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STATISTICAL PROPERTIES OF COMPACT GROUP EXTENSIONS OF HYPERBOLIC FLOWS AND THEIR TIME ONE MAPS

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Abstract. Recent work of Dolgopyat shows that "typical" hyperbolic flows exhibit rapid decay of correlations. Melbourne and Török used this result to derive statistical limit laws such as the central limit theorem and the almost sure invariance principle for the time-one map of such flows.

In this paper, we extend these results to equivariant observations on compact group extensions of hyperbolic flows and their time one maps.

1. Introduction.

Let $\Lambda \subset M$ be a hyperbolic basic set for a smooth flow T_t on a compact manifold M. Let μ denote an equilibrium measure supported on Λ, corresponding to a Hölder continuous potential [5]. The flow has *exponential decay of correlations* if given sufficiently regular observations $\phi, \psi : M \to \mathbb{R}$, there are constants $C > 0$, $\beta > 0$ (depending on ϕ and ψ) such that

$$\Big|\int_M \phi \cdot \psi \circ T_t \, d\mu - \int_M \phi \, d\mu \int_M \psi \, d\mu\Big| \le Ce^{-\beta t}, \text{ for all } t > 0.$$

Unlike the case of hyperbolic diffeomorphisms, it turns out that general hyperbolic flows do not have exponential decay of correlations, even when they are topologically mixing. Moreover, examples of Ruelle [21] and Pollicott [19] show that the rate of decay of correlations may be arbitrarily slow.

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Recently, Dolgopyat [7] proved that "typical" hyperbolic flows are better behaved. In particular, if it is possible to choose two periodic points in Λ whose periods have a ratio that is Diophantine (which is almost certainly the case), then sufficiently regular observations decay superpolynomially fast (faster than any specified polynomial rate). We say that such observations satisfy *rapid decay of correlations.*

Melbourne and Török [13] used this result to establish statistical limit laws such as the central limit theorem (CLT), the weak invariance principle (WIP), the law of the iterated logarithm (LIL), and the almost sure invariance principle (ASIP) for the time-one maps of such rapidly mixing hyperbolic flows. For example, the CLT in [13] guarantees that for typical hyperbolic flows and sufficiently regular observations $\phi : M \to \mathbb{R}$ of mean zero, $\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\phi\circ T_j$ converges in distribution to a normal distribution with mean zero and variance $\sigma^2 \geq 0$. Moreover, $\sigma^2 = 0$ if and only if $\sum_{j=0}^{n-1}\phi\circ T_j$ is uniformly bounded.

(The corresponding statistical limit laws for the hyperbolic flow itself, with $\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\phi\circ T_j$ replaced by $\frac{1}{\sqrt{t}}\int_0^t \phi\circ T_s\,ds$, are well-known [20, 24, 6, 14] and do not require any mixing conditions.)

In the remainder of this paper, we shall speak only of the CLT and ASIP, but all statements about the CLT carry through to the WIP, and all statements about the ASIP imply the corresponding statements for the LIL.

1.1. ***G*-extensions; discrete time.** The aim of this paper is to generalise the results in [7, 13] to the equivariant context. It is convenient to recall the analogous results for the case of diffeomorphisms [9, 12]. Suppose that $\Lambda \subset M$ is a topologically mixing hyperbolic basic set for a diffeomorphism $f : M \to M$ and let μ denote an equilibrium measure supported on Λ corresponding to a Hölder potential. Let G be a compact connected Lie group with Haar measure ν and let $h : \Lambda \to G$ be Hölder. The cocycle h induces a G-extension $f_h : \Lambda \times G \to \Lambda \times G$ defined by

$$f_h(x, g) = (f(x), gh(x)).$$

Recently, we showed [10] that the G-extension $f_h : \Lambda \times G \to \Lambda \times G$ is mixing with respect to $\mu \times \nu$ for an open and dense set of cocycles $h : \Lambda \to G$ in the C^r topology for all r. Moreover, we obtain C^2-openness and C^∞-density. (For related references, see [1, 11, 17].) A natural question is to determine the rate of mixing.

Dolgopyat [8] considered stably mixing G-extensions and showed that under certain hypotheses, sufficiently regular observations $\phi : \Lambda \times G \to \mathbb{R}$ satisfy rapid decay of correlations and the CLT. However, Dolgopyat also gave examples where the rate of decay is arbitrarily slow (even when $\Lambda \times G$ is stably mixing).

On the other hand, in the context of equivariant dynamical systems it is natural [15] to consider vector-valued observations $\phi : \Lambda\times G \to \mathbb{R}^d$ that are G-equivariant in the following sense. Let $\rho : G \to \mathbf{GL}(\mathbb{R}^d)$ denote a representation of G on $\mathbb{R}^d$. Then $\phi : \Lambda \times G \to \mathbb{R}^d$ is *G-equivariant* if $\phi(x, ag) = \rho_a\phi(x, g)$ for all $a \in G$. Equivalently, $\phi(x, g) = \rho_g v(x)$ for some $v : \Lambda \to \mathbb{R}^d$. From now on we suppress the ρ, writing $\phi(x, g) = gv(x)$ and so on.

In [9], we showed that if $f_h : \Lambda \times G \to \Lambda \times G$ is mixing (not necessarily stably mixing), then Hölder equivariant observations $\phi : \Lambda \times G \to \mathbb{R}^d$ satisfy exponential decay of correlations, the CLT, and the ASIP. In [12], we show that ergodicity of f_h suffices for the CLT and ASIP. For example, the CLT in [9, 12] states that if $\sigma_h : \Lambda\times G \to \Lambda\times G$ is ergodic for a Hölder cocycle $h : \Lambda \to G$, and $\phi : \Lambda \times G \to \mathbb{R}^d$ is a Hölder equivariant observation of mean zero, then $\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\phi\circ\sigma_h^j$ converges in distribution to a d-dimensional normal distribution with mean zero and covariance matrix Σ. Moreover, Σ commutes with the action of G on $\mathbb{R}^d$. Finally, the CLT is degenerate ($\det\Sigma = 0$) if and only if at least one coordinate of $\sum_{j=0}^{n-1}\phi\circ\sigma_h^j$ is uniformly bounded, and typically this case can be ruled out [15].

1.2. **G-extensions; continuous time.** In this paper, we consider continuous time G-extensions of the form $\Lambda\times G$ where Λ is a mixing basic set for a hyperbolic flow and G is a compact connected Lie group. Let T_t denote the flow on Λ and let $h_t:\Lambda\to G$ be a Hölder cocycle (so $h_{s+t}=h_s\cdot h_t\circ T_s$). Then we construct the Hölder G-extension $T_{h,t}:\Lambda\times G\to\Lambda\times G$. Again, it follows from [10] that $\Lambda\times G$ is mixing for an open and dense set of C^r cocycles h_t. As before, we say that $\phi:\Lambda\times G\to\mathbb{R}^d$ is G-equivariant if $\phi(x,g)=gv(x)$ where $v:\Lambda\to\mathbb{R}^d$.

Suppose that Λ is a mixing hyperbolic basic set for the flow T_t and that $\Lambda\times G$ is ergodic. By [14], the CLT and ASIP hold for Hölder G-equivariant observations $\phi:\Lambda\times G\to\mathbb{R}^d$. As in the nonequivariant context, decay of correlations for the flow is more delicate, as are the CLT and ASIP for the time-one map $T=T_1$. It is these issues that we address in this paper.

The precise statements of our main results are quite technical and are deferred until Section 2. Roughly speaking, we have the following generalisation of the results in [7, 13].

Theorem 1.1 (Rough statement of main results)**.** *Let Λ be a mixing basic set for a hyperbolic flow and let G be a compact connected Lie group. Consider G-extensions $T_{h,t}:\Lambda\times G\to\Lambda\times G$ and G-equivariant observations $\phi:\Lambda\times G\to\mathbb{R}^d$. For typical basic sets Λ and Hölder G-extensions $\Lambda\times G$, it is the case that sufficiently regular G-equivariant observations satisfy rapid decay of correlations.*

If we consider the time-one map $T_{h,1}$ of such flows, then sufficiently regular G-equivariant observations ϕ of mean zero satisfy statistical limit laws such as the CLT and the ASIP, and typically these limit laws are nondegenerate (nonsingular covariance).

In Theorem 1.1, we speak of typical G-extensions. Typical can be taken to have the meaning of prevalence, governed by a Diophantine condition, as in Dolgopyat [7], but stronger results are sometimes possible depending on the representation of G on $\mathbb{R}^d$. In proving rapid decay, we may decompose $\mathbb{R}^d$ into its G-irreducible subspaces and hence may suppose without loss that G acts irreducibly and faithfully on $\mathbb{R}^d$. Note that $d=1$ or $d=2$ if G is abelian, and $d\geq 3$ if G is not abelian. Also, when G is abelian, we can assume that $d=2$, since otherwise $d=1$ and we are in the situation of [7].

In the abelian case, $G=T^m$ is a torus. Write $h(x)=(\theta_1(x),\dots,\theta_m(x))$. Choose four periodic orbits in Λ. It turns out that $\Lambda\times G$ is rapid mixing provided that a certain measure one Diophantine condition involving the periods and the values of θ_j along these orbits is satisfied. Hence, we obtain rapid mixing and so on for a prevalent set of G-extensions as in [7]. (We note that rapid mixing of Λ and $\Lambda\times G$ are unrelated in these results.)

When G is not abelian, the situation is somewhat simpler (just as the question of stable ergodicity of $\Lambda\times G$ is simpler for G semisimple than G abelian [11]) and we obtain rapid mixing for an open and dense prevalent set of G-extensions.

In Section 2, we give a precise statement of the results described roughly in Theorem 1.1. The remainder of the paper is concerned with proving this theorem. By [13], it will suffice to prove the results on rapid decay. Much of the proof is a straightforward generalisation of the approach in Dolgopyat [7]; these parts of the proof are carried out in Sections 3 and 4. Eventually the proof diverges from that in [7], particularly in the case when G is nonabelian, and this part of the proof is contained in Section 5.

2. **Statement of the main results.** In this section, we give a precise statement of the results described in Theorem 1.1. We do this within the context of G-extensions of one-sided (noninvertible) symbolic flows. This is no loss of generality, see Remark 2.8.

Let $\sigma : X \to X$ be a one-sided subshift of finite type and let G be a compact connected Lie group. If $h : X \to G$ is a measurable cocycle, we define the G-extension $\sigma_h : X \times G \to X \times G$. If $r : X \to \mathbb{R}$ is a positive measurable roof function, then we define the suspension $X_r = \{(x,u) \in X \times \mathbb{R} : 0 \le u \le r(x)\}/\sim$ where $(x, r(x)) \sim (\sigma x, 0)$, and the suspension flow is given by $T_t(x,u) = (x, u+t)$.

Identifying $r : X \to \mathbb{R}$ with a G-invariant function $r : X \times G \to \mathbb{R}$, we construct

$$X_r \times G = (X \times G)_r = \{(x,g,u) \in X \times G \times \mathbb{R} : 0 \le u \le r(x)\}/\sim$$

where $(x,g,r(x)) \sim (\sigma x, gh(x), 0)$. Note that $X_r \times G$ can be viewed as a G-extension of a suspension or a suspension of a G-extension. The suspension flow is $T_{h,t}(x,g,u) = (x,g,u+t)$ computed subject to the identifications.

Let $\theta \in (0,1)$. We define the Hölder spaces $F^\theta(X,\mathbb{R})$ and $F^\theta(X,G)$ in the usual way [9, 16]. Recall that $\|v\|_\theta = |v|_\infty + |v|_\theta$. We suppose that $r \in F^\theta(X,\mathbb{R})$ and $h \in F^\theta(X,G)$. Given a continuous potential function $F : X_r \to \mathbb{R}$, we define $f(x) = \int_0^{r(x)} F(x,u)du$. Suppose that $f \in F^\theta(X,\mathbb{R})$ and let μ be the equilibrium measure on X corresponding to f. Set $\bar{r} = \int_X r d\mu$ and define the invariant measure $\mu_r = \mu \times \ell/\bar{r}$ where ℓ is Lebesgue measure. Then μ_r is the equilibrium measure on X_r corresponding to F. Denoting Haar measure on G by ν, we form the invariant product measures $m = \mu \times \nu$ on $X \times G$ and $m_r = \mu_r \times \nu$ on $X_r \times G$. Again, $m_r = m \times \ell/\bar{r}$.

As in [9], we consider the Hölder space $F_G^\theta(X_r \times G, \mathbb{R}^d)$ of equivariant observations $\phi(x,g,u) = gv(x,u)$ where $v \in F^\theta(X_r, \mathbb{R}^d)$. Define $F_G^{k,\theta}(X_r \times G) \approx F^{k,\theta}(X_r)$ to consist of those equivariant observations ϕ such that $\partial_t^j \phi \in F_G^\theta(X_r \times G)$ for $j = 0,1,\dots,k$ where $\partial_t \phi = \frac{d}{dt}\phi \circ T_{h,t}|_{t=0}$ denotes the derivative of ϕ along the flow. [Recall that the metric on $\{(x,t) \in X \times \mathbb{R} \mid 0 \le t < r(x)\} \subset X_r$ is given by $d\big((x,t),(x',t')\big) = d_\theta(x,x') + |t-t'|$, with d_θ the metric on the shift space. Note that functions in $F_G^{k,\theta}$, $k \ge 1$, are continuous along the flow, but this is not the case for functions in F_G^θ.]

Given $\phi \in L^\infty(X_r \times G, \mathbb{R}^d)$, $\psi \in L^1(X_r \times G, \mathbb{R}^d)$, we define the correlation function $\rho_{\phi,\psi} : [0,\infty) \to \mathrm{Mat}_d(\mathbb{R})$ to be

$$\rho_{\phi,\psi}(t) = \int_{X_r \times G} \phi(\psi \circ T_t)^T dm_r - \int_{X_r \times G} \phi\, dm_r \int_{X_r \times G} \psi^T\, dm_r.$$

The aim is to prove rapid decay of $\rho_{\phi,\psi}(t)$ as $t \to \infty$.

Recall from the introduction that we may suppose without loss that G acts irreducibly and faithfully on $\mathbb{R}^d$, $d \ge 2$. Moreover G is abelian if $d = 2$ and nonabelian if $d \ge 3$. Recall also that the Gibbs measure μ_r on X_r corresponds to a continuous potential $F : X_r \to \mathbb{R}$ for which $f(x) = \int_0^{r(x)} F(x,u)du$ lies in $F_\theta(X,\mathbb{R})$.

Remark 2.1. *In the following, we say that a constant is* uniform *in F_θ if it varies continuously with f, r, and h in the Hölder topology F_θ for a fixed value of θ.*

Theorem 2.2. *Suppose that $X_r \times G$ is mixing and that G is nonabelian. Then there exists an open and dense (and prevalent) set of cocycles $h \in F^\theta(X,G)$ and a uniform constant $C>0$ with the following property: For any $n \geq 1$, there exists a uniform integer $k \geq 1$ such that*

$$|\rho_{\phi,\psi}(t)| \leq C\|\phi\|_{k,\theta}|\psi|_1/t^n, \text{ for all } t>0$$

for all $\phi \in F_G^{k,\theta}(X_r \times G)$ and $\psi \in L^1_G(X_r \times G)$.

Theorem 2.3. *Suppose that $X_r \times G$ is mixing and that G is abelian. Choose four fixed points $x_j \in X$, $j=1,\dots,4$ for the subshift $\sigma : X \to X$. Then there is a full measure set $Q \subset \mathbb{R}^4 \times G^4$ with the following property:*

If $r \in F^\theta(X,G)$, $h \in F^\theta(X,G)$ are such that $(r(x_1),\dots r(x_4), h(x_1),\dots,h(x_4)) \in Q$, then there exists a constant $C>0$ and for any $n \geq 1$, there exists an integer $k \geq 1$ such that

$$\rho_{\phi,\psi}(t) \leq C\|\phi\|_{k,\theta}|\psi|_1/t^n, \text{ for all } t>0$$

for all $\phi \in F_G^{k,\theta}(X_r \times G)$ and $\psi \in L^1_G(X_r \times G)$.

Remark 2.4. *In the nonabelian case, the validity of rapid mixing depends only on the cocycle h (independent of the roof function r). Moreover, the constant C and required differentiability k are uniform.*

In contrast, the abelian case is analogous to the case $G=1$ in Dolgopyat [7]: rapid mixing depends also on the roof function r, and the constant C depends very delicately on r and h (and is uniform in f).

In both cases, rapid mixing of $X_r \times G$ is independent of rapid mixing of X_r. (Rapid mixing of X_r is detected by the trivial representations of G which we have excluded.)

Lemma 2.5 (cf. [13]). *Let (Y,m) be a measure space, $S: Y \to Y$ an ergodic measure preserving transformation, and $\phi : Y \to \mathbb{R}^d$ a mean zero L^∞ observation. Suppose that there exist constants $C>0$ and $n>2$ such that*

$$|\int_Y \phi\,(\psi \circ S^j)^T dm| \leq C|\psi|_1/j^n,$$

for all $j \geq 1$ and all $\psi \in L^1(Y,\mathbb{R}^d)$. Define $\phi_N = \sum_{j=0}^{N-1} \phi \circ S^j$. Then

(a) The limit $\Sigma = \lim_{N\to\infty} \frac{1}{N}\int_Y \phi_N \phi_N^T\, dm$ exists and defines a $d \times d$ covariance matrix. Moreover, Σ is singular if and only if there exists $c \in \mathbb{R}^d$ nonzero and $\chi \in L^\infty(Y,\mathbb{R})$ such that $c \cdot \phi = \chi \circ S - \chi$ almost everywhere.

(b) **(CLT):** $\frac{1}{\sqrt{N}}\phi_N \longrightarrow_d N(0,\Sigma)$ *as $N \to \infty$.*

Proof. A similar result was proved in [13], except that decay was assumed only against L^∞ observations and the conclusions were correspondingly weaker. But arguing as in [13, Theorem 2], we find that $\phi = \widetilde{\phi} + \chi \circ S - \chi$ where $\chi \in L^\infty(Y,\mathbb{R}^d)$ and $\{\widetilde{\phi} \circ T^j\}$ is a sequence of reverse martingale differences. Moreover, the covariance matrix for ϕ is identical to the one for $\widetilde{\phi}$ and is given by $\Sigma = \int_Y \widetilde{\phi}\widetilde{\phi}^T$. This establishes part (a).

It follows from the martingale approximation argument in [13, Theorem 2] that if $c \in \mathbb{R}^d$, then $c \cdot \phi$ satisfies the CLT with variance $c^T \Sigma c$. Hence part (b) follows from the Cramer-Wold technique. ∎

Theorem 2.6. *Let G be a compact connected Lie group acting (not necessarily irreducibly or faithfully) on $\mathbb{R}^d$. Suppose that the suspension $X_r \times G$ is mixing and let $\phi \in F_G^{k,\theta}(X_r \times G)$ be an equivariant observation with mean zero ($\int_{X_r \times G} \phi \, dm = 0$). Define $\phi_N = \sum_{j=0}^{N-1} \phi \circ T_{h,j}$.*

Suppose that $\phi : X_r \times G \to \mathbb{R}^d$ satisfies rapid mixing as in Theorems 2.2 and 2.3 with $n > 2$. Then

(a) The limit $\Sigma = \lim_{N\to\infty} \frac{1}{N} \int_{X_r \times G} \phi_N \phi_N^T \, dm$ exists and defines a $d \times d$ covariance matrix that commutes with the action of G on $\mathbb{R}^d$. Moreover, Σ is singular if and only if there exists $c \in \mathbb{R}^d$ nonzero such that $c \cdot \phi_N$ is uniformly bounded.

(b) **(CLT):** $\frac{1}{\sqrt{N}} \phi_N \longrightarrow_d N(0, \Sigma)$ *as $N \to \infty$.*

(c) **(ASIP):** *For each $c \in \mathbb{R}^d$, after possibly enriching the probability space $X_r \times G$ without changing the distribution of the sequence $\{c \cdot \phi_N\}$, there exists a one-dimensional Brownian motion W_c with variance $c^T \Sigma c$ such that for any $\alpha > 0$,*

$$c \cdot \phi_N = W_c(N) + O(N^{1/4+\alpha}) \quad \text{as } N \to \infty$$

almost everywhere. □

Proof. Parts (a) and (b) are immediate from Lemma 2.5 except for the statement about nondegeneracy. But if Σ is singular, then it follows from Livšic regularity that $c \cdot \phi = \chi \circ T - \chi$ for $c \in \mathbb{R}^d$ nonzero and χ Hölder. Part (c) requires more work and we only sketch the details which are completely analogous to those in [9] combined with [13]. The idea is to prove the ASIP for observations on $\widehat{X}_r \times G$, where $\widehat{X}$ is the two-sided version of the subshift X, and without loss of generality to prove this in *backwards time*. This ensures that the martingale approximation yields a genuine martingale (rather than a sequence of reverse martingale differences). The next step is to approximate the observation by one that depends only on future coordinates, at which point the technique of Strassen [23] can be applied. See [13, Section 3] and [9] for further details. □

Remark 2.7. *(a) In fact, the CLT and ASIP hold for all $n > 1$. However, the limit in part (a) need not hold, so the covariance matrix Σ is defined somewhat differently. (For example, $\Sigma = \int_{X_r \times G} \widetilde{\phi}\, \widetilde{\phi}^T \, dm$ where $\widetilde{\phi}$ is the martingale approximation for ϕ.)*

(b) An argument in [15] *shows that Σ is nonsingular for an open, dense, and prevalent set of observations $\phi \in F_G^{k,\theta}(X_r \times G)$.*

(c) We conjecture that a d-dimensional version of the ASIP (without taking one-dimensional projections) is valid.

(d) We note once again that the corresponding results for the flow itself (with $\phi_N = \sum_{j=0}^{N-1} \phi \circ T_{h,j}$ replaced by $\int_0^t \phi \circ T_{h,t}$) are considerably more elementary [14] *and do not require mixing hypotheses on $X_r \times G$.*

Remark 2.8. *It follows from standard arguments that the results stated above for G-extensions of suspensions of one-sided symbolic flows apply equally to G-extensions of basic sets for (invertible) hyperbolic flows. We now sketch these arguments.*

When $G = 1$, we can reduce from a general hyperbolic basic set first to a suspension of a two-sided subshift of finite type [2, 3, 5]*, and second to a suspension of a one-sided subshift of finite type* [22, 4] *(the second step was already mentioned in* *the proof of Theorem 2.6. The same reduction for G-extensions follows as in* [9]. *The results for basic sets are then immediate from the results for suspensions of two-sided subshifts. The almost sure invariance principle passes between one-sided and two-sided subshifts by combining the arguments in* [9, 13]. *The rapid decay result passes from one-sided to two-sided subshifts by an approximation argument, cf.* [9] *(which requires further that $\psi \in F_G^\theta(X_r \times G)$ with $|\psi|_1$ replaced by $\|\psi\|_\theta$).*

3. **Complex transfer operators and rapid mixing.** In this section, we show that the absence of approximate eigenfunctions for a family of complex equivariant Ruelle transfer operators implies rapid mixing. The proof is a straightforward generalisation of an argument of Dolgopyat [7] to the equivariant context. A good background reference is Parry and Pollicott [16].

In Section 2, we introduced the potential $f \in F^\theta(X,\mathbb{R})$ with corresponding equilibrium measure μ on X. The Ruelle transfer operator $L_f : F^\theta(X,\mathbb{R}) \to F^\theta(X,\mathbb{R})$ is defined by $(L_f v)(x) = \sum_{\sigma y = x} e^{f(y)} v(y)$. Without loss [16], we assume that f is normalised so that $L_f 1 = 1$.

Recall that $r \in F^\theta(X,\mathbb{R})$ is the roof function defining the suspension flow X_r and that $h \in F^\theta(X,G)$ is the cocycle defining the G-extension $\sigma_h : X \times G \to X \times G$. Suppose that $\mathbb{R}^d$ is a fixed representation of G. For each $s \in \mathbb{C}$, we define the complex equivariant Ruelle operator $L_{f+sr,h}$ on $F^\theta(X,\mathbb{C}^d)$ to be

$$(L_{f+sr,h} v)(x) = \sum_{\sigma y = x} e^{(f+sr)(y)} h(y)^{-1} v(y).$$

The operator $L_{f+sr,h}$ is a combination of the equivariant Ruelle operator $L_{f,h}$ and the complex Ruelle operator L_{f+sr} both of which can be found in [16].

Theorem 3.1. *Suppose that $X_r \times G$ is mixing. Of the following statements, (a) implies (b).*

(a) Absence of approximate eigenfunctions for $L_{f+ibr,h}$: *$\exists \alpha > 0$ such that $\|(I - L_{f+ibr,h})^{-1}\|_\theta \le |b|^\alpha$, for all $b \in \mathbb{R}$ with $|b| > 2$.*

(b) Rapid mixing: *For any $n \ge 1$, $\exists k \ge 1$, $C > 0$ such that for all $\phi \in F_G^{k,\theta}(X_r \times G)$ and $\psi \in L_G^1(X_r \times G)$,*

$$\Big|\int \phi (\psi \circ T_t)^T dm_r - \int \phi\, dm_r \int \psi^T dm_r\Big| \le C\|\phi\|_{k,\theta}|\psi|_1/t^n, \ \text{for all } t > 0.$$

In the remainder of this section, we prove Theorem 3.1. We begin with three preliminary propositions.

Proposition 3.2. *Suppose that there exist constants $\alpha_1 > 1$, $C_1 > 1$ such that*

$$\|(I - L_{f+ibr,h})^{-1}\|_\theta \le C_1 |b|^{\alpha_1},$$

for all $b \in \mathbb{R}$ with $|b|$ large enough. Choose $\alpha_2 > \alpha_1 + 1$. Then there is a constant $b_0 > 1$ such that

$$\|(I - L_{f-sr,h})^{-1}\|_\theta \le 2C_1 |b|^{\alpha_1},$$

for all $s = a + ib \in \mathbb{C}$ satisfying $|b| \ge b_0$, $|a| \le |b|^{-\alpha_2}$.

Proof. Recall the identity

$$(I - L_{f-sr,h})^{-1} = (I - A)^{-1}(I - L_{f-ibr,h})^{-1},$$

where

$$A = (I - L_{f-ibr,h})^{-1}(L_{f-sr,h} - L_{f-ibr,h}) = (I - L_{f-ibr,h})^{-1} L_{f-ibr,h} M,$$

where $Mv = e^{-ar}v - v$.

For $|a|$ small enough, $|e^{-ar}-1|_\infty \le 2|a||r|_\infty$, and $|e^{-ar}-1|_\theta \le 2e^{|a||r|_\infty}|a||r|_\theta$. It follows that $\|M\|_\theta \le 4e^{|a||r|_\infty}|a||r|_\theta$. Provided $|a| \le |b|^{-\alpha_2}$, we have $\|M\|_\theta \le D|b|^{-\alpha_2}$ for some constant D. From the basic inequality (Proposition 4.2(ii)), it follows that $\|L_{f-ibr,h}\|_\theta \le 2C_6|b|$. Combining these estimates with the hypothesis of the proposition yields

$$\|A\|_\theta \le E|b|^{\alpha_1+1-\alpha_2},$$

and hence $(I-A)$ is invertible for $|b|$ large enough provided $\alpha_2 > \alpha_1 + 1$. Indeed we can arrange that $\|(I-A)^{-1}\|_\theta \le 2$ on this region proving the result. □

Proposition 3.3. *Given $v \in L^1(X_r, \mathbb{R}^d)$ and $s \in \mathbb{C}$, define*

$$\widehat{v}_s(x) = \int_0^{r(x)} v(x,u)e^{-su}du.$$

Then $\widehat{v}_s \in L^1(X, \mathbb{C}^d)$ and $|\widehat{v}_s|_1 \le D|v|_1$ where $D = |r|_\infty e^{|\operatorname{Re} s||r|_\infty}$.

Similarly, if $v \in L^\infty(X_r, \mathbb{R}^d)$ then $\widehat{v}_s \in L^\infty(X, \mathbb{C}^d)$ and $|\widehat{v}_s|_\infty \le D|v|_\infty$.

Moreover, if $v \in F^\theta(X_r, \mathbb{R}^d)$ then $\widehat{v}_s \in F^\theta(X, \mathbb{C}^d)$ and $\|\widehat{v}_s\|_\theta \le (D+E)\|v\|_\theta$ where $E = |r|_\theta e^{|\operatorname{Re} s||r|_\infty}$.

Proof. To prove the L^1 estimate, compute that

$$\int_X |\widehat{v}_s(x)|d\mu \le \int_X \int_0^{r(x)} |v(x,u)e^{-su}|du d\mu \le |r|_1 \int_{X_r} |v(x,u)|d\mu_r e^{|\operatorname{Re} s||r|_\infty},$$

so that $|\widehat{v}_s|_1 \le D|v|_1$. The L^∞ estimate is similar. Moreover,

$$\begin{aligned} |\widehat{v}_s(x) - \widehat{v}_s(y)| &= |\int_0^{r(x)} v(x,u)e^{-su}du - \int_0^{r(y)} v(y,u)e^{-su}du| \\ &\le |r(x)-r(y)|e^{|\operatorname{Re} s||r|_\infty}|v|_\infty + |r|_\infty e^{|\operatorname{Re} s||r|_\infty}|v(x,u)-v(y,u)| \\ &\le |r|_\theta d_\theta(x,y)e^{|\operatorname{Re} s||r|_\infty}|v|_\infty + D|v|_\theta d_\theta(x,y), \end{aligned}$$

so that $|\widehat{v}_s|_\theta \le E|v|_\infty + D|v|_\theta$. Hence $\|\widehat{v}_s\|_\theta \le (D+E)\|v\|_\theta$. □

In the next proposition, we write Y_r instead of $X_r \times G$, since the result does not depend upon the structure of being a G-extension. Thus Y_r is the suspension of a map $T : Y \to Y$ with suspension flow T_t.

Proposition 3.4. *Let $\phi, \psi \in L^\infty(Y_r, \mathbb{R}^d)$. Suppose that ϕ is k-times differentiable along the flow direction and that ψ has support lying within $\{(y,s) \in Y \times \mathbb{R} : s \in [\epsilon, r(y)-\epsilon]\}$ for some $\epsilon > 0$. Then $\rho^{(k)}_{\phi,\psi} = (-1)^k \rho_{(\partial_t^k \phi),\psi}$ where ∂_t denotes differentiation with respect to the flow direction.*

Proof. To simplify the notation, we suppose that ϕ or ψ has mean zero, and we suppose that $d = 1$.

Write $\rho_{\phi,\psi}(t) = \int_Y S(y,t)dm$, where

$$S(y,t) = \int_0^{r(y)} \phi(y,u)\psi(T_t(y,u))du = \int_0^{r(y)} \phi(y,u)\psi(T^n y, u+t-r_n(y))du.$$

Here, the lap number $n = n(y, u+t)$ is defined by the condition $r_n(y) \le u+t < r_{n+1}(y)$. Note that the condition on the support of ψ guarantees that n is locally constant with respect to u in the above integral. Moreover, by assuming that ϕ and ψ are extended to be zero outside $\{(y,s) \in Y \times \mathbb{R} \mid 0 \le s \le r(y)\}$, we can write

$$\begin{aligned} S(y,t) &= \int_{-\infty}^{\infty} \phi(y,u)\psi(T^n y, u+t-r_n(y))du \\ &= \int_{-\infty}^{\infty} \phi(y,u-t)\psi(T^n y, u-r_n(y))du. \end{aligned}$$

Hence

$$\begin{aligned} \partial_t S(y,t) &= -\int_{-\infty}^{\infty} \partial_t \phi(y,u-t)\psi(T^n y, u-r_n(y))du \\ &= -\int_0^{r(y)} \partial_t \phi(y,u)\psi(T_t(y,u))du. \end{aligned}$$

This proves the required result for $k=1$, and the general case follows by induction. □

The strategy behind the proof of Theorem 3.1 is to show that the Laplace transform of ρ satisfies a Paley-Wiener Theorem so that ρ decays rapidly as $t \to \infty$. First, we write ρ, which is defined for $t \ge 0$, as the restriction to $t \in [0,\infty)$ of a convenient L^∞ function $R : \mathbb{R} \to \mathbb{R}$ such that $R(t) = 0$ for $t < -|r|_\infty$. This function R is given in Proposition A.1. The Laplace transform $\widehat{R}(s) = \int_0^\infty e^{-ts}R(t)dt$ is defined in the right-hand complex plane, and our hypotheses guarantee a suitable analytic extension of $\widehat{R}$ across the imaginary axis. (In fact, a general result of Pollicott [19] implies that such an analytic continuation exists, but we shall not require this result.) Using the Laplace inversion formula, the existence of this extension into the left-half-plane gives the desired decay of R (and hence ρ) as $t \to \infty$.

Lemma 3.5. *Suppose that $X_r \times G$ is mixing. Assume the conditions of Proposition 3.2 and let $\phi \in F_G^\theta(X_r \times G)$, $\psi \in L_G^1(X_r \times G)$ with $\int_{X_r \times G} \phi\, dm_r = 0$. Write*

$$\phi(x,g,u) = gv(x,u), \qquad \psi(x,g,u) = gw(x,u),$$

where $v \in F^\theta(X_r, \mathbb{R}^d)$, $w \in L^1(X_r, \mathbb{R}^d)$. Then

(i) *$\widehat{R}(s) = \widehat{R}(a+ib)$ has an analytic continuation to the region $A \cup \{a > 0\}$ where*

$$A = \{|a| < \epsilon,\ |b| < b_0 + 1\} \cup \{|a| < |b|^{-\alpha_2},\ |b| > b_0\}.$$

(ii) *The analytic continuation is given by*

$$\widehat{R}(s) = \frac{1}{\bar{r}} \int_G g \int_X \{(I - L_{f-sr,h})^{-1}\widehat{v}_{-s}\}\widehat{w}_s^T d\mu\, g^T d\nu, \tag{3.1}$$

where $\widehat{v}_s \in F^\theta(X, \mathbb{C}^d)$, $\widehat{w}_s \in L^\infty(X, \mathbb{C}^d)$ are defined as in Proposition 3.3.

(iii) *There exists a constant C such that $|\widehat{R}(s)| \le C\|\phi\|_\theta |\psi|_1 \max\{1, |b|^{\alpha_1}\}$ for all $s = a+ib \in A$.*

Proof. Since R is bounded, $\widehat{R}$ is analytic for $\operatorname{Re} s > 0$. It is well-known (see the appendix) that formula (3.1) is valid for $\operatorname{Re} s > 0$. Since $X_r \times G$ is ergodic and $\operatorname{Fix}(G) = 0$, the usual convexity argument [16, 9] shows that 1 is not in the spectrum of $L_{f,h}$. Similarly, weak mixing of $X_r \times G$ guarantees that 1 is not in the spectrum of $L_{f+ibr,h}$ for all $b \ne 0$. It follows that the right-hand-side of (3.1) continues analytically to an open set that contains the closed half-plane $\operatorname{Re} s \ge 0$.

Proposition 3.2 guarantees that the right-hand-side of (3.1) is analytic on A. Altogether, we have that the right-hand-side continues analytically to the desired region and hence formula (3.1) is valid.

It follows immediately from Proposition 3.2 that

$$|\widehat{R}(s)| \le C'\|\widehat{v}_{-s}\|_\theta |\widehat{w}_s|_1 \max\{1, |b|^{\alpha_1}\},$$

on the region A. Hence, part (iii) follows from Proposition 3.3 since $\operatorname{Re} s$ is bounded on A. □

Proof of Theorem 3.1 By an elementary approximation argument, we may suppose that ψ satisfies the support condition of Proposition 3.4. By condition (a) of Theorem 3.1, we may assume that the conclusions of Proposition 3.2 and hence Lemma 3.5 are valid.

By Taylor's Theorem, for any $k \ge 1$, $R_{\phi,\psi}(t) = P_{k-1}(t) + R_k(t)$ where

$$P_{k-1}(t) = \sum_{j=0}^{k-1} \frac{R^{(j)}_{\phi,\psi}(0)}{j!} t^j, \qquad R_k(t) = \int_0^t R^{(k)}_{\phi,\psi}(s) \frac{(t-s)^{k-1}}{(k-1)!} ds.$$

Note that $R_k(t)$ is the convolution of $R^{(k)}_{\phi,\psi}$ and the function $g(t) = \chi_{[0,\infty)}(t) \cdot t^{k-1}/(k-1)!$ (χ_S stands for the characteristic function of the set S). By Proposition 3.4 and Lemma 3.5(iii),

$$|\widehat{R^{(k)}_{\phi,\psi}}| = |\widehat{R_{(\partial_t^k \phi),\psi}}| \le C_1 \|\partial_t^k \phi\|_\theta |\psi|_1 \max\{1, |b|^{\alpha_1}\} \le C_1 \|\phi\|_{k,\theta} |\psi|_1 \max\{1, |b|^{\alpha_1}\}$$

in the region A, while $|\widehat{g}(s)| = 1/|s|^k \le 1/|b|^k$. Hence

$$|\widehat{R}_k(a+ib)| \le C_1 \|\phi\|_{k,\theta} |\psi|_1 \max\{1, |b|^{\alpha_1 - k}\}.$$

By Lemma 3.5(i), $\widehat{R}$ has an analytic continuation beyond the imaginary axis to a region including the contour Γ defined by

$$a = \max\{-\epsilon, -|b|^{-\alpha_2}\}.$$

Hence, by the Laplace inversion formula,

$$R(t) = C_2 \int_\Gamma e^{st} \widehat{R}(s) ds = C_2(A(t) + B(t))$$

where

$$A(t) = \int_\Gamma e^{st} \widehat{P}_{k-1}(s) ds, \qquad B(t) = \int_\Gamma e^{st} \widehat{R}_k(s) ds.$$

Now, $\widehat{P}_{k-1}(s) = \sum_{j=0}^{k-1} R^{(j)}_{\phi,\psi}(0)/s^{j+1}$ is analytic in the left-half-plane. Moving the contour of integration to $a = -\epsilon$, it is easily verified that $|A(t)| \le C_3 \|\phi\|_{k,\theta} |\psi|_1 e^{-\epsilon t}$. The second term satisfies

$$|B(t)| \le C_4 \|\phi\|_{k,\theta} |\psi|_1 \Big\{ e^{-\epsilon t} + \int_{b_0}^\infty \exp\{-|b|^{-\alpha_2} t\} |b|^{\alpha_1 - k} db \Big\}. \tag{3.2}$$

Since α_1, α_2 are fixed, for any n we can choose k sufficiently large that $R(t)$ converges to zero faster than $1/t^n$. □

Remark 3.6. *In this section, it is clear that $k = k(\alpha, n)$ depends only on α and n. In fact, it suffices that $k > (\alpha+1)(n+2)$. To see this, recall that $\alpha_2 > \alpha_1 + 1$ suffices in Proposition 3.2 and make the substitution $y = |b|^{-\alpha_2} t$ in the integrand in* (3.2).

The constant C is uniform in the Hölder functions f, r, h, as well as the exponents α and n.

4. **Approximate eigenfunctions.** We define the operator $M_{sr,h}$ on $F^\theta(X,\mathbb{C}^d)$ to be

$$M_{sr,h}v = e^{-sr}hv\circ\sigma.$$

Note that $L_{f+sr,h}M_{sr,h} = I$ for all $s\in\mathbb{C}$.

Theorem 4.1. *Suppose that G is abelian. Of the following statements, (c) implies (d).*

- **(c) Approximate eigenfunctions for $L_{f+ibr,h}$:** *For any $\alpha>0$, there exists b arbitrarily large such that $\|(I-L_{f+ibr,h})^{-1}\|_\theta \ge |b|^\alpha$.*
- **(d) Approximate eigenfunctions for $M_{ibr,h}$:** *For any $\alpha>0$ there exists a fixed $\beta>0$, b arbitrarily large, and $w\in F^\theta$ with $|w(x)|=1$ for all $x\in X$ such that*

$$|M_{ibr,h}^{[\beta\ln|b|]}w-w|_\infty \le 1/|b|^\alpha.$$

In the remainder of this section, we prove Theorem 4.1. We require the following standard estimates, of which the first is trivial, the second is the "basic inequality", and the third is the "Ruelle-Perron-Frobenius property". Throughout this section, notation such as C_i and α_j has been chosen to conform with the notation in [7] wherever possible.

Proposition 4.2. *There exist constants $C_6, C_7>0$, $\rho\in(0,1)$ such that for all $v\in F^\theta(X,\mathbb{C}^d)$, $n\ge1$, $b\in\mathbb{R}$, $|b|>1$,*

(i) $|L^n_{f+ibr,h}|_\infty \le 1$,
(ii) $|L^n_{f+ibr,h}v|_\theta \le C_6|b||v|_\infty + \theta^n|v|_\theta$,
(iii) $\|L^n_f v\|_\theta \le \int|v|d\mu + C_7\rho^n\|v\|_\theta$.

Remark 4.3. *If we define $\|v\|_b = \max\{|v|_\infty, |v|_\theta/C_6|b|\}$, then $\|L^n_{f+ibr,h}\|_b\le 2$ for all b.*

Lemma 4.4 (Dolgopyat [7, Lemma 2])**.** *If $|\lambda|=1$, then for all $n,N\ge1$,*

$$\|(\lambda-L_f^n)^{-1}\|_N \le \frac{C_8}{|1-\lambda|}\Big(\ln\frac{1}{|1-\lambda|}+\ln N+n\Big),$$

where $\|v\|_N=\max\{|v|_\infty,|v|_\theta/N\}$.

The following lemma states, roughly speaking, that if $L_{f+ibr,h}$ has an approximate eigenfunction v corresponding to eigenvalue one, then the iterates of v remain close to the unit circle for a long time.

Lemma 4.5. *Let G be a compact connected Lie group (not necessarily abelian) acting on $\mathbb{R}^d$. Suppose that condition (c) of Theorem 4.1 holds. Then, for any $\alpha',\beta>0$, there exists a fixed $\alpha>0$ and b arbitrarily large such that*

(1) *There exists $v\in F^\theta(X,\mathbb{C}^d)$ with $|v|_\infty\le1$ and $|v|_\theta\le C_6|b|$, such that*

$$|(L^n_{f+ibr,h}v)(x)|\ge 1-1/|b|^{\alpha'},\ \text{ for all } x\in X \text{ and all } n,\ 0\le n\le 3\beta\ln|b|,$$

AND

(2) $\|(I-L_{f+ibr,h})^{-1}\|_\theta\ge|b|^\alpha$.

Moreover, $\alpha=\alpha(\alpha')$ can be chosen so that $\alpha\to\infty$ as $\alpha'\to\infty$.

Proof. This is identical to Dolgopyat [7, Lemma 3] (or actually the slightly stronger result obtained by examining the proof). □

For the remainder of this section we restrict to the abelian case, so $G = \mathbb{T}^m$. We have already reduced to the case where G acts irreducibly on $\mathbb{R}^2$. This leads via the approach in Section 3 to a (nonirreducible) action of G on $\mathbb{C}^2$. In proving Theorem 4.1, we may reduce once more to an irreducible action of G on $\mathbb{C}$.

Lemma 4.6. *Let G be a compact connected abelian Lie group acting on $\mathbb{C}$. For any $\alpha'' > 0$, there exists $\alpha', \beta > 0$ such that if condition (1) of Lemma 4.5 holds for some large enough b, then there exist $\overline{w} \in F^\theta(X, \mathbb{C})$ and $\varphi \in \mathbb{R}$ such that $|\overline{w}(x)| = 1$ for all $x \in X$, $|\overline{w}|_\theta \le 4C_6|b|$ and*

$$|M^{\bar{n}}_{ibr,h}\overline{w} - e^{i\varphi}\overline{w}|_\infty \le 8/|b|^{\alpha''},$$

where $\bar{n} = [\beta \ln |b|]$.

Proof. Given $\alpha'' > 1$, choose α' and β so that

$$\beta = (-\alpha'' - 2)/\ln \theta, \qquad \alpha' = 2\alpha'' + \beta|f|_\infty.$$

For b we require that $|b| \ge \max\{4C_6, 1/\theta\}$ and $|b|^{\alpha'} \ge 2$.

Choose v as in Lemma 4.5(1) and write $v = sw$ where $0 < s(x) \le 1$ and $|w(x)| = 1$ for all x. Similarly, write $L^{\bar{n}}_{f+ibr,h}v = \bar{v} = \bar{s}\bar{w}$. By definition,

$$\bar{s}(x)\overline{w}(x) = \sum_{\sigma^{\bar{n}}y=x} e^{(f+ibr)_{\bar{n}}(y)} h_{\bar{n}}(y)^{-1} s(y)w(y).$$

Hence

$$\sum_{\sigma^{\bar{n}}y=x} e^{f_{\bar{n}}(y)}\big[1 - e^{ibr_{\bar{n}}(y)} h_{\bar{n}}(y)^{-1} s(y)w(y)(\overline{w}(x))^{-1}\big] = 1 - \bar{s}(x) \le 1/|b|^{\alpha'}.$$

Since each term has positive real part, we deduce that

$$e^{f_{\bar{n}}(y)}\,\mathrm{Re}\big[1 - e^{ibr_{\bar{n}}(y)} h_{\bar{n}}(y)^{-1} w(y)(\overline{w}(x))^{-1}\big] \le 1/|b|^{\alpha'},$$

for each x, y with $\sigma^{\bar{n}}y = x$. Now, $e^{-f_{\bar{n}}(x)} \le e^{\bar{n}|f|_\infty} \le e^{\beta|f|_\infty \ln|b|} = |b|^{\beta|f|_\infty}$, for all $x \in X^+$, so that

$$\mathrm{Re}(1 - e^{ibr_{\bar{n}}(y)} h_{\bar{n}}(y)^{-1} w(y)(\overline{w}(x))^{-1}) \le 1/|b|^{2\alpha''},$$

Hence,

$$|e^{ibr_{\bar{n}}(y)} h_{\bar{n}}(y)^{-1} w(y) - \overline{w}(x)| \le 2/|b|^{\alpha''}. \tag{4.3}$$

Similarly, defining $L^{2\bar{n}}_{f+ibr,h}v = \bar{\bar{v}} = \bar{\bar{s}}\,\overline{\overline{w}}$,

$$|e^{ibr_{\bar{n}}(y)} h_{\bar{n}}(y)^{-1} \overline{w}(y) - \overline{\overline{w}}(x)| \le 2/|b|^{\alpha''}, \tag{4.4}$$

for all $x, y \in X^+$ with $\sigma^{\bar{n}}y = x$.

Fix $y_0 \in X^+$, and define $w(y_0) = e^{i\varphi_1}$, $\overline{w}(y_0) = e^{i\varphi_2}$. The choice of β guarantees that $\theta^{\bar{n}} \le 1/|b|^{\alpha''+1}$. Hence $\bar{n}$ is sufficiently large that given any $x \in X^+$ there exists y with $\sigma^{\bar{n}}y = x$ such that $d_\theta(y, y_0) < 1/|b|^{\alpha''+1}$. Since $|w|_\theta, |\overline{w}|_\theta < 4C_6|b|$ (because $|\bar{v}(u)|, |v(u)| \ge 1/2$ at each u), it follows that

$$|w(y) - e^{i\varphi_1}| \le 1/|b|^{\alpha''}, \qquad |\overline{w}(y) - e^{i\varphi_2}| \le 1/|b|^{\alpha''}.$$

Hence, for this choice of y,

$$|e^{ibr_{\bar{n}}(y)} h_{\bar{n}}(y)^{-1} e^{i\varphi_1} - \overline{w}(x)| \le 3/|b|^{\alpha''}, \qquad |e^{ibr_{\bar{n}}(y)} h_{\bar{n}}(y)^{-1} e^{i\varphi_2} - \overline{\overline{w}}(x)| \le 3/|b|^{\alpha''},$$

so that $|\overline{w}(x) - e^{i\varphi}\overline{\overline{w}}(x)| \le 6/|b|^{\alpha''}$ for all $x \in X^+$, where $\varphi = \varphi_1 - \varphi_2$.

Substituting into (4.4) yields

$$|e^{ibr_{\bar{n}}(y)}h_{\bar{n}}(y)^{-1}\overline{w}(y) - e^{-i\varphi}\overline{w}(x)| \le 8/|b|^{\alpha''}$$

for all x, y with $\sigma^{\bar{n}}y = x$. This completes the proof. □

Corollary 4.7. *Suppose that condition (c) of Theorem 4.1 holds. Then, for any $\alpha > 0$, there exists a fixed $\beta > 0$ and there exists b arbitrarily large such that*

(1) *There exists $\overline{w} \in F^\theta(X, \mathbb{C})$ with $|\overline{w}(x)| = 1$ for all $x \in X$ and $|\overline{w}|_\theta \le 4C_6|b|$, and there exists $\varphi \in \mathbb{R}$, such that*

$$|M^{\bar{n}}_{ibr,h}\overline{w} - e^{i\varphi}\overline{w}|_\infty \le 1/|b|^\alpha,$$

where $\bar{n} = [\beta \ln |b|]$, AND

(2) $\|(I - L_{f+ibr,h})^{-1}\|_\theta \ge |b|^\alpha$.

Proof. The estimate for $M_{ibr,h}$ remains true with α'' replaced by α, and α can be assumed to be arbitrarily large. □

For each $N > 1$, we define $\|v\|_N = |v|_\infty + |v|_\theta/N$.

Proposition 4.8. *Suppose that condition (1) in Corollary 4.7 is valid. Define $Kv = e^{-i\varphi}\overline{w}L^{\bar{n}}_f(v\overline{w}^{-1})$. Then there exists a constant $C_{20} > 0$ such that*

$$\|L^{\bar{n}}_{f+ibr,h} - K\|_N \le C_{20}/|b|^\alpha,$$

where $N = |b|^{\alpha+1}$.

Proof. Compute that

$$\begin{aligned}
([L^{\bar{n}}_{f+ibr,h} - K]v)(x) &= \sum_{\sigma^{\bar{n}}y=x} e^{f_{\bar{n}}(y)}\big(e^{ibr_{\bar{n}}(y)}h_{\bar{n}}(y)^{-1} - e^{-i\varphi}\overline{w}(\sigma^{\bar{n}}y)/\overline{w}(y)\big)v(y) \\
&= \sum_{\sigma^{\bar{n}}y=x} e^{f_{\bar{n}}(y)}e^{ibr_{\bar{n}}(y)}h_{\bar{n}}(y)^{-1}\big(\overline{w}(y) - e^{-i\varphi}e^{-ibr_{\bar{n}}(y)}h_{\bar{n}}(y)\overline{w}(\sigma^{\bar{n}}y)\big)v(y)/\overline{w}(y) \\
&= \sum_{\sigma^{\bar{n}}y=x} e^{f_{\bar{n}}(y)}e^{ibr_{\bar{n}}(y)}h_{\bar{n}}(y)^{-1}\big((I - e^{-i\varphi}M^{\bar{n}}_{ibr,h})\overline{w}\big)(y)v(y)/\overline{w}(y)
\end{aligned}$$

It follows immediately that

$$|(L^{\bar{n}}_{f+ibr,h} - K)v|_\infty \le |v|_\infty/|b|^\alpha. \tag{4.5}$$

The computation of $|(L^{\bar{n}}_{f+ibr,h} - K)v|_\theta$ leads to six terms involving $|f|_\theta$, $|b||r|_\theta$, $|h|_\theta$, $|(I - M^{\bar{n}}_{ibr,h})\overline{w}|_\theta$, $|v|_\theta$ and $|1/\overline{w}|_\theta$. These terms have estimates (up to a universal constant) of the form

$$\begin{gathered}
|f|_\theta(1-\theta)^{-1}|v|_\infty/|b|^\alpha, \quad |r|_\theta(1-\theta)^{-1}|v|_\infty/|b|^{\alpha-1}, \quad |h|_\theta(1-\theta)^{-1}|v|_\infty/|b|^\alpha, \\
\big(1 + |b||r|_\theta(1-\theta)^{-1}\big)|v|_\infty + 2|\overline{w}|_\theta|v|_\infty, \quad |v|_\theta/|b|^\alpha, \quad |1/\overline{w}|_\theta|v|_\infty/|b|^\alpha
\end{gathered}$$

Note that $|1/\overline{w}|_\theta \le |\overline{w}|_\theta \le C_6|b|$. The fourth and fifth terms are the crucial ones, demonstrating that

$$|(L^{\bar{n}}_{f+ibr,h} - K)v|_\theta \le C_{17}|b||v|_\infty + |v|_\theta/|b|^\alpha. \tag{4.6}$$

By (4.5) and (4.6),

$$\|L^{\bar{n}}_{f+ibr,h} - K\|_N \le 1/|b|^\alpha + C_{17}|b|/N,$$

for all $N \ge 1$. The result follows. □

Proof of Theorem 4.1 Choose $\alpha^* = (\alpha-2)/2$. There exists b arbitrarily large such that conditions (1) and (2) of Corollary 4.7 are simultaneously valid. Let $\overline{w}$, $\bar{n}$, K and φ be as in condition (1). In particular, by Proposition 4.8,

$$\|L^{\bar{n}}_{f+ibr,h} - K\|_N \le C_{20}/|b|^{\alpha^*}, \tag{4.7}$$

where $N = |b|^{\alpha^*+1}$.

Choosing b sufficiently large, we may suppose that $N > 4C_6|b|$. We claim that

$$|e^{i\varphi}-1| \le 1/|b|^{\alpha^*-2} = 1/|b|^{(\alpha-6)/2}. \tag{4.8}$$

Suppose for contradiction that $|e^{i\varphi}-1| \ge 1/|b|^{\alpha^*-2}$. Then by Lemma 4.4,

$$\|(I - e^{-i\varphi}L^{\bar{n}}_f)^{-1}\|_N \le C_{18}|b|^{\alpha^*-1}.$$

Moreover, multiplication by $\overline{w}$ has norm less than 2 (since $N > 2C_6|b|$), and so

$$\|(I-K)^{-1}\|_N \le C_{19}|b|^{\alpha^*-1}. \tag{4.9}$$

Substituting (4.7),(4.9) into the identity $A^{-1} = B^{-1} + A^{-1}(B-A)B^{-1}$ yields $\|(I - L^{\bar{n}}_{f+ibr,h})^{-1}\|_N \le C_{21}|b|^{\alpha^*-1}$ and so

$$\|(I - L^{\bar{n}}_{f+ibr,h})^{-1}\|_\theta \le C_{21}|b|^{2\alpha^*}.$$

It follows from the identity $(I-A)^{-1} = (I + A + \cdots + A^{n-1})(I-A^n)^{-1}$ that

$$\|(I - L_{f+ibr,h})^{-1}\|_\theta \le C_{22}|b|^{2\alpha^*+1}\ln|b| = C_{22}|b|^{\alpha-1}\ln|b|.$$

This contradicts condition (2) of Corollary 4.7, thus proving the claim.

Combining (4.8) with condition (1) of Corollary 4.7 implies that $|M^{\bar{n}}_{ibr,h}\overline{w} - \overline{w}|_\infty \le 2/|b|^{(\alpha-6)/2}$. Since α is arbitrary, we have proved the theorem. □

Remark 4.9. *In this section, the constants C_j are uniform in f, r and h (in the fixed Hölder topology). The exponent α depends continuously on f (via Lemma 4.5). The same is true of the magnitude of b in the expression "there exists b arbitrarily large".*

5. **Proof of Theorems 2.2 and 2.3.**

In this section, we prove first Theorem 2.3 (the abelian case) and then Theorem 2.2 (the nonabelian case).

5.1. **The abelian case.**

Lemma 5.1. *Suppose that G is abelian. Consider suspensions $X_r \times G$ of one-sided subshifts of finite type, defined by the roof function $r \in F^\theta(X,\mathbb{R})$ and the cocycle $h \in F^\theta(X,G)$. Suppose that X possesses four fixed points x_j, $j = 1,\dots,4$, and write $r(x_j) = \ell_j$, $h(x_j) = (e^{i\theta_{1j}},\dots,e^{i\theta_{dj}})$. Then for almost every $\ell_j, \theta_{ij} \in \mathbb{R}$, the operator $M_{ibr,h}$ does not have approximate eigenfunctions in the sense of Theorem 4.1(d).*

Proof. Without loss, we may restrict to the case when $\ell_j \ne 0$ for $j = 1,\dots,4$ and $\ell_3\theta_4 - \ell_4\theta_3 \ne 0$. Suppose that $M_{ibr,h}$ has approximate eigenfunctions. Fix $\alpha > 4$. Then there is a sequence of functions $w_k \in F^\theta(X,\mathbb{R})$ with $|w_k(x)| \equiv 1$ satisfying

$$|M^{\bar{n}_k}_{ib_k r,h} w_k - w_k|_\infty = O(|b_k|^{-\alpha}),$$

where $|b_k| \to \infty$ as $k \to \infty$ and $\bar{n}_k = [\beta \ln |b_k|]$.

Since the irreducible representations of G are one-dimensional, we may suppose without loss that $d = 1$. Inserting the data for the four fixed points yields the conditions

$$b_k\bar{n}_k\ell_j + \bar{n}_k\theta_j - 2\pi m_j = O(|b_k|^{-\alpha}), \tag{5.10}$$

where $m_j = m_j(b_k) \in \mathbb{Z}$ for $j = 1, \ldots, 4$. In particular,

$$|m_j| = O(|b_k| \ln |b_k|), \tag{5.11}$$

for each j. Write $m = (m_1, m_2, m_3, m_4)$ and $\omega = (-\theta_2, \theta_1, \tau\theta_4, -\tau\theta_3)$ where $\tau = (\ell_1\theta_2 - \ell_2\theta_1)/(\ell_3\theta_4 - \ell_4\theta_3)$. This choice of τ together with (5.10) leads to cancellations in $\langle m, \omega\rangle$ and we compute that $\langle m, \omega\rangle = O(|b_k|^{-\alpha})$. Let $\alpha' \in (4, \alpha)$. By (5.11), $\langle m, \omega\rangle = O(|m|^{-\alpha'})$. This Diophantine condition holds for $|m| \to \infty$ and hence defines a set of measure zero of $\omega \in \mathbb{R}^4$ since $\alpha' > 4$. It follows easily that the corresponding set of $\ell_j, \theta_j \in \mathbb{R}^8$ has measure zero. □

Corollary 5.2. *Typically, suspensions $X_r \times G$ of toral extensions of one-sided subshifts of finite type are rapid mixing in the sense of Theorem 3.1(b).*

Proof. By Lemma 5.1, typically the condition of Theorem 4.1(d) fails. This implies the failure of Theorem 4.1(c), equivalently the validity of Theorem 3.1(a) and hence the validity of Theorem 3.1(b). □

5.2. **The nonabelian case.** In this section, we suppose that G is a compact connected nonabelian Lie group acting irreducibly and faithfully on $\mathbb{R}^d$ (so $d \geq 3$). In particular, if ξ is a nonzero vector in $\mathbb{C}^d$, then the subspace $\mathbb{C}\{\xi\}$ is not invariant under G. Moreover:

Lemma 5.3. *Suppose that G acts irreducibly and unitarily on $\mathbb{C}^d$, $d \geq 2$. Let $D \subset G \times G$ consist of those elements (g_1, g_2) such that there exists a unit vector ξ such that $g_j\xi \in \mathbb{C}\{\xi\}$ for $j = 1, 2$. Then the complement of D is open and of full measure in $G \times G$.*

Proof. Clearly, D is closed. Almost all pairs of elements (g_1, g_2) topologically generate G. If such a pair lies in D, then the corresponding unit vector ξ has the property that $\mathbb{C}\{\xi\}$ is an invariant subspace for G, contradicting irreducibility with $d \geq 2$. Hence almost all pairs do not lie in D. □

Theorem 2.2 is an immediate consequence of the next Proposition.

Proposition 5.4. *Suppose that G acts irreducibly and unitarily on $\mathbb{C}^d$, $d \geq 2$, and $X_r \times G$ is mixing, where h, r, f are θ-Hölder.*

1. *There are group elements $(g_1, g_2) \in G \times G$, determined only by the cocycle h, such that if statement (1) of Lemma 4.5 is valid for fixed $\beta > 1/(-\ln\theta)$, $\alpha' > \beta|f|_\infty$ and arbitrarily large b, then $(g_1, g_2) \in D$.*
2. *The pair (g_1, g_2) varies continuously with h in the Hölder topology. Moreover, $(g_1, g_2) \notin D$ for an open, dense and prevalent set of cocycles $h \in F^\theta(X, G)$.*
3. *If $(g_1, g_2) \notin D$, then the suspension flow is rapidly mixing.*

Proof. We describe first the elements g_1, g_2 and prove parts (2) and (3), assuming (1). We then prove (1).

Suppose for simplicity that $x \in X$ is a fixed point with symbol $x = x_0x_0\cdots = x_0^\infty$. Let W_1, W_2 be words of the same length (q say) such that $x_0W_jx_0$ are admissible sequences for $j = 1, 2$, and such that the three words W_1, W_2, x_0^q are distinct. Define $E_m = \{x, \sigma^k(x_0^mW_jx_0^\infty), j = 1, 2, k \geq 0\}$ for $m \geq 0$ and let $E = \bigcup E_m$. By construction, all points in E are isolated except for x. Moreover, $W_jx_0^\infty$ lies in E_m for all m and is achieved precisely once for each m.

Set $g_1 = \lim_{m\to\infty} h_m(x)^{-1}h_m(x_0^{m-q}W_1x_0^\infty)$. To see that the limit exists, note that

$$h_m(x)^{-1}h_m(x_0^{m-q}W_1x_0^\infty) = h_q(x)^{-1}h_{m-q}(x)^{-1}h_{m-q}(x_0^{m-q}W_1x_0^\infty)h_q(W_1x_0^\infty),$$

and since h is Hölder, the sequence $h_m(x)^{-1}h_m(x_0^mW_1x_0^\infty)$ is Cauchy. Similarly, define $g_2 = \lim_{m\to\infty} h_m(x)^{-1}h_m(x_0^{m-q}W_2x_0^\infty)$.

Note that g_1 and g_2 depend only on the underlying skew product $\sigma_h : X\times G \to X\times G$, and in particular on the values of h at points in the set E defined above. We have complete control over g_1 and g_2, by perturbing h at the points $W_jx_0^\infty$ say. Hence, by Lemma 5.3, $(g_1, g_2) \notin D$ for a dense (and clearly prevalent) set of $h \in F^\theta(X, G)$. A standard argument using Hölder continuity shows that g_1 and g_2 depend continuously on h. Hence $(g_1, g_2) \notin D$ for an open and dense, and prevalent, set of $h \in F^\theta(X, G)$. This proves (2).

Part (1) of the Proposition implies that either rapid mixing holds, or $(g_1, g_2) \in D$. Indeed, suppose that $X_r\times G$ is not rapid mixing. Then it follows from Theorem 3.1 that there exist approximate eigenfunctions, hence condition (c) of Theorem 4.1 holds. It follows from Lemma 4.5 that statement (1) of Lemma 4.5 is valid, hence $(g_1, g_2) \in D$. This proves (3).

We now proceed with the proof of (1). Given $\beta > 1/(-\ln\theta)$, $\alpha' > \beta|f|_\infty$, choose b arbitrarily large and $v \in F^\theta(X, \mathbb{C}^d)$ as in Lemma 4.5. Write $\bar n = [\beta \ln |b|]$. Keeping with the notation used in the proof of Lemma 4.6, denote by $w(x) = v(x)/|v(x)|$ the "phase" of $v : X \to \mathbb{C}^d \setminus \{0\}$.

It follows from an early stage of Lemma 4.6 (see relation (4.3), which can be obtained for nonabelian G as well) that if $\sigma^{\bar n}y' = \sigma^{\bar n}y''$, then

$$|e^{ibr_{\bar n}(y')}h_{\bar n}(y')^{-1}w_b(y') - e^{ibr_{\bar n}(y'')}h_{\bar n}(y'')^{-1}w_b(y'')| \le 4/|b|^{\alpha''},$$

where $\alpha' = 2\alpha'' + \beta|f|_\infty$ and we have chosen to stress the dependence of w on b. Taking $y' = x$ and $y'' = x_0^{\bar n-q}W_1x_0^\infty$,

$$|h_{\bar n}(x_0^{\bar n-q}W_1x_0^\infty)h_{\bar n}(x)^{-1}w_b(x) - e^{ib[r_{\bar n}(x_0^{\bar n-q}W_1x_0^\infty)-r_{\bar n}(x)]}w_b(x_0^{\bar n-q}W_1x_0^\infty)| \le 4/|b|^{\alpha''}.$$

Now,

$$|w_b(x_0^{\bar n-q}W_1x_0^\infty) - w_b(x)| \le |w_b|_\theta\theta^{\bar n-q} \le \widetilde C_6|b|\theta^{\beta\ln|b|} = \widetilde C_6|b|^{1+\beta\ln\theta},$$

which converges to zero by the choice of β. Therefore

$$|h_{\bar n}(x_0^{\bar n-q}W_1x_0^\infty)h_{\bar n}(x)^{-1}w_b(x) - e^{ib[r_{\bar n}(x_0^{\bar n-q}W_1x_0^\infty)-r_{\bar n}(x)]}w_b(x)| \to 0 \tag{5.12}$$

as $b\to\infty$. Passing to a subsequence of b's, we may suppose that $h_{\bar n}(x)^{-1}w_b(x)$ converges to a fixed unit vector ξ. Then (5.12) implies that

$$h_{\bar n}(x)^{-1}h_{\bar n}(x_0^{\bar n-q}W_1x_0^\infty)\xi \to \mathbb{C}\{\xi\}$$

as $b\to\infty$, hence g_1 preserves the space $\mathbb{C}\{\xi\}$. Starting with W_2 instead of W_1, we obtain that g_2 preserves the subspace $\mathbb{C}\{\xi\}$ as well, and conclude that $(g_1, g_2) \in D$, as claimed. □

Appendix A. Laplace transform of the correlation function. In this appendix, we derive formula (3.1) for $\widehat{R}$ for $\operatorname{Re} s > 0$. Suppose that $\phi = gv, \psi = gw : X_r \times G \to \mathbb{R}^d$ are equivariant bounded observations with $\int_{X_r \times G} \phi \, dm_r = 0$. We use the labeling $y \in X \times G$, $(y, b) \in (X \times G)_r = X_r \times G$.

Proposition A.1. *Let*

$$R(t) = \sum_{n=0}^{\infty} \int_{X_r \times G} \phi(y,a) \Big(\int_0^{r(\sigma^n y)} \psi^T(\sigma_h^n y, b) \delta(t + a - r_n(y) - b) db \Big) dm_r.$$

where the sum converges absolutely and is bounded by $|\phi|_\infty |\psi|_\infty$.

Then $R(t) = \rho(t)$ *for* $t \geq 0$, *and* $R(t) = 0$ *for* $t < -|r|_\infty$.

Proof. Consider the expression

$$E(y, a, t) = \sum_{n=0}^{\infty} \int_0^{r(\sigma^n y)} \psi(\sigma_h^n y, b) \delta(t + a - r_n(y) - b) db,$$

defined for $y \in X \times G$, $a \in [0, r(y)]$, $t \in \mathbb{R}$.

For fixed y, a, the n'th term in the sum is nonzero if and only t lies in the interval $[r_n(y) - a, r_{n+1}(y) - a]$ in which case the n'th term is given by $\psi(\sigma_h^n y, t + a - r_n(y))$.

Hence, when $t \geq 0$ and $t \in [r_n(y) - a, r_{n+1}(y) - a]$, the n'th term coincides with $\psi(T_t(y, a))$. It follows that $E(y, a, t) = \psi(T_t(y, a))$, provided that $t \geq 0$ and $t \neq r_n(y) - a$ for some n. At the same time, if $t < -|r|_\infty$, then $t + a < 0$ for all $(y, a) \in X_r \times G$ so that $E(y, a, t) = 0$.

Now $R(t) = \int_{X_r} \phi(y, a) E^T(y, a, t) \, dm_r$ so it is immediate that $R(t) = 0$ for $t < -|r|_\infty$. If $t \geq 0$, then $E(y, a, t) = \psi(T_t(y, a))$ for almost all (y, a) so that $R(t) = \rho(t)$. □

Since R is bounded, the Laplace transform $\widehat{R}$ is defined for all $\operatorname{Re} s > 0$. Let $U : L^2_G(X \times G, \mathbb{R}^d) \to L^2_G(X \times G, \mathbb{R}^d)$ be the isometry defined by $U\phi = \phi \circ \sigma_h$, and let U^* be the adjoint of U. We compute that

$$\begin{aligned}
\widehat{R}(s) &= \int_0^\infty e^{-ts} R(t) dt \\
&= \sum_{n=0}^{\infty} \int_{X_r \times G} \phi(y,a) \Big(\int_0^{r(\sigma^n y)} \psi^T(\sigma_h^n y, b) e^{-s(-a + r_n(y) + b)} db \Big) \, dm_r \\
&= \frac{1}{\bar{r}} \sum_{n=0}^{\infty} \int_{X \times G} e^{-s r_n} \widehat{\phi}_{-s} U^n \widehat{\psi}_s^T dm = \frac{1}{\bar{r}} \sum_{n=0}^{\infty} \int_{X \times G} (U^*)^n (e^{-s r_n} \widehat{\phi}_{-s}) \widehat{\psi}_s^T dm,
\end{aligned}$$

where we use the notation $\widehat{f}_s(y) = \int_0^{r(y)} f(y, a) e^{-sa} da$ for $f : Y_r \to \mathbb{R}^m$ and $y \in Y$.

Now, we specialise to the context of suspensions of subshifts of finite type. Recall [9] that $U^* = g L_{f,h} g^{-1}$ and so $(U^*)^n (e^{-s r_n} \widehat{\phi}_{-s}) = g L^n_{f - sr, h} \widehat{v}_{-s}$. Hence

$$\widehat{R}(s) = \frac{1}{\bar{r}} \sum_{n=0}^{\infty} \int_G g \int_X \{ L^n_{f-sr,h} \widehat{v}_{-s} \} \widehat{w}_s^T d\mu \, g^T d\nu. \tag{A.13}$$

Let $\tau(A)$ denote the spectral radius of A. It follows from [18] that $\tau(L_{f-sr,h}) \leq \tau(L_{f - \operatorname{Re}(s) r})$. On the other hand, it is evident that $\tau(L_{f-ar}) < \tau(L_f) = 1$ for all $a > 0$ (since r is strictly positive). Hence, for $\operatorname{Re} s > 0$,

$$(I - L_{f-sr,h})^{-1} \widehat{v}_{-s} = \sum_{n=0}^{\infty} L^n_{f-sr,h} \widehat{v}_{-s},$$

and therefore (A.13) becomes

$$\widehat{R}(s) = \frac{1}{\bar{r}} \int_G g \int_X \{(I - L_{f-sr,h})^{-1} \widehat{v}_{-s}\} \widehat{w}_s^T d\mu \, g^T d\nu.$$

This proves the validity of formula (3.1) for $\mathrm{Re}\, s > 0$.

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