KHOVANOV HOMOLOGY FROM ALE SPACES

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Abstract. We describe a short construction of Khovanov homology of links via derived categories of coherent sheaves on deformations of the Hilbert schemes of points on ALE surfaces.

1. Introduction

Cautis and Kamnitzer gave a celebrated construction [CK] of Khovanov homology which can be thought of as mirror to the symplectic Khovanov homology of [SS]. Their model of the mirror is motivated by the geometric Langlands programme, and has the advantage of generalising to more general Lie algebras.

Here we use a mirror motivated by Manolescu's description [Ma] of the space used in [SS]. This has the advantage that the proofs of the necessary braid relations are simpler, since they arise from standard two dimensional derived equivalences [ST].

Here we describe the results rather briefly, leaving all geometric and mirror-symmetric motivations to [Th]. After 4 years of this paper sitting on our desks we have decided to use [Hi] to simplify the deformation theory and make it available for someone with more energy to contemplate. In particular the deformation theory of Section 4 is rather sketchy and ad hoc, using tricks to deduce the noncompact results we need from the better understood compact setting. And we gave up before really forcing this all through anyway. A more direct approach would be much preferable.

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2. ALE spaces and maps between them

Let A_{k-1} denote the standard surface singularity

$$A_{k-1} := \mathbb{C}^2/(\mathbb{Z}/k\mathbb{Z}),$$

where the generator 1 of $\mathbb{Z}/k\mathbb{Z}$ acts as $\mathrm{diag}(e^{2\pi i/k}, e^{-2\pi i/k}) \in SL(2,\mathbb{C})$. Equivalently A_{k-1} is the hypersurface $\{x^k = yz\} \subset \mathbb{C}^3$.

Let S_{k-1} be the minimal resolution of A_{k-1}. It has an A_{k-1}-chain of -2-curves $C_i \cong \mathbb{P}^1,\ i = 1, \ldots, k-1$, and is holomorphic symplectic.

Maps between spaces. Crucial to our construction will be the observation that there is a natural inclusion $S_{k-1} \subset S_k$ taking the A_{k-1}-chain of curves C_i in the former to the first $k-1$ curves of the A_k-chain $C_1, \ldots, C_{k-1}, C_k$ in the latter. We see this as follows; cf. Figure 1.

Let $\overline{A}_{k-1}$ denote the blow up of $\mathbb{C}^2$ in the ideal (x^k, y). It can be constructed via blow ups and a blow down in smooth centres as follows.

(1) Blow up the (reduced) origin in $\mathbb{C}^2$, giving an exceptional divisor $E_1 \cong \mathbb{P}^1$.
(2) Blow up the point $\infty \in E_1$ (its intersection with the proper transform of the x-axis). We get a new exceptional divisor E_2, and the proper transform of E_1 which is a -2-curve C_1.
(r) At the rth stage, blow up $\infty \in E_{r-1}$ to produce a new exceptional divisor E_r, and the proper transform of E_{r-1} is a -2-curve C_r.

After the kth step we get a surface $\overline{S}_{k-1}$ with an A_{k-1}-chain of -2-curves C_i and a -1-curve E_k. Then blow down the $C_i,\ i = 1, \ldots, k-1$ to get $\overline{A}_{k-1}$.

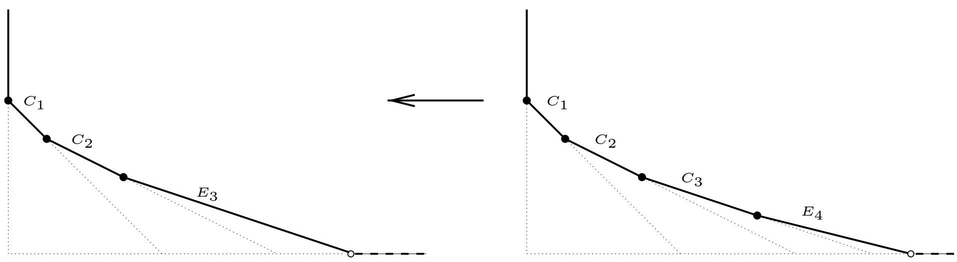

Figure 1. Newton polygon diagram of the blow up map $\overline{S}_2 \leftarrow \overline{S}_3$. On removing the divisors corresponding to the dashed lines (the proper transforms of the x-axis) we get an inclusion $S_2 \subset S_3$ in the opposite direction.

Now $\overline{A}_{k-1} = \mathrm{Bl}_{(x^k,y)}\, \mathbb{C}^2 = \{\mu x^k = \lambda y\} \subset \mathbb{C}^2_{x,y} \times \mathbb{P}^1_{[\lambda:\mu]}$. Therefore if we remove the proper transform $\overline{\{y=0\}} = \{\mu = 0\}$ of the x-axis we can set $[\lambda : \mu] = [z : 1]$ to get the affine variety

$$\{x^k = yz\} \subset \mathbb{C}^2_{x,y} \times \mathbb{C}_z,$$

which is precisely A_{k-1}. Thus $\overline{A}_{k-1}$ and $\overline{S}_{k-1}$ are partial compactifications of A_{k-1} and S_{k-1} respectively (since $\overline{S}_{k-1}$ is the minimal resolution of $\overline{A}_{k-1}$).

We obtained $\overline{S}_k$ from $\overline{S}_{k-1}$ by blowing up the latter in the point $\infty \in E_k$. But $\infty = \overline{\{y=0\}} \cap E_k$ lies in the divisor $\overline{\{y=0\}}$ that we remove from $\overline{S}_{k-1}$ to get S_{k-1}, so the inclusion $S_{k-1} \subset \overline{S}_{k-1}$ lifts to the blow up: $S_{k-1} \subset \overline{S}_k$. Its image is clearly contained in the open subset S_k, and maps the curves $C_i \subset S_{k-1}$ to the corresponding curves $C_i \subset S_k$, as claimed.

3. Derived categories

Throughout this paper $D(Y)$ will denote the bounded derived category of coherent sheaves *with compact support* on a smooth quasiprojective variety[1] Y. By [BKR, Hai] the category

$$D_n := D(\operatorname{Hilb}^n S_{2n-1})$$

has a canonical identification with the Σ_n-equivariant derived category of the n-fold product S_{2n-1}^n, where the symmetric group Σ_n permutes the factors:

$$R\pi_{1*}\,\pi_2^*\colon D(\operatorname{Hilb}^n S_{2n-1}) \xrightarrow{\sim} D(S_{2n-1}^n)^{\Sigma_n}. \tag{3.1}$$

Here the π_i are the projections from the underlying *reduced* variety Z_{red} of the fibre product Z of $\operatorname{Hilb}^n S_{2n-1}$ and S_{2n-1}^n over S_{2n-1}^n/Σ_n:

$$\begin{tikzcd} Z_{\mathrm{red}} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & S_{2n-1}^n \arrow[d, "q"] \\ \operatorname{Hilb}^n S_{2n-1} \arrow[r, "\pi"] & S_{2n-1}^n/\Sigma_n. \end{tikzcd} \tag{3.2}$$

Any $E \in D(S_{2n-1}^n)$ defines an element (with its obvious Σ_n-linearisation) by

$$\Sigma_n.E := \bigoplus_{\sigma\in\Sigma_n} \sigma^* E \ \in\ D(S_{2n-1}^n)^{\Sigma_n}. \tag{3.3}$$

Thus from the spherical objects $L_i := \mathcal{O}_{C_i}(-1) \in D(S_{2n-1})$ we define

$$\mathscr{L} = \mathscr{L}_n := \Sigma_n.(L_1 \boxtimes L_3 \boxtimes \ldots \boxtimes L_{2n-1}) \in D(S_{2n-1}^n)^{\Sigma_n}. \tag{3.4}$$

Since none of the L_{2i-1} intersect, the support of $\mathscr{L}$ is disjoint from the big diagonal in S_{2n-1}^n, so its image in $D(\operatorname{Hilb}^n S_{2n-1})$ is easily calculated from (3.2). Namely, via the map q (3.2), $C_1 \times C_3 \times \ldots \times C_{2n-1}$ embeds into S_{2n-1}^n/Σ_n with image contained in the locus where π is an isomorphism. Therefore we can think of it as embedded in $\operatorname{Hilb}^n S_{2n-1}$, whereupon

$$\mathscr{L} = \mathcal{O}_{C_1\times C_3\times\ldots\times C_{2n-1}}(-1,-1,\ldots,-1) \in D(\operatorname{Hilb}^n S_{2n-1}). \tag{3.5}$$

Later we will need the computation of Homs between objects such as (3.4):

$$\begin{aligned}&\operatorname{Ext}^*_{D(S_{2n-1}^n)^{\Sigma_n}}\Big(\Sigma_n.(E_1\boxtimes\ldots\boxtimes E_n), \Sigma_n.(F_1\boxtimes\ldots\boxtimes F_n)\Big)\\ &\qquad\qquad = \bigoplus_{\sigma\in\Sigma_n}\bigotimes_{i=1}^n \operatorname{Ext}^*_{D(S_{2n-1})}(E_i, F_{\sigma(i)}).\end{aligned} \tag{3.6}$$

[1] Therefore when in Section 4 we work with smooth families $Y_k \to \operatorname{Spec}\mathbb{C}[t]/(t^{k+1})$ over Artinian rings, $D(Y_k)$ will denote the bounded derived category of perfect complexes with compactly supported cohomology.

Braid group action. A canonical homomorphism $\Phi\colon \mathrm{Aut}(D(S_{2n-1}) \hookrightarrow \mathrm{Aut}(D(S^n_{2n-1})^{\Sigma_n})$ is constructed in [Pl]. We only need to know its rather natural action on objects of the form (3.4):

$$\Phi(T)\Big(\Sigma_n.(E_1 \boxtimes \ldots \boxtimes E_n)\Big) \;=\; \Sigma_n.\big(T(E_1) \boxtimes \ldots \boxtimes T(E_n)\big). \tag{3.7}$$

The spherical twists T_{L_i} [ST] in the L_i generate a (faithful) braid group action $B_{2n} \hookrightarrow \mathrm{Aut}(D(S_{2n-1}))$ [KS, ST]. Therefore setting

$$\mathsf{T}_i := \Phi(T_i)[1] \;\in\; \mathrm{Aut}\left(D(S^n_{2n-1})^{\Sigma_n}\right) \tag{3.8}$$

gives generators of a braid group $B_{2n} \hookrightarrow \mathrm{Aut}(D_n)$. (Since the braid relations are homogeneous the extra shift [1] makes no difference.) Thus any $\beta \in B_{2n}$ gives an autoequivalence $\mathsf{T}_\beta \in \mathrm{Aut}(D_n)$. We define the braid invariant

$$H^*(\beta) := \mathrm{Ext}^*_{D_n}(\mathsf{T}_\beta \mathscr{L}, \mathscr{L}[n]). \tag{3.9}$$

In Section 5 we will put a bigrading on H^* via a natural $\mathbb{C}^*$-action on S_{2n-1}.

Link invariants. We would like $H^*(\beta)$ of (3.9) to be an invariant of the isotopy class of the link given the plait closure of β. (In fact it is not; we will have to deform $\mathrm{Hilb}^n S_{2n-1}$ in the next Section to achieve this.) By a result of Birman [Bir], modified slightly in [Big], and the fact that the T_βs are *functors* (so that $\mathrm{Ext}^*_{D_n}(\mathsf{T}_\alpha\mathsf{T}_\beta \mathscr{L}, \mathscr{L}[n]) = \mathrm{Ext}^*_{D_n}(\mathsf{T}_\beta \mathscr{L}, \mathsf{T}_{\alpha^{-1}}\mathscr{L}[n])$, for instance), it would be sufficient to prove the following; see Figure 2.

(1) $\mathsf{T}_1\mathscr{L} \cong \mathscr{L}$,
(2) $\mathsf{T}_{2i-1}\mathsf{T}_{2i}\,\mathscr{L} \cong \mathsf{T}^{-1}_{2i-1}\mathsf{T}^{-1}_{2i}\mathscr{L}$,
(3) $\mathsf{T}_{2i}\mathsf{T}_{2i-1}\mathsf{T}_{2i+1}\mathsf{T}_{2i}\,\mathscr{L} \cong \mathscr{L}$, and
(4) $\mathrm{Ext}^*_{D_n}(\mathsf{T}_\beta \mathscr{L}_n, \mathscr{L}_n[n]) \cong \mathrm{Ext}^*_{D_{n+1}}(\mathsf{T}_\beta \mathscr{L}_{n+1}, \mathsf{T}^{\pm 1}_{2n}\mathscr{L}_{n+1}[n+1])$.

In the last relation (stabilisation as we increase the number of strands in our braid, or Markov II as it is called in [SS]), β is an element of B_{2n} which on the right hand side of the equation is considered as an element of B_{2n+2} via the standard inclusion $B_{2n} \hookrightarrow B_{2n+2}$.

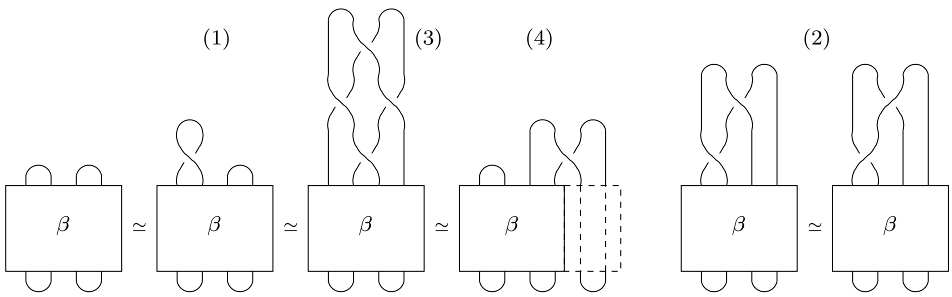

FIGURE 2. Equivalent plait closures of a braid $\beta \in B_4$

Theorem 3.10. *The relations (1), (3) and (4) hold in the categories D_n, but (2) does not.*

Proof.[2] On the surface S_{2n-1} an elementary calculation gives $T_1L_1 \cong L_1[-1]$ and $T_1L_{2i+1} \cong L_{2i+1}$ for $i \geq 1$ (because $R\operatorname{Hom}(L_1, L_{2i+1}) = 0$). Therefore by (3.7), $\mathsf{T}_1\mathscr{L} \cong \mathscr{L}[-1][1] = \mathscr{L}$, proving relation (1).

For (2) we calculate on S_{2n-1} that both $T_{2i-1}T_{2i}$ and $T_{2i-1}^{-1}T_{2i}^{-1}$ leave L_{2j+1} alone for $j \neq i, i-1$. Both also take L_{2i-1} to L_{2i}, but

$$T_{2i-1}T_{2i}L_{2i+1} \cong \mathcal{O}_{C_{2i-1}\cup C_{2i}\cup C_{2i+1}}(-1,0,0), \tag{3.11}$$

$$T_{2i-1}^{-1}T_{2i}^{-1}L_{2i+1} \cong \mathcal{O}_{C_{2i-1}\cup C_{2i}\cup C_{2i+1}}(0,0,-1). \tag{3.12}$$

Since these are *not* isomorphic it follows from (3.7) that $\mathsf{T}_{2i-1}\mathsf{T}_{2i}\mathscr{L} \ncong \mathsf{T}_{2i-1}^{-1}\mathsf{T}_{2i}^{-1}\mathscr{L}$, i.e. (2) does not hold.

Another calculation on S_{2n-1} shows that $T_{2i}T_{2i-1}T_{2i+1}T_{2i}$ also leaves L_{2j+1} alone for $j \neq i, i-1$, but swaps $L_{2i\pm1}$:

$$T_{2i}T_{2i-1}T_{2i+1}T_{2i}L_{2i\pm1} = L_{2i\mp1}.$$

Relation (3) then follows again from (3.7).

Finally to prove (4) we consider S_{2n-1} as an open subvariety of S_{2n+1} using the inclusion map of Section 2. Since this takes the A_{2n-1}-chain of curves C_i to the first $2n-1$ curves C_i of the A_{2n+1}-chain on S_{2n+1}, it intertwines the action of B_{2n} on $D(S_{2n-1})$ and that of $B_{2n} \subset B_{2n+2}$ on $D(S_{2n+1})$.

Using (3.7) and (3.6) we compute the right hand side of (4) (in the T_{2n} case; T_{2n}^{-1} is very similar) as the direct sum over $\tilde{\sigma} \in \Sigma_{n+1}$ of the terms

$$\left(\bigotimes_{i=1}^{n} \operatorname{Ext}^*_{S_{2n+1}}\left(T_\beta L_{2i-1}, T_{2n}L_{2\tilde{\sigma}(i)-1}\right)\right) \otimes \operatorname{Ext}^{*+n+1}_{S_{2n+1}}(L_{2n+1}, T_{2n}L_{2\tilde{\sigma}(n+1)-1}).$$

The last term vanishes unless $\tilde{\sigma}(n+1) = n+1$ or n. The first case occurs if and only if $\tilde{\sigma}$ is the image of $\sigma \in \Sigma_n \hookrightarrow \Sigma_{n+1}$, in which case we get

$$\bigoplus_{\sigma\in\Sigma_n}\left(\bigotimes_{i=1}^{n} \operatorname{Ext}^*_{S_{2n+1}}\left(T_\beta L_{2i-1}, T_{2n}L_{2\sigma(i)-1}\right)\right)[n+1] \tag{3.13}$$

using the easy calculation on the surface S_{2n+1} that

$$\operatorname{Ext}^*_{S_{2n+1}}(L_{2n+1}, T_{2n}L_{2n+1}) = \mathbb{C}[0]. \tag{3.14}$$

Similarly $\tilde{\sigma}(n+1) = n$ if and only if $\tilde{\sigma} = (n\ n{+}1)\circ\sigma$ for some $\sigma \in \Sigma_n \hookrightarrow \Sigma_{n+1}$. In this case we get

$$\bigoplus_{\sigma\in\Sigma_n}\left(\bigotimes_{i=1}^{n} \operatorname{Ext}^*_{S_{2n+1}}\left(T_\beta L_{2i-1}, T_{2n}L_{2\tilde{\sigma}(i)-1}\right)\right)[n] \tag{3.15}$$

from the easy calculation

$$\operatorname{Ext}^*_{S_{2n+1}}(L_{2n+1}, T_{2n}L_{2n-1}) = \mathbb{C}[-1]. \tag{3.16}$$

[2]For geometry (and pictures!) explaining this proof, see [Th, Section 2.8].

Since T_{2n} leaves L_{2j-1} alone unless $j=n$, the terms in (3.13) and (3.15) simplify except when $\sigma(i)=n$ (corresponding in (3.15) to $\tilde{\sigma}(i)=n+1$). The sum of (3.13) and (3.15) therefore becomes the sum over $\sigma\in\Sigma_n$ of

$$\bigotimes_{\substack{i=1\\ i\neq\sigma^{-1}(n)}}^{n} \operatorname{Ext}^*\left(T_\beta L_{2i-1}, L_{2\sigma(i)-1}\right) \tag{3.17}$$

$$\otimes \operatorname{Ext}^*\left(T_\beta L_{2\sigma^{-1}(n)-1}, T_{2n}L_{2n-1}\oplus T_{2n}L_{2n+1}[-1]\right)[n+1].$$

From $R\operatorname{Hom}(L_{2n},L_{2n-1})=\mathbb{C}[-1]$ we get the extension exact triangle

$$L_{2n-1}\to T_{2n}L_{2n-1}\to L_{2n} \tag{3.18}$$

and so

$$\operatorname{Ext}^*(E,L_{2n})[-1]\to\operatorname{Ext}^*(E,L_{2n-1})\to\operatorname{Ext}^*(E,T_{2n}L_{2n-1})\to\operatorname{Ext}^*(E,L_{2n})$$

for any $E\in D(S_{2n+1})$. Since $\operatorname{Ext}^*(L_j,L_{2n})=0$ for $j<2n-1$, the final arrow is zero for $E=L_j$, $j<2n-1$. It is also zero for $E=L_{2n-1}$ because the generator of $R\operatorname{Hom}(L_{2n-1},T_{2n}L_{2n-1})=\mathbb{C}$ is the first arrow of (3.18) which is in the image of the identity in $\operatorname{Ext}^*(L_{2n-1},L_{2n-1})$. Therefore for all E in the subcategory generated by L_j, $j\leq 2n-1$, we have a splitting

$$\begin{aligned}\operatorname{Ext}^*(E,L_{2n-1}) &\cong \operatorname{Ext}^*(E,L_{2n})[-1]\oplus\operatorname{Ext}^*(E,T_{2n}L_{2n-1})\\ &\cong \operatorname{Ext}^*(E,T_{2n}L_{2n+1})[-1]\oplus\operatorname{Ext}^*(E,T_{2n}L_{2n-1}).\end{aligned} \tag{3.19}$$

The second isomorphism follows from $R\operatorname{Hom}(E,L_{2n+1})=0$. Applying this to $E=T_\beta L_{2\sigma^{-1}(n)-1}$ and substituting into (3.17) shows the right hand side of relation (4) is

$$\bigoplus_{\sigma\in\Sigma_n}\bigotimes_{i=1}^{n}\operatorname{Ext}^*_{S_{2n+1}}\left(T_\beta L_{2i-1},L_{2\sigma(i)-1}\right)[n+1]. \tag{3.20}$$

Since this can equally be computed in $D(S_{2n-1})$ by the inclusion map, we see from (3.6) that it equals $\operatorname{Ext}^*_{D(S^n_{2n-1})^{\Sigma_n}}(\mathsf{T}_\beta\mathscr{L}_n,\mathscr{L}_n[n])$, thus proving (4). □

4. Deformation

Throughout this Section, n is fixed. Let E denote the exceptional divisor of the crepant resolution $H_n:=\operatorname{Hilb}^n(S_{2n-1})\to\operatorname{Sym}^n(S_{2n-1})$ with class[3] $[E]\in H^1(\Omega_{H_n})$. Via the isomorphism $\Omega_{H_n}\cong T_{H_n}$ induced by the holomorphic symplectic form we get a canonical class $\kappa_0\in H^1(T_{H_n})$, the space of first order deformations of H_n.

This deformation can be globalised using twistor families, or by direct construction via the hyperkähler quotient construction. We find it convenient to use the holomorphic Poisson deformation theory of Hitchin [Hi],

[3]This is easily defined despite the noncompactness of H_n; for instance the exact sequence $0\to\Omega_{H_n}\to\Omega_{H_n}(\log D)\to\mathcal{O}_D\to 0$ has extension class in $\operatorname{Ext}^1(\mathcal{O}_D,\Omega_{H_n})$; its image in $\operatorname{Ext}^1(\mathcal{O}_{H_n},\Omega_{H_n})=H^1(\Omega_{H_n})$ is $[E]$.

using the Poisson structure on S_{2n-1}. Now all of our calculations, and the invariant (3.9), depend only on a formal neighbourhood (or germ of S_{2n-1})

$$S^{\mathrm{o}}_{2n-1} \subset S_{2n-1}$$

of the A_{2n-1}-chain of curves $C_i \subset S_{2n-1}$, and the part of the Hilbert scheme $H^{\mathrm{o}}_n \subset H_n$ parameterising points supported on S^{o}_{2n-1}. So we fix any projective Poisson compactification

$$S^{\mathrm{o}}_{2n-1} \subset \mathbb{S}.$$

So $\mathbb{S}$ could be a compactification of all of S_{2n-1} to which the holomorphic Poisson structure of S_{2n-1} extends, like the minimal resolution of $\mathbb{P}^2/(\mathbb{Z}/2n\mathbb{Z})$, where $\mathbb{Z}/2n\mathbb{Z} \subset PSL(3,\mathbb{C})$ acts with weights $(1,-1,0)$. But for $n \le 10$, $\mathbb{S}$ could also be a $K3$ surface; this will be important below.

The Poisson structure on $\mathbb{S}$ induces one on its Hilbert schemes of points. Thus we get a compactification

$$H^{\mathrm{o}}_n \subset \mathbb{H} := \mathrm{Hilb}^n(\mathbb{S}) \tag{4.1}$$

where the holomorphic symplectic structure θ on the left is induced by the Poisson structure $\sigma \in H^0(\Lambda^2 T_{\mathbb{H}})$ on the right: $(\sigma|_{H^{\mathrm{o}}_n})^{-1} = \theta$.

The exceptional divisor $\mathbb{E}$ of $\mathrm{Hilb}^n(\mathbb{S}) \to \mathrm{Sym}^n(\mathbb{S})$ compactifies $E^{\mathrm{o}} \subset H^{\mathrm{o}}_n$. Now [Hi] gives, for all small $t \in \mathbb{C}$, a family (J_t, σ_t) of complex structures and compatible holomorphic Poisson structures on $\mathbb{H}$ (and so a family $\mathbb{H}_t$ with central fibre $\mathbb{H}$) whose Kodaira-Spencer class at any time t is

$$\kappa_t := [\mathbb{E}]^{1,1}_t \lrcorner\, \sigma_t \in H^1(T_{\mathbb{H}_t}). \tag{4.2}$$

Here $[\mathbb{E}]^{1,1}_t$ denotes the projection of $[\mathbb{E}] \in H^2(\mathbb{H})$ to $H^{1,1}(\mathbb{H}_t) \subset H^2(\mathbb{H}_t) = H^2(\mathbb{H})$. Either $H^{2,0}(\mathbb{H}) = 0$, in which case this is just $[\mathbb{E}]$, or $\mathbb{H}$ is holomorphic symplectic in which case $H^{2,0}(\mathbb{H}_t)$ is generated by the symplectic form $\theta_t := \sigma_t^{-1}$. In this latter case the (1,1) projection removes multiples of $\mathrm{Re}\,\theta_t, \mathrm{Im}\,\theta_t$, so in either case we find that

$$[\mathbb{E}]^{1,1}_t \in \langle\, \mathrm{Re}\,\theta, \mathrm{Im}\,\theta, [\mathbb{E}]\,\rangle \subset H^2(\mathbb{H}_t) = H^2(\mathbb{H}). \tag{4.3}$$

Hitchin shows that the degeneracy locus of the tensors σ_t are all the same, so can be removed to give a quasi-projective holomorphic symplectic family $\mathbb{H}^{\mathrm{o}}_t$ containing H^{o}_n in its central fibre. Finally, by [Hi, Proposition 7], in this family the cohomology class of this symplectic form θ_t is $[\theta_t] = [\theta] - 2t[\mathbb{E}] \in H^2(\mathbb{H}^{\mathrm{o}}_t)$ if $H^{2,0}(\mathbb{H}) = 0$, while more generally

$$[\theta_t] \in \langle\, \mathrm{Re}\,\theta, \mathrm{Im}\,\theta, [\mathbb{E}]\,\rangle. \tag{4.4}$$

Lemma 4.5. *The sheaf $\mathscr{L}_n$ deforms uniquely to a sheaf $\mathscr{L}_t$ on $\mathbb{H}^{\mathrm{o}}_t$.*

Proof. Let $\mathbb{H}^{\mathrm{o}}_k$ denote the pullback of the family $\mathbb{H}^{\mathrm{o}}_t$ to the base $B_k := \mathrm{Spec}\,\mathbb{C}[t]/t^{k+1}$, and let its fibrewise holomorphic 2-form be denoted θ. Suppose inductively that the support $C_1 \times \ldots \times C_{2n-1}$ of $\mathscr{L}_n$ (3.5) deforms to a B_k-family $\mathcal{C}_k = (\mathbb{P}^1)^n \times B_k \hookrightarrow \mathbb{H}^{\mathrm{o}}_k$. This is trivially true for $k = 0$.

Let $\kappa_k \in H^1(T_{\mathbb{H}^\circ_k/B_k})$ denote the Kodaira-Spencer class of $\mathbb{H}^\circ_{k+1}$. Its projection to $H^1(N_{\mathcal{C}_k/\mathbb{H}^\circ_k})$ is the obstruction to deforming $\mathcal{C}_k \subset \mathbb{H}^\circ_k$ to some $\mathcal{C}_{k+1} \subset \mathbb{H}^\circ_{k+1}$. But $(\mathbb{P}^1)^n$ has no holomorphic 2-forms, so $\mathcal{C}_k$ is fibrewise Lagrangian. Thus $N_{\mathcal{C}_k/\mathbb{H}^\circ_k} \cong \Omega_{\mathcal{C}_k/\mathbb{H}^\circ_k}$ and this obstruction is the same as the restriction of $\kappa_k \lrcorner\, \theta \in H^1(\Omega_{\mathbb{H}^\circ_k/B_k})$ to $H^1(\Omega_{\mathcal{C}_k/\mathbb{H}^\circ_k})$.

But by construction (4.2), this is the cohomology class $[\mathbb{E}]^{1,1}$ restricted to $\mathcal{C}_k/\mathbb{H}^\circ_k$. Since $\mathcal{C}_0$ is disjoint from $\mathbb{E}$ and Lagrangian with respect to θ, this vanishes by (4.3) and we can produce $\mathcal{C}_{k+1} \subset \mathbb{H}^\circ_{k+1}$. The rigidity $H^1(T_{(\mathbb{P}^1)^n}) = 0$ ensures that $\mathcal{C}_{k+1} \cong (\mathbb{P}^1)^n \times B_{k+1}$. Since we can deform to all orders (and the Hilbert scheme of the family $\mathbb{H}_t$ is projective) an actual deformation $\mathcal{C}_t \hookrightarrow \mathbb{H}^\circ_t$ exists for small t.

Finally, pushing forward $\mathcal{O}_{\mathbb{P}^1}(-1)^{\boxtimes n}$ via $\mathcal{C}_t \hookrightarrow \mathbb{H}^\circ_t$ defines the deformation $\mathscr{L}_t$. Uniqueness follows from the vanishing of $\mathrm{Ext}^1_{H_n}(\mathscr{L}_n, \mathscr{L}_n)$. $\square$

Deforming the autoequivalences, I: compact case. Showing that the generators T_i deform is more involved. We first restrict to the special case that $\mathbb{S}$ is a $K3$ surface, where compactness simplifies things. In particular $\mathbb{H}^\circ_t = \mathbb{H}_t$ in this case since the Hilbert scheme and its deformations are holomorphic symplectic. We use the notation above, so that $p_k \colon \mathbb{H}_k \to B_k$ is the basechange of $\mathbb{H}_t$ to the base $B_k := \operatorname{Spec} \mathbb{C}[t]/t^{k+1}$, and θ is its fibrewise holomorphic 2-form.

Starting at $k = 0$, suppose inductively that we have uniquely extended the autoequivalence T_i to $\mathbb{H}_k$. Let $\kappa_k \in H^1(T_{\mathbb{H}_k/B_k})$ denote the Kodaira-Spencer class of $\mathbb{H}_{k+1}$. Its contraction with θ is a fibrewise $(1,1)$-form whose image in $R^2 p_{k*}\mathbb{C} \cong H^2(\mathbb{H}) \otimes \mathbb{C}[t]/t^{k+1}$ lies in the span of $\operatorname{Re}\theta, \operatorname{Im}\theta, [\mathbb{E}]$ by (4.3).

Now the action[4] of T_{L_i} on $D(\mathbb{S})$ induces a very simple action on $H^*(\mathbb{S})$: it simply reflects in the class $[C_i] \in H^2(\mathbb{S}) \subset H^*(\mathbb{S})$. (In particular, it preserves the grading, unusually[5].) The induced action of $\mathsf{T}_i[-1]$ on $H^*(\mathbb{H}) = H^*(\operatorname{Hilb}^n \mathbb{S})$ also preserves the grading, and, on H^2,

$$H^2(\mathbb{H}) = H^2(\mathbb{S}) \oplus [\mathbb{E}]/2,$$

its action is also reflection in $[C_i]$ on the first summand and the identity on the second. It follows that T_i fixes $[\mathbb{E}], \operatorname{Re}\theta, \operatorname{Im}\theta$ in $H^2(\mathbb{H})$ (the latter because C_i is Lagrangian), and thus also $\kappa_k \lrcorner\, \theta$.

[4]Any Fourier-Mukai transform $D(X) \to D(Y)$ induces a map $H^*(X) \to H^*(Y)$ by using the Mukai vector of the Fourier-Mukai kernel as a convolution.

[5]This is a reflection of the fact that the autoequivalence T_{L_i} can be seen as the limit of a family of symplectomorphisms; see for instance [Th]. The same is true for T_i acting on $D(\mathbb{H})$.

We now use the modified HKR isomorphisms[6] relative to B_k:

$$\begin{aligned} HH^*(\mathbb{H}_k/B_k) &\cong \bigoplus_{i+j=*} R^i p_{k*}\Lambda^j(T_{\mathbb{H}_k/B_k}), \\ HH_*(\mathbb{H}_k/B_k) &\cong \bigoplus_{j-i=*} R^i p_{k*}(\Omega^j_{\mathbb{H}_k/B_k}). \end{aligned}$$

Via these isomorphisms we think of κ_k as lying in $HH^2(\mathbb{H}_k/B_k)$, θ as lying in $HH_2(\mathbb{H}_k/B_k)$ and $\kappa_k \lrcorner\, \theta$ as lying in $HH_0(\mathbb{H}_k/B_k)$, with T_i fixing the latter two. It therefore also fixes $\kappa_k \in HH^2(\mathbb{H}_k/B_k)$ since

$$\lrcorner\, \theta \colon HH^2 \longrightarrow HH_0$$

is an injection. (Under HKR it corresponds to the inclusion $\lrcorner\, \theta \colon H^0(\Lambda^2 T) \oplus H^1(T) \oplus H^2(\mathcal{O}) \to H^0(\mathcal{O}) \oplus H^1(\Omega) \oplus H^2(\Omega^2)$ followed by the inclusion of the latter into $\bigoplus_p H^p(\Omega^p)$.)

Thus, by[7] [HMS], the Fourier-Mukai kernel of T_i deforms from $\mathbb{H}_k \times_{B_k} \mathbb{H}_k$ to $\mathbb{H}_{k+1} \times_{B_{k+1}} \mathbb{H}_{k+1}$. The deformation is also unique, since $HH^1(\mathbb{H}) \cong H^0(T_{\mathbb{H}}) \oplus H^1(\mathcal{O}_{\mathbb{H}}) = 0$.

Thus the Fourier-Mukai kernel of T_i deforms uniquely to all orders, giving an autoequivalence of the derived category of perfect sheaves on $\mathbb{H}_k$ for any k. We next use this information to prove the same in the noncompact case.

Deforming the autoequivalences, II: noncompact case. Having worked out the compact case, we now work more generally with a noncompact holomorphic symplectic family $\mathbb{H}^\circ_t$ of Section 4 (given by removing the degeneracy locus from the holomorphic Poisson family $\mathbb{H}_t$ that deforms the Hilbert scheme of points on a Poisson surface compactification of S_{2n-1}).

The HKR isomorphism $HH^2(\mathbb{H}^\circ_t) \cong H^0(\Lambda^2 T_{\mathbb{H}^\circ_t}) \oplus H^1(T_{\mathbb{H}^\circ_t}) \oplus H^2(\mathcal{O}_{\mathbb{H}^\circ})$ still holds in this noncompact setting, and the equivalence T_i acts on HH^2 as in [Ca]. The third summand vanishes, and we write $\mathsf{T}_i \kappa_t = (a, b)$ with respect to the first two. We wish to know, just as in the compact case, that T_i preserves our deformation class $\kappa_t = [\mathbb{E}^\circ_t]^{1,1} \lrcorner\, \sigma_t \in H^1(T_{\mathbb{H}^\circ_t}) \subset HH^2(\mathbb{H}^\circ_t)$ of (4.2), i.e. that $a = 0$ and $b = \kappa_t$.

Working infinitesimally as before, we basechange $\mathbb{H}^\circ_t$ back to $\mathbb{H}^\circ_k \to B_k = \operatorname{Spec} \mathbb{C}[t]/(t^{k+1})$, and assume inductively that T_i has been extended to $D(\mathbb{H}^\circ_k)$. Restrict to the open locus U in $\mathbb{H}^\circ$ of points such that the corresponding subscheme of $\mathbb{S}$ does not intersect any of the curves C_i. This has the property that the Fourier-Mukai kernel of T_i restricted to $\mathbb{H} \times U$ lies in $U \times U$

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[6] These are the standard HKR isomorphisms composed with $\mathrm{Td}^{-1/2} \lrcorner\, (\,\cdot\,)$ acting on $\bigoplus H^i(\Lambda^j T)$ and $\mathrm{Td}^{1/2} \wedge (\,\cdot\,)$ acting on $\bigoplus H^i(\Omega^j)$. They intertwine the action of HH^* on HH_* [Ca] with the interior multiplication of $H^*(\Lambda^* T)$ on $H^*(\Omega^*)$ [CV1, CV2]. And given an autoequivalence, they intertwine the induced map on HH_* [Ca] with the map on cohomology of footnote 4, by [MS, Theorem 1.2]. The relative versions we use here are described carefully in [HMS].

[7] In [HMS] the *unmodified* HKR isomorphisms are used, but this makes no difference since $\mathrm{Td}^{-1/2} \lrcorner\, (\,\cdot\,)$ acts trivially on $H^1(T)$ when $c_1 = 0$.

(both being $\mathcal{O}_{\Delta_U}[1]$). Thus the kernel of the extension of T_i to $D(\mathbb{H}_k^o)$ also has support disjoint from $(\mathbb{H}_k^o \backslash U) \times U$. Therefore $\mathsf{T}_i(\kappa_t)|_U = (\mathsf{T}_i|_U)(\kappa_t|_U)$. Further shrinking U to be an affine open, κ_k becomes isomorphic to 0. We conclude that $a|_U = 0$, and so $a = 0$.

To show that $b = \kappa_k$ in $R^1 p_{k*} T_{\mathbb{H}_k^o/B_k}$ we use the isomorphism provided by the holomorphic symplectic form to equivalently compare $b \lrcorner \theta$ and $\kappa_k \lrcorner \theta$ in $R^1 p_{k*} \Omega_{\mathbb{H}_k^o/B_k}$. Knowing the cohomology of $\mathbb{H}^o$ and its deformations[8], it is sufficient to know that $b \lrcorner \theta$ and $\kappa_k \lrcorner \theta$ have the same integrals over the cycles

$$C_j \times \{p_1\} \times \ldots \times \{p_{n-1}\} \quad \text{and} \quad \mathbb{P}^1_{p_1} \times \{p_2\} \times \ldots \times \{p_{n-1}\}$$

in the Hilbert scheme. Here the $p_i \in S_{2n-1} \subset \mathbb{S}$ are distinct points disjoint from all C_i, and $\mathbb{P}^1_{p_1}$ is the cycle of all length-2 subschemes of S_{2n-1} supported at p_1.

Since κ_k is supported on E, the integral of $\kappa_k \lrcorner \theta$ over the first cycle is zero. When $|i-j| > 1$, so that C_i and C_j are disjoint, T_i is the identity away from C_i and so $b \lrcorner \theta$ is also zero over the first cycle. When $|i-j| \leq 1$ we pass back to the $K3$ case to see that the two integrals agree, and the same applies to the integrals over the second cycle. **Fixme: how do we know our T_i deforms the same way as on the $K3$? Uniqueness of deformations problems ?**

Alternative approach. Deforming the autoequivalences to first order. Another approach is to use the description of D_n as $D(S_{2n-1}^n)^{\Sigma_n}$. As in [Ca, Prop 8.2] the equivalence (3.1), set up by the Fourier-Mukai kernel $\mathcal{O}_{Z_{\rm red}}$, induces an equivalence $D(H_n \times H_n) \cong D(S_{2n-1}^n \times S_{2n-1}^n)^{\Sigma_n \times \Sigma_n}$ set up by the kernel $\mathcal{O}_{Z_{\rm red}}^\vee[2n] \boxtimes \mathcal{O}_{Z_{\rm red}}$. This takes $\mathcal{O}_\Delta$ (the identity Fourier-Mukai functor) to $\Sigma_n^\nabla.\mathcal{O}_\Delta$ (the identity Fourier-Mukai functor for $D(S_{2n-1}^n)^{\Sigma_n}$), where

$$\Sigma_n^\nabla := (\Sigma_n \times \Sigma_n)/\Sigma_n^\Delta$$

is the quotient by the diagonal copy of Σ_n (which acts in the obvious way on $\mathcal{O}_\Delta$ so that $\Sigma_n^\nabla.\mathcal{O}_\Delta$ indeed carries an action of $\Sigma_n \times \Sigma_n$). This identifies the deformations HH^2 of D_n:

$$\mathrm{Ext}^2_{H_n \times H_n}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong \mathrm{Ext}^2_{S_{2n-1}^n \times S_{2n-1}^n}(\Sigma_n^\nabla.\mathcal{O}_\Delta, \Sigma_n^\nabla.\mathcal{O}_\Delta)^{\Sigma_n \times \Sigma_n}.$$

By a tedious calculation this latter group is a direct sum over the conjugacy classes $[g]$ of Σ_n of

$$\mathrm{Ext}^2_{S_{2n-1}^n \times S_{2n-1}^n}(\mathcal{O}_\Delta, g^*\mathcal{O}_\Delta)^{Z(g)},$$

where $g \in \Sigma_n^\nabla$ is a representative of the conjugacy class, and its centraliser $Z(g)$ acts diagonally on both $\mathcal{O}_\Delta$ and $g^*\mathcal{O}_\Delta$ and so on Ext^2 by conjugation.

[8]The key point being that $H^{1,1}$ is generated by the cohomology classes of the curves C_i and the exceptional divisor $\mathbb{E}$, and that these have a perfect pairing with the H_2 classes $[C_i]$ and a vertical $\mathbb{P}^1$ fibre in $\mathbb{E}$ so that $H^{1,1}$ injects into H^2.

The codimension inside Δ of the intersection of Δ and $g.\Delta$ is twice the sum of (length(cycle)-1) over the cycle decomposition of g. In particular it is >2 except for two conjugacy classes: those of the identity and transpositions such as $(1\,2)$. Therefore only these contribute to Ext^2, giving

$$\mathrm{Ext}^2_{S^n_{2n-1}\times S^n_{2n-1}}(\mathcal{O}_\Delta,\mathcal{O}_\Delta)^{\Sigma_n} \;\oplus\; \mathrm{Ext}^2_{S^n_{2n-1}\times S^n_{2n-1}}(\mathcal{O}_\Delta,(1\,2)^*\mathcal{O}_\Delta)^{Z((1\,2))}.$$

The first summand is easily computed to be $\mathrm{Ext}^2_{S_{2n-1}\times S_{2n-1}}(\mathcal{O}_\Delta,\mathcal{O}_\Delta)$. **Fixme: need to use the compactification $\mathbb{S}$ here. Or just not state this, it's only important that it contains $\mathrm{Ext}^2_{S_{2n-1}\times S_{2n-1}}(\mathcal{O}_\Delta,\mathcal{O}_\Delta)$** By a simple Koszul resolution argument the second summand is $H^0(\Lambda^2 N)$, where N is the rank-2, trivial determinant normal bundle to $\Delta\cap(1\,2)\Delta$ inside $(1\,2)\Delta$. Thus it is $H^0(\mathcal{O}_{\Delta\cap(1\,2)\Delta})\cong\mathbb{C}$ with trivial $Z((1\,2))$-action. Therefore

$$HH^2(D_n) \;=\; HH^2(D(S_{2n-1})) \;\oplus\; \mathbb{C}. \tag{4.6}$$

Via the HKR isomorphism, $HH^2(D_n)$ can also be described as $H^2(\mathcal{O}_{H_n})\oplus H^1(T_{H_n})\oplus H^0(\Lambda^2 T_{H_n})$. The first summand is zero; the third is spanned by the Poisson structure σ **Again need to use compactification here[9] or not state this. Key is just to show that this copy of $\mathbb{C}$ is contained in Ext^2 and matches with the one in (4.6)**. Applying the symplectic form to the second gives[10] $H^1(T_{H_n})\cong H^1(\Omega_{H_n})\cong H^2(H_n)\cong H^2(S_{2n-1})\oplus\langle E\rangle$. All told we get

$$HH^2(D_n) \;=\; H^2(S_{2n-1})\oplus\langle\sigma\rangle \;\oplus\; \mathbb{C}\langle E\rangle. \tag{4.7}$$

Similarly writing $HH^2(D(S_{2n-1}))$ as $H^2(S_{2n-1})\oplus\langle\sigma\rangle$ in (4.6), it is natural that the splittings (4.6) and (4.7) should correspond. In fact all we shall need is that the final summands are the same (up to scale). This is easily seen by noting that they span the subspace of elements of HH^2 whose action on objects supported away from the big diagonal in S^n_{2n-1} – or the exceptional locus E in $\mathrm{Hilb}^n S_{2n-1}$ – is trivial[11].

Let $P_\beta\in D(S_{2n-1}\times S_{2n-1})$ denote the Fourier-Mukai kernel of T_β, so that the kernel for T_β is $\Sigma_n^\nabla.P_\beta^{\boxtimes n}[1]\in D(S^n_{2n-1}\times S^n_{2n-1})^{\Sigma_n\times\Sigma_n}$ [Pl]. The induced action on Fourier-Mukai functors on D_n is given by the kernel

$$\Big(\Sigma_n^\nabla.P_\beta^{\boxtimes n}\Big)^\vee[2n]\boxtimes\big(\Sigma_n^\nabla.P_\beta^{\boxtimes n}\big) \;\in\; D\big((S^n_{2n-1})^{\times 4}\big)^{\Sigma_n^{\times 4}}. \tag{4.8}$$

---

[9] If we decide to use the Poisson compactification $\mathbb{S}$, we should find an extra $\Lambda^2 H^0(T_{\mathbb{S}})$ in $\mathrm{Ext}^2_{S_{2n-1}\times S_{2n-1}}(\mathcal{O}_\Delta,\mathcal{O}_\Delta)$, and an extra $H^0(\Lambda^2 T_{\mathbb{S}})$ in $H^0(\Lambda^2 T_{\mathbb{H}})\subset HH^2(D_n)$; presumably these can be identified.

[10] By direct calculation of both sides the second isomorphism here is easily checked to hold, despite the noncompactness of H_n.

[11] Here we use the Fourier-Mukai functor to make any element of $HH^2=\mathrm{Ext}^2(\mathcal{O}_\Delta,\mathcal{O}_\Delta)$ induce a morphism $\mathcal{F}\to\mathcal{F}[2]$ for each element $\mathcal{F}$ of our category. In this way the Poisson form in $H^0(\Lambda^2 T)$ gives a nontrivial morphism $\mathcal{O}_p\to\mathcal{O}_p[2]$ for any point away from the big diagonal (or on it). And given a nonzero element of $H^2(S_{2n-1})$ we can find one of the -2-curves C_i on which it is nonzero. Pick distinct points $p_j\in S\backslash C_i$. Then $C:=C_i\times\{p_1\}\times\ldots\times\{p_{n-1}\}\hookrightarrow H_n$ is disjoint from E and the corresponding element of $HH^2(D_n)$ gives a nonzero morphism $\mathcal{O}_C\to\mathcal{O}_C[2]$.

See [Ca, Prop 8.2]; this takes the identity kernel $\Sigma_n^\nabla.\mathcal{O}_\Delta$ to itself, and so induces a map on $HH^2 = \mathrm{Ext}^2_{S^n_{2n-1}\times S^n_{2n-1}}(\Sigma_n^\nabla.\mathcal{O}_\Delta, \Sigma_n^\nabla.\mathcal{O}_\Delta)^{\Sigma_n\times\Sigma_n}$. We must show that this map preserves $\kappa_0 \in HH^2$ to deduce that T_β deforms to an autoequivalence of the first order deformation of D_n along e by [To, HMS].

Lemma 4.9. *The Fourier-Mukai kernel* (4.8) *takes* κ_0 *to itself.*

Proof. Because of the kernel's form as the $\Sigma_n\times\Sigma_n$ orbit of an object already equivariant under the diagonal copy of $\Sigma_n\times\Sigma_n$ (4.8) it is easy to see how it respects Σ_n orbits, in the following sense. Pick any Σ_n^Δ-linearised object $X \in D(S^n_{2n-1}\times S^n_{2n-1})$, and let $X' \in D(S^n_{2n-1}\times S^n_{2n-1})$ denote its image under the Fourier-Mukai kernel $P_\beta^{\boxtimes n}[2n]\boxtimes P_\beta^{\boxtimes n}$. Then the kernel (4.8) takes $\Sigma_n^\nabla.X$ to $\Sigma_n^\nabla.X'$. Furthermore a morphism $X\to Y$ induces morphisms $\Sigma_n^\nabla.X\to\Sigma_n^\nabla.Y$ and $X'\to Y'$, and the image of the former under (4.8) is the morphism $\Sigma_n^\nabla.X'\to\Sigma_n^\nabla.Y'$ induced by the latter.

Apply this to $X=\mathcal{O}_\Delta$ and $Y=(1\,2)^*\mathcal{O}_\Delta[2]$ and our morphism κ_0 between them. We find that (4.8) takes κ_0 to a map $\Sigma_n^\nabla.\mathcal{O}_\Delta\to\Sigma_n^\nabla.(1\,2)^*\mathcal{O}_\Delta[2]\cong(1\,2)^*\Sigma_n^\nabla.\mathcal{O}_\Delta[2]$ which is Σ_n^∇ applied to a map $\mathcal{O}_\Delta\to(1\,2)^*\mathcal{O}_\Delta[2]$. We already noted (4.6) that there is only one such nonzero map up to scale, so κ_0 is taken to a multiple of itself.

It is sufficient to check that the multiple is 1 for each generator T_i. In fact, T_1 will suffice, since the multiples are the same for each generator. Hence we are interested in a formal neighbourhood of a single -2-curve, which we can compactify inside a smooth $K3$ surface S. Since S is holomorphic symplectic, so is its Hilbert scheme. **Now use usual argument by comparing to action on H^2, which we know. Need to know it fixes σ** □

The functor T_β induces a map on cohomology $H^*(H_n)\to H^*(H_n)$ by pullback to the product, cup product with the Mukai vector of its Fourier-Mukai kernel, and pushdown to the other factor.

Lemma 4.10. *The action of* $\mathsf{T}_\beta[-1]$ *on* $H^*(H_n,\mathbb{C})$ *fixes* $\mathrm{Re}\,\sigma$, $\mathrm{Im}\,\sigma$ *and* $[E]$.

Proof. To show that $\mathsf{T}_\beta[-1]$ preserves $[\sigma]$ it is sufficient to show that the generators $\mathsf{T}_i[-1]$ of the braid group action preserve it.

Since the kernel P_i for T_{L_i} sits inside an exact sequence $0\to\mathcal{O}_\Delta\to P_i\to\mathcal{O}_{C_i}(-1)\boxtimes\mathcal{O}_{C_i}(-1)\to 0$ we get an induced map from the identity kernel $\Sigma_n^\nabla.\mathcal{O}_\Delta$ to the kernel for $\mathsf{T}_i[-1]$. It is sufficient to show that the cone on this map takes $[\sigma]$ to 0, since $\Sigma_n^\nabla.\mathcal{O}_\Delta$ induces the identity map on cohomology.

Follows from the support being Lagrangian so when pull-up, wedge with Mukai vector and pushdown, we get zero. Or, since σ on H_n and the Σ_n-invariant holomorphic form on $\mathbb{C}^{2n}$ both pull back to the same form on Z_{red} (3.2) it is sufficient to show that the latter form, which we also call σ, is preserved by the cohomological action. Now on the product everything is induced by the original action on the surface, which trivially preserves σ.

Since $\mathsf{T}_\beta[-1]$ fixes both $e\in HH^2$ (Lemma 4.9) and $\sigma\in H^2$, we claim that it also fixes $[E]=e\lrcorner\,\sigma$. To prove this we use the HKR isomorphism *twisted* by $\mathrm{Td}^{1/2}(H_n)$. Since $c_1(H_n) = 0$ this does not change the image of e in $H^1(T_{H_n})$. We therefore identify σ with a class in Hochschild homology HH_2, and this class is also fixed by the natural action of $\mathsf{T}_\beta[-1]$ on HH_* [Ca] since that action is compatible with its action on H^* by [MS, Thm 1.2]. The twisted HKR isomorphism also intertwines contraction $\lrcorner$ between HH^* and HH_* and contraction $\lrcorner$ between $H^*(\Lambda^* T)$ and $H^*(\Omega^*)$, by [MNS]. And finally, the actions of $\mathsf{T}_\beta[-1]$ on HH^* and HH_* commute with contractions $\lrcorner$ between them, by [Ca]. ☐

Proposition 4.11. *The autoequivalence $\mathsf{T}_\beta \in \mathrm{Aut}(D_n)$ deforms uniquely to an autoequivalence $\mathsf{T}_\beta \in \mathrm{Aut}(D(\mathcal{H}_t))$ on nearby fibres $\mathcal{H}_t$.*

Proof. By induction we assume that T_β deforms to the nth order over the base $\mathbb{P}^1$; the initial case $n = 0$ is trivial. (In other words we assume that there is a perfect complex on the pullback of $\mathcal{H} \times \mathcal{H} \to \mathbb{P}^1 \times \mathbb{P}^1$ to the obvious diagonal inclusion $\operatorname{Spec} \mathbb{C}[t]/(t^{n+1}) \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ which restricts to the kernel for T_β over the closed point.)

Let $\kappa \in H^1(T_{\mathcal{H}_n/B_n}) \subset HH^2(\mathcal{H}_n/B_n)$ denote the Kodaira-Spencer class of the deformation to order $n+1$. By the choice of our family $\mathcal{H}$, the contraction of κ with the holomorphic 2-form on $\mathcal{H}$ is contained in the span of $\langle \mathrm{Re}\,\sigma, \mathrm{Im}\,\sigma, [E] \rangle \subset H^2(H_n)$ (4.3). Here we use the natural trivialisation of the cohomologies of the fibres of $\mathcal{H}$ over the base $\operatorname{Spec} \mathbb{C}[t]/(t^{n+1})$. Since the action on cohomology of the deformation of $\mathsf{T}_\beta[-1]$ is constant, it follows that it fixes this contraction, *and* the holomorphic symplectic form. So by the same compatibility of the action with HH_* and HH^* as in the proof of Lemma 4.10 we find that it preserves κ too. Therefore, by [HMS, Prop 6.4, Cor 5.3], T_β deforms to order $n+1$.

So T_β deforms to all orders. Therefore by [Li, Prop 3.6.1] the kernel for T_β in fact deforms over Spec of the complete local ring at the origin in $\mathbb{P}^1$. So we get a formal point in the stack of complexes with no negative self-Exts on the fibres of $\mathcal{H} \times_{\mathbb{P}}^1 \mathcal{H} \to \mathbb{P}^1$ (since T_β is an equivalence the self-Exts equal those of $\mathcal{O}_\Delta$). Lieblich shows this is an Artin stack of local finite presentation, so this formal point lies in a smooth scheme (over $\mathbb{P}^1$) in the stack. By taking an étale slice we can get a Euclidean open neighbourhood of the origin in $\mathbb{P}^1$ over which the kernel deforms and defines an autoequivalence (by openness of the invertibility condition).

Finally, uniqueness follows form the fact that the kernel P_β for T_β is rigid: $\mathrm{Ext}^1_{H_n \times H_n}(P_\beta, P_\beta) \cong \mathrm{Ext}^1_{H_n \times H_n}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \cong H^0(T_{H_n}) \oplus H^1(\mathcal{O}_{H_n}) = 0$. **Not true in noncompact case! Perhaps use $\mathbb{C}^*$-actions, deformations fixed by them, etc ? this might be the weakest point: ideally we'd have a compactly-supported version of HKR to ensure uniqueness** ☐

5. Khovanov homology

Theorem 5.1. *The relations (1), (2), (3) and (4) hold in the above deformations of the categories D_n.*

Proof. Since the objects on both sides of (1) and (3) are isomorphic and rigid on H_n, they deform uniquely with $\mathcal{H}$ and so remain isomorphic.

To deal with (4) we work on its right hand side, and move T_{2n} across the Ext group to rewrite it as $\operatorname{Ext}^*_{D_{n+1}}(\mathsf{T}_{2n}^{-1}\mathsf{T}_\beta\mathscr{L}_{n+1}, \mathscr{L}_{n+1}[n+1])$. This has the advantage that the $\mathscr{L}_{n+1}$ on the right hand side is disjoint from the exceptional divisor $E \subset H_{n+1}$, so the Ext group can be calculated Σ_{n+1}-equivariantly away from the large diagonal $\Delta \subset S^{n+1}_{2n+1}$. By construction, the deformation of Section 4 is zero on the complement of Δ, so we can compute with the natural product structure on $S^{n+1}_{2n+1}\backslash\Delta$ even after deformation. By rigidity the sheaves $\mathscr{L}_n$ and $\mathscr{L}_{n+1}$ do not change on this space either, though of course the restriction of $\mathsf{T}_{2n}^{-1}\mathsf{T}_\beta\mathscr{L}_{n+1}$ will change with the deformation.

So we now work through the proof of relation (4) in Theorem 3.10 (with all T_{2n}s moved across the Ext groups they appear in) as $\mathsf{T}_{2n}^{-1}\mathsf{T}_\beta\mathscr{L}_{n+1}$ changes but the space remains the same. Use the product structure to push down the last factor from S^{n+1}_{2n+1} to S^n_{2n+1} (and from there to the image of S^n_{2n-1} via the inclusion of Section 2) just as in (3.13–3.16). Before we deform, this pushdown used the following computations:

$$\begin{aligned}
\operatorname{Ext}^*_{S_{2n+1}}(T_{2n}^{-1}L_{2n+1}, T_{2n}L_j) &= 0, \quad j < 2n-1,\\
\operatorname{Ext}^*_{S_{2n+1}}(T_{2n}^{-1}L_{2n+1}, L_{2n+1}) &= \mathbb{C}[0], \quad \text{and}\\
\operatorname{Ext}^*_{S_{2n+1}}(T_{2n}^{-1}L_{2n+1}, L_{2n+1}) &= \mathbb{C}[-1].
\end{aligned}$$

On deformation, these can only get smaller by upper semicontinuity, but they are already as small as their Euler characteristics will allow. Thus they remain unchanged, as does the equality $\operatorname{Ext}^*(L_{2n}, L_{2n-1}) = \mathbb{C}[-1]$ used to prove (3.19). Thus we quickly simplify to an expression on the image of $S^n_{2n-1} \hookrightarrow S^n_{2n+1}$ which deforms the expression (3.20) that we got before deformation. Compatibility of the deformation of H_{n+1} and that of H_n means that expression is precisely what the left hand side of relation (4) deforms to.

Finally we prove (2). From (3.11, 3.12) it is sufficient to show that the spherical sheaves $\mathcal{O}_{C_{2i-1}\cup C_{2i}\cup C_{2i+1}}(-1,0,0)$ and $\mathcal{O}_{C_{2i-1}\cup C_{2i}\cup C_{2i+1}}(0,0,-1)$ become isomorphic on deforming by κ_0. (Throughout this proof we will omit to say "take $\boxtimes$ with $L_1 \boxtimes L_3 \boxtimes \ldots \boxtimes L_{2i-3} \boxtimes L_{2i} \boxtimes L_{2i+3} \boxtimes \ldots \boxtimes L_{2n-1}$ then apply Σ_n (3.17)" when referring to these sheaves (3.11, 3.12) as elements of D_n.) Since they are both rigid these deformations are unique and equal to the deformations of $\mathsf{T}_{2i-1}\mathsf{T}_{2i}\mathscr{L}$ and $\mathsf{T}_{2i-1}^{-1}\mathsf{T}_{2i}^{-1}\mathscr{L}$ along κ_0.

There is an obvious nonzero map

$$\Phi\colon \mathcal{O}_{C_{2i-1}\cup C_{2i}\cup C_{2i+1}}(-1,0,0) \longrightarrow \mathcal{O}_{C_{2i-1}\cup C_{2i}\cup C_{2i+1}}(0,0,-1)$$

factoring through $\mathcal{O}_{C_{2i-1}}(-1)$. It deforms with κ_0 since

$$\operatorname{Ext}^1\big(\mathcal{O}_{C_{2i-1}\cup C_{2i}\cup C_{2i+1}}(-1,0,0), \mathcal{O}_{C_{2i-1}\cup C_{2i}\cup C_{2i+1}}(0,0,-1)\big) = 0.$$

It has kernel $\mathcal{O}_{C_{2i}\cup C_{2i+1}}(-1,0)$ and cokernel $\mathcal{O}_{C_{2i}\cup C_{2i+1}}(0,-1)$. Both components C_{2i} and C_{2i+1} (times by $C_1 \times C_3 \times \ldots \times C_{2i-3} \times C_{2i} \times C_{2i+3} \times \ldots \times C_{2n-1}$)

of their support do not deform under the first order deformation κ_0 because $C_{2i} \cdot C_{2i} \neq 0 \neq C_{2i} \cdot C_{2i+1}$. Therefore the deformation of Φ has no kernel or cokernel, and is an isomorphism. $\square$

Bigrading. There are also $\mathbb{C}^*$-actions on the spaces S_{2n-1} with respect to which the inclusion maps $S_{2n-1} \subset S_{2n+1}$ are equivariant **check**. Thinking of S_{2n-1} as the minimal resolution of $\{x^{2n} = yz\}$, we give x weight -1, and y and z weights $-n$; this action then lifts to S_{2n-1}. This gives the natural holomorphic symplectic form (which is $dxdy/y$ in these coordinates) weight -1.

The working above was all equivariant with respect to this $\mathbb{C}^*$-action (**including the canonical deformation of Hilb^n of Section 4 – so this still carries a $\mathbb{C}^*$-action ?**). To make relations (1) and (3) hold equivariantly we tensor the definition (3.8) by the weight ± 1 character of $\mathbb{C}^*$. The upshot is an extra $\mathbb{C}^*$-action (i.e. grading) on the link invariant....etc.

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