M3S4/M4S4: Applied probability: 2007-8
Assessed Coursework 2: SOLUTIONS

1. Matrices with non-negative entries in which the rows sum to 1 are sometimes called *stochastic matrices*. The transition matrices of Markov chains are stochastic matrices.

   (a) Show that stochastic matrices have at least one eigenvalue equal to 1.

   Answer:

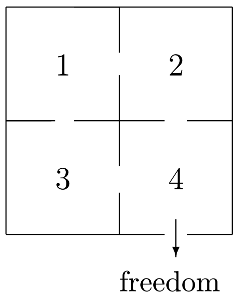
   From the definition, stochastic matrices must satisfy $P\mathbf{1} = 1 \cdot \mathbf{1}$ so they have eigenvalue 1. $\boxed{2}$

   (b) Hence show that if P is a stochastic matrix, then so is P^n .

   Answer:

   $P^n\mathbf{1} = P^{n-1}P\mathbf{1} = P^{n-1}\mathbf{1} = P^{n-2}P\mathbf{1} = \ldots = \mathbf{1}$ $\boxed{2}$

2. A rat runs through the following maze:



   The rat starts in a given cell and at each step it moves to a neighbouring cell (chosen with equal probability from those available, independently of the past). It continues moving between cells in this way until it escapes to the outside. Assume that the outside (freedom) is denoted state 0 (and can only be reached from cell 4). We assume that once the rat has escaped it remains escaped forever.

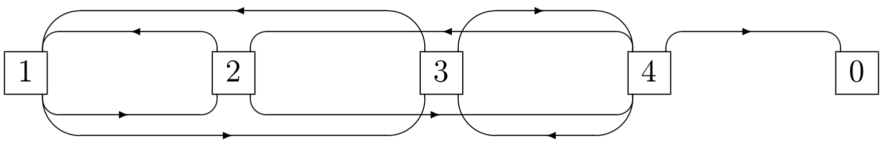
   (a) Determine the transition matrix; draw the transition diagram and find the communicating classes associated with this chain. **Answer:**

   Transition matrix:

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

$\boxed{4}$

Transition diagram:



Communicating classes: $\{1,2,3,4\}$ (open), $\{F\}$ (closed). $\boxed{2}$

(b) fairly obvious that stationary distribution is $(0,0,0,0,1)$ as the rat will eventually escape, this is also the solution to $\boldsymbol{\pi} = \boldsymbol{\pi}P$, this is unique as the chain has only one closed communicating class. $\boxed{2}$ $\boxed{2}$

(c) Find m_i: the expected number of steps taken to escape given that the rat starts in cell i, for $i = 1,2,3,4$.

Conditioning on the first step we have

$$\begin{aligned} m_1 &= 1 + \frac{1}{2}m_2 + \frac{1}{2}m_3 && (1)\\ m_2 &= 1 + \frac{1}{2}m_1 + \frac{1}{2}m_4 && (2)\\ m_3 &= 1 + \frac{1}{2}m_1 + \frac{1}{2}m_4 && (3)\\ m_4 &= 1 + \frac{1}{3}m_2 + \frac{1}{3}m_3 && (4) \end{aligned}$$

giving

$$\begin{aligned} m_2 &= m_3 \quad \text{from (2) and (3)} \\ \Rightarrow m_4 &= 1 + \frac{2}{3}m_2 \quad \text{from (4)} && (5)\\ \Rightarrow m_1 &= 1 + m_2 \quad \text{from (1)} && (6)\\ \Rightarrow m_2 &= 3 + m_4 \quad \text{subs for } m_1 \text{ from (6) in (2)} && (7)\\ \Rightarrow m_2 &= 12 \quad \text{subs for } m_4 \text{ from (5) in (7)} && (8) \end{aligned}$$

giving $m_1 = 13,\ m_2 = 12,\ m_3 = 12,\ m_4 = 9.$

$\boxed{6}$