International Journal of Theoretical and Applied Finance
Vol. 13, No. 7 (2010) 1131–1147

DOI: 10.1142/S0219024910006145

ASYMPTOTIC ANALYSIS FOR FOREIGN EXCHANGE DERIVATIVES WITH STOCHASTIC VOLATILITY

CHARLES CUTHBERTSON

Department of Statistics, University of Oxford
OX1 3TG, UK

GRIGORIOS PAVLIOTIS*

Department of Mathematics, Imperial College London
SW7 2AZ, UK
g.pavliotis@imperial.ac.uk

AVRAAM RAFAILIDIS

Department of Mathematics, King's College London
WC2R 2LS, UK

PETTER WIBERG

Mathematics Institute, The University of Warwick
Coventry CV4 7AL, UK

Received 27 July 2009
Accepted 6 September 2010

We consider models for the valuation of derivative securities that depend on foreign exchange rates. We derive partial differential equations for option prices in an arbitrage-free market with stochastic volatility. By use of standard techniques, and under the assumption of fast mean reversion for the volatility, these equations can be solved asymptotically. The analysis goes further to consider specific examples for a number of options, and to a considerable degree of complexity.

Keywords: Derivatives pricing; FX options; stochastic volatility; multiscale analysis; singular perturbation theory.

1. Introduction

The foreign exchange world has not escaped the recent financial turbulence. The disorder, which manifested at the time across asset classes and geographical regions, has affected the business massively in terms of liquidity and risk appetite by institutional investors, hedge funds and market makers alike. The credit and equity world was quick to pass the virus to currencies fluctuations, both of emerging market and

*Corresponding author.

more established types. Some volatile, or indeed jump events are worth mentioning, without going into any detail: the depreciation of the South African rand, Turkish lira, Australian dollar and other currencies against the US dollar, most in double digits and in just a few days time in late 2008. The Icelandic krona also slid a massive 50% against the euro within a mere two days. More recently Europe's sovereign debt crisis and the consequent slide of the common currency has resulted in new investor anxiety. The unwind of positions in terms of spot and structured FX products has been for a long period the rule rather than the exception.

Where this leaves the industry is subject of discussion, what is evident however, is the preference of vanilla derivatives against the exotic counterpart, as a means of return to clarity. Whether or not a discussed "double dip" type of scenario is to be realized, an additional systemic volatility is suggested for the long run and transparency along with simplicity are more likely to persist. Foreign exchange derivatives as a whole have been popular for many years, providing a route for speculation and at the same time being an invaluable tool for the risk management of positions exposed to currency fluctuation risk. This will continue to be the case, albeit on the risk management side, and from a mathematical perspective, the valuation of such assets and the modeling of volatilities are as important as ever.

Under this environment we present in what follows a study for the pricing of standard foreign exchange options under the assumption of stochastic volatilities between currencies. Stochastic volatility models are motivated by statistical studies of historical asset prices which show that the size of returns are autocorrelated [3, 15]. They have been proven to fit smiles and skews and this fact alone has contributed to their popularity. In foreign exchange markets, the smile manifests itself in a much more pronounced way, the convexity measure is always positive, as empirical studies confirm. Stochasticity is therefore "embedded" in the dynamics of FX option volatilities. The literature is substantial, the pioneers being [8] and [10] followed by others. Vanilla option prices have been corrected first under the assumption of stochastic evolution for volatility. Later, a number of other types of options have been incorporated into the picture. For FX options it has been suggested that in addition, stochastic skewness should be incorporated as well, [4].

The economic model we present here considers stochastic evolution for the FX processes as well as the process of an "equity" asset in the domestic economy. As a main result we provide corrections to Black Scholes prices for foreign exchange products when volatility evolves under specific stochastic dynamics. We take this further by considering a bigger family of options, and the analysis is general enough to be valid for both foreign exchange as well as FX/Equity type of hybrid derivatives. The mathematical machinery used is asymptotic expansion techniques. While numerical considerations form a major part of finance nowadays, asymptotic techniques have not been explored yet to the full of their capacity. The first to use asymptotics in finance were Fouque, Papanicolaou and Sircar (FPS thereafter) in a series of interesting papers, see [5–7] and the references therein. They provide corrections to Black Scholes for standard equity options, when volatility follows an Ornstein Uhlenbeck process. These results have been generalised further in [14] for a more general family of stochastic volatility models and in calculating higher order terms. Later, [9] have applied the same techniques to price pure volatility products. For a general review on foreign exchange derivatives, see [11].

The structure of the paper is as follows. The model is introduced in Sec. 2. We consider stochastic equations for the evolution of the share price and the exchange rate between the two currencies, and their respective stochastic volatilities, one for the share price and one for the exchange rate. Two deterministic processes give the prices of bonds in each currency. The PDE for the risk-neutral price in this incomplete stochastic volatility environment is derived, and analyzed by asymptotic methods. We show that the asymptotic method gives a correction equation and that the option price can be viewed as a perturbation of the price for a constant volatility model. In Sec. 3, we apply this result to two types of derivatives, (i) a call option on a foreign stock with a strike in the foreign currency, and (ii) a call option on a foreign stock with a strike in the domestic currency (a so called *compo* or *composite* option in finance jargon). In Sec. 4, we investigate the impact of correlation between the exchange rate and it's stochastic volatility process. The impact of correlation is exemplified using a binary option on the exchange rate. Section 5 investigates a slightly modified model with the share replaced by a second exchange rate for a third currency. Section 6 summarizes the conclusions.

2. Asymptotic Analysis for Currency Derivatives

2.1. *The model*

Let us consider a two-currency model with a spot exchange rate measured in units of domestic currency per unit of foreign currency, for instance pounds per dollar. We assume that the domestic short rate r_d and foreign short rate r_f are constants, and denote the corresponding riskless asset prices as B_f and B_d respectively.

The exchange rate follows a geometric Brownian motion with stochastic volatility. In addition we enrich the economy with a "share" that is denominated in the foreign currency. The price process of this asset will be considered to follow similar dynamics. The volatilities considered are functions of Ornstein–Uhlenbeck (OU) processes, properties of which will be discussed. Consequently, the stochastic process for the volatility is mean-reverting, the process oscillates around its long term mean. We summarize the above according to the following scheme:

Assumption 2.1. The market operates by the following dynamics under the objective probability measure

$$\text{exchange rate:} \qquad dX_t = X_t\mu_1 dt + X_t f(Y_t) dW_t^{(0)}, \tag{2.1}$$

$$\text{stock price:} \qquad dS_t = S_t\mu_2 dt + S_t g(Z_t) dW_t^{(1)}, \tag{2.2}$$

$$dY_t = \frac{1}{\epsilon}(m_1 - Y_t)dt + \frac{\sigma_1}{\sqrt{\epsilon}}dW_t^{(2)}, \tag{2.3}$$

$$dZ_t = \frac{1}{\epsilon}(m_2 - Z_t)dt + \frac{\sigma_2}{\sqrt{\epsilon}}dW_t^{(3)}, \tag{2.4}$$

domestic bond: $$dB_d = r_d B_d dt, \tag{2.5}$$

foreign bond: $$dB_f = r_f B_f dt. \tag{2.6}$$

The correlation between the Wiener processes $W^{(i)}$ and $W^{(j)}$ for $i, j = 0, \ldots, 3$ is denoted by ρ_{ij}. For $i = 1, 2$, m_i, μ_i and σ_i are constants. Furthermore, $\frac{1}{\epsilon}$ is the rate of mean reversion of the OU process, with $0 < \epsilon \ll 1$.

The objective is to price a derivative that pays $h(x, S)$ at the time of maturity T, for a very general family of functions h. By standard no-arbitrage theory [2, 1] the price at time t is given by the discounted expectation of the payoff under Q, the risk neutral measure,

$$P^\epsilon(t, X_t, Y_t) = e^{-r_d(T-t)}\mathbb{E}^Q_{t,x}[h(X(T), S(T))]. \tag{2.7}$$

Here we use the notation $\mathbb{E}^Q_{t,x}[\bullet] = \mathbb{E}[\bullet|\mathcal{F}_t, X_t = x]$. The Q-dynamics of (2.1)–(2.4) are then

$$dX_t = X_t(r_d - r_f)dt + X_t f(Y_t)dW_t^{(0)*} \tag{2.8}$$

$$dS_t = S_t(r_f - \rho_{01} f(Y_t)g(Z_t))dt + g(Z_t)dW_t^{(1)*} \tag{2.9}$$

$$dY_t = \left(\frac{1}{\epsilon}(m_1 - Y_t) - \frac{\sigma_1}{\sqrt{\epsilon}}\Lambda(Y_t)\right)dt + \frac{\sigma_1}{\sqrt{\epsilon}}d\overline{W}_t^{(2)*} \tag{2.10}$$

$$dZ_t = \left(\frac{1}{\epsilon}(m_2 - Z_t) - \frac{\sigma_2}{\sqrt{\epsilon}}\Gamma(Y_t, Z_t)\right)dt + \frac{\sigma_2}{\sqrt{\epsilon}}d\overline{W}_t^{(3)*} \tag{2.11}$$

The diffusion coefficient in (2.9) is different from the one in (2.2). This is because S_t is an asset in the foreign economy, and we transfer to the risk neutral measure of the *domestic* economy. In the above we have explicitly found the market prices of risk concerning the first two Wiener processes. The new market prices of risk Λ and Γ are:

$$\begin{aligned} \Lambda(y) &= \frac{\rho_{02}(r_f + \mu_1 - r_d)}{f(y)} + \lambda_1(y)\sqrt{1 - \rho_{01}^2} \\ \Gamma(y, z) &= \frac{(r_f + \mu_2 + f(y)g(z)\rho_{01} - r_d)}{g(z)} + \lambda_2(z)\sqrt{1 - \rho_{01}^2}, \end{aligned} \tag{2.12}$$

and are chosen in such a way that the correlation structure remains the same.

To begin with, we assume the following correlations to be zero:

$$\rho_{23} = \rho_{02} = \rho_{03} = 0, \tag{2.13}$$

and we relax this assumption later. From (2.7) and the Feynman–Kac formula we deduce that $P^\epsilon(t,x,s,y,z)$ is the solution to the following partial differential equation,

$$\left(\mathscr{L}_2 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_1 + \frac{1}{\epsilon}\mathscr{L}_0\right)P^\epsilon = 0, \tag{2.14a}$$

with final condition:

$$P^\epsilon(T,x,s,y,z) = h(x,s). \tag{2.14b}$$

where the partial differential operators are defined as follows:

$$\mathscr{L}_0 = (m_1 - y)\frac{\partial}{\partial y} + (m_2 - z)\frac{\partial}{\partial z} + \frac{1}{2}\sigma_1^2\frac{\partial^2}{\partial y^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2}{\partial z^2} \tag{2.15}$$

$$\mathscr{L}_1 = sg(z)\left(\rho_{12}\sigma_1\frac{\partial}{\partial s\partial y} + \rho_{13}\sigma_2\frac{\partial}{\partial s\partial z}\right) - \left(\Lambda(y)\frac{\partial}{\partial y} + \Gamma(y)\frac{\partial}{\partial z}\right) \tag{2.16}$$

$$\begin{aligned}\mathscr{L}_2 = {} & \frac{\partial}{\partial t} + (r_d - r_f)x\frac{\partial}{\partial x} + \frac{1}{2}x^2 f^2(y)\frac{\partial^2}{\partial x^2} \\ & + (r_f - \rho_{01}f(y)g(z))s\frac{\partial}{\partial s} + \frac{1}{2}s^2 g^2(z)\frac{\partial^2}{\partial s^2} - r_d\cdot\end{aligned} \tag{2.17}$$

Note that $\mathscr{L}_2$ is the operator corresponding to the normal pricing problem of a currency derivative without stochastic volatility [2]. So, if y and z are kept fixed then this prices a currency derivative where $f(y)$ is the volatility for the exchange rate and $g(z)$ is the volatility for the stock price. We define $\mathscr{L}_{(\sigma_{x,s})}$ to be the partial differential operator for the normal currency derivative price:

$$\begin{aligned}\mathscr{L}_{(\sigma_{x,s})} = {} & \frac{\partial}{\partial t} + (r_d - r_f)x\frac{\partial}{\partial x} + \frac{1}{2}x^2\sigma_x^2\frac{\partial^2}{\partial x^2} \\ & + (r_f - \rho_{01}\sigma_{x,s})s\frac{\partial}{\partial s} + \frac{1}{2}s^2\sigma_s^2\frac{\partial}{\partial s^2} - r_d\cdot\end{aligned} \tag{2.18}$$

The operator $\mathscr{L}_2$ in this case corresponds to $\mathscr{L}_{(\sigma_{x,s})}$ with $\sigma_x = f(y)$, $\sigma_y = g(z)$ and $\sigma_{x,s} = f(y)g(z)$. Our original problem is thus a singular perturbation problem for the operator $\mathscr{L}_{(\sigma_{x,s})}$. Notice furthermore that, in order to avoid problems with boundary layers, we have taken the initial conditions (2.14b) to be independent of y and z.

2.2. *Asymptotic analysis*

Our goal now is to solve (2.14a) perturbatively. Our analysis is based on standard multiscale techniques, see e.g. [5–7, 12, 13] and the references therein. To this end, we assume the following ansatz for the solution to (2.14a)

$$P^{\epsilon}(t,x,s,y,z) = P_0(t,x,s,y,z) + \sqrt{\epsilon}P_1(t,x,s,y,z) + \epsilon P_2(t,x,s,y,z) + \epsilon^{3/2}P_3(t,x,s,y,z) + \cdots$$

Inserting the above into (2.14a) we get the following equation for P^{ϵ} we obtain:

$$0 = \frac{1}{\epsilon}\mathscr{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}}(\mathscr{L}_0 P_1 + \mathscr{L}_1 P_0) + (\mathscr{L}_0 P_2 + \mathscr{L}_1 P_1 + \mathscr{L}_2 u_0) + \sqrt{\epsilon}(\mathscr{L}_0 P_3 + \mathscr{L}_1 P_2 + \mathscr{L}_2 P_1) + O(\epsilon)$$

We equate equal powers of $\sqrt{\epsilon}$ in the above equation to obtain the following sequence of partial differential equations:

$$\mathscr{L}_0 P_0 = 0, \tag{2.19}$$

$$\mathscr{L}_0 P_1 = -\mathscr{L}_1 P_0, \tag{2.20}$$

$$\mathscr{L}_0 P_2 = -\mathscr{L}_1 P_1 - \mathscr{L}_2 P_0, \tag{2.21}$$

$$\mathscr{L}_0 P_3 = -\mathscr{L}_1 P_2 - \mathscr{L}_2 P_1. \tag{2.22}$$

The goal is now to obtain an equation for the first term in the expansion, P_0, and to calculate the $\mathcal{O}(\sqrt{\epsilon})$ correction P_1.

2.2.1. *The leading order term*

By (2.19), P_0 belongs to the null space of $\mathscr{L}_0$. But $\mathscr{L}_0$ is the generator of the two dimensional OU process (Y_t, Z_t), which is ergodic, so we have:

$$\mathbb{E}^{\phi}\{P_0\} = P_0,$$

where ϕ denotes the invariant distribution of the two-dimensional OU process. Therefore, P_0 is a constant with respect to (y,z), so $P_0 = P_0(x,s,t)$ is a function of x, s and t alone.

We proceed with the analysis of equation (2.20). P_0 is independent of y and z, so

$$\mathcal{L}_1 P_0 = 0,$$

and equation (2.20) becomes

$$\mathcal{L}_0 P_1 = 0,$$

from which it follows that $P_1 = P_1(x,s,t)$.

Next we consider (2.21). In order for this equation to be well-posed, it is necessary that its right hand side belongs to the range of $\mathscr{L}_0$. By the Fredholm Alternative, this is true if and only if the right hand side is in the orthogonal complement of the null space of $\mathscr{L}_0^*$:

$$\mathbb{E}^{\phi}(\mathcal{L}_1 P_1 + \mathcal{L}_2 P_0) = 0.$$

From the above solvability condition, and using the fact that $\mathcal{L}_1 P_1 = 0$, we obtain the partial differential equation for P_0:

$$\begin{aligned}\mathbb{E}^{\phi}\{\mathscr{L}_2 P_0\} &= \frac{\partial P_0}{\partial t} + (r_d - r_f)x\frac{\partial P_0}{\partial x} + \frac{1}{2}x^2\mathbb{E}^{\phi}\{f^2(y)\}\frac{\partial^2 P_0}{\partial x^2} \\ &\quad + (r_f - \rho_{01}\mathbb{E}^{\phi}\{f(y)g(z)\})s\frac{\partial P_0}{\partial s} + \frac{1}{2}s^2\mathbb{E}^{\phi}\{g^2(z)\}\frac{\partial^2 P_0}{\partial s^2} - r_d P_0 \\ &= 0 \end{aligned} \tag{2.23}$$

This is the same partial differential equation that is obtained in a constant volatility model, see e.g. [2]. The difference is that the diffusion coefficients in the new PDE are given by the second moments over the OU process, instead of given a priori as constants part of the original model formulation.

Introducing the notation

$$\sigma_f^2 := \mathbb{E}^{\phi}\{f^2(y)\}, \quad \sigma_g^2 := \mathbb{E}^{\phi}\{g^2(z)\} \quad \text{and} \quad \sigma_{f,g} := \mathbb{E}^{\phi}\{f(y)g(z)\},$$

we can write

$$\mathbb{E}^{\phi}\{\mathscr{L}_2 P_0\} = \mathscr{L}_{(\sigma_{f,g})} P_0,$$

where the operator $\mathcal{L}_{\sigma_{f,g}}$ was defined in (2.18) We remark incidentally that in this case we *define* $\sigma_{f,g}$. Normally in the case with constant volatility, $\sigma_{f,g}$ is equal to σ_f multiplied by σ_g but this is not true in this case as σ_f multiplied by σ_g is $\sqrt{\mathbb{E}^{\phi}\{f^2(y)\}\mathbb{E}^{\phi}\{g^2(z)\}}$. This is not important for the analysis to follow, but it does demonstrate the connection with the constant volatility model.

To sum up, the first term in the expansion $P_0(x, s, t)$ satisfies the following partial differential equation, with final condition:

$$\mathscr{L}_{(\sigma_{f,g})} P_0 = 0, \tag{2.24}$$

$$P_0(T, x, s) = h(x, s). \tag{2.25}$$

We now attempt to obtain the first order correction to P_0 by finding an explicit formula for P_1.

2.2.2. *The* $\mathcal{O}(\sqrt{\epsilon})$ *correction*

Equation (2.21) becomes:

$$\begin{aligned}\mathscr{L}_2 P_0 &= \mathscr{L}_2 P_0 - \mathbb{E}^{\phi}\{\mathscr{L}_2 P_0\} \\ &= x^2(f^2(y) - \mathbb{E}^{\phi} f^2(y))\frac{\partial^2 P_0}{\partial x^2} + s^2(g^2(z) - \mathbb{E}^{\phi} g^2(z))\frac{\partial^2 P_0}{\partial s^2} \\ &\quad + \rho_{01}(\mathbb{E}^{\phi}\{f(y)g(z)\} - f(y)g(z))\frac{\partial P_0}{\partial s}.\end{aligned}$$

The solution of this equation is:

$$P_2 = -\mathscr{L}_0^{-1}(\mathscr{L}_2 - \mathbb{E}^{\phi}\{\mathscr{L}_2\})P_0, \tag{2.26}$$

On the other hand, the solvability condition for equation (2.22) reads:

$$\mathbb{E}^{\phi}(\mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) = 0.$$

We substitute (2.26) into the above equation to obtain

$$\mathbb{E}^{\phi}\{\mathscr{L}_2 P_1 - \mathscr{L}_1 \mathscr{L}_0^{-1}(\mathscr{L}_2 - \mathbb{E}^{\phi}\{\mathscr{L}_2\})P_0\} = 0.$$

Rearranging we get

$$\mathbb{E}^{\phi}\{\mathscr{L}_2 P_1\} = \mathbb{E}^{\phi}\{\mathscr{L}_1 \mathscr{L}_0^{-1}(\mathscr{L}_2 - \mathbb{E}^{\phi}\{\mathscr{L}_2\})P_0\}.$$

If we can find solutions to the problems

$$\mathscr{L}_0 \alpha(y) = f^2(y) - \mathbb{E}^{\phi}\{f^2(y)\} \tag{2.27}$$

$$\mathscr{L}_0 \beta(z) = g^2(z) - \mathbb{E}^{\phi}\{g^2(z)\} \tag{2.28}$$

$$\mathscr{L}_0\{\gamma(y,z)\} = \mathbb{E}^{\phi}\{f(y)g(z)\} - f(y)g(z) \tag{2.29}$$

then the equation for P_1 can be represented as

$$\mathscr{L}_{(\sigma_{f,g})} P_1 = \mathscr{A} P_0$$

where

$$\mathscr{A} = \mathbb{E}^{\phi}\left\{\mathscr{L}_1\left(x^2\alpha(y)\frac{\partial^2}{\partial x^2} + s^2\frac{\partial^2}{\partial s^2}\beta(z) + \left(\rho_{01}s\frac{\partial}{\partial s}\right)\gamma(y,z)\right)\right\}.$$

Solutions to (2.27)–(2.29) do exist since, by taking expectations, we note that they are orthogonal to the null space of the OU process. Thus by the Fredholm Alternative they have solutions and we are justified in using this representation. Using the fact that P_0 does not depend on y or z and the definition of $\mathscr{L}_1$ we can write $\mathscr{A}$ as

$$\mathscr{A} = Ax^2 s\frac{\partial^3}{\partial x^2 \partial s} + Bs\frac{\partial}{\partial s}\left(s^2\frac{\partial^2}{\partial s^2}\right) + Cs\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}\right),$$

where

$$A = \rho_{12}\sigma_1 \mathbb{E}^{\phi}\left\{g(z)\frac{\partial \alpha}{\partial y}\right\} \tag{2.30a}$$

$$B = \rho_{12}\sigma_2 \mathbb{E}^{\phi}\left\{g(z)\frac{\partial \beta}{\partial z}\right\} \tag{2.30b}$$

$$C = \rho_{12}\sigma_1 \mathbb{E}^{\phi}\left\{\frac{\partial \gamma}{\partial y}g(z)\right\} + \rho_{13}\sigma_2 \mathbb{E}^{\phi}\left\{\frac{\partial \gamma}{\partial z}g(z)\right\}. \tag{2.30c}$$

To solve the equation for P_1 note that $\mathscr{L}_{(\sigma_{f,g})}$ has only the $\frac{\partial}{\partial t}$ term with derivatives in t and so the following identity holds by the product rule:

$$\mathscr{L}_{(\sigma_{f,g})}(-(T-t)\mathscr{A} P_0) = \mathscr{A} P_0 - (T-t)\mathscr{L}_{(\sigma_{f,g})}\mathscr{A} P_0.$$

But this last term is zero because $\mathscr{L}(\sigma_{f,g})P_0 = 0$ and the two operators $\mathscr{L}(\sigma_{f,g})$ and $\mathscr{A}$ commute. This can be seen from the following fact:

$$x^n \frac{\partial^n P_0}{\partial x^n}\left(x^m \frac{\partial^m P_0}{\partial x^m}\right) = x^m \frac{\partial^m P_0}{\partial x^m}\left(x^n \frac{\partial^n P_0}{\partial x^n}\right)$$

which holds for all integers n, m, and noting that all derivatives in the two operators are in this form. Therefore,

$$P_1 = -(T-t)\mathscr{A}P_0. \tag{2.31}$$

Putting everything together, we obtain the following formula for P_1:

$$P_1(x,s,t) = (T-t)\left(A\,x^2 s \frac{\partial^3 P_0}{\partial x^2 \partial s} + Bs\frac{\partial}{\partial s}\left(s^2 \frac{\partial^2 P_0}{\partial s^2}\right) + Cs\frac{\partial}{\partial s}\left(s\frac{\partial P_0}{\partial s}\right)\right), \tag{2.32}$$

where the coefficients A, B and C are given by equation (2.30) and $P_0(x,s,t)$ solves (2.24)–(2.25). Hence, the $\mathcal{O}(\sqrt{\epsilon})$ accurate approximation to P^ϵ reads

$$P^\epsilon \approx P_0 - \sqrt{\epsilon}(T-t)\left(A\,x^2 s \frac{\partial^3 P_0}{\partial x^2 \partial s} + B\,s\frac{\partial}{\partial s}\left(s^2 \frac{\partial^2 P_0}{\partial s^2}\right) + C\,s\frac{\partial}{\partial s}\left(s\frac{\partial P_0}{\partial s}\right)\right).$$

This is the corrected formula for FX derivatives. The preceding multiscale analysis can be justified rigorously by standard methods, see e.g. [12, 13].

3. Option Pricing

Having obtained the asymptotic correction to the Black Scholes prices we now provide a few examples of currency options. The explicit formulae of these, when considering deterministic volatility are well known [2].

3.1. *Foreign call struck in the foreign currency*

This is a European option to buy one unit of foreign stock at maturity time T, by paying K units of domestic currency. We assume constant volatility σ_f and the stock price volatility σ_g so that the claim in the foreign currency is represented as

$$h(T, s, \sigma_g) = \max[S(T) - K, 0].$$

Assuming no arbitrage, the value in domestic currency of this claim is

$$P(T, s, x, \sigma_f, \sigma_g) = x \cdot P_{BS}(T, s, \sigma_g),$$

where P_{BS} is the Black Scholes price of the stock option with short rate r_f, i.e.

$$P(T, s, x, \sigma_f, \sigma_g) = xsN(d_1) - xe^{-r_f(T-t)}KN(d_2)$$

with

$$d_1(t,s) = \frac{1}{\sigma_g\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r_f + \frac{1}{2}\sigma_g^2\right)(T-t)\right\}$$

and

$$d_2(t,s) = d_1(t,s) - \sigma_g\sqrt{T-t}.$$

Since we have $P_0(t,s,x) = P(t,s,x,\sigma_f,\sigma_g)$ with $\sigma_f^2 = \mathbb{E}^{\phi}\{f^2\}$, $\sigma_g^2 = \mathbb{E}^{\phi}\{g^2\}$ we can derive the following derivatives

$$\begin{aligned}
\frac{\partial^2 P_0}{\partial^2 x \partial s} &= 0 \\
\frac{\partial P_0}{\partial s} &= xN(d_1) \\
\frac{\partial^2 P_0}{\partial s^2} &= \frac{xe^{-d_1^2/2}}{s\sigma_g\sqrt{2\pi(T-t)}} \\
\frac{\partial^3 P_0}{\partial s^3} &= \frac{xe^{-d_1^2/2}}{s^2\sigma_g\sqrt{2\pi(T-t)}}\left(1 + \frac{d_1}{\sigma_f\sqrt{T-t}}\right)
\end{aligned} \tag{3.1}$$

Using the formulae for the derivatives of the option price and equation (2.31) we can derive the correction

$$P_1 = Bxs\frac{e^{\frac{-d_1^2}{2}}}{\sqrt{2\pi}\sigma_g}\left(d_1 - \frac{\sqrt{T-t}}{\sigma_g}\right) + Csx\left(\frac{e^{\frac{-d_1^2}{2}}}{\sqrt{2\pi(T-t)}} + N(d_1)\right). \tag{3.2}$$

It is now possible to estimate the parameters B^{ϵ} and C^{ϵ} for observed option prices by an appropriate linear regression for each strike price and maturity.

3.2. *Foreign stock call option struck in domestic currency*

As in the previous example, we have a European option to buy one unit of foreign equity at time T but we pay in the domestic currency. So the claim is represented in domestic currency by:

$$h(T,s,\sigma_g) = \max[X(T)\cdot S(T) - K, 0]$$

We can interpret $X(t)\cdot S(t) = \overline{S}(t)$ as the price of a domestically traded asset whose dynamics are readily obtained both in the objective and risk-neutral measure. If we now run through a similar asymptotic analysis to the one considered in the previous section where we consider the dynamics of $\overline{S}_t$ only, with no reference to S and X, it can be seen that P_0 is the solution to the Black Scholes equation with volatility $\mathbb{E}^{\phi}\{f^2(y)+g^2(z)+2\rho f(y)g(z)\} =: \sigma_{f,g}$. Thus we have the following pricing formula for P_0:

$$P_0(t,\overline{s}) = \overline{s}N(d_1) - e^{-r_d(T-t)}KN(d_2) \tag{3.3}$$

where

$$d_1(t,\bar{s}) = \frac{1}{\sigma_{f,g}\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r_f + \frac{1}{2}\sigma_{f,g}^2\right)(T-t)\right\}$$
$$d_2(t,\bar{s}) = d_1(t,\bar{s}) - \sigma_{f,g}\sqrt{T-t}. \tag{3.4}$$

It is then an exercise to derive the final expression for the corrected price:

$$\begin{aligned} P^\epsilon &\approx P_0 + \sqrt{\epsilon}P_1 \\ &= P_0 - \sqrt{\epsilon}\left((T-t)B\,\bar{s}\frac{\partial}{\partial\bar{s}}\left(\bar{s}^2\frac{\partial P_0^2}{\partial\bar{s}^2}\right) + C\,\bar{s}\frac{\partial}{\partial\bar{s}}\left(\bar{s}\frac{\partial P_0}{\partial\bar{s}}\right)\right), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \frac{\partial P_0}{\partial\bar{s}} &= N(d_1) \\ \frac{\partial^2 P_0}{\partial\bar{s}^2} &= \frac{e^{\frac{-d_1^2}{2}}}{\bar{s}\sigma_{f,g}\sqrt{2\pi(T-t)}} \\ \frac{\partial^3 P_0}{\partial\bar{s}^3} &= \frac{e^{\frac{-d_1^2}{2}}}{\bar{s}^2\sigma_{f,g}\sqrt{2\pi(T-t)}}\left(1 + \frac{d_1}{\sigma_{f,g}\sqrt{T-t}}\right). \end{aligned} \tag{3.6}$$

Note that the only difference between pricing a domestic option in domestic currency is the volatility and the starting asset value $\bar{s} = sx$. The parameters A and B can be estimated now through linear regression analysis.

4. The Correlation Effect

Nonzero correlation between the exchange rate and it's volatility process can be incorporated into the picture; the only operator that is changed is $\mathscr{L}_1$. This now becomes:

$$\mathscr{L}_1 = sg(z)\left(\rho_{12}\sigma_1\frac{\partial}{\partial s\partial y} + \rho_{13}\sigma_2\frac{\partial}{\partial s\partial z}\right) + xf(y)\rho_{02}\sigma_1\frac{\partial}{\partial x\partial y} - \left(\Lambda(y)\frac{\partial}{\partial y} + \Gamma(y)\frac{\partial}{\partial z}\right).$$

Going through the analysis we see that $\mathscr{L}_2$ and $\mathscr{L}_0$ are unchanged and results regarding $\mathscr{L}_1$ remain true. However the operator $\mathscr{A}$ is now:

$$\begin{aligned} \mathscr{A} = A\,x^2 s\frac{\partial^3}{\partial x^3} + B\,s\frac{\partial}{\partial s}\left(s^2\frac{\partial^2}{\partial s^2}\right) + C\,s\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}\right) \\ + D\,x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2}{\partial x^2}\right) + E\,xs\frac{\partial^2}{\partial x\partial s} \end{aligned} \tag{4.1}$$

where A, B and C are still given by formulae (2.30) and we further have

$$D = \rho_{02}\sigma_1\mathbb{E}^\phi\left(f(y)\frac{\partial\alpha}{\partial y}\right), \tag{4.2a}$$

$$E = \rho_{12}\sigma_1\mathbb{E}^\phi\left(\frac{\partial\gamma}{\partial y}f(y)\right) + \rho_{02}\sigma_2\mathbb{E}^\phi\left(\frac{\partial\gamma}{\partial z}f(y)\right). \tag{4.2b}$$

If the option does not depend on the stock price then we can ignore partial derivatives involving s and $\mathscr{A}$ is:

$$\mathscr{A} = A\,x^3 \frac{\partial^3}{\partial x^3} + D\,x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2}{\partial x^2}\right),$$

in which case the pricing expression to order $O(\sqrt{\epsilon})$ becomes

$$P_0 + \sqrt{\epsilon}P_1 = P_0 - \sqrt{\epsilon}(T-t)A\,x^3 \frac{\partial^3 P_0}{\partial x^3} + D\,x \frac{\partial P_0}{\partial x}\left(x^2 \frac{\partial^2 P_0}{\partial x^2}\right).$$

The pricing formula for P_0 does not depend on s so the pricing partial differential equation is

$$\frac{\partial P_0}{\partial t} + (r_d - r_f)x\frac{\partial P_0}{\partial x} + \frac{1}{2}x^2\mathbb{E}\{f^2(y)\}\frac{\partial^2 P_0}{\partial x^2} = 0. \tag{4.3}$$

Note however that, if we take X_t to be an asset price with domestic rate r_d, then the corresponding Black Scholes equation is

$$\frac{\partial P_{BS}}{\partial t} + r_d x\frac{\partial P_{BS}}{\partial x} + \frac{1}{2}x^2\mathbb{E}\{f^2(y)\}\frac{\partial^2 P_{BS}}{\partial x^2} = 0.$$

Therefore, if we substitute in $x\exp(-r_f(T-t))$ for x we get the partial differential equation (4.3). So if we know the solution to the Black Scholes pricing equation we know the solution to P_0 by making the substitution

$$P_0(t,x) = P_{BS}(t, x\exp(-r_f(T-t))). \tag{4.4}$$

We exploit this property in the next example.

4.1. *Binary option on an exchange rate*

The pricing function under consideration is on the exchange rate of the form

$$h(x,s,y,z) = 1_{[a,b]}(X(T))$$

That is, if the exchange rate is between a and b the owner will obtain one unit of domestic currency, and nothing otherwise. The Black Scholes price for the option can be calculated explicitly to be:

$$P_{BS}(t,x) = e^{-r_d(T-t)}(N(d_2^*) - N(d_1^*)),$$

with

$$d_1^* = \frac{\ln(x/(b)) + r_d - \frac{1}{2}\sigma_f^2)(T-t)}{\sigma_f\sqrt{T-t}}$$

and

$$d_2^* = \frac{\ln(x/(a)) + r_d - \frac{1}{2}\sigma_f^2)(T-t)}{\sigma_f\sqrt{T-t}}.$$

We have also used the notation N for the standard normal cumulative density function. Hence we have

$$P_0(t,x) = P_{BS}(t, x\exp(-r_f(T-t))) = e^{-r_d(T-t)}(N(d_2) - N(d_1)),$$

with

$$d_1 = \frac{\ln(xe^{-r_f(T-t)}/b) + r_d - \frac{1}{2}\sigma_f^2)(T-t)}{\sigma_f\sqrt{T-t}}$$

and

$$d_2 = \frac{\ln(xe^{-r_f(T-t)}/a) + r_d - \frac{1}{2}\sigma_f^2)(T-t)}{\sigma_f\sqrt{T-t}}.$$

To calculate the corrected price we need to calculate the derivatives, i.e. the Greeks of P_0. These are

$$\frac{\partial P_0}{\partial x} = \frac{e^{-(r_f+r_d)(T-t)}(e^{-d_2^2/2} - e^{-d_1^2/2})}{x\sigma\sqrt{2\pi(T-t)}},$$

$$\frac{\partial^2 P_0}{\partial x^2} = -e^{-(r_d+r_f)(T-t)}\frac{A_2e^{-d_2^2/2} - A_1e^{-d_1^2/2}}{x^2\sigma\sqrt{2\pi(T-t)}},$$

$$\frac{\partial^3 P_0}{\partial x^3} = e^{-(r_d+r_f)(T-t)}\frac{e^{-d_2^2/2}}{x^3\sigma\sqrt{2\pi(T-t)}}\left(2 + \frac{3d_2+2}{\sigma\sqrt{T-t}} + \frac{d_2^2-1}{\sigma^2(T-t)}\right),$$

$$- \frac{e^{-d_1^2/2}}{x^3\sigma\sqrt{2\pi(T-t)}}\left(2 + \frac{3d_1+2}{\sigma\sqrt{T-t}} + \frac{d_1^2-1}{\sigma^2(T-t)}\right).$$

where $A_1 = 1 + \frac{d_1}{\sigma\sqrt{(T-t)}}$ and $A_2 = 1 + \frac{d_2}{\sigma\sqrt{(T-t)}}$.

5. A Two Foreign Currency Model

In this section we briefly consider an economy that supports three currencies: a domestic currency with short rate r_d and two foreign currencies with exchange rates X_1 and X_2, viewed as domestic currency per unit of foreign currency. The foreign short rates are r_1 and r_2, respectively. We denote the exchange rate between X_1 and X_2 as $X_{12} = X_1/X_2$. The model operates under the following dynamics.

Assumption 5.1. (Three Currency Exchange Rate Model). The market operates by the following dynamics under the objective probability measure

$$\text{(exchange rate)} \quad dX_t^i = X_t^i\mu_i\,dt + X_t^i f(Y_t^i)\,dV_t^i, \quad i = 1,2,\ldots \tag{5.1a}$$

$$dY_t^i = \frac{1}{\epsilon}(m_1 - Y_t^i)\,dt + \frac{\sigma_1}{\sqrt{\epsilon}}dW_t^i, \quad i = 1,2,\ldots \tag{5.1b}$$

$$dB_t^i = r_iB_t^i\,dt, \quad i = 1,2,\ldots \tag{5.1c}$$

$$dB_t = r_dB_t, \qquad i = 1,2,\ldots \tag{5.1d}$$

where $V_i, W^{(i)}, i = 1, 2$ are standard one-dimensional Brownian motions. Furthermore, we assume the following correlation structure between these Brownian motions:

$$d\langle V^1, V^2\rangle = \rho\, dt,$$
$$d\langle W^i, V^i\rangle = \rho_i\, dt, \quad i = 1, 2, \dots$$
$$d\langle W^1, V^2\rangle = d\langle W^2, V^1\rangle = 0,$$
$$d\langle W^1, W^2\rangle = 0.$$

We remark that this is exactly the Itô representation we had for the currency model including a stock.

In the risk-neutral measure equations (5.1) become

$$\text{(exchange rate)} \quad dX_t^i = X_t^i(r_d - r_i)\, dt + X_t^i f(Y_t^i) dV_t^{*i}, \quad i = 1, 2, \dots \tag{5.2a}$$

$$dY_t^i = \left(\frac{1}{\epsilon}(m_i - Y_t^i) - \frac{\sigma_1}{\sqrt{\epsilon}}\Lambda^i(Y_t^i)\right) dt + \frac{\sigma_i}{\sqrt{\epsilon}} d\overline{W}_t^i, \quad i = 1, 2, \dots \tag{5.2b}$$

with $\Lambda^i(Y_t^i), i = 1, 2$ being some appropriate market price of risk.

Let us introduce the notation $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The pricing PDE of $P^\epsilon(t, x, y)$ for a typical contract with payoff function $h(x)$ is, by the Feynman-Kac formula

$$\left(\mathscr{L}_2 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_1 + \frac{1}{\epsilon}\mathscr{L}_0\right) P^\epsilon = 0, \quad P^\epsilon(T, x, y) = h(x), \tag{5.3}$$

where the operators are defined by

$$\mathscr{L}_0 = \sum_{i=1}^{2}(m_i - y_i)\frac{\partial}{\partial y_i} + \sum_{i=1}^{2}\frac{1}{2}\sigma_i^2\frac{\partial^2}{\partial y^2}, \tag{5.4}$$

$$\mathscr{L}_1 = \sum_{i=1}^{2} x_i f_i(y_i)\rho_{i2}\sigma_i\frac{\partial}{\partial x_i \partial y_i} - \sum_{i=1}^{2}\Lambda(y_i)\frac{\partial}{\partial y_i}, \tag{5.5}$$

$$\mathscr{L}_2 = \frac{\partial}{\partial t} + \sum_{i=1}^{2}(r_d - r_i)x_i\frac{\partial}{\partial x_i} + \frac{1}{2}x_i^2 f^2(y_i)\frac{\partial^2}{\partial x_i^2}$$
$$+ \rho_1 f_1(y_1) f_2(y_2)\frac{\partial^2}{\partial x_1 \partial x_2} - r_d. \tag{5.6}$$

The asymptotic analysis is very similar to the previous case. We look for a solution to (5.3) in the form of multiscale expansion:

$$P^\epsilon(t, x, y) = P_0(t, x, y) + \sqrt{\epsilon}P_1(t, x, y) + \epsilon P_2(t, x, y) + \cdots.$$

Calculations similar to the ones presented in Sec. 2.2 enable us to obtain the first two terms in the expansion

$$\begin{aligned} P^\epsilon &\approx P_0 + \sqrt{\epsilon} P_1 \\ &= P_0 - \sqrt{\epsilon}(T-t)\sum_{i=1}^{2} A_i x_i \frac{\partial^2 P_0}{\partial x_i^2} + B_i x_i^2 \frac{\partial^3 P_0}{\partial x_i^3} \\ &\quad + C_i x_1 x_2 \frac{\partial^2 P_0}{\partial x_1 \partial x_2} + D_1 x_1^2 x_2 \frac{\partial^3 P_0}{\partial x_1^2 \partial x_2} + D_2 x_1 x_2^2 \frac{\partial^3 P_0}{\partial x_1 \partial x_2^2} \end{aligned}$$

where

$$\begin{aligned} A_i &= \rho_{12}\sigma_i \mathbb{E}^\phi\{f_i(y_i)\alpha_i'(y_i)\} - \frac{1}{2}\mathbb{E}^\phi(\Lambda_i(y_i)\alpha_i'(y_i)) \\ B_i &= \frac{1}{2}\rho_{12}\sigma_i \mathbb{E}^\phi\{f_i(y_i)\alpha_i'(y_i)\} \\ C_i &= \rho_{01}\rho_{12}\sigma_i \mathbb{E}^\phi\left\{f_i(y_i)\frac{\partial}{\partial y_i}\gamma(y_1,y_2)\right\} - \rho_{01}\mathbb{E}^\phi\left(\Lambda_i(y_i)\frac{\partial}{\partial y_i}\gamma(y_1,y_2)\right) \\ D_i &= \rho_{01}\rho_{12}\sigma_i \mathbb{E}^\phi\left((f_i(y_i)\frac{\partial}{\partial y_i}\gamma(y_1,y_2)\right) \end{aligned}$$

and $\alpha_i(y_i), \gamma(y_1, y_2)$ are the solutions to the problems

$$\begin{aligned} \mathscr{L}_0\{\alpha_i(y_i))\} &= f_i^2(y_i) - \mathbb{E}^\phi\{f_i^2(y_i)\} \\ \mathscr{L}_0\{\gamma(y_1,y_2)\} &= \mathbb{E}^\phi\{f_1(y_1)f_2(y_2)\} - f_1(y_1)f_2(y_2). \end{aligned}$$

The leading order term P_0 satisfies the following final value problem

$$\begin{aligned} &\frac{\partial P_0}{\partial t} + \frac{1}{2}\sum_{i=1}^{2}(r_d - r_i)x_i\frac{\partial P_0}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{2} x_i^2 \mathbb{E}\{f_i^2(y_i)\}\frac{\partial^2 P_0}{\partial x_i^2} \\ &\quad + \rho_{01}x_1x_2\mathbb{E}\{f_1(y_1)f_2(y_2)\}\frac{\partial^2 P_0}{\partial x_1 \partial x_2} - r_d P_0 = 0, \\ &P_0(T,x) = h(x). \end{aligned}$$

As an application consider the following payoff structure

$$h(x_i, T) = \max(X_T^{12} - K, 0) \tag{5.7}$$

where $X^{12} = X^1/X^2$. That is, the owner has the right but not the obligation to exchange K units of currency 1 for one unit of currency 2. This can be viewed as a call option on the exchange rate between two foreign currencies. The volatility of which can be calculated by Itô's formula:

$$\sigma_{12}^2(y) = f_1^2(y_1) + f^2(y_2) - 2\rho f_1(y_1)f_2(y_2) \tag{5.8}$$

This is similar to the case of a foreign call option struck in a domestic currency. The call option price thus satisfies the Black Scholes partial differential equation with volatility $\mathbb{E}^\phi\{\sigma_{12}^2(y)\}$ and this is the first order term in the asymptotic expansion.

6. Discussion

Given the increasing number of desks willing to undertake more rigorous model testing in all asset classes, and the general trend of risk managing via multiple model comparison, the work presented here should be of interest to banks and hedge funds alike. We have used the theory of asymptotic expansions to study stochastic volatility models for the pricing of derivative securities with foreign exchange risk. For international investors, the sensitivity of the value of a portfolio to changes in the currency market can be both a source of unwanted risks and speculative opportunities. We present a stochastic volatility model and derive the relevant partial differential equations for the pricing of FX derivatives. By applying asymptotic methods to these equations, we showed that the price in the stochastic volatility model can be approximated by calculating a correction term to the price from the Black Scholes, constant volatility model. These results are exemplified by derivatives commonly traded in the financial markets. More general types of options can be priced under the framework.

We discussed the impact of correlation between exchange rate and its volatility process. In practice, for financial models with several variables, the issue of dependency is important. One particularly interesting direction is to generalize (2.13) and investigate the leverage effect between the variables of the model. Future work includes an econometric study of the ideas presented in this note.

Acknowledgment

We thank H. O. Rasmussen for stimulating discussions.

References

[1] M. Baxter and A. Rennie, *Financial Calculus: An Introduction to Derivatives Pricing* (Cambridge University Press, 1996).
[2] T. Björk, *Arbitrage Theory in Continuous Time* (Oxford University Press, Oxford, 1998).
[3] J. Y. Campbell, A. W. Lo and A. C. MacKinlay, *The Econometrics of Financial Markets* (Princeton University Press, 1996).
[4] P. Carr and L. Wu, Stochastic skew models for fx options, Courant Institute, New York University and Graduate School of Bussines, Fonlham University Preprint (2004).
[5] J.-P. Fouque, G. Papanicolaou and K. R. Sircar, Asymptotics of a two-scale stochastic volatility model, in *Équations Aux Dérivées Partielles et Applications*, Éd. Gauthier-Villars, (Elsevier, Paris, 1998), pp. 517–525.
[6] J.-P. Fouque, G. Papanicolaou and K. R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility* (Cambridge University Press, Cambridge, 2000).
[7] J.-P. Fouque, G. Papanicolaou and K. R. Sircar, Multiscale stochastic volatility asymptotics, *SIAM Journal on Multiscale Modeling and Simulation* **43**(2) (2003) 22–42 (electronic).
[8] S. L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies* **6** (1993) 327–343.

[9] S. Howison, A. Rafailidis and H. Rasmussen, On the pricing and hedging of volatility derivatives, *Applied Mathematical Finance* **11**(4) (2004) 317–346.

[10] J. C. Hull and A. White, The pricing of options on assets with stochastic volatilities, *Journal of Finance* **42**(8) (1987) 281–300.

[11] A. Lipton, *Mathematical Methods for Foreign Exchange Derivatives*, World Scientific Publishing, 2001.

[12] G. C. Papanicolaou, Asymptotic analysis of stochastic equations, in *Studies in Probability Theory*, Vol. 18 of MAA Stud. in Math. (American Mathematical Society, 1977), pp. 111–179.

[13] G. C. Papanicolaou, Introduction to the asymptotic analysis of stochastic equations, in *Modern Modeling of Continuum Phenomena (Ninth Summer Sem. Appl. Math., Rensselaer Polytech. Inst., Troy, N.Y., 1975*) (1977) 109–147.

[14] H. Rasmussen and P. Wilmott, Assymptotic analysis of stochastic volatility models, in *New Directions in Mathematical Finance*, P. Wilmott and H. Rasmussen, eds. (Wiley Finance, 2002).

[15] S. Taylor, *Asset Price Dynamics, Volatility, and Prediction* (Princeton University Press, Berlin, 2007).