Section I: Introduction

Material we assume known

- Definition of an n-dimensional manifold (always Hausdorff, second countable). Nearly always C^∞. Our basic language is *differential topology*. May sometimes mention topological and PL manifolds. Orientations.
- Manifolds with boundary ∂M. If $\partial M = 0$ and M compact say M is closed. This is the default setting.
- Submanifolds $P \subset M$, closed subsets.
- Vector bundles $V \to M$, in particular tangent and normal bundles.

- Sections of $TM \to M$ are *vector fields*. If M is closed (say) a time dependent vector field V_t defines an ODE $\frac{dx}{dt} = V_t(x)$ which has a solution for all time with any given initial condition. This defines a family of *diffeomorphisms* $F_t : M \to M$.
- Applications of the implicit function theorem. For example

1. If $f : M \to N$ is surjective we say that $y \in N$ is a *regular value* if for all $x \in f^{-1}(y)$ the derivative df_x is surjective. In this case $f^{-1}(y)$ is a submanifold of dimension $\mathrm{dim} M - \mathrm{dim} N$.
2. If $P, Q \subset M$ are submanifolds we say they meet transversally at $x \in P \cap Q$ if $TM_x = TP_x + TQ_x$. If this is so for all points in $P \cap Q$ then $P \cap Q$ is a submanifold of dimension $\mathrm{dim} M - \mathrm{dim} P - \mathrm{dim} \mathrm{Q}$.
3. A submanifold $P \subset M$ has a *tubular neighbourhood*, diffeomeorphic to the unit ball bundle in the normal bundle of P in M.
4. A map $\iota : M \to N$ is an *immersion* if the derivative is injective at each point. If M is closed and ι is an injective immersion then $\iota(M)$ is a submanifold.

- Transversality:"general position", results. For example
  1. If $f : M \to N$ then almost all points $y \in N$ are regular values. (Open, dense, set if M is compact) In particular if $\mathrm{dim} M < \mathrm{dim} N$ then $f^{-1}(y)$ is empty for almost all y.
  2. If P, Q are submanifolds of M then there is a diffeomorphism $\phi : M \to M$ arbitrarily close to the identity such that $\phi(M)$ is transverse to Q.

- Cut-off functions, partitions of unity etc.
- Example of applications of some of the above: Any closed n-manifold M can be embedded as a submanifold in $\mathbf{R}^{2n+1}$.

1. Choose charts $\chi_a : U_a \to \mathbf{R}^n$ (for $1 \leq a \leq A$) and smooth functions β_a supported in U_a, equal to 1 on $\overline{V_a} \subset U_a$, where $(V_a)_{1 \leq a \leq A}$ forms an open cover of M.
2. Let
$$\iota = (\beta_1, \ldots, \beta_A, \beta_1\chi_1, \ldots, \beta_A\chi_A) : M \to \mathbf{R}^N$$
with $N = (n+1)A$. This is an injective immersion, so ιM is a submanifold. Thus now we think of $M \subset \mathbf{R}^N$.
3. Now suppose $N > 2n+1$ and we have a $M \subset \mathbf{R}^N$. There is an obvious map $f : TM \to \mathbf{R}^N$. If $p \in \mathbf{R}^N$ does not lie in the image of f then projection from p immerses M in $\mathbf{R}^{N-1}$. We can also define $g : M \times M \times \mathbf{R} \to \mathbf{R}^N$ by $g(x_1, x_2, t) = tx_1 + (1-t)x_2$. If p does not lie in the image of g then projection from p is an injective map from M to $\mathbf{R}^{N-1}$. Since $\dim\,(M \times M \times \mathbf{R}) = 2n+1$ and $\dim TM = 2n$ the images of f, g have dense complements so we can find a suitable point p to embed in dimension $N-1$.
4. Note that this argument shows we can immerse M in $\mathbf{R}^{2n}$.

There are many books which cover the above. One good one is

M. Spivak *A comprehensive introduction to differential geometry, Vol. 1* Publish or Perish

- We may sometimes discuss *structures on manifolds*. Complex, symplectic.
- We assume some algebraic topology: $\pi_1(M), H_*(M)$.

Possible goals one might have;

- *Modest*: study examples.
- *Ambitious*: classify.
- *Intermediate*: “systematic understanding”.

The general plan of the course is to outline some of the theory of high dimensional manifolds as developed circa 1960 and also to discuss some results on low dimensional manifolds.

Section II: Surgery

Most generally we mean that we have an M^n and an n-dimensional submanifold with boundary $\Omega \subset M$. Suppose we have another manifold with boundary Ω' and a diffeomorphism $\phi : \partial\Omega' \to \partial\Omega$. Then we form a new manifold

$$M' = (M \setminus \mathrm{int}\Omega) \cup_\phi \Omega'.$$

A more restricted use of the term (surgery) occurs when we have an embedded sphere $\Sigma \subset M^n$, where $\Sigma \cong S^p$ with trivial normal bundle. If we fix a trivialisation of the normal bundle then there is a tubular neighbourhood N of Σ with closure $\overline{N} = \Omega$ where Ω is identified with $S^p \times B^{n-p}$.
Now

$$\partial(S^p \times B^{n-p}) = S^p \times \partial B^{n-p} = S^p \times S^{n-p-1},$$

and this is the same as $\partial\Omega'$ where $\Omega' = B^{p+1} \times S^{n-p-1}$. So we get a new manifold M': the result of surgery on Σ. (But the construction may also depend essentially on the trivialisation of the tubular neighbourhood.)

- In $\Omega' = B^{p+1} \times S^{n-p-1}$ we have an embedded sphere $\{0\} \times S^{n-p-1}$ with a given trivialisation of its normal bundle. Thus we get $\Sigma' \subset M'$ and if we do surgery on Σ' we recover M.
- Take $S^p \times \{\mathrm{pt.}\}$ in $S^p \times S^q$. Then surgery (with the obvious framing) gives S^{p+q}. This can be seen by observing that

$$S^{p+q} = S^p \times B^q \cup_{S^p \times S^{q-1}} B^{p+1} \times S^{q-1}.$$

If $\mathbf{R}^{p+q+1} = \mathbf{R}^{p+1} \oplus \mathbf{R}^q$ then the decomposition corresponds to taking vectors (ξ, η) with $|\xi| \geq |\eta|$ or $|\xi| \leq |\eta|$ respectively.

Application

Any finitely presented group can occur as $\pi_1(M^n)$ for $n \geq 4$.
For connected n-dimensional manifolds M_1, M_2 we have the notion of the *connected sum* $M_1 \sharp M_2$.
Note One should be careful with orientations here. If M_i are orientable then $M_1 \sharp M_2$ will not always be diffeomorphic to $M_1 \sharp \overline{M_2}$.
For $p \geq 1$ let M be the connected sum of N copies of $S^1 \times S^p$.
We have elements $\gamma_1, \ldots, \gamma_N$ in $\pi_1(M)$ corresponding to the S^1 factors in the summands.
If $p \geq 2$ then $\pi_1(M)$ is the free group generated by the γ_i.
Let $W_1, \ldots W_r$ be words in the γ_i. If $p \geq 2$ then these words can be represented by disjoint embedded circles.
The normal bundles to each of these are trivial since M is orientable and $SL(n, \mathbf{R})$ is connected. Fix trivialisations. Doing surgeries we get a new manifold M'.

Proposition If $p \geq 3$ then $\pi_1(M')$ is the group with presentation

$$\langle \gamma_i : W_j = 1 \rangle.$$

This follows from two applications of the Van Kampen theorem. The point is that when we remove the loops we do not change π_1 (if $p \geq 3$).

Consequence: Since there is no algorithmic procedure for determining if two groups given by presentations are isomorphic there can be no such procedure for determining if two n-manifolds are diffeomorphic, once $n \geq 4$.

For 2-manifolds there is such a procedure, for example by computing homology.

There are known restrictions on fundamental groups of 3-manifolds.

Dehn surgery

Now let $K \subset S^3$ be an embedded circle (or "knot"). The normal bundle is trivial but two trivialisations differ up to homotopy by a class in $\pi_1(SO(2)) = \mathbf{Z}$. Any trivialisation determines a "parallel" copy (or longitude) K' of K. We can fix a standard trivialisation by specifying that the *linking number* of K, K' is zero. More generally the trivialisations are indexed by the linking number in $\mathbf{Z}$. This depends on an orientation of S^3 but not an orientation of K.

Thus for each integer r we can do "r-framed surgery" on K to get another 3-manifold M'.

- If $r = 0$ then M' has the integral homology of $S^1 \times S^2$;
- If $r = \pm 1$ then M' has the integral homology of S^3;
- If $|r| > 1$ then $H_1(M'; \mathbf{Z}) = \mathbf{Z}/r\mathbf{Z}$.

(There is a generalisation to the case when r is rational but we will not go into that.)

Example

$+1$ surgery on the right handed trefoil.
Wirtinger presentation of $\pi_1(S^3 \setminus K)$.

- Think of $K \subset \mathbf{R}^3 \subset S^3$.
- Choose a planar projection of the knot and fix an orientation.
- For each "arc" of the picture (i.e. between undercrossings) take a generator γ_i which passes from the base point under the arc, in a direction fixed by the orientation.
- For each crossing where an arc γ_k passes over another strand of the knot cutting it into arcs γ_i, γ_{i+1} we have a relation of the form
$$\gamma_{i+1} = \gamma_k \gamma_i \gamma_k^{-1}.$$
- This gives a system of generators and relations for $\pi_1(S^3 \setminus K)$

In the case of the trefoil we get generators $a = \gamma_1, b = \gamma_2, c = \gamma_3$ with

$$b = cac^{-1} \ , \ a = bcb^{-1} \ , \ c = aba^{-1}.$$

For each r the class $baca^{r-3}$ is represented by a meridian with linking number r. Doing the surgery kills this class, so $\pi_1(M')$ has a presentation with one extra relation $baca^{r-3} = 1$.
We take $r = +1$. Eliminate c to get

$$bab = aba \quad , \quad ba^2ba^{-3} = 1.$$

Write $a = x, b = x^{-1}y$ we get

$$x^{-1}y^2 = yx$$

which is equivalent to $(yx)^2 = x^5$, and

$$x^{-1}y^2 = yx$$

which is equivalent to $(yx)^2 = z^3$.

Conclusion

In this case $\pi_1(M')$ has a presentation $y^3 = x^5 = (yx)^2$.
Let $\Gamma \subset SO(3)$ be the group of symmetries of the icosahedron: it has order 60.

- X = rotation about $2\pi/5$ at a vertex p;
- Y = rotation about $2\pi/3$ at centre of a face with p a vertex;
- Then one sees that YX is rotation about π at midpoint of opposite edge.

So $X^5 = Y^3 = (XZ)^2 = 1$. We get a surjective homomorphism $\rho : \pi_1(M') \to \Gamma$.

In fact (as we may see later) M' is the "Poincaré 3-manifold" and may alternatively be described as $SO(3)/\Gamma$. Then ρ is the induced map $\pi_1(M') \to \Gamma$ and the kernel of ρ is $\pi_1(SO(3)) = \mathbf{Z}/2$. So $\pi_1(M')$ has order 120.

Futher examples (1)

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Seifert fibrations

Consider S^1 as the unit circle in $\mathbf{C}$.
Consider the solid torus

$$N = S^1 \times B^2 = \{(z, w) \in \mathbf{C}^2 : |z| = 1, |w| \leq 1\}.$$

Then we have an obvious free S^1 action on N

$$\lambda(z, w) = (\lambda z, w).$$

A 3-manifold M is a (principle) S^1 bundle if there is an action of S^1 on M which is locally modelled on this one.

Now given non-zero $r \in \mathbf{Z}$. We can consider the action on N

$$\lambda(z,w) = (\lambda^r z, \lambda w).$$

This is not free: points with $w = 0$ have stabiliser the cyclic group of order r: $C_r \subset S^1$. The quotient of N by the action is the same as the quotient of B^2 by the obvious action of C_r which is again a copy of B_2. A 3-manifold M has a *Seifert fibration* if there is an S^1 action on M which is locally modelled on one of these, for suitable r. The quotient M/S^1 is a 2-manifold.

Suppose M_0 is an S^1 bundle and let K be a fibre. Thus K has an obvious 0-framing and for any integer r we get another framing. For $r \neq 0$, performing Dehn surgery with this framing we get a new manifold M_1 which has a Seifert fibration with a fibre of multiplicity r.
This is just because when we re-glue the solid torus we take the standard action on the boundary to the other one.

Now consider the Poincaré manifold $M = SO(3)/\Gamma$. We can think of $SO(3)$ as the set of unit tangent vectors to S^2. Thus there is an action of S^1 on $SO(3)$ which commutes with the action of Γ, so we get an induced action on M. Contemplating the icosahedron one see this that this is a Seifert fibration with three multiple fibres of multiplicities 2,3,5 and quotient space S^2.

Now we would like to see that the manifold M', obtained by +1 Dehn surgery on the trefoil, also has such a Seifert fibration.

Think of $S^3 \subset \mathbf{C}^2$. Then we have an S^1 action

$$\lambda(z_1, z_2) = (\lambda^2 z_1, \lambda_3 z_2).$$

The map

$$(z_1, z_2) \mapsto z_1^3 z_2^{-3} \in \mathbf{C} \cup \{\infty\}$$

shows that the quotient is S^2. There are two multiple fibres, of multiplicities 2,3, corresponding to $z_2 = 0, z_1 = 0$.

We have the “Clifford torus”in S^3.

$$T = \{(z_1, z_2) : |z_1| = |z_2| = 1/\sqrt{2}\}.$$

Writing $S^3 = \mathbf{R}^3 \cup \infty$, this goes over to the standard torus in $\mathbf{R}^3$.
The trefoil is the (2,3)-torus knot: any orbit in T under our action.
The action defines a framing of the knot, as above. (i.e. a nearby fibre gives a parallel copy (longitude) of the knot.
EXERCISE. These two fibres have linking number 6=2.3.

Thus $+1$ Dehn surgery relative to the standard framing corresponds to taking $r = 6 - 1 = 5$ relative to the fibration framing. We create a new multiple fibre of multiplicity 5.

From this it is not hard to see that M is diffeomorphic to the Poincaré manifold $SO(3)/\Gamma$.
Note however that it is not true that the base and the multiplicities of singular fibres determine the total space.
Example: S^3 (the Hopf fibration) and $S^1 \times S^2$ are two S^1 bundles over S^2.

A good reference for much of the above is *Rolfsen: Knots and links*

Further Examples (2)

Connections with complex geometry: blowing up.

Recall that $\mathbf{CP}^n$ is the quotient of $\mathbf{C}^{n+1} \setminus \{0\}$ by the obvious action of $\mathbf{C}^*$. It is a complex manifold and so has a natural orientation. Let $\overline{\mathbf{CP}^n}$ be the same manifold with the opposite orientation.
We can also write $\mathbf{CP}^n = \mathbf{C}^n \cup \mathbf{CP}^{n-1}$.

In complex geometry the *blow up* $\widehat{\mathbf{C}^n}$ of $\mathbf{C}^n$ at the origin is defined as the subset of $\mathbf{CP}^{n-1} \times \mathbf{C}^n$ satisfying the equations

$$u_i z_j = z_i u_j.$$

Here $(z_i) \in \mathbf{C}^n$ and $[u_i] \in \mathbf{CP}^{n-1}$. There is a projection map $\pi : \widehat{\mathbf{C}^n} \to \mathbf{C}^n$ with $\pi^{-1}(0) = \mathbf{CP}^{n-1}$ but otherwise a diffeomorphism.
More generally, if X is any complex manifold of complex dimension n and $p \in X$ we construct a new manifold $\hat{X}$.
Proposition $\hat{X}$ is diffeomorphic to the connected sum of X and $\overline{\mathbf{CP}^n}$

To see this we construct an orientation-reversing diffeomorphism h from $\widehat{\mathbf{C}^n}$ to $\mathbf{CP}^n \setminus \{0\}$.
We start be defining

$$h_0 : \mathbf{C}^n \setminus \{0\} \to \mathbf{C}^n \setminus \{0\}$$

$$h_0(z) = \frac{1}{|z|^2} z.$$

This acts as inversion on each complex line through the origin and reverses orientation. Now regarding $\mathbf{C}^n \setminus \{0\} \subset \widehat{\mathbf{C}^n}$ and also $\mathbf{C}^n \setminus \{0\} \subset \mathbf{CP}^n$ we can write

$$h_0(u, z) = [u, \sum u_i \overline{z_i}].$$

In this form it is clear that h_0 extends to the desired diffeomorphism h.

Further Examples (3)

: Elliptic fibrations and logarithmc transformations.

Take a generic homogenous polynomial f_1 of degree 3 in z_1, z_2, z_3. Then the zero set of f is a 2-dimensional submanifold C_1 of $\mathbf{CP}^2$.
Basic fact: C is a complex torus. It is convenient for us to write this as $\mathbf{C}^*/\mathbf{Z}$ where the action is generated by multiplication by μ with $|\mu| \neq 1$.
Now take another generic polynomial f_2 with zero set C_2. Then C_1, C_2 meet in 9 points. Away from these points the quotient f_1/f_2 is well-defined in $\mathbf{C} \cup \{\infty\}$. Blow up these 9 points to get a manifold $X = \mathbf{CP}^2 \sharp 9\overline{\mathbf{CP}^2}$.

The fundamental point is that the map f_1/f_2 extends to a well-defined holomorphic map $F : X \to S^2$.
The fibre $F^{-1}(\lambda)$ can be identified with the curve defined by the polynomial $f_1 - \lambda f_2$. For generic λ this is again a torus. (in fact there are exactly $\chi(X) = 12$ singular fibres). Such a structure is called an “elliptic fibration”.

For any $k \geq 1$ we can take the “fibrewise connected sum” of k copies of X to get a new manifold with an elliptic fibration (in fact this can be done in such a way that the result is again a complex manifold, with a holomorphic fibration).

Consider a neighbourhood of a torus fibre in X. This can be written (differentiably) as $S^1 \times N$ where $N = S^1 \times B^2$ as before. For non-zero r we can make the same construction we did before, to add a multiple fibre, but now we multiple everything by S^1, so we get a new manifold X_r say. A more careful study shows that this is compatible with the complex structure, so X_r is again a complex manifold. It has an elliptic fibration but now there are multiple fibres. In this context the construction is called logarithmic transformation.

The manifold X is simply connected. The blowing up construction gives us 9 disjoint 2-spheres in X. Each meets a fibre in just one point. This implies that the complement of a fibre is also simply connected and in turn that X_r is.

EXERCISE. If we perform two logarithmic transforms with multiplicites r_1, r_2 then the resulting manifold is simply connected provided r_1, r_2 are co-prime. If we perform more than two transforms then the manifold is not simply connected.

A good reference for much of the above is *Gompf and Stipsicz: 4-manifolds and Kirby calculus*.

Section 3. Morse functions, hendles and cobordism

This is a central section in the course. *Inter alia* we are working towards a proof of the “h-cobordism theorem”. Many of the ideas can be traced a long way back (some perhaps to Möbius 1865).
There is a wide variety of paths we could take.

We want to have discussions at three levels:

- spaces (differential topoology);
- homotopy;
- homology.

We will begin with homology.

Let M be a compact n-manifold. A function f on M is a *Morse function* if the derivative df is transverse to the zero section in T^*M.
What this says is that at *critical points* p in M, where $df = 0$ the Hessian is non-degenerate. In local co-ordinates

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) \neq 0.$$

A basic fact is that Morse functions exist (and are "dense" in a suitable sense).

By the classification of quadratic forms we can choose co-ordinates centred at p so that

$$f(x) = f(p) - \left(\sum i = 1^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2\right) + O(x^3).$$

The number λ is called the *index* of the critical point.

By a slightly subtle lemma (Morse lemma) we can choose the co-ordinates so that the $O(x^3)$ term is 0. In practice we can avoid appealing to this, because it is easy to show that we can change f slightly to have the given form.

Fix a Morse function f on M. We will describe the "Witten complex" which computes the homology of M.
Let g be a Riemannian metric on M. This defines the gradient vector field $\mathrm{grad} f$, vanishing at the critical points. We have a gradient flow

$$\frac{dx}{dt} = -\mathrm{grad} f_x.$$

Each solution $x(t)$ has forward and backward limits $\lim_{t\to\pm\infty} x(t)$, which are critical points.

Fundamental fact, I For generic metrics g; if $p_{\pm}$ are critical points of indices $\mu_{\pm}$ then the set of flow lines which travel from p_- (at $t = -\infty$) to p_+ (at $t = +\infty$) is a manifold (possibly empty) of dimension $\mu_- - \mu_+ - 1$.
Important note: here we take the quotient by the obvious action of **R** on the flow lines.

Explanation

There is "descending manifold" $V(p_-)$ from p_-, of dimension μ_-, and an "ascending manifold" $U(p_+)$ from p^+ of dimension $n-\mu_+$.

Without loss of generality $f(p_-) > f(p_+)$. Fix generic c such that $f(p_-) < c < f(p_+)$. Then the level set $N_c = f^{-1}(c)$ is a manifold of dimension $n-1$. The intersection $V(p_-) \cap N_c$ is locally a submanifold of dimension $\mu_- - 1$ and the intersection $U(p_+) \cap N_c$ is locally a submanifold of dimension $n-\mu_+ - 1$. If the intersection is transverse then it is (locally) a submanifold of dimension

$$(\mu_- - 1) + (n - \mu_+ - 1) - (n-1) = \mu_- - \mu_+ - 1.$$

The intersection points correspond to flow lines from μ_- to μ_+.

It is not hard to show that this transverality can be achieved, for generic metrics g (The “Morse-Smale condition”.) Fix such a metric.

Let $\mathcal{M}(p_-, p_+)$ be the set of flow lines from p_- to p_+ (divided by the **R**-action). In general this is not compact but the failure of compactness arises solely from "factorisations" through intermediate critical points. In particular

- If $\mu_- - \mu_+ = 1$ then $\mathcal{M}(p_-, p_+)$ is compact: a finite set of points.
- $\mu_- - \mu_+ = 2$ then $\mathcal{M}(p_-, p_+)$ is a 1-manifold with boundary points corresponding to factorisations through an intermediate critical point of index $\mu_- - 1 = \mu_+ + 1$.

The general statement is that $\mathcal{M}(p_-, p_+)$ has a compactification which is a "manifold with corners".

Granted this we can proceed to define the Witten complex. For simplicity we use $\mathbf{Z}/2$ co-efficients, to avoid discussing signs.
Let C_p be the $\mathbf{Z}/2$-vector space with basis corresponding to critical points of index p.
Let $\partial : C_p \to C_{p-1}$ be the linear map with matrix entries given by counting flow lines.
Then the fact that a 1-manifold with boundary has an even number of boundary points shows that $\partial^2 = 0$.
Let H_* be the homology of this complex.

FACT: This computes the singular homology $H_*(M, \mathbf{Z}/2)$.

Taking account of orientations and signs we can compute the integral homology in a similar way.

We will see soon that the homology computed by the Witten complex of a Morse function is indeed the ordinary singular homology of M. But it is interesting to imagine that we do not know any other definition of homology and see why this gives a good approach, i.e. independent of the choice of Morse function f.

Suppose that f_{-1}, f_1 are two Morse functions on M (satisfying also the Morse-Smale condition). For $t \in \mathbf{R}$ let f_t be the function given by

1. $f_t = f_{-1}$ if $t \leq -1$;
2. $f_t = f_1$ if $t \geq 1$;
3. $f_t = \beta(t) f_{-1} + (1 - \beta(t)) f_1$ if $-1 \leq t \leq 1$;

where $\beta(t)$ is a smooth function equal to 1 if $t \leq -1/2$ and to 0 if $t \geq 1/2$.

Now we have a time-dependent vector field on M defined by $\mathrm{grad} f_t$. We consider solutions of the equation $\frac{dx}{dt} = \mathrm{grad} f_t$. These have forward limits as $t \to +\infty$, which are critical points of f_1, and backward limits as $t \to -\infty$, which are critical points of f_{-1}.

Similar considerations to those before show that, after perhaps slightly perturbing the vector field, there are for each μ a finite number of solutions which have forward and backward limits of the same index μ. Counting these defines a map

$$I : C_*^+ \to C_*^-,$$

where $C_*^\pm$ are the complexes defined by $f_{\pm 1}$. An argument like the proof that $\partial^2 = 0$ shows that this is a map of chain complexes and so induces a map on homology.

More generally we can consider any time dependent vector field, equal to $\mathrm{grad} f_{-1}$ for $t << 0$ and to $\mathrm{grad} f_1$ for $t >> 0$. Again we get a map of complexes and we argue that this is independent of the vector field up to chain homotopy. Then a "gluing argument" shows that I induces an isomorphism on homology. So the "Morse homology" is independent of the choice of f.

This leads to one proof of the Poincaré Duality Theorem: for a compact oriented n-manifold M;

$$H_p(M : \mathbf{Z}) = H^{n-p}(M; \mathbf{Z}).$$

The proof is to replace f by $-f$.

Digression on Poincaré Duality

Let P and Q be oriented submanifolds of dimension p, q respectively in an oriented manifold M of dimension n. Then, after perhaps making a small perturbation, we can suppose P, Q intersect transversally in a manifold of dimension $p+q-n$ (empty if $p+q<n$).

1. The homology class of the intersection is independent of the perturbation.
2. The homology class of the intersection depends only on the homology classes $[P] \in H_p(M), [Q] \in H_q(M)$;
3. While it is not quite true that any homology class can be represented by a submanifold, the construction extends to define a bilinear map

$$H_p(M) \times H_q(M) \to H_{n-p-q}(M).$$

A detailed treatment of duality is given in the book *Characteristic classes* Milnor and Stasheff.

If for simplicity we work with co-efficients in a field F then when $p+q=n$ this pairing

$$H_p(M) \times H_{n-p} \to H_n(M) = F,$$

defines a duality $H_p(M) = (H_{n-p}(M))^* = H^{n-p}(M)$ which is the same as that we saw above via the Witten complex.

Under this, the *intersection pairing* $H_p \times H_q \to H_{n-p-q}$ goes over to the algebraic topologists *cup product* $H^i \times H^j \to H^{i+j}$.

Classical example. Let M be the complex projective plane. Then $H_2(M) = \mathbf{Z}$ with generator the class $[L]$ of a complex line L. Then $L.L = 1$ (two lines meet in a point).

Let C_d be a complex curve defined by a polynomial of degree d. Then it is clear that $C_d.L = d$, so $[C_d] = d[L]$. Thus $C_{d_1}.C_{d_2} = d_1 d_2$ which is *Bezout's Theorem*.

Other duality theorems. **Lefschetz** For an oriented n-manifold M with boundary

$$H_p(M) = H^{n-p}(M, \partial M).$$

Alexander For a “reasonable” subset $A \subset S^n$

$$\hat{H}_p(S^n \setminus A) = H^{n-p-1}(A).$$

For example a Jordan curve A in S^2, knot A in S^3 (linking number).

Return to our Morse function f on a compact oriented n-manifold M.

We now make another discussion at the level of *homotopy*.

Suppose for simplicity that as p runs over the critical points the values $f(p)$ are distinct (the *critical values*).
For $c \in \mathbf{R}$ let $M_c = f^{-1}(-\infty, c]$.

- If c is not a critical value then M_c is a compact manifold with boundary.
- If $c << 0$ then M_c is empty and if $c >> 0$ then $M_c = M$.
- If the interval $[c_1, c_2]$ does not contain any critical values than M_{c_1}, M_{c_2} are homotopy equivalent (in fact diffeomorphic).
- As c increases across a critical value corresponding to a critical point p of index μ the set M_c changes at the level of homotopy by the attachment of a μ-cell.

This gives a (partial) description of M as a *CW complex*. The information needed to give a complete description consists of the attaching maps of the successive cells. (It is best here to consider the case when the index increases with the critical value, so suitable sets M_c are the skeleta of the CW complex.)

For details see *J. Milnor Morse Theory Princeton UP.*

Cellular homology

Suppose that X is a finite CW complex with skeleta X_j. Then by excision $H_i(X_j, X_{j-1})$ vanishes for $i \neq j$ and has one generator for each j cell when $i = j$. From the long exact sequences of pairs we get maps

$$\partial : H_{j+1}(X_{j+1}, X_j) \to H_j(X_j, X_{j-1})$$

with $\partial^2 = 0$ and such that the homology of the resulting complex is $H_*(X)$. Tracing through the definitions you find that this is exactly the Witten complex.

Handles

Now we want to describe at the level of differential topology how M_c changes as c crosses a critical value.

Recall the decomposition

$$S^n = B^{n-\lambda} \times S^\lambda \cup S^{n-\lambda-1} \cup B^{\lambda+1}.$$

A way of seeing this is to write the ball B^{n+1} (at the level of homoeomorphism) as

$$B^{n+1} = B^{n+1-\lambda} \times B^\lambda$$

whence

$$S^n = \partial B^{n+1} = B^{n+1-\lambda} \times \partial B^\lambda \cup \partial B^{n+1-\lambda} \times B^\lambda.$$

The $n+1$-dimensional λ-handle is the product $B^{n+1-\lambda} \times B^{\lambda}$ a "manifold with corners".
Suppose W is an $n+1$-dimensional manifold with boundary and Σ is an embedded $\lambda - 1$ dimensional sphere in the boundary ∂W. Suppose we are given a trivialisation of the normal bundle of Σ in ∂W. Thus a tubular neighbourhood $N \subset \partial W$ is identified with $S^{\lambda-1} \times B^{n+1-\lambda}$. We form a new topological space W' by adjoining $B^{n+1-\lambda} \times B^{\lambda}$ to W along $S^{\lambda-1} \times B^{n+1-\lambda}$. This can be given the structure of a manifold with boundary (rounding off the corners). We say that W' is obtained from W by adjoining a λ-handle along Σ.

It is clear that $\partial W'$ is obtained from ∂W by surgery on Σ, using the given trivialisation of the normal bundle.

When W is empty and $\lambda = 0$ we interpret this saying that $W' = B^{n+1}$.

When $\lambda = n + 1$ so Σ is a component of ∂W the effect is to fill in that component with a ball.

The basic point is that if f is a Morse function on an (n+1)-dimensional manifold M then the set M_c changes by attaching a λ-handle as c increases through a critical value belonging to a critical point of index λ.
To see this one easily reduces to the local quadratic model.

Suppose we are given a framed link $L = \bigcup_i L_i$ in $S^3 = \partial B^4$. So the L_i are disjoint embedded circles and each has a framing, specified by an integer r_i. We attach 2-handles to B^4 to get a 4-manifold with boundary X.

Then $H_2(X; \mathbf{Z})$ has generators corresponding to the components L_i which we can represent by embedded surfaces $\Sigma_i \subset X$. The intersection numbers are given by the linking numbers

$$\Sigma_i.\Sigma_j = \mathrm{lk}(L_i, L_j) \ \ \mathrm{for} i \neq j$$

and the framings

$$\Sigma_i.\Sigma_i = r_i.$$

Let Q be the intersection matrix $Q_{ij} = \Sigma_i.\Sigma_j$.

It is not hard to see that X is simply connected. We have a long exact homology sequence

$$0 \to H_2(\partial X) \to H_2(X) \to H_2(X, \partial X) \to H_1(\partial X) \to 0.$$

By duality $H_2(X, \partial X) = H^2(X)$ and the map in the middle is given by Q.
If $\mathrm{det} Q \neq 0$ then ∂X is a *rational homology 3-sphere* and $|H_1(\partial X)| = |\mathrm{det} Q$. In particular if $\mathrm{det Q} = \pm 1$ we get an integral homology 3-sphere.

Examples

1. Consider the function f on $\mathbf{CP}^2$ defined by

$$f(z_0, z_1, z_2) = \frac{\sum_i a_i |z_i|^2}{\sum_i |z_i|^2},$$

where $a_0 < a_1 < a_2$. This is a Morse function with a minimum at $[1, 0, 0]$ a maximum at $[0, 0, 1]$ and a critical point of index 2 at $[0, 1, 0]$. Let X be the manifold with boundary obtained by removing a small ball around the maximum. Then X is obtained by attaching a 2-handle along the "unknot"with framing $r = 1$.

2. Let L be the standard 2-component link with framings 0. Then X is $(S^2 \times S^2) \setminus B^4$.

3. Let L be the 8 component link associated to the "E_8 diagram". and all framings -2. Then $\mathrm{det Q} = \pm 1$ and ∂X is a homology sphere. In fact this is the Poincaré manifold we discussed before.

Remark In this case we can take the Σ_i to be 2-spheres. We can also describe X by "plumbing".

There is a useful analogy with the case when we add 1-handles to B^2 to construct surfaces. The interesting situation then is the nonorientable one when we have "framings" $0, 1$ modulo 2.

Example. $S^2 \times S^2 \sharp \overline{\mathbf{CP}}^2$ is diffeomorphic to $\mathbf{CP}^2 \sharp 2\overline{\mathbf{CP}}^2$.
One way to see this uses complex geometry. Let $V \subset \mathbf{CP}^3$ be the quadric surface defined by $z_0 z_1 = z_2 z_3$. Then the map $(s, t) \mapsto [1, st, s, t]$ for $s, t \in \mathbf{C} \cup \{\infty\}$ shows that V is diffeomorphic to the $S^2 \times S^2$.
Consider the projection from the point $[1, 0, 0, 0]$ of V to a plane. This becomes well-defined on the blow up $S^2 \times S^2 \sharp \overline{\mathbf{CP}}^2$.
On the other hand there are two lines $\{s = 0\}, \{t = 0\}$ on V which are collapsed to points by the projection. Studying the situation you see that the inverse map becomes well-defined on the blow-up of $\underline{\mathbf{P}}^2$ at two points.

The “real” version of this is

$$S^1 \times S^1 \sharp \mathbf{RP}^2 = \mathbf{RP}^2 \sharp 2\mathbf{RP}^2.$$

The proof is the same. On the other hand we can see this directly using “handle slides” .
In a similar vein, “Kirby calculus” gives a systematic way to manipulate handle and surgery descriptions of 4 and 3 manifolds.

For all this see the book of Gompf and Stipsicz.

Cobordism

A cobordism W between n-manifolds M_0, M_1 is an $(n+1)$-manifold with boundary the disjoint union $M_0 \sqcup M_1$. There are oriented and an unoriented versions of the theory. We allow M_0 to be empty, then we say M_1 is null cobordant. In either case we get a *cobordism ring* $\Omega_* = \bigoplus_{n\geq 0} \Omega_n$ with operations given by disjoint union (or connected sum) and products. The group Ω_n is the set of equivalence classes of n-manifolds, under the equivalence relation defined by cobordism.

The notion can be motivated by considering a family of "equations" $f_t(x) = 0$. As t varies continuously the solutions sets can change as manifolds, but they are cobordant.

Basic example: $\mathbf{RP}^2$ is not null cobordant.
In fact if $M = \partial W$ we claim that the Euler characteristic of M is 0 modulo 2. This follows from Poincaré duality and the exact sequence of (W, M).
From now on we concentrate on the oriented case.

Let W be a cobordism from M_0 to M_1. A Morse function on W is a smooth function $f : W \to [0,1]$ with

- $f(M_i) = i$;
- $\mathrm{grad} f$ non-vanishing on M_i;
- Non-degenerate critical points in the interior of W.

Considering the sets $f^{-1}(-\infty, c]$ we see that M_1 is obtained from M_0 by a sequence of surgeries.
Conversely it is easy to see that if M' is obtained from M by a surgery then M' is cobordant to M.
So cobordiem is the equivalence relation generated by surgery.

Section 4 Cobordism, Pontrayagin-Thom and characteristic classes.

The reference for this section is the book of Milnor and Stasheff.

This is a big topic and we will not be able to cover too much.

One basic point that emerges is that cobordism can be translated into homotopy questions which can largely be reduced to homology.

We begin with characteristic classes. These are bound up with the topology of Lie groups, but the only groups we we really need to consider are $O(n), SO(n), U(n)$.

It is useful to have the general notion of a principle bundle $P \to B$ with structure group G.

- If $G = O(n)$ this is the same as considering real vector bundles with fibre $\mathbf{R}^n$ and Euclidean metrics on the fibres.
- If $G = SO(n)$ this is the same as considering real vector bundles with fibre $\mathbf{R}^n$ together with Euclidean metrics and an orientation on the fibres.
- If $G = U(n)$ this is the same as considering complex vector bundles with fibre $\mathbf{C}^n$ together with Hermitian metrics and an orientation on the fibres.

In fact the metrics will play no real role. We could equally well consider the consider the groups $GL(n, \mathbf{R}), SL(n, \mathbf{R}), GL(n, \mathbf{C})$.

Such a bundle can be specified by an open cover $B = \bigcup_\alpha U_\alpha$ and *transition functions*

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G,$$

such that $g_{\alpha\gamma} = g_{\alpha\beta} g_{\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.
Consider the case when $B = S^n$. It is an easy fact that we can take a cover U_1, U_2 by enlarged hemispheres so that $U_1 \cap U_2$ is a tubular neighbourhood of the equator S^{n-1}.
It is similarly an easy fact that the bundle, up to isomorphism, is determined by the homotopy class of the transition function

$$g : S^{n-1} \to G.$$

So studying equivalence classes of G bundles over S^n is the same as studying the homotopy group $\pi_{n-1}(G)$.

Example 0

Take $n = 1$ and $G = O(1) = \pm 1$. Then $\pi_0(G)$ is a set with two elements.

Application Consider an embedded circle C in a 2-manifold. The normal bundle is determined by an element in $\{0, 1\}$–depending whether the bundle is trivial or a Möbius band. This is the same as the (mod 2) self-intersection number $C.C$.

Example 1

Take $n = 2$ and $G = S^1 = U(1) = SO(2)$. Then $\pi_1(G) = \mathbf{Z}$.
Application Consider an embedded 2-sphere Σ in an oriented 4-manifold. The normal bundle is determined by an integer. This is the same as the self-intersection number $\Sigma.\Sigma$.

Now take $n = 4$ and $G = SO(3)$. We need a digression. Recall that $SU(2)$ is the group of complex unitary 2×2 matrices with determinant 1. Simple algebra shows that these have the form

$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}$$

with $|z|^2 + |w|^2 = 1$. Thus $SU(2)$ can be identified with the 3-sphere S^3.

Now let V be the set of 2×2 skew-Hermitian matrices with trace 0. This is a 3-dimensional real vector space. The group $SU(2)$ acts on V by $v \mapsto gvg^{-1}$. You can check that this action preserves orientation and a Euclidean metric on V. So we get a homomorphism $SU(2) \to SO(3)$. You can check that this is surjective and has kernel the centre$\{\pm 1\}$ of $SU(2)$. The conclusion is that $SO(3) = S^3/\pm 1$.
Thus we have

- $\pi_1(SO(3)) = \mathbf{Z}/2$;
- $\pi_2(SO(3)) = 0$;
- $\pi_3(SO(3)) = \mathbf{Z}$.

Thus $SO(3)$ bundles over S^4 are classified by an integer.
Application
The normal bundle of an embedded 4-sphere in an (oriented) 7-manifold is determined by an integer.

Note that this is *not* given by a self-intersection number.

Now take $n = 4$ and $G = SO(4)$. We construct a surjective group homomorphism

$$SU(2) \times SU(2) \to SO(4).$$

Let $\tilde{V} = V \oplus \mathbf{R}1$. This is the set of matrices of the form for $z, w \in \mathbf{C}$. So $\tilde{V}$ is a 4-dimensional real vector space. You can check that $(g_1, g_2)(v) = g_1 v g_2^{-1}$ defines an action of $SU(2) \times SU(2)$ on $\tilde{V}$ which preserves a Euclidean metric and orientation. Thus we get a homomorphism to $SO(4)$ which you can check is surjective with kernel $(1, 1), (-1, -1)$.
This construction can also be expressed using quaternions.

It follows that
$\pi_1(SO(4) = \mathbf{Z}/2, \pi_2(SO(4)) = 0, \pi_3(SO(4) = \mathbf{Z} \oplus \mathbf{Z}$. So $SO(4)$ bundles over S^4 are classified by a pair of integers (k_1, k_2).
Application. Let Σ be an embedded 4-sphere in an 8-manifold. The normal bundle is determined by a pair (k_1, k_2). The self-intersection number is $k_1 - k_2$.

Now consider $SO(d)$ bundles over S^4 for $d \geq 5$. The fibration

$$SO(d-1) \to SO(d) \to S^{d-1}$$

gives a long exact homotopy sequence

$$\dots \pi_4(S^{d-1}) \to \pi_3(SO(d-1)) \to \pi_3(SO(d)) \to 0.$$

If $d \geq 6$ this shows that $\pi_3(SO(d)) = \pi_3(SO(d-1))$. The interesting case is when $d = 5$. One finds that the map $\mathbf{Z} = \pi_4(S^4) \to \pi_3(SO(4)) = \mathbf{Z} \oplus \mathbf{Z}$ takes 1 to $(1, 1)$. It follows that $\pi_3(SO(d)) = \mathbf{Z}$ for $d \geq 5$. So $SO(d)$ bundles over S^4 are determined by a single integer, for $d \geq 5$.

Now we develop the theory more systematically.
Let $E \to B$ be a real rank d vector bundle over a compact base. For each $x \in B$ we can find a neighbourhood U and sections s_i which form a basis for the fibre over each point of U. Multiply by a cut-off function, equal to 1 on a smaller neighbourhood U', to get sections defined over all of B.
Using compactness we get a finite collection of sections $\sigma_1, \dots \sigma_N$ which generate all fibres of E. Thus for each x in B we have a surjective evaluation map $e_x : \mathbf{R}^N \to E_x$ whose kernel is a $(N-d)$-dimensional subspace of $\mathbf{R}^N$. The annihiliator in the dual space is a d-dimensional subspace of $\mathbf{R}^N$. So we get a map

$$f : B \to \mathrm{Gr}(d, N),$$

to the *Grassmann manifold* of d-dimensional subspace of $\mathbf{R}^n$.

There is a tautological bundle U over the Grassmann manifold. Let H be its dual. One finds that E is canonically isomorphic to $f^*(H)$.
(If we use metrics we do not need to distinguish between U, H.)
If $f' : B \to \mathrm{Gr}(d, N')$ is a map defined by other choices one shows that f, f' become homotopic when we embed $\mathrm{Gr}(d, N), \mathrm{Gr}(d, N')$ in some suitably large Grassmannian $\mathrm{Gr}(d, N'')$.
Conclusion: isomorphism classes of real vector bundles over B are in 1-1 correspondence with homotopy classes of maps $B \to \mathrm{Gr}(d, \infty)$. In practice we can always replace ∞ by some large N.

What we have constructed is the *classifying space* BG for $G = O(d)$.
The construction applies equally well to $SO(d)$ (oriented subspaces) and $U(d)$. (In the complex case we do need to distinguish between U, H.)

From this point of view, *characteristic classes* for a group G are just cohomology classes $c \in H^*(BG)$. Then for any bundle $E \to B$ we get a class $f^*(c) \in H^*(B)$, (independent of the choice of f).

Example Take $G = U(1) = SO(2) = S^1$. Then $BG = \mathbf{CP}^\infty$ and $H^*(BG)$ is freely generated as a ring by a single class $h \in H^2$. For an S^1 bundle $L \to B$ we get a characteristic class $c_1(L) \in H^2(B; \mathbf{Z})$. This is the first Chern class. In fact one can show that this gives a 1-1 correspondence between such isomorphism classes of such bundles and $H^2(B; \mathbf{Z})$. This extends what we saw in the case $B = S^2$.

Now let $E \to B$ be a complex vector bundle of rank d. We form the projective bundle $\mathbf{P}(E)$, with fibres $\mathbf{CP}^{d-1}$. There is a tautological line bundle over $\mathbf{P}(E)$: let H be its dual. Then we get a first Chern class $h \in H^2(\mathbf{P}(E))$ of H. Now we have a map

$$\mu : H^*(\mathbf{CP}^{d-1}) \otimes H^*(B) \to H^*(\mathbf{P}(E)).$$

Lemma The map μ is an isomorphism of additive groups. One can prove this by an inductive argument using Mayer-Vietoris and a suitable finite covering of B. Alternatively, it follows easily from the "Serre spectral sequence".

By the lemma we can write $h^d \in H^{2d}(\mathbf{P}(E))$ in the form

$$h^d = -\sum c_i h^{d-i},$$

for certain classes $c_i \in H^{2i}(B)$. This is one way to define the Chern classes $c_i(E)$ for $i = 1, \dots, d$.
There are various formulae one can establish. For example $c(E \oplus E') = c(E)c(E')$ where $c(E) = 1 + c_1(E) + c_2(E) + \dots$.
In particular $c(E \oplus \mathbf{R}) = c(E)$.

There is a completely parallel discussion for real vector bundles, using co-efficients $\mathbf{Z}/2$, since $H^*(\mathbf{RP}^\infty, \mathbf{Z}/2$ is $\mathbf{Z}_2[h]$ for $h \in H^1$. We get *Stiefel-Whitney classes* $w_i(E) \in H^i(B; \mathbf{Z}/2)$. The class $w_1(E)$ vanishes if and only if E can be oriented.

If $E \to B$ is a real vector bundle we define the Pontrayagin classes $p_i(E) \in H^{4i}$ to be $c_{2i}(E \otimes_{\mathbf{R}} \mathbf{C})$.

If $E \to B$ is an oriented real vector bundle of even rank $2k$ there is another important characteristic class, the *Euler class*. Let $E_0 \subset E$ be the complement of the zero-section. Then one shows that there is a *Thom class* $\tau \in H^{2k}(E, E_0)$ which is characterised by the fact that on each fibre it restricts to the generator of $H^{2k}(\mathbf{R}^{2k}, \mathbf{R}^{2k} \setminus \{0\})$ chosen by the orientation. Then the Euler class $e(E) \in H^{2k}(B)$ is the pull back of τ by the zero section $B \to W$. (When B is a smooth manifold and one works with real co-efficients then the Thom class can be represented by a compactly supported closed $2k$-form on E with integral 1 over each fibre.)

In the case when B is a smooth oriented n-manifold the Euler class of $E \to B$ can be defined geometrically as follows. One takes a generic section s of E meeting the zero section transversally in a $(n-2k)$-dimensional submanifold $Z \subset B$ then $e(E)$ is the Poincaré dual of $[Z] \in H_{n-2k}(B)$.
Example: for an oriented manifold M^{2k} the Euler class $e(TM) \in H^{2k}(M) = \mathbf{Z}$ is given by counting (with signs) the zeros of a generic vector field which gives the Euler characteristic $\chi(M)$.
Remark: One way to see that the count of zeros of a vector field gives $\sum(-1)^i \mathrm{dim} H_i$ *is to consider a Morse function.*

Example If Σ^{2k} is a submanifold of M^{4k} then the Euler class of the normal bundle is the self-intersection number $\Sigma.\Sigma$.

To match up with the previous discussion of $SO(d)$ bundles over S^4 $d \geq 3$

- If $d \neq 4$ we have an integer invriant given by p_1;
- If $d = 4$ we have two integer invariants given by p_1, e. In terms of the previous discussion one finds that $e = k_1 - k_2, p_1 = 2(k_1 + k_2)$, so the only constraint is that $2e = p_1$ modulo 4.

What does this have to do with cobordism? Suppose M is an oriented manifold of dimension $4k$ and $a_1 + 2a_2 + 3a_3 + \cdots = k$. write p_i for $p_i(TM)$. Then the class $p_1^{a_1} p_2^{a_2} \dots$ is in the top dimension so we can evaluate to get an integer. These are called Pontrayagin numbers. For example:

- when $k = 1$ we have a number p_1;
- when $k = 2$ we have p_1^2, p_2;
- when $k = 3$ we have $p_1^3, p_2 p_1, p_3$.

Let W be a $(4k+1)$-dimensional cobordism from M_0 to M_1. Then the restriction of TW to TM_i is $TM_i \oplus \mathbf{R}$. This means that the Pontrayagin classes of TW restrict to those of TM_i and hence the Pontrayagin numbers of M_i are equal. So for each decomposition $k = a_1 + 2a_2 + \dots$ we get a map $P_{\underline{a}} : \Omega_{4k} \to \mathbf{Z}$, where Ω_{4k} is the oriented cobordism group. If N_k is the number of decompositions we have a map $P : \Omega_{4k} \to \mathbf{Z}^{N_k}$.
Theorem of Thom The $\Omega_n \otimes \mathbf{Q}$ are zero for n not divisible by 4 and $P : \Omega_{4k} \otimes \mathbf{Q} \to \mathbf{Q}^{N_k}$ is an isomorphism. Futhermore, generators of $\Omega_{4k} \otimes \mathbf{Q}$ are given by products of projective spaces $\mathbf{CP}^{2m}$.

We do not have time to say much about the proof of this Theorem but we try explain some of the ideas. We begin with a digression.
Let M^{n+k} be a compact manifold. We consider a submanifold $Z \subset M$ of dimension n with a trivialisation of the normal bundle. If Z_0, Z_1 are two such, we say Z_0, Z_1 are framed cobordant if there is a submanifold $W \subset M \times [0,1]$ with boundary $Z_0 \sqcup Z_1$ and with a trivialisation of the normal bundle of W which restrictions to the given trivialisations on the Z_i. This defines an equivalence relation and set of equivalence classes is denoted $\Omega_n(M)$.
Pontrayagin's Theorem $\Omega_n(M)$ can be identified with cohomtopy group $[M, S^k]$.

In one direction suppose we have $Z \subset M$ with a tubular neighbourhood $N = Z \times B^k$. We define $f_0 : N \to B^k$ to be the projection. Now regard S^k as B^k with boundary collapsed to a point ∞ Composing with collapsing map we get $f : N \to S^k$ and we extend to M by defining $f(x) = \infty$ for $x \in M \setminus N$.
In the other direction given a homotopy class of maps $M \to S^k$ we choose a smooth representative f and take a regular value $y \in S^k$. Then $Z = f^{-1}(y)$ is a framed submanifold of M.

In particular if $M = S^{n+k}$ we identify the framed cobordism group with $\pi_{n+k}(S^k)$. The "limit"as $k \to \infty$ gives the stable homotopy groups of spheres. A basic fact from homotopy theory is that these are all *finite*, for $k \geq 1$.

Example Take $n = 1, k = 2$ so we are considering framed 1-dimensional submanifolds of S^3. It is not hard to see that all cobordism classes can be represented by a standard circle with some framing of the normal bundle, determined by an integer since $\pi_1(SO(2)) = \mathbf{Z}$. This corresponds to the Hopf invariant $\pi_3(S^2) = \mathbf{Z}$.

Now take $n = 1, k \geq 3$. We again take a standard circle but the framings are given by $\pi_1(SO(k)) = \mathbf{Z}/2$. This corresponds to $\pi_{k+1}(S^k) = \mathbf{Z}/2$ for $k \geq 3$.

Thom's construction is a little more complicated. Suppose we have a submanifold $Z^n \subset \mathbf{R}^{n+k} \subset S^{n+k}$. At each point of Z the normal bundle determines a point in $Gr_k(\mathbf{R}^{n+k})$. Recall that we have a tautological bundle $U \to Gr_k(\mathbf{R}^{n+k})$. Let N be a tubular neighbourhood of Z and let N_U be a tubular neighbourhood of the zero section in U. Then there is an obvious way to define a map $f_0 : N \to N_U$. Define the *Thom space* $T(U)$ to be the space obtained from U by collapsing the complement of N_U to a point. Then we get a map $f : S^{n+k} \to T(U)$ in the same fashion as before.

Any manifold Z^n can be embedded in $\mathbf{R}^{n+k}$ for $k \geq n+1$. The basic fact is that this construction sets up a 1-1 correspondence

$$\Omega_n = \pi_{n+k}(T(U)).$$

Thus the computation of the cobordism groups Ω_n is translated into a homotopy problem.
Now Thom's theorem is proved by homotopy theory. The basic input is that $H^n(Gr_k(\mathbf{R}^{n+k}); \mathbf{Q})$ ($k \geq n+1$) is generated by products of the Pontrayagin classes $p_i \in H^{4i}$. (Notice that the Euler class, when k is even, lies in H^k and $k > n$.)
Then one uses arguments comparing homology and homotopy and the finiteness of certain homotopy groups. See Milnor and Stasheff for details.

The signature

Let M be an oriented manifold of dimension $4k$. We have a cup-product $H^{2k} \times H^{2k} \to Z$. Taking real co-efficients, this defines a nondegenerate quadratic form on $H^{2k}(M; \mathbf{R})$. Thus we can write $b^{2k} = b^{2k}_{+} + b^{2k}_{-}$ where $b^{2k}_{\pm}$ are the dimensions of maximal positive/negative subspaces. The *signature* $\sigma(M)$ is defined to be a $b^{2k}_{+} - b^{2k}_{-}$.

Lemma

If $M = \partial W$ then $\sigma(M) = 0$. Consider the exact sequence in real cohomology:

$$\dots H^{2k}(W) \to H^{2k}(M) \to H^{2k+1}(W, M) \dots$$

By Lefschetz duality the last term is the dual of the first. One easily sees that the seciond map is the adjoint of the first. This implies that if $I \subset H^{2k}(M)$ is the image of the first map then I is its own anhilliator with respect to the quadratic form. Thus I is an isotropic subspace for the form, of dimension $(1/2)\mathrm{dim} H^{4k}(M)$. It follows that the signatre is zero.

This shows that the signature is a cobordism invriant and defines a homomorphism $\sigma : \Omega_{4k} \to \mathbf{Z}$.

Thom's theorem implies that for each dimension $4k$ the signature is given by some universal linear combination of the Pontrayagin numbers.

Examples$k = 1$ We have $\sigma(\mathbf{CP}^2) = 1$ and $p_1(\mathbf{CP}^2) = 3$. So the formula is $\sigma = p_1/3$.

$k = 2$. Suppose the formula is

$$\sigma = Ap_1^2 + Bp_2$$

We have $\sigma(\mathbf{CP}^4) = \sigma(\mathbf{CP}^2 \times \mathbf{CP}^2) = 1$. Calculations show that for $\mathbf{CP}^4$ we have $p_1^2 = 5^2 = 25 \; , \; p_2 = 10$ and for $\mathbf{CP}^2 \times \mathbf{CP}^2$ we have $p_1^2 = 23^2 = 18 \; , \; p_2 = 3^2 = 9$. So

$$25A + 10B = 1 \quad 18A + 9B = 1.$$

The formula is $\sigma = \frac{1}{45}(7p_2 - p_1^2)$.

The general case. Let $f(t)=\frac{t}{\tanh t}$. This is an even function of t. Take an arbitrary number of variables t_i and consider $F=\prod_i f(t_i)$. This can be written in terms of the elementary symmetric functions in t_i^2 i.e.

$$S_1=\sum t_i^2 \quad S_2=\sum t_i^2 t_j^2 \dots$$

Let $F=\sum_k F_k$ where F_k has total degree $2k$ in the t_i. Define a polynomial L_k by $F_k=L_k(S_1,S_2,\dots)$.
Hirzebruch's Signature Theorem For a manifold M of dimension $4k$

$$\sigma(M)=\langle L_k(p_1,p_2,\dots),[M]\rangle.$$

It is an exercise to see that this gives the right answer when $k=1,2$ using

$$f(t)=1+\frac{1}{3}t^2-\frac{1}{45}t^4+\dots.$$

Exotic spheres

We assume here knowledge of the quaternions. They form a 4-dimensional non-commutative field $\mathbf{H}$. The usual construction allows us to define projective spaces $\mathbf{HP}^n$ and $\mathbf{HP}^1 = S^4$. Then $\mathbf{HP}^2$ is an oriented 8 manifold with homology $\mathbf{Z}$ in dimensions $0, 4, 8$. The generator in H_4 is represented by an embedded 4-sphere $\Sigma = \mathbf{HP}^1 \subset \mathbf{HP}^2$ and $\Sigma.\Sigma = 1$. We have $\mathbf{HP}^2 = \Sigma \cup \mathbf{R}^8$ and the boundary of a tubular neighbourhood N of Σ is a 7-sphere. The fibration $\partial N \to S^4$ is the *quaternionic Hopf fibration* $p : S^7 \to S^4$ defined by $p(Z_0, Z_1) = [Z_0, Z_1]$. All of this is completely analogous to the real and complex cases.

Now consider the normal bundle ν of $\Sigma = S^4$ in $\mathbf{HP}^2$. This is an $SO(4)$ bundle over S^4 and so is determined by a pair of integers (k_1, k_2), as discussed above. Calculations show that ν corresponds to the pair $(1, 0)$, so $p_1(\mathbf{HP}^2, \Sigma) = 2$. Thus $p_1^2 = 4$ and the signature formula gives $p_2(\mathbf{HP}^2) = 7$. (i.e $1 = \frac{1}{45}(7.7 - 4)$).
Let k_1, k_2 be integers such that $k_1 - k_2 = 1$. Let $E \to S^4$ be the corresponding oriented $\mathbf{R}^4$ bundle, $X = X(k_1, k_2)$ be the unit ball bundle and $Y = \partial X = Y(k1, k_2) \subset E$ be the unit sphere bundle. Thus Y is a 7-manifold which fibres over S^4 with fibre S^3. When $k_1 = 1, k_2 = 0$ we see from the discussion above that Y is the sphere S^7.

Whatever pair (k_1, k_2) we take with $k_1 - k_2 = 1$ the 7-manifold Y is a *homotopy* 7-*sphere*. This follows easily from the Serre spectral sequence for the fibration

$$S^3 \to Y \to S^4.$$

(The only potentially interesting differential is $d_4 : H^3(S^3) \to H^4(S^4)$ and one sees that this is given by the Euler class, which we have supposed to be 1. One can equally well use the long exact homotopy sequence.)
Fixing a pair (k_1, k_2) with $k_1 - k_2 = 1$, we consider the Pontrayagin class p_1 of the tangent space to E evaluated on S^4. Since $p_1(TS^4) = 0$ we get $p_1 = 2(k_1 + k_2)$.

Suppose that Y is diffeomorphic to S^7. Then we can attach a 7 ball to X to get a closed 8-manifold M^8. When $k_1 = 1, k_2 = 0$ this recovers $\mathbf{HP}^2$. In the general case the putative manifold M must look homologically like $\mathbf{HP}^2$, in that $H^4(M)$ is generated by the class of the 4-sphere. Thus $p_1^2(M) = 4(k_1 + k_2)^2$ and $\sigma(M) = 1$. The signature theorem gives

$$1 = \frac{1}{45}(7p_2 - 4(k_1 + k_2)^2).$$

Thus $4(k_1 + k_2)^2 = -45 = 4 \mathrm{mod} 7$. Writing $k_1 + k_2 = 1 + 2k_2$ we see that we must have $k_2 = 0, -1 \ \mathrm{mod} 7$.
On the contrary, take (say) $k_2 = 1, k_1 = 2$. Then $4(k_1 + k_2)^2 = 1 \mathrm{mod} 7$. So we conclude that the 7-manifold Y is homotopy equivalent, but not diffeomorphic to S^7.

All this is from the famous paper of Milnor (1956). Milnor also shows by an explicit construction that Y is *homeomorphic* to S^7. The main point is that the Pontrayagin numbers and the signature are integers but Hirzebruch's formula involves rational numbers. This, and related ideas, lead to a large body of results in late 20th. century differential topology (also related to results in homotopy theory).

Section 5

The h-cobordism theorem.

References for this section are Milnor *Lectures on the h-cobordism theorem*, Rourke *Introduction to piecewise linear topology* and Kosinski *Differential manifolds*.

We begin with a fundamental result of Whitney. For simplicity we do not give the sharpest statement.

Theorem Suppose that $p+q=n$ and that P^p, Q^q are connected submanifolds of M^n intersecting transversally in a finite number of points. Suppose the homological intersection number $P.Q$ is zero. Under the assumptions

- M is simply connected;
- $p, q \geq 3$;

there is an *isotopy* of M taking P to a submanifold disjoint from Q.

(An *isotopy* is a smooth map $H : M \times [0,1] \to M$ such that for each $t \in [0,1]$ the map $h_t : M \to M$ defined by $h_t(x) = H(x,t)$ is a diffeomorphism.)

We can essentially reduce to the case when P, Q meet in two points x, y with local intersection numbers ± 1. Choose arcs γ_P, γ_Q in P, Q respectively running from x to y. Since M is simply connected the composite bounds a disc $\iota : B^2 \to M$. We show that ι can be chosen to be an embedding and to extend to an embedding of a "standard model" U in M.

To construct the standard model start with two curves in the plane:

$$\Gamma_P = \{y = x^2 - 1\},$$

$$\Gamma_Q = \{y = 1 - x^2\}.$$

Let $V \subset \mathbf{R}^2$ be an open neighbourhood of $\{|y| \leq 1 - x^2, |x| \leq 1\}$ and let $V_0 \subset V$ be a small neighbourhood of the $(\Gamma_P \cup \Gamma_Q) \cap V$. Consider $V \times \mathbf{R}^{p-1} \times \mathbf{R}^{q-1}$ and let $\tilde{P} = \Gamma_P \times \mathbf{R}^{p-1} \times \{0\}$, $\tilde{Q} = \Gamma_Q \times \{0\} \times \mathbf{R}^{q-1}$. Let U be a neighbourhood of $V \times \{0\} \times \{0\}$ and (with a slight abuse of notation) $\tilde{P}, \tilde{Q} \subset U$.

We want to embed U in M, taking $\tilde{P}, \tilde{Q}$ to P, Q.
If we do this it is not hard to construct the desired isotopy, working within the image of U.

To construct the embedding of U, let $V_0 \subset V$ be a neighbourhood of the two arcs with a smooth "inner boundary" γ and let $U_0 \subset U$ be a neighbourhood of $V_0 \times \{0\} \times \{0\}$.

- Choose embedded arcs in P, Q joining x, y.
- Now we have an embedding of $\Gamma_P \cup \Gamma_Q$ mapping to these arcs.
- By a local study, we can extend this to an embedding ι of V_0, taking $\tilde{P}, \tilde{Q}$ to P, Q. (This uses the fact that the intersection numbers at x, y are opposite.)

Now $\iota(\gamma)$ is a loop in M disjoint from P, Q. Since $\pi_1(M)$ is trivial this bounds a disc D.

- Since $n \geq 4$ we can suppose D is immersed.
- Since $n \geq 5$ we can suppose D is embedded.
- Since $p, q \geq 3$ we can suppose that D does not meet P, Q.
- In fact we can suppose that D meets $\iota(V_0)$ only in $\iota(\gamma)$.

So we have an embedded disc $D \subset M$ with normal bundle trivialised over the boundary.

The obstruction to extending this over this disc lies in $\pi_1(SO(n-2)) = \mathbf{Z}/2$ since $n > 4$.
We still apparently have a problem in $\mathbf{Z}/2$. But further thought shows that we can change our trivialisation over the boundary to remove this.
Now use this trivialisation of the normal bundle to extend to an embedding of U.

The h-cobordism theorem addresses the question: How can we show that a pair of manifolds are diffeomorphic?
So far the only approach we have is by careful inspection and good fortune—recall for example our discussion of the Poincaré homology sphere.

Let M_0, M_1 be (oriented) n-manifolds. An *h-cobordism* between M_0, M_1 is an oriented cobordism W^{n+1} from M_0 to M_1 such that the inclusions $M_0 \subset W, M_1 \subset W$ are homotopy equivalences. We say that M_0, M_1 are *h-cobordant*.

Smale's h-cobordism theorem

If M_0, M_1 are simply connected n-manifolds, with $n \geq 5$ then any h-cobordism W from M_0 to M_1 is diffeomorphic to a product. In particular M_0, M_1 are diffeomorphic.

Corollary

"The high dimensional Poincaré conjecture"
If $n \geq 5$ and M^{n+1} is homotopy equivalent to S^{n+1} then M is homeomorphic to S^{n+1}.

To prove the h-cobordism theorem it suffices to show that there is a Morse function on W with no critical points.

There are different, equivalent languages: Morse functions or handle decompositions. We will use a hybrid approach.

Choose a Morse function f on W. The "Witten complex" now describes the relative homology $H_*(W, M_0) = 0$.
There cannot be a single critical point, because that would give the wrong homology. So the simplest case to consider is when there are just two critical points with adjacent indices.
To fix ideas and give the main idea suppose that $n = 6$ and we have critical points w_3, w_4 of index $3, 4$. (Critical points of very small or high index involve some extra tricks and complications.)

To get the homology right the number of flow lines from w_4 to w_3, counted with signs must be ± 1.
We must have $f(w_4) > f(w_3$. Say $f(w_4) = 3/4, f(w_3) = 1/4$.
Then $M = f^{-1}(1/2)$ is a 6-manifold.
In M there are a pair of embedded 3-spheres with trivial normal bundles:

- P: the points that flow up to w_4;
- Q: the points that flow down to w_3.

The flow-line count says that the homological intersection number P.Q is ± 1.

We construct W from $M_0 \times [0, \epsilon]$ by attaching a 3-handle and then a 4-handle. The "handle cancellation theorem" asserts that if P, Q meet transversally in a single point then W is a product. In other language: if there is just one flow line from w_4 to w_3 then we can modify the Morse function f to get one with no critical points.

From another point of view: start with $M \times [1/2 - \epsilon, 1/2 + \epsilon]$. We get W by attaching a 4-handle to $M \times (1/2 + \epsilon)$ along $P \times (1/2 + \epsilon)$ and a 4-handle to $M \times (1/2 - \epsilon)$ along $Q \times (1/2 - \epsilon)$. We need to show that if P, Q meet transversally in a single point then the result is a product $M \times [0, 1]$.

Basic observation: a neighbourhood of $P \cup Q \subset M$ is standard, so it suffices to do this in a standard model.

For the standard model let $g(x) = x^3 - 3x$, with critical points at $x = \pm 1$. "Suspend" this to the function on $\mathbf{R}^7$

$$F = g(x_0) + (x_1^2 + x_2^2 + x_3^2) - (x_4^2 + x_5^2 + x_6^2).$$

We see a model pair of spheres in $F^{-1}(0)$. Deforming g to $x^3 + x$ shows what we need.

Now in our original set-up, there is no reason why P, Q should meet in a single point. But by the Whitney theorem [with an obvious generalisation of the statement we made] we can deform P (say) to have this property. This deformation does not affect the handle decomposition picture.
In this way we establish the h-cobordism theorem (in a model case).

Very rough outline of the general picture:
To determine whether simply connected manifolds M_0^n, M_1^n are diffeomorphic ($n \geq 5$):

1. See that M_i are homotopy equivalent (homology).
2. See that M_i are cobordant (Pontrayagin classes etc.)
3. See that a cobordism can be modified to be an h-cobordism (surgery...).

In practice this has only been done systematically in a few cases: e.g. simply connected 5-manifolds (Smale, Barden)

There is an extension of the theory to non simply connected manifolds, which involves interesting algebra.

Final Section: 4-manifolds

The only standard algebraic topology invariant of a simply connected (oriented) 4-manifold X is the intersection form on $H_2(X)$. This gives numbers b^2_+, b^2_-. Manifolds with the same intersection form are homotopy equivalent, h-cobordant and even homeomorphic (Freedman).

A class $c \in H_2(X;\mathbf{Z})$ is called *characteristic* if $c.\alpha = \alpha^2$ mod 2 for all $\alpha \in H_2(X,\mathbf{Z})$. Let $\mathcal{C}_X$ be the set of classes $c \in H_2(X)$ which are characetristic and with $c.c = 4 + 5b_2^+ - b_2^-$. This can be identified with the set of homotopy classes of *almost complex structures* on X. The *Seiberg-Witten invariant* is a map $SW : \mathcal{C}_X \to \mathbf{Z}$ which is an oriented diffeomorphism invariant. (Really, we should distinguish between the cases $b^+ > 1$ and $b^+ \leq 1$—in the latter case the theory is more complicated.)

Example (Fintushel and Stern)

The *K3 surface* X is the 4-manifold obtained by fibre sum of two copies of $\mathbf{CP}^2 \sharp 9 \overline{\mathbf{CP}^2}$. So X fibres over S^2 with 24 singular fibres. One finds $b^-(X) = 19, b^+(X) = 3$. Let $F \subset X$ be a smooth fibre so the boundary of a tubular neighbourhood of X is T^3. Let K be a knot in S^3 and Y be the complement of a tubular neighbourhood, so Y has boundary T^2. Let X_K be the 4-manifold obtained from X by removing the neighbourhood of F and replacing with $Y \times S^1$ in such a way that $H_1(X_K) = 0$. Then X_K is simply connected and homotopy equivalent to X.

For this manifold X_K, the map SW vanishes except on multiples of $[F]$ and $SW(\lambda[F])$ is the co-efficient of t^λ in the normalised *Alexander polynomial* of K.
For example, if K is a fibred knot, so we have a monodromy $\alpha : H^1(\Sigma) \to H^1(\Sigma)$ where Σ is a surface of genus g, then the normalised Alexander polynomial is $t^{-g}\det(\alpha - t1)$.
In this way (and many others) one gets a huge variety of distinct h-cobordant, homeomorphic, smooth 4-manifolds.