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Isoperimetric inequalities for the logarithmic potential operator ☆

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ABSTRACT

In this paper we prove that the disc is a maximiser of the Schatten p-norm of the logarithmic potential operator among all domains of a given measure in $\mathbb{R}^2$, for all even integers $2 \le p < \infty$. We also show that the equilateral triangle has the largest Schatten p-norm among all triangles of a given area. For the logarithmic potential operator on bounded open or triangular domains, we also obtain analogies of the Rayleigh–Faber–Krahn or Pólya inequalities, respectively. The logarithmic potential operator can be related to a nonlocal boundary value problem for the Laplacian, so we obtain isoperimetric inequalities for its eigenvalues as well.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set. We consider the logarithmic potential operator on $L^2(\Omega)$ defined by

$$\mathcal{L}_\Omega f(x) := \int\limits_\Omega \frac{1}{2\pi} \ln \frac{1}{|x-y|} f(y)dy, \quad f \in L^2(\Omega), \tag{1.1}$$

where ln is the natural logarithm and $|x-y|$ is the standard Euclidean distance between x and y. Clearly, $\mathcal{L}_\Omega$ is compact and self-adjoint. Therefore, all of its eigenvalues and characteristic numbers are discrete and real. We recall that the characteristic numbers are the inverses of the eigenvalues. The characteristic numbers of $\mathcal{L}_\Omega$ may be enumerated in ascending order of their modulus,

$$|\mu_1(\Omega)| \le |\mu_2(\Omega)| \le \dots$$

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where $\mu_i(\Omega)$ is repeated in this series according to its multiplicity. We denote the corresponding eigenfunctions by $u_1, u_2, \ldots$, so that for each characteristic number μ_i there is a unique corresponding (normalized) eigenfunction u_i,

$$u_i = \mu_i(\Omega)\mathcal{L}_\Omega u_i, \quad i = 1, 2, \ldots.$$

It is known, see for example Mark Kac [12] (see also [15]), that the equation

$$u(x) = \mathcal{L}_\Omega f(x) = \int\limits_\Omega \frac{1}{2\pi} \ln \frac{1}{|x-y|} f(y)dy$$

is equivalent to the equation

$$-\Delta u(x) = f(x), \quad x \in \Omega, \tag{1.2}$$

with the nonlocal integral boundary condition

$$-\frac{1}{2}u(x) + \int\limits_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{1}{2\pi} \ln \frac{1}{|x-y|} u(y) dS_y - \int\limits_{\partial\Omega} \frac{1}{2\pi} \ln \frac{1}{|x-y|} \frac{\partial u(y)}{\partial n_y} dS_y = 0, \ x \in \partial\Omega, \tag{1.3}$$

where $\frac{\partial}{\partial n_y}$ denotes the outer normal derivative at a point y on the boundary $\partial\Omega$, which is assumed piecewise C^1 here.

In general, the boundary value problem (1.2)–(1.3) has several interesting applications (see Kac [12,14], Saito [23] and [15]).

Spectral properties of the logarithmic potential have been considered in many papers (see [1,2,5,9,13, 24,25]). In this paper we are interested in isoperimetric inequalities of the logarithmic potential $\mathcal{L}_\Omega$, that is also, in isoperimetric inequalities of the nonlocal Laplacian (1.2)–(1.3). For a recent general review of isoperimetric inequalities for the Dirichlet, Neumann and other Laplacians we refer to Benguria, Linde and Loewe in [4]. Isoperimetric inequalities for Schatten norms for double layer potentials have been recently considered by Miyanishi and Suzuki [19].

In Rayleigh's famous book "Theory of Sound" (first published in 1877), by using some explicit computation and physical interpretations, he stated that the disc minimizes (among all domains of the same area) the first eigenvalue of the Dirichlet Laplacian. The proof of this conjecture was obtained about 50 years later, simultaneously (and independently) by G. Faber and E. Krahn. Nowadays, the Rayleigh–Faber–Krahn inequality has been established for many other operators; see e.g. [11] for further references (see also [3] and [21]). Among other things, in this paper we also prove the Rayleigh–Faber–Krahn theorem for the integral operator $\mathcal{L}_\Omega$, i.e. it is proved that the disc is a minimizer of the first eigenvalue of the Laplacian (1.2)–(1.3) among all domains of a given measure in $\mathbb{R}^2$.

By using the Feynman–Kac formula and spherical rearrangement Luttinger [18] proved that the disc D is a maximizer of the partition function of the Dirichlet Laplacian among all domains of the same area as D for all positive values of time, i.e.

$$\sum_{i=1}^{\infty} \exp(-t\mu_i^{\mathcal{D}}(\Omega)) \le \sum_{i=1}^{\infty} \exp(-t\mu_i^{\mathcal{D}}(D)), \quad \forall t > 0, \ |\Omega| = |D|,$$

where $\mu_i^{\mathcal{D}}$, $i = 1, 2, \ldots$, are the characteristic numbers of the Dirichlet Laplacian. From here by using the Mellin transform one obtains

$$\sum_{i=1}^{\infty} \frac{1}{[\mu_i^{\mathcal{D}}(\Omega)]^p} \le \sum_{i=1}^{\infty} \frac{1}{[\mu_i^{\mathcal{D}}(D)]^p}, \quad |\Omega| = |D|, \tag{1.4}$$

when $p > 1$, $\Omega \subset \mathbb{R}^2$. We prove an analogy of this Luttinger's inequality for the integral operator $\mathcal{L}_\Omega$. In our note [22] we obtained similar results for convolution type integral operators with positive nonincreasing kernels. In the present setting the main difficulty arises from the fact that the logarithmic kernel is not positive and that we cannot use the Brascamp–Lieb–Luttinger type rearrangement inequalities directly.

In Section 2 we present main results of this paper. Their proofs will be given in Section 4 and Section 3. In Section 5 we discuss shortly about isoperimetric inequalities for polygons and show that the Schatten p-norm is maximised on the equilateral triangle centered at the origin among all triangles of a given area.

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2. Main results and examples

Let H be a separable Hilbert space. By $\mathcal{S}^\infty(H)$ we denote the space of compact operators $P : H \to H$. Recall that the singular values $\{s_n\}$ of $P \in \mathcal{S}^\infty(H)$ are the eigenvalues of the positive operator $(P^*P)^{1/2}$ (see e.g. [10]). The Schatten p-classes are defined as

$$\mathcal{S}^p(H) := \{P \in \mathcal{S}^\infty(H) : \{s_n\} \in \ell^p\}, \quad 1 \leq p < \infty.$$

In $\mathcal{S}^p(H)$ the Schatten p-norm of the operator P is defined by

$$\|P\|_p := \left(\sum_{n=1}^{\infty} s_n^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty. \tag{2.1}$$

For $p = \infty$, we can set

$$\|P\|_\infty := \|P\|$$

to be the operator norm of P on H. As outlined in the introduction, we assume that $\Omega \subset \mathbb{R}^2$ is an open bounded set and we consider the logarithmic potential operator on $L^2(\Omega)$ of the form

$$\mathcal{L}_\Omega f(x) = \int_\Omega \frac{1}{2\pi} \ln \frac{1}{|x-y|} f(y)dy, \quad f \in L^2(\Omega). \tag{2.2}$$

We also assume that the operator $\mathcal{L}_\Omega$ is positive:

Remark 2.1. In Landkof [16, Theorem 1.16, p. 80] the positivity of the operator $\mathcal{L}_\Omega$ is proved in domains $\overline{\Omega} \subset U$, where U is the unit disc. In general, $\mathcal{L}_\Omega$ is not a positive operator. For any bounded open domain Ω the logarithmic potential operator $\mathcal{L}_\Omega$ can have at most one negative eigenvalue, see Troutman [24] (see also Kac [13]).

Note that for positive self-adjoint operators the singular values equal the eigenvalues. It is known that $\mathcal{L}_\Omega$ is a Hilbert–Schmidt operator. By $|\Omega|$ we will denote the Lebesgue measure of Ω.

Theorem 2.2. *Let D be a disc centered at the origin. Then*

$$\|\mathcal{L}_\Omega\|_p \leq \|\mathcal{L}_D\|_p \tag{2.3}$$

for any even integer $2 \leq p < \infty$ and any bounded open domain Ω with $|\Omega| = |D|$.

Note that for even integers p we do not need to assume the positivity of the logarithmic potential operator. For odd integers we have the following:

Theorem 2.3. *Let D be a disc centered at the origin and let Ω be a bounded open domain with $|\Omega| = |D|$. Assume that the logarithmic potential operator is positive for Ω and D. Then*

$$\|\mathcal{L}_\Omega\|_p \le \|\mathcal{L}_D\|_p \tag{2.4}$$

for any integer $2 \le p < \infty$.

Let us give several examples calculating explicitly values of the right hand side of (2.3) for different values of p.

Example 2.4. Let $D \equiv U$ be the unit disc. Then by Theorem 2.2 we have

$$\|\mathcal{L}_\Omega\|_p \le \|\mathcal{L}_U\|_p = \left(\sum_{m=1}^{\infty} \frac{3}{j_{0,m}^{2p}} + \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{2}{j_{l,m}^{2p}} \right)^{\frac{1}{p}}, \tag{2.5}$$

for any even $2 \le p < \infty$ and any bounded open domain Ω with $|\Omega| = |D|$. Here j_{km} denotes the mth positive zero of the Bessel function J_k of the first kind of order k.

The right hand sight of the formula (2.5) can be confirmed by a direct calculation of the logarithmic potential eigenvalues in the unit disc, see Theorem 3.1 in [1].

We also obtain the following Rayleigh–Faber–Krahn inequality when $p = \infty$:

Theorem 2.5. *The disc D is a minimizer of the characteristic number of the logarithmic potential $\mathcal{L}_\Omega$ with the smallest modulus among all domains of a given measure, that is,*

$$\|\mathcal{L}_\Omega\| \le \|\mathcal{L}_D\|$$

for an arbitrary bounded open domain $\Omega \subset \mathbb{R}^2$ with $|\Omega| = |D|$. Here $\|\cdot\|$ is the operator norm on the space L^2.

Example 2.6. Let $D \equiv U$ be the unit disc. Then by Theorem 2.5 we have

$$\|\mathcal{L}_\Omega\| \le \|\mathcal{L}_U\| = \frac{1}{j_{01}^2} \tag{2.6}$$

for any bounded open domain Ω with $|\Omega| = |D|$.

From Corollary 3.2 in [1] we calculate explicitly the operator norm in the right hand sight of (2.6).

3. Proof of Theorem 2.5

Let us first prove Theorem 2.5. To do it we first prove the following:

Lemma 3.1. *The characteristic number μ_1 of the logarithmic potential $\mathcal{L}_\Omega$ with the smallest modulus is simple, and the corresponding eigenfunction u_1 can be chosen nonnegative.*

Proof. The eigenfunctions of the logarithmic potential $\mathcal{L}_\Omega$ may be chosen to be real as its kernel is real. First let us prove that u_1 cannot change sign in the domain Ω, that is,

$$u_1(x)u_1(y) = |u_1(x)u_1(y)|, \; x, \, y \in \Omega.$$

In fact, in the opposite case, by virtue of the continuity of the function $u_1(x)$, there would be neighborhoods $U(x_0, r) \subset \Omega$ such that

$$|u_1(x)u_1(y)| > u_1(x)u_1(y), \quad x, y \in U(x_0, r) \subset \Omega.$$

On the other hand we have

$$\int_\Omega \frac{1}{2\pi} \ln \frac{1}{|x_0 - z|} \frac{1}{2\pi} \ln \frac{1}{|z - x_0|} dz > 0, \quad x_0 \in \Omega. \tag{3.1}$$

From here by continuity it is simple to check that there exists $\rho > 0$ such that

$$\int_\Omega \frac{1}{2\pi} \ln \frac{1}{|x - z|} \frac{1}{2\pi} \ln \frac{1}{|z - y|} dz > 0, \quad x, y \in U(x_0, \rho) \subset U(x_0, r). \tag{3.2}$$

Now let us introduce a new function

$$\widetilde{u}_1(x) := \begin{cases} |u_1(x)|, \ x \in U(x_0, \rho), \\ u_1(x), \ x \in \Omega \backslash U(x_0, \rho). \end{cases} \tag{3.3}$$

Then we obtain

$$\begin{aligned} \frac{(\mathcal{L}_\Omega^2 \widetilde{u}_1, \widetilde{u}_1)}{\|\widetilde{u}_1\|^2} &= \frac{1}{\|\widetilde{u}_1\|^2} \int_\Omega \int_\Omega \int_\Omega \frac{1}{2\pi} \ln \frac{1}{|x - z|} \frac{1}{2\pi} \ln \frac{1}{|z - y|} dz \widetilde{u}_1(x) \widetilde{u}_1(y) dx dy \\ &> \frac{1}{\|u_1\|^2} \int_\Omega \int_\Omega \int_\Omega \frac{1}{2\pi} \ln \frac{1}{|x - z|} \frac{1}{2\pi} \ln \frac{1}{|z - y|} dz u_1(x) u_1(y) dx dy = \frac{1}{\mu_1^2}, \end{aligned} \tag{3.4}$$

where μ_1^2 is the smallest characteristic number of $\mathcal{L}_\Omega^2$ and u_1 is the eigenfunction corresponding to μ_1^2, i.e.

$$u_1 = \mu_1^2 \mathcal{L}_\Omega^2 u_1.$$

Therefore, by the variational principle we also have

$$\frac{1}{\mu_1^2} = \sup_{f \in L^2(\Omega)} \frac{(\mathcal{L}_\Omega^2 f, f)}{\|f\|^2}. \tag{3.5}$$

This means that the strong inequality (3.4) contradicts the variational principle (3.5) because $\|\widetilde{u}_1\|_{L^2} = \|u_1\|_{L^2} < \infty$.

Since u_1 is nonnegative it follows that μ_1 is simple. Indeed, if there were an eigenfunction v_1 linearly independent of u_1 and corresponding to μ_1, then for all real c the linear combination $u_1 + cv_1$ also would be an eigenfunction corresponding to μ_1 and therefore, by what has been proved, it could not become negative in Ω. As c is arbitrary, this is impossible. $\square$

Proof of Theorem 2.5. Let Ω be a bounded open set in $\mathbb{R}^2$. Its symmetric rearrangement $\Omega^* \equiv D$ is an open disc centered at 0 with the measure equal to the measure of Ω, i.e. $|D| = |\Omega|$. Let u be a nonnegative measurable function in Ω, such that all its positive level sets have finite measure. In the definition of the symmetric-decreasing rearrangement of u one can use the so-called layer-cake decomposition (see [17]), which expresses a nonnegative function u in terms of its level sets as

$$u(x)=\int_0^\infty \chi_{\{u(x)>t\}}dt, \tag{3.6}$$

where χ is the characteristic function of the corresponding domain.

Definition 3.2. Let u be a nonnegative measurable function in Ω. The function

$$u^*(x):=\int_0^\infty \chi_{\{u(x)>t\}^*}dt \tag{3.7}$$

is called the (radially) symmetric-decreasing rearrangement of a nonnegative measurable function u.

As in the proof of Lemma 3.1 $\mu_1^2(\Omega)$ is the smallest characteristic number of $\mathcal{L}_\Omega^2$ and u_1 is the eigenfunction corresponding to μ_1^2, i.e.

$$u_1=\mu_1^2(\Omega)\mathcal{L}_\Omega^2 u_1.$$

By Lemma 3.1 the first characteristic number μ_1 of the operator $\mathcal{L}_\Omega$ is simple; the corresponding eigenfunction u_1 can be chosen positive in Ω, and in view of Lemma 3.1 we can apply the above construction to the first eigenfunction u_1. Recall[1] the rearrangement inequality for the logarithmic kernel (cf. Lemma 2 in [7])

$$\begin{aligned}&\int_\Omega\int_\Omega\int_\Omega u_1(y)\frac{1}{2\pi}\ln\frac{1}{|y-z|}\frac{1}{2\pi}\ln\frac{1}{|z-x|}u_1(x)dzdydx\le\\ &\quad\int_D\int_D\int_D u_1^*(y)\frac{1}{2\pi}\ln\frac{1}{|y-z|}\frac{1}{2\pi}\ln\frac{1}{|z-x|}u_1^*(x)dzdydx.\end{aligned} \tag{3.8}$$

In addition, for each nonnegative function $u\in L^2(\Omega)$ we have

$$\|u\|_{L^2(\Omega)}=\|u^*\|_{L^2(D)}. \tag{3.9}$$

Therefore, from (3.8), (3.9) and the variational principle for the positive operator $\mathcal{L}_D^2$, we get

$$\begin{aligned}\mu_1^2(\Omega)&=\frac{\int_\Omega|u_1(x)|^2dx}{\int_\Omega\int_\Omega\int_\Omega u_1(y)\frac{1}{2\pi}\ln\frac{1}{|y-z|}\frac{1}{2\pi}\ln\frac{1}{|z-x|}u_1(x)dzdydx}\ge\\ &\frac{\int_D|u_1^*(x)|^2dx}{\int_D\int_D\int_D u_1^*(y)\frac{1}{2\pi}\ln\frac{1}{|y-z|}\frac{1}{2\pi}\ln\frac{1}{|z-x|}u_1^*(x)dzdydx}\ge\\ &\inf_{v\in L^2(D)}\frac{\int_D|v(x)|^2dx}{\int_D\int_D\int_D v(y)\frac{1}{2\pi}\ln\frac{1}{|y-z|}\frac{1}{2\pi}\ln\frac{1}{|z-x|}v(x)dzdydx}=\mu_1^2(D).\end{aligned}$$

[1] For the proof of the rearrangement inequality (3.8) for the logarithmic kernel see Lemma 5.4. The proof is the same with the difference that is in this case the symmetric-decreasing rearrangement is used instead of the Steiner symmetrization.

Finally, note that 0 is not a characteristic number of $\mathcal{L}_D$ (cf. Corollary 1 in [24]). Therefore,

$$0 < |\mu_1(D)|.$$

This completes the proof. $\square$

4. Proofs of Theorem 2.2 and Theorem 2.3

First we prove the Brascamp–Lieb–Luttinger type rearrangement inequality for the logarithmic kernel (cf. [6]).

Lemma 4.1. *Let D be a disc centered at the origin. Then*

$$\int_\Omega \cdots \int_\Omega \frac{1}{2\pi} \ln \frac{1}{|y_1 - y_2|} \cdots \frac{1}{2\pi} \ln \frac{1}{|y_p - y_1|} dy_1 \dots dy_p \leq$$
$$\int_D \cdots \int_D \frac{1}{2\pi} \ln \frac{1}{|y_1 - y_2|} \cdots \frac{1}{2\pi} \ln \frac{1}{|y_p - y_1|} dy_1 \dots dy_p, \tag{4.1}$$

for any $p = 2, 3, \dots,$ and for any bounded open set Ω with $|\Omega| = |D|$.

Proof. Here we prove it for $p = 2$ and the proof is based on the proof of Lemma 2 in [7]. The proof for arbitrary p is essentially the same as the case $p = 2$. Let us fix $r_0 > 0$ and consider the function

$$f(r) := \begin{cases} \frac{1}{2\pi} \ln \frac{1}{r}, & r \leq r_0, \\ \frac{1}{2\pi} \ln \frac{1}{r_0} - \frac{1}{2\pi} \int_{r_0}^{r} s^{-1} \frac{1+r_0^2}{1+s^2} ds, & r > r_0. \end{cases} \tag{4.2}$$

Let us see that the function $f(r)$ is strictly decreasing and has a limit as $r \to \infty$. If $r \leq r_0$ then

$$f(r_1) = \frac{1}{2\pi} \ln \frac{1}{r_1} > \frac{1}{2\pi} \ln \frac{1}{r_2} = f(r_2)$$

for $r_1 < r_2$. If $r > r_0$ then

$$f(r) = \frac{1}{2\pi} \ln \frac{1}{r_0} - \frac{1}{2\pi} \int_{r_0}^{r} s^{-1} \frac{1 + r_0^2}{1 + s^2} ds =$$
$$\frac{1}{2\pi} \ln \frac{1}{r_0} - \frac{1}{2\pi}(1 + r_0^2)[\ln r - \frac{1}{2} \ln(1 + r^2) - \ln r_0 + \frac{1}{2} \ln(1 + r_0^2)]. \tag{4.3}$$

Thus $f(r_1) > f(r_2)$ for $r_1 < r_2$, that is, $f(r)$ is strictly decreasing. From (4.3) it is easy to see that

$$\lim_{r\to\infty} f(r) = \frac{1}{2\pi} \ln \frac{1}{r_0} - \frac{1}{2\pi}(1 + r_0^2)[-\ln r_0 + \frac{1}{2} \ln(1 + r_0^2)]. \tag{4.4}$$

We use the notation

$$f_\infty := \frac{1}{2\pi} \ln \frac{1}{r_0} - \frac{1}{2\pi}(1 + r_0^2)[-\ln r_0 + \frac{1}{2} \ln(1 + r_0^2)].$$

By construction $\frac{1}{2\pi} \ln \frac{1}{r} - f(r)$ is decreasing. Thus if we define

$$h_1(r) = f(r) - f_\infty$$

we have the decomposition

$$\frac{1}{2\pi}\ln\frac{1}{r} = h_1(r) + h_2(r)$$

where h_1 is positive strictly decreasing function and h_2 is decreasing. Hence by the Brascamp–Lieb–Luttinger rearrangement inequality we have

$$\int_\Omega\int_\Omega h_1(|y_1-y_2|)h_1(|y_2-y_1|)dy_1dy_2 \le \int_D\int_D h_1(|y_1-y_2|)h_1(|y_2-y_1|)dy_1dy_2 \tag{4.5}$$

and

$$\int_\Omega\int_\Omega h_2(|y_1-y_2|)h_2(|y_2-y_1|)dy_1dy_2 \le \int_D\int_D h_2(|y_1-y_2|)h_2(|y_2-y_1|)dy_1dy_2. \tag{4.6}$$

Thus it remains to show

$$\int_\Omega\int_\Omega h_1(|y_1-y_2|)h_2(|y_2-y_1|)dy_1dy_2 \le \int_D\int_D h_1(|y_1-y_2|)h_2(|y_2-y_1|)dy_1dy_2 \tag{4.7}$$

which does not follow directly from the Brascamp–Lieb–Luttinger rearrangement inequality since h_2 is not positive. Define for $R > 0$

$$q_R(r) := \begin{cases} h_2(r) - h_2(R), \ r \le R, \\ 0, \ r > R, \end{cases} \tag{4.8}$$

and note that by monotone convergence

$$I_\Omega(h_1, h_2) = \lim_{R\to\infty}[I_\Omega(h_1, q_R) + h_2(R)\int_\Omega\int_\Omega h_1(|y_1-y_2|)dy_1dy_2] \tag{4.9}$$

with the notation

$$I_\Omega(f, g) = \int_\Omega\int_\Omega f(|y_1-y_2|)g(|y_2-y_1|)dy_1dy_2. \tag{4.10}$$

Since h_1 and q_R are positive and nonincreasing

$$I_\Omega(h_1, q_R) \le I_D(h_1, q_R)$$

by the Brascamp–Lieb–Luttinger rearrangement inequality. Noting that

$$\int_\Omega\int_\Omega h_1(|y_1-y_2|)dy_1dy_2 \le \int_D\int_D h_1(|y_1-y_2|)dy_1dy_2$$

we obtain

$$I_\Omega(h_1,h_2)=\lim_{R\to\infty}[I_\Omega(h_1,q_R)+h_2(R)\int_\Omega\int_\Omega h_1(|y_1-y_2|)dy_1dy_2]\le$$
$$\lim_{R\to\infty}[I_D(h_1,q_R)+h_2(R)\int_D\int_D h_1(|y_1-y_2|)dy_1dy_2]=I_D(h_1,h_2), \tag{4.11}$$

completing the proof. □

Proof of Theorem 2.2. Since the logarithmic potential operator is a Hilbert–Schmidt operator, by using bilinear expansion of its iterated kernels (see, for example, [26]) we obtain for $p\ge 2$, $p\in\mathbb{N}$,

$$\sum_{j=1}^{\infty}\frac{1}{\mu_j^p(\Omega)}=\int_\Omega\dots\int_\Omega\frac{1}{2\pi}\ln\frac{1}{|y_1-y_2|}\dots\frac{1}{2\pi}\ln\frac{1}{|y_p-y_1|}dy_1\dots dy_p. \tag{4.12}$$

Recalling the inequality (4.1) stating that

$$\int_\Omega\dots\int_\Omega\frac{1}{2\pi}\ln\frac{1}{|y_1-y_2|}\dots\frac{1}{2\pi}\ln\frac{1}{|y_p-y_1|}dy_1\dots dy_p\le$$
$$\int_D\dots\int_D\frac{1}{2\pi}\ln\frac{1}{|y_1-y_2|}\dots\frac{1}{2\pi}\ln\frac{1}{|y_p-y_1|}dy_1\dots dy_p, \tag{4.13}$$

we obtain

$$\sum_{j=1}^{\infty}\frac{1}{\mu_j^p(\Omega)}\le\sum_{j=1}^{\infty}\frac{1}{\mu_j^p(D)},\quad p\ge 2,\ \ p\in\mathbb{N}, \tag{4.14}$$

for any bounded open domain $\Omega\subset\mathbb{R}^2$ with $|\Omega|=|D|$. Taking even p in (4.14) we complete the proof of Theorem 2.2. □

Proof of Theorem 2.3. The inequality (4.14) also proves Theorem 2.3 when the logarithmic potential operator is positive (see also Remark 2.1).

Remark 4.2. It follows from the properties of the kernel that the Schatten p-norm of the operator $\mathcal{L}_\Omega$ is finite when $p>1$ see e.g. the criteria for Schatten classes in terms of the regularity of the kernel in [8]. The above techniques do not allow us to prove Theorem 2.2 for all $p>1$. In view of the Dirichlet Laplacian case, it seems reasonable to conjecture that the Schatten p-norm is still maximised on the disc also for all $p>1$.

5. On the case of polygons

We can ask the same question of maximizing the Schatten p-norms in the class of polygons with a given number n of sides. We denote by $\mathcal{P}_n$ the class of plane polygons with n edges. We would like to identify the maximizer for Schatten p-norms of the logarithmic potential $\mathcal{L}_\Omega$ in $\mathcal{P}_n$. According to Section 2, it is natural to conjecture that it is the n-regular polygon. Currently, we can prove this only for $n=3$:

Theorem 5.1. *The equilateral triangle centered at the origin has the largest Schatten p-norm of the operator $\mathcal{L}_\Omega$ for any even integer $2\le p<\infty$ among all triangles of a given area. More precisely, if Δ is the equilateral triangle centered at the origin, we have*

$$\|\mathcal{L}_\Omega\|_p \le \|\mathcal{L}_\Delta\|_p \tag{5.1}$$

for any even integer $2 \le p < \infty$ *and any bounded open triangle* Ω *with* $|\Omega| = |\Delta|$.

Similarly, we have the following analogy of Theorem 2.3:

Theorem 5.2. *Let* Δ *be an equilateral triangle centered at the origin and let* Ω *be a bounded open triangle with* $|\Omega| = |\Delta|$. *Assume that the logarithmic potential operator is positive for* Ω *and* Δ. *Then*

$$\|\mathcal{L}_\Omega\|_p \le \|\mathcal{L}_\Delta\|_p \tag{5.2}$$

for any integer $2 \le p < \infty$.

Let u be a nonnegative, measurable function on $\mathbb{R}^2$, and let x^2 be a line through the origin of $\mathbb{R}^2$. Choose an orthogonal coordinate system in $\mathbb{R}^2$ such that the x^1-axis is perpendicular to x^2.

Definition 5.3. (See [6].) A nonnegative, measurable function $u^\star(x|x^2)$ on $\mathbb{R}^2$ is called a *Steiner symmetrization* with respect to x^2 of the function $u(x)$, if $u^\star(x^1,x^2)$ is a symmetric decreasing rearrangement with respect to x^1 of $u(x^1,x^2)$ for each fixed x^2.

The Steiner symmetrization (with respect to the x^1-axis) $\Omega^\star$ of a measurable set Ω is defined in the following way: if we write (x^1,z) with $z \in \mathbb{R}$, and let $\Omega_z = \{x^1 : (x^1,z) \in \Omega\}$, then

$$\Omega^\star = \{(x^1,z) \in \mathbb{R} \times \mathbb{R} : x^1 \in \Omega_z^*\}$$

where Ω^* is a symmetric rearrangement of Ω (see the proof of Theorem 2.5). We obtain:

Lemma 5.4. *For a positive function* u *and a measurable* $\Omega \subset \mathbb{R}^2$ *we have*

$$\begin{aligned}&\int_\Omega\int_\Omega\int_\Omega u(y)\frac{1}{2\pi}\ln\frac{1}{|y-z|}\frac{1}{2\pi}\ln\frac{1}{|z-x|}u(x)dzdydx \le \\ &\quad\int_{\Omega^\star}\int_{\Omega^\star}\int_{\Omega^\star} u^\star(y)\frac{1}{2\pi}\ln\frac{1}{|y-z|}\frac{1}{2\pi}\ln\frac{1}{|z-x|}u^\star(x)dzdydx,\end{aligned} \tag{5.3}$$

where $\Omega^\star$ *and* $u^\star$ *are Steiner symmetrizations of* Ω *and* u, *respectively.*

Proof. The proof is based on the proof of Lemma 2 in [7]. Let us fix $r_0 > 0$ and consider the function

$$f(r) := \begin{cases} \frac{1}{2\pi}\ln\frac{1}{r}, & r \le r_0, \\ \frac{1}{2\pi}\ln\frac{1}{r_0} - \frac{1}{2\pi}\int_{r_0}^r s^{-1}\frac{1+r_0^2}{1+s^2}ds, & r > r_0. \end{cases} \tag{5.4}$$

The function $f(r)$ is strictly decreasing and has a limit as $r \to \infty$ (see the proof of Lemma 4.1)

$$\lim_{r\to\infty} f(r) = f_\infty.$$

Since $f(r)$ is strictly decreasing $\frac{1}{2\pi}\ln\frac{1}{r} - f(r)$ is decreasing. Thus if we define

$$h_1(r) = f(r) - f_\infty$$

we have the decomposition

$$\frac{1}{2\pi}\ln\frac{1}{r} = h_1(r) + h_2(r)$$

where h_1 is positive strictly decreasing function and h_2 is decreasing. Hence by the Brascamp–Lieb–Luttinger rearrangement inequality for the Steiner symmetrization (see Lemma 3.2 in [6]) we have

$$\int_\Omega\int_\Omega\int_\Omega u(y)h_1(|y-z|)h_1(|z-x|)u(x)dzdydx \leq$$
$$\int_{\Omega^\star}\int_{\Omega^\star}\int_{\Omega^\star} u^\star(y)h_1(|y-z|)h_1(|z-x|)u^\star(x)dzdydx. \tag{5.5}$$

Thus it remains to show that

$$\int_\Omega\int_\Omega\int_\Omega u(y)h_2(|y-z|)h_2(|z-x|)u(x)dzdydx \leq$$
$$\int_{\Omega^\star}\int_{\Omega^\star}\int_{\Omega^\star} u^\star(y)h_2(|y-z|)h_2(|z-x|)u^\star(x)dzdydx \tag{5.6}$$

and

$$\int_\Omega\int_\Omega\int_\Omega u(y)h_1(|y-z|)h_2(|z-x|)u(x)dzdydx \leq$$
$$\int_{\Omega^\star}\int_{\Omega^\star}\int_{\Omega^\star} u^\star(y)h_1(|y-z|)h_2(|z-x|)u^\star(x)dzdydx, \tag{5.7}$$

which does not follow directly from the Brascamp–Lieb–Luttinger rearrangement inequality since h_2 is not positive. Define for $R > 0$

$$q_R(r) := \begin{cases} h_2(r) - h_2(R), \ r \leq R, \\ 0, \ r > R, \end{cases} \tag{5.8}$$

and note that by monotone convergence we have

$$I_\Omega(u, h_2) = \lim_{R\to\infty}[I_\Omega(u, q_R) + 2h_2(R)J_\Omega(u, q_R) + h_2^2(R)(\int_\Omega u(x)dx)^2] \tag{5.9}$$

with the notations

$$I_\Omega(u, g) = \int_\Omega\int_\Omega\int_\Omega u(y)g(|y-z|)g(|z-x|)u(x)dzdydx \tag{5.10}$$

and

$$J_\Omega(u, g) = \int_\Omega\int_\Omega\int_\Omega u(y)g(|z-x|)u(x)dzdydx. \tag{5.11}$$

Since q_R is positive and nonincreasing and noting that

$$\int_{\Omega} u(x)dx = \int_{\Omega^\star} u^\star(x)dx$$

we obtain

$$I_\Omega(u, q_R) \le I_{\Omega^\star}(u^\star, q_R),$$
$$J_\Omega(u, q_R) \le J_{\Omega^\star}(u^\star, q_R),$$

by the Brascamp–Lieb–Luttinger rearrangement inequality. Therefore,

$$I_\Omega(u, h_2) = \lim_{R\to\infty}[I_\Omega(u, q_R) + 2h_2(R)J_\Omega(u, q_R) + h_2^2(R)(\int_{\Omega} u(x)dx)^2] \le$$
$$\lim_{R\to\infty}[I_{\Omega^\star}(u^\star, q_R) + 2h_2(R)J_{\Omega^\star}(u^\star, q_R) + h_2^2(R)(\int_{\Omega^\star} u^\star(x)dx)^2] = I_{\Omega^\star}(u^\star, h_2). \quad (5.12)$$

This proves the inequality (5.6). Similarly, now let us show that the inequality (5.7) is valid. We have

$$\widetilde{I}_\Omega(u, h_2) = \lim_{R\to\infty}[\widetilde{I}_\Omega(u, q_R) + h_2(R)\widetilde{J}_\Omega(u, h_1)] \quad (5.13)$$

with the notations

$$\widetilde{I}_\Omega(u, g) = \int_{\Omega}\int_{\Omega}\int_{\Omega} u(y)h_1(|y-z|)g(|z-x|)u(x)dzdydx \quad (5.14)$$

and

$$\widetilde{J}_\Omega(u, h_1) = \int_{\Omega}\int_{\Omega}\int_{\Omega} u(y)h_1(|y-x|)u(x)dzdydx. \quad (5.15)$$

Since both q_R and h_1 are positive and nonincreasing

$$\widetilde{I}_\Omega(u, q_R) \le \widetilde{I}_{\Omega^\star}(u^\star, q_R), \quad (5.16)$$
$$\widetilde{J}_\Omega(u, h_1) \le \widetilde{J}_{\Omega^\star}(u^\star, h_1), \quad (5.17)$$

by the Brascamp–Lieb–Luttinger rearrangement inequality. Therefore, we obtain

$$\widetilde{I}_\Omega(u, h_2) = \lim_{R\to\infty}[\widetilde{I}_\Omega(u, q_R) + h_2(R)\widetilde{J}_\Omega(u, h_1)] \le$$
$$\lim_{R\to\infty}[\widetilde{I}_{\Omega^\star}(u^\star, q_R) + h_2(R)\widetilde{J}_{\Omega^\star}(u^\star, h_1)] = \widetilde{I}_{\Omega^\star}(u^\star, h_2).$$

This proves the inequality (5.7). □

Lemma 5.4 implies the following analogy of the Pólya theorem [20] for the operator $\mathcal{L}_\Omega$.

Theorem 5.5. *The equilateral triangle Δ centered at the origin is a minimizer of the first characteristic number of the logarithmic potential $\mathcal{L}_\Omega$ among all triangles of a given area, i.e.*

$$0 < |\mu_1(\Delta)| \leq |\mu_1(\Omega)|$$

for any triangle $\Omega \subset \mathbb{R}^2$ with $|\Omega| = |\Delta|$.

Remark 5.6. In other words Theorem 5.5 says that the operator norm of $\mathcal{L}_\Omega$ is maximized in an equilateral triangle among all triangles of a given area.

Proof of Theorem 5.5. By Theorem 2.5 and Lemma 3.1 the first characteristic number μ_1 of the operator $\mathcal{L}_\Omega$ is positive and simple; the corresponding eigenfunction u_1 can be chosen positive in Ω. Using the fact that by applying a sequence of the Steiner symmetrizations with respect to the mediator of each side, a given triangle converges to an equilateral one (see Fig. 3.2 in [11]), from (5.3) we obtain

$$\int\limits_\Omega \int\limits_\Omega \int\limits_\Omega u_1(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} u_1(x) dz dy dx \leq$$

$$\int\limits_\Delta \int\limits_\Delta \int\limits_\Delta u_1^\star(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} u_1^\star(x) dz dy dx. \tag{5.18}$$

Therefore, from (5.18) and the variational principle for the positive operator $\mathcal{L}_\Delta^2$, we get

$$\mu_1^2(\Omega) = \frac{\int_\Omega |u_1(x)|^2 dx}{\int_\Omega \int_\Omega \int_\Omega u_1(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} u_1(x) dz dy dx} \geq$$

$$\frac{\int_\Delta |u_1^\star(x)|^2 dx}{\int_\Delta \int_\Delta \int_\Delta u_1^\star(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} u_1^\star(x) dz dy dx} \geq$$

$$\inf_{v \in L^2(\Delta)} \frac{\int_\Delta |v(x)|^2 dx}{\int_\Delta \int_\Delta \int_\Delta v(y) \frac{1}{2\pi} \ln \frac{1}{|y-z|} \frac{1}{2\pi} \ln \frac{1}{|z-x|} v(x) dz dy dx} = \mu_1^2(\Delta).$$

Here we have used the fact that the Steiner symmetrization preserves the L^2-norm. $\square$

Proofs of Theorem 5.1 and Theorem 5.2. The proofs of Theorem 5.1 and Theorem 5.2 rely on the same techniques as the proofs of Theorem 2.2 and Theorem 2.3 with the difference that now the Steiner symmetrization is used instead of the symmetric-decreasing rearrangement. Since the Steiner symmetrization has the same property (4.13) (cf. Lemma 3.2 in [6]) as the symmetric-decreasing rearrangement, it is clear that any Steiner symmetrization increases (or at least does not decrease) the Schatten p-norms for integer $p \geq 2$. Thus, for the proof we only need to recall the fact that a sequence of Steiner symmetrizations with respect to the mediator of each side, a given triangle converges to an equilateral one. The rest of the proof is exactly the same as the proofs of Theorem 2.2 and Theorem 2.3. $\square$

Remark 5.7. A sequence of three Steiner symmetrizations allows us to transform any quadrilateral into a rectangle (see Fig. 3.3 in [11]). Therefore, it suffices to look at the maximization problem among rectangles for $\mathcal{P}_4$. Unfortunately, for $\mathcal{P}_5$ (pentagons and others), the Steiner symmetrization increases, in general, the number of sides. This prevents us from using the same technique for general polygons.

References

[1] J.M. Anderson, D. Khavinson, V. Lomonosov, Spectral properties of some integral operators arising in potential theory, Q. J. Math. Oxford Ser. (2) 43 (172) (1992) 387–407.

[2] J. Arazy, D. Khavinson, Spectral estimates of Cauchy's transform in $L^2(\Omega)$, Integral Equations Operator Theory 15 (6) (1992) 901–919.

[3] C. Bandle, Isoperimetric Inequalities and Applications, Monographs and Studies in Mathematics, vol. 7, Pitman (Advanced Publishing Program), Boston, MA, London, 1980.

[4] R.D. Benguria, H. Linde, B. Loewe, Isoperimetric inequalities for eigenvalues of the Laplacian and the Schrödinger operator, Bull. Math. Sci. 2 (1) (2012) 1–56.

[5] M.Š. Birman, M.Z. Solomjak, Estimates for the singular numbers of integral operators, Uspekhi Mat. Nauk 32 (1(193)) (1977) 17–84, 271.

[6] H.J. Brascamp, E.H. Lieb, J.M. Luttinger, A general rearrangement inequality for multiple integrals, J. Funct. Anal. 17 (1974) 227–237.

[7] E. Carlen, M. Loss, Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on S^n, Geom. Funct. Anal. 2 (1) (1992) 90–104.

[8] J. Delgado, M. Ruzhansky, Schatten classes on compact manifolds: kernel conditions, J. Funct. Anal. 267 (3) (2014) 772–798.

[9] M.R. Dostanić, The asymptotic behavior of the singular values of the convolution operators with kernels whose Fourier transforms are rational functions, J. Math. Anal. Appl. 395 (2) (2012) 496–500.

[10] I.C. Gohberg, M.G. Kreĭn, Introduction to the Theory of Linear Nonselfadjoint Operators, translated from Russian by A. Feinstein, Translations of Mathematical Monographs, vol. 18, American Mathematical Society, Providence, RI, 1969.

[11] A. Henrot, Extremum Problems for Eigenvalues of Elliptic Operators, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2006.

[12] M. Kac, On some connections between probability theory and differential and integral equations, in: Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, University of California Press, Berkeley, Los Angeles, 1951, pp. 189–215.

[13] M. Kac, On some probabilistic aspects of classical analysis, Amer. Math. Monthly 77 (1970) 586–597.

[14] M. Kac, Integration in Function Spaces and Some of Its Applications: Lezioni Fermiane (Fermi Lectures), Accademia Nazionale dei Lincei, Pisa, 1980.

[15] T.Sh. Kal'menov, D. Suragan, On spectral problems for the volume potential, Dokl. Math. 80 (2) (2009) 646–649.

[16] N.S. Landkof, Foundations of Modern Potential Theory, translated from Russian by A.P. Doohovskoy, Die Grundlehren der Mathematischen Wissenschaften, Band 180, Springer-Verlag, New York–Heidelberg, 1972.

[17] E.H. Lieb, M. Loss, Analysis, second edition, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.

[18] J.M. Luttinger, Generalized isoperimetric inequalities, Proc. Natl. Acad. Sci. USA 70 (1973) 1005–1006.

[19] Y. Miyanishi, T. Suzuki, Eigenvalues and eigenfunctions of double layer potentials, arXiv:1501.03627, 2015.

[20] G. Pólya, On the characteristic frequencies of a symmetric membrane, Math. Z. 63 (1955) 331–337.

[21] G. Pólya, G. Szegö, Isoperimetric Inequalities in Mathematical Physics, Annals of Mathematics Studies, vol. 27, Princeton University Press, Princeton, NJ, 1951.

[22] M. Ruzhansky, D. Suragan, On Schatten norms of integral operators of convolution type, Russian Math. Surveys (2015), in press.

[23] N. Saito, Data analysis and representation on a general domain using eigenfunctions of Laplacian, Appl. Comput. Harmon. Anal. 25 (1) (2008) 68–97.

[24] J.L. Troutman, The logarithmic potential operator, Illinois J. Math. 11 (1967) 365–374.

[25] J.L. Troutman, The logarithmic eigenvalues of plane sets, Illinois J. Math. 13 (1969) 95–107.

[26] V.S. Vladimirov, Equations of Mathematical Physics, translated from Russian by Eugene Yankovsky, Mir, Moscow, 1984.