to the critical point  $c_0$  of  $G$ , the itinerary of  $c_t$  is the same as the itinerary of  $c$ , where  $c_t \in \text{Crit}(G_t)$  naturally corresponds to  $c \in \text{Crit}(G)$ .

*Proof of Claim.* First note that  $g_t = h_t \circ g_0$  where each  $h_t \in \mathcal{C}^k(I)$  is a diffeomorphism. Since  $\mathcal{R}^N(G) = g_0$ , there exist  $p > 0$  and closed, pairwise disjoint intervals  $0 \in \Delta_0, \Delta_1, \dots, \Delta_{p-1} \subset I$  with  $G(\Delta_i) \subset \Delta_{i+1}$  for  $0 \leq i \leq p-1$ , and  $G(\Delta_{p-1}) \subset \Delta_0$ , such that

$$g_0 = \mathcal{R}^N(G) = \Lambda_G^{-1} \circ G^p \circ \Lambda_G,$$

where  $\Lambda_G : I \rightarrow \Delta_0$  is an affine mapping.

Let  $\bar{h}_t : \Delta_0 \rightarrow \Delta_0$  be the  $C^k$  diffeomorphism given by  $\bar{h}_t = \Lambda_G \circ h_t \circ \Lambda_G^{-1}$ . Let  $\Delta_{-1}$  denote the union of immediate basins of attraction of super-attracting cycles of  $G$ . Consider a  $C^k$  extension of  $\bar{h}_t$  to a diffeomorphism  $H_t : I \rightarrow I$  with the property that  $H_t|_{\Delta_i}$  is the identity for all  $i \neq 0$ . Then let  $G_t \in \mathcal{A}_b^{3+\alpha}(I)$  be the map  $G_t = H_t \circ f$ . Note that  $G_t^i(0) = G^i(0)$  for all  $0 \leq i \leq p$ , that  $G_t$  is  $N$ -times renormalizable under  $\mathcal{R}$  and that  $\mathcal{R}^N(G_t) = h_t \circ g_0 = g_t$ . ✓

Now let us show that the claim proves the proposition. Observe that the condition that  $b - 1$  of the critical points of  $G$  lie in an periodic cycle defines a codimension  $b - 1$  subspace of  $\mathcal{A}_b^{3+\alpha}(I)$ . Setting

$$w = \left. \frac{d}{dt} \right|_{t=0} G_t,$$

we obtain that

$$D(\Phi \circ \mathcal{R}^N)(G)w = D\Phi(g_0)v \neq 0.$$

Therefore  $\Phi \circ \mathcal{R}^N$  is a  $\mathcal{C}^1$  local submersion at  $G$ . By the Implicit Function Theorem,  $(\Phi \circ \mathcal{R}^N)^{-1}(0)$  is a codimension-one,  $\mathcal{C}^1$  Banach submanifold of  $\mathbb{O}_1$  an open subset of  $\mathcal{A}_b^{3+\alpha}(I)$ . Furthermore, if  $h \in (\Phi \circ \mathcal{R}^N)^{-1}(0)$ , then  $\mathcal{R}^N(h) \in W_\varepsilon^{s,2+\alpha'}(F_*)$ , and so  $h$  belongs to the global stable set  $W^{s,2+\alpha'}(F_*)$ . By Proposition 5.14, we have that  $h$  in fact belongs to  $W^{s,r}(g)$ . The proposition follows.  $\square$

*Proof of Theorem D.* By Lemma 5.15, we have that the family of mappings with all critical points non-degenerate is open and dense in  $\Gamma = \Gamma_{\mathcal{A}_{even,b}^r(I)}$ . Let  $\Gamma_1$  be the subset of mappings in  $\Gamma$  that has exactly one solenoidal attractor. Arguing as in the proof of Theorem 5.7, using Theorem C in place of Theorem A, we have that  $\Gamma_1$  is open and dense in  $\Gamma$ . Let  $\mathcal{X} \subset \Gamma$  denote the open, dense set of mappings with all critical points non-degenerate and exactly one solenoidal attractor. We need to show that any mapping  $f \in \mathcal{X}$  can be approximated by mappings in  $\Gamma$  with exactly one solenoidal attractor containing exactly one critical point, which is non-degenerate.

Let  $c$  be a recurrent critical point of  $f$  such that  $\omega(c)$  is a solenoidal attractor, and let  $F : U \rightarrow V$  be an asymptotically holomorphic polynomial-like renormalization of  $f$  at  $c$ . By Proposition 5.14, there exists a compact set of polynomial-like germs  $\{f_n\}$  such that for all  $\varepsilon > 0$  and all  $n$  sufficiently large

$$\|\mathcal{R}^n(F) - f_n\|_{\mathcal{C}^r(I)} < \varepsilon,$$

which are infinitely renormalizable with the same combinatorics as  $\mathcal{R}^n(F)$ . By Theorem B we have that we can approximate each  $f_n$  by polynomial-like germs  $g_m$  which are infinitely renormalizable at one critical point and with  $b - 1$  periodic critical points. By Proposition 5.18 for each  $g_m$ , there is a codimension- $b$ , submanifold,  $\mathcal{H}_{g_m}$ , of  $\mathcal{A}_{2,b}^r(I)$  consisting of mappings topologically conjugate to  $g_m$ , and any sequence of mappings,  $\hat{g}_m \in \mathcal{H}_{g_m}$ , accumulates on the topological conjugacy class of  $f_n$  in  $\mathcal{A}_{2,b}(I)$ . Arguing as in the proof of Theorem C any mapping in the topological conjugacy class of  $f_n$  in  $\mathcal{A}_{2,b}(I)$  can be approximated by such mappings,  $\hat{g}_m$ . By Theorem 2.7, for  $\varepsilon > 0$ , sufficiently small, these manifolds laminate  $B_\varepsilon(f_n) \subset \mathcal{A}_{even,b}^r(I)$ . So for any neighbourhood  $\mathcal{U}' \subset \mathcal{A}_{even,b}^r(I)$  of  $F$ , there exists  $n$  so that  $\mathcal{R}^n(\mathcal{U}' \cap \mathcal{T}_n)$  intersects such a topological conjugacy class, where  $\mathcal{T}_n$  is the set of mappings which are  $n$  times renormalizable. We can conclude by arguing as in the proof of Theorem 5.6.  $\square$

Finally we obtain:

**Theorem 5.19.** *Let  $r > 3$  and  $b \in \mathbb{N}$ . Let  $\underline{b}$  be a  $b$ -tuple consisting of even integers. Each connected component of  $\Gamma_{\mathcal{A}_{\underline{b}}^r(I)}$  is locally connected.*

*Proof.* Let  $\Gamma$  denote a connected component of  $\Gamma_{\mathcal{A}^r(I)}$ , and suppose that there exists  $f \in \Gamma$ , so that  $\Gamma$  is not locally connected at  $f$ . Then there is an arbitrarily small open set



$\mathcal{V} \subset \mathcal{A}^r(I)$ , with  $f \in \mathcal{V}$ , such that for every open set  $\mathcal{U} \subset \mathcal{V}$ , with  $\mathcal{U} \ni f$ , we have that  $\mathcal{U} \cap \Gamma$  is not connected. Take  $\varepsilon > 0$  small enough so that  $B_\varepsilon(f) \supset \mathcal{V}$ , and set  $\mathcal{U} = B_\varepsilon(f)$ . Since  $\Gamma$  is a closed set, since  $\Gamma$  is not locally connected at  $f$ ,  $\Gamma \cap \mathcal{U}$  has infinitely many components: If  $\Gamma \cap \mathcal{U}$  contained only finitely many components,  $\Gamma_0 \cup \dots \cup \Gamma_k$ , with  $f \in \Gamma_0$ , then  $\Gamma_1 \cup \dots \cup \Gamma_k$  is a relatively closed subset of  $\mathcal{U}$ , but now, there is an open set  $\mathcal{U}' \subset \mathcal{U}$  so that  $\mathcal{U}' \cap \Gamma = \Gamma_0$  is connected, which contradicts the choice of  $\mathcal{V}$ . Thus we have that  $\Gamma \cap \mathcal{U}$  consists of infinitely many connected components, which must accumulate on  $f$ .

Since  $\Gamma$  is connected, by Theorem D, there are codimension-one components  $\Gamma_n$  of  $\Gamma \cap \mathcal{U}$ , so that  $\text{dist}(\Gamma_n, f)$  is arbitrarily small, and with  $\text{diam}(\Gamma_n) \geq \varepsilon/2$ . Evenmore, we can assume that each  $\Gamma_n \ni x_n$  so that  $\partial(B_{\varepsilon/4}(x_0) \cap \Gamma_n) \subset \partial B_{\varepsilon/4}(x_0)$ .

Let  $\mathcal{Z} \subset \Gamma$ , denote the set of mappings with exactly one solenoidal attractor, which contains exactly one non-degenerate critical point and no others. By Theorem D, we have that  $\mathcal{Z}$  is a union of codimension-one open sets, which is dense in  $\Gamma$ .

Suppose first that  $f \in \mathcal{Z}$ . Then there is a neighbourhood  $\mathcal{U}_1$  of  $f$ , and a renormalization  $\mathcal{R}^n$ , so that  $\mathcal{R}^n(\mathcal{U}_1)$  is contained in the space of asymptotically holomorphic polynomial-like mappings with non-degenerate critical points. Moreover, taking a deeper renormalization if necessary, we can assume that  $\mathcal{R}^{n'}(\mathcal{U}_1)$  is contained in an arbitrarily small neighbourhood of the quadratic-like fixed point of renormalization.

By the claim in the proof of Proposition 5.18, there exists a  $\varepsilon' > 0$  and a neighbourhood  $N$  of  $f$  in  $\mathcal{Z}$  so that for each  $g \in N$ , there is a transverse family  $\{g_t\}_{|t| < \varepsilon'}$  to  $g = g_0$ , so that  $\mathcal{R}^{n'}(g_t)$ ,  $|t| < \varepsilon'$ , is a transverse family to the local stable manifold of renormalization, and  $\mathcal{R}^{n'}$  is injective on  $\{g_t\}_{|t| < \varepsilon'}$ . But now, since each  $\Gamma_n$  has codimension-one, and they accumulate on  $f$ , there exist arbitrarily large  $n$  so that for  $g_0$  close to  $f$ ,  $\Gamma_n \cap \{g_t\}_{|t| < \varepsilon'} \neq \emptyset$ . So there exist  $g_n \in \Gamma_n \cap \{g_t\}_{|t| < \varepsilon'}$ , converging to  $f$ , so that  $\mathcal{R}^{n'}(g_n)$  is an infinitely renormalizable quadratic-like mapping. This contradicts the injectivity of  $\mathcal{R}^{n'}$  on each transverse family. Thus  $\Gamma$  is locally connected at  $f \in \mathcal{Z}$ .

Now assume that  $f$  is an arbitrary mapping in  $\Gamma$ . In each  $\Gamma_n$ , there is a dense set of relatively open manifolds consisting of mappings in  $\mathcal{Z}$ . Since each  $\Gamma_n$  has the property that  $\partial(B_{\varepsilon/4}(f) \cap \Gamma_n) \subset \partial B_{\varepsilon/4}(f)$ , we have that the set  $\mathcal{Y}$  of all limit points of the  $\Gamma_n$  contains a codimension-1 connected submanifold of  $\Gamma$ , contained in  $\overline{\mathcal{U} \cap \Gamma}$ . Thus  $\mathcal{Z}$  is dense in  $\mathcal{Y}$ , and points in  $\mathcal{Z} \cap \mathcal{Y}$  are accumulated by points in  $\Gamma_n$ . But this contradicts the fact that  $\Gamma$  is locally connected at  $f \in \mathcal{Z}$ .  $\square$

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