7 Continuous time Markov processes

$X(t)$ develops in continuous time ($t \geq 0$) (state space still discrete).

Markov Property

$$\mathrm{P}(X(t) = j \,|\, X(t_1) = i_1, X(t_2) = i_2, \ldots, X(t_n) = i_n) = \mathrm{P}(X(t) = j \,|\, X(t_n) = i_n)$$

for any $n > 1$ and $0 \leq t_1 < t_2 < \ldots < t_n < t$.

Time Homogeneity

For $0 \leq s < t$,

$$\mathrm{P}(X(t) = j \,|\, X(s) = i) = \mathrm{P}(X(t-s) = j \,|\, X(0) = i).$$

Define

$$\begin{aligned} p_{ij}(s, t+s) &= \mathrm{P}(X(t+s) = j \,|\, X(s) = i) \\ p_{ij}(0, t) &= \mathrm{P}(X(t) = j \,|\, X(0) = i) = p_{ij}(t) \end{aligned}$$

$$P(t) = \left(\quad p_{ij}(t) \quad \right) \qquad t \geq 0$$

$$p_{ij}(0) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad P(0) = I \quad \text{(identity matrix)}$$

Continuous time analogue of C-K equations:

$$p_{ij}(0, s+t) = \sum_k p_{ik}(0, s) p_{kj}(s, s+t) \qquad 0 < s < t,$$

alternatively,

$$P(s+t) = P(s)P(t)$$

Condition: $p_{ij}(t)$ continuous and differentiable.

$$\begin{aligned} q_{ij} &= \left. \frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(t) \right|_{t=0} \qquad \text{right derivatives} \\ &= \lim_{h \to 0} \left(\frac{p_{ij}(h) - p_{ij}(0)}{h} \right) \qquad \text{from above.} \end{aligned}$$

Therefore,

$$p_{ij}(h) = \begin{cases} 1 + hq_{ii} + o(h) & i = j \\ hq_{ij} + o(h) & i \neq j \end{cases} \quad \text{small } h$$

Check

$i = j$

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t} p_{ii}(t)\right|_{t=0} &= \lim_{h\to 0}\left(\frac{p_{ii}(h)-p_{ii}(0)}{h}\right) \\
&= \lim_{h\to 0}\left(\frac{1+hq_{ii}+o(h)-1}{h}\right) \\
&= \lim_{h\to 0}\left(q_{ii}+\frac{o(h)}{h}\right) \\
&= q_{ii}
\end{aligned}$$

$i \neq j$

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(t)\right|_{t=0} &= \lim_{h\to 0}\left(\frac{p_{ij}(h)-p_{ij}(0)}{h}\right) \\
&= \lim_{h\to 0}\left(\frac{hq_{ij}+o(h)-0}{h}\right) \\
&= \lim_{h\to 0}\left(q_{ij}+\frac{o(h)}{h}\right) \\
&= q_{ij}
\end{aligned}$$

Let Q be the matrix of q_{ij}.

Note

$$\begin{aligned}
\sum_j p_{ij}(t) &= 1 \quad \forall i \\
\left.\frac{\mathrm{d}}{\mathrm{d}t}\sum_j p_{ij}(t)\right|_{t=0} &= \left.\frac{\mathrm{d}}{\mathrm{d}t} 1\right|_{t=0} \\
\Rightarrow \sum_j q_{ij} &= 0 \qquad \text{i.e. rows of } Q \text{ sum to zero}
\end{aligned}$$

Also

$$\begin{aligned}
q_{ii} &\leq 0 & & (p_{ii}(h) = 1 + \; hq_{ii} + o(h), h \geq 0) \\
q_{ij} &\geq 0 \quad i \neq j & & (p_{ij}(h) = \; hq_{ij} + o(h), h \geq 0)
\end{aligned}$$

A continuous time Markov process may be specified by stating its Q matrix.

Description of process

Let T_i be the time spent in state i before moving to another state.

For any $i \geq 1$

$$\begin{aligned}
\mathrm{P}(T_i > t) &= \mathrm{P}(X(s) = i,\ 0 \leq s \leq t \,|\, X(0) = i) \\
&= \mathrm{P}\left(X(s) = i,\ 0 \leq s \leq \frac{t}{n} \,\middle|\, X(0) = i\right) \times \mathrm{P}\left(X(s) = i, \frac{t}{n} \leq s \leq \frac{2t}{n} \,\middle|\, X\left(\frac{t}{n}\right) = i\right) \\
&\quad \times \ldots \times \mathrm{P}\left(X(s) = i, \frac{(n-1)t}{n} \leq s \leq t \,\middle|\, X\left(\frac{(n-1)t}{n}\right) = i\right)
\end{aligned}$$

By time homogeneity,

$$\begin{aligned}
\mathrm{P}(T_i > t) &= \mathrm{P}\left(X(s) = i, 0 \leq s \leq \frac{t}{n} \,\middle|\, X(0) = i\right)^n \quad \forall n \\
&= \lim_{n\to\infty}\left[\mathrm{P}\left(X(s) = i, 0 \leq s \leq \frac{t}{n} \,\middle|\, X(0) = i\right)\right]^n \\
&= \lim_{n\to\infty}\left[1 + q_{ii}\frac{t}{n} + o\left(\frac{t}{n}\right)\right]^n \\
\Rightarrow \mathrm{P}(T_i > t) &= e^{tq_{ii}} \\
\Rightarrow \mathrm{P}(T_i \leq t) &= 1 - e^{tq_{ii}}.
\end{aligned}$$

[Recall, $X \sim \mathit{Exponential}(\lambda)$, $\mathrm{P}(X \leq t) = 1 - \exp(-\lambda t)$,and $\mathrm{E}(X) = 1/\lambda$]

Thus, T_i is exponentially distributed with mean $-1/q_{ii}$.

Suppose the process is known to change state at time t, where does it jump to?

Probability jump is from i to j is

$$\begin{aligned}
&\lim_{h\to 0}\mathrm{P}(X(t+h) = j \,|\, X(t) = i, X(t+h) \neq i) \\
&\lim_{h\to 0}\frac{\mathrm{P}(X(t+h) = j \,|\, X(t) = i)}{\mathrm{P}(X(t+h) \neq i \,|\, X(t) = i)} \qquad (j \neq i) \\
= &\lim_{h\to 0}\frac{p_{ij}(h)}{\sum_{k\neq i} p_{ik}(h)} = \frac{q_{ij}}{\sum_{k\neq i} q_{ik}} \\
= &-\frac{q_{ij}}{q_{ii}}
\end{aligned}$$

[As $\sum_k q_{ik} = 0$; $q_{ii} = -\sum_{k\neq i} q_{ik}$]

Therefore, the process acts like this:

It remains in state i for a period exponentially distributed with mean $\boxed{-1/q_{ii}}$, and then jumps to another state. The state is j ($\neq i$) with probability $\boxed{-q_{ij}/q_{ii}}$.

It then stays in this state for a period exponentially distributed with mean $\boxed{-1/q_{jj}}$ etc.

Thus the process only depends on the Q matrix.

7.1 Embedded Markov Chain

If the process is observed only at jumps, then a Markov chain is observed with transition matrix

$$P = \begin{pmatrix} \ddots & -\frac{q_{ij}}{q_{ii}} & -\frac{q_{ij}}{q_{ii}} & \cdots \\ -\frac{q_{ij}}{q_{ii}} & 0 & -\frac{q_{ij}}{q_{ii}} & \cdots \\ \vdots & -\frac{q_{ij}}{q_{ii}} & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

known as the **Embedded Markov Chain**.

States of a Markov process may be defined (as persistent, transient etc) in accordance with their properties in the embedded Markov chain (with the exception of periodicity, which is not applicable to continuous processes).

7.2 Forward and Backward Equations

Given Q, how do we get $P(t), t \geq 0$?

1. $t, h \geq 0$ $P(t+h) = P(t)P(h)$ (C-K equations).

$$\begin{aligned} \frac{P(t+h) - P(t)}{h} &= \frac{P(t)P(h) - P(t)}{h} \\ &= P(t)\left(\frac{P(h) - I}{h}\right) = P(t)\left(\frac{P(h) - P(0)}{h}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}P(t) &= P(t)\left(\lim_{h\to 0}\left(\frac{P(h) - P(0)}{h}\right)\right) \\ &= P(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}P(t)\bigg|_{t=0}\right) \\ &= P(t)Q \end{aligned}$$

Giving a set of Forward Differential Difference equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}p_{ij}(t) = \sum_k p_{ik}(t)q_{kj} \qquad \forall i, j$$

2. $t, h \geq 0$ $P(t+h) = P(h)P(t)$ (C-K equations).

$$\begin{aligned} \frac{P(t+h) - P(t)}{h} &= \frac{P(h)P(t) - P(t)}{h} \\ &= \left(\frac{P(h) - P(0)}{h}\right)P(t) \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}P(t) &= QP(t) \end{aligned}$$

Giving a set of Backward Differential Difference equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(t) = \sum_k q_{ik} p_{kj}(t) \qquad \forall i, j$$

Can in principle solve these equations to obtain $P(t)$. (which we will for various processes in due course!)

7.3 Stationary Distributions

Suppose $\boldsymbol{\pi}$ satisfies

$$\boldsymbol{\pi} = \boldsymbol{\pi} P(t), \quad \sum_j \pi_j = 1, \quad \pi_j \geq 0 \qquad \forall t$$

then $\boldsymbol{\pi}$ is a stationary (equilibrium) distribution.

So if $\mathrm{P}(X(0) = i) = \pi_i$ then
$\mathrm{P}(X(t) = i) = \pi_i \quad \forall i.$

Note:

$$\begin{aligned} \boldsymbol{\pi} P(t) &= \boldsymbol{\pi} \quad \forall t \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\pi} P(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\pi}) \\ \Rightarrow \boldsymbol{\pi} \left(\frac{\mathrm{d}}{\mathrm{d}t} P(t) \right) &= \mathbf{0} \quad \forall t. \end{aligned}$$

In particular,

$$\begin{aligned} \boldsymbol{\pi} \left(\frac{\mathrm{d}}{\mathrm{d}t} P(t) \bigg|_{t=0} \right) &= \mathbf{0} \\ \boldsymbol{\pi} Q &= \mathbf{0} \end{aligned}$$

So, we can find $\boldsymbol{\pi}$ from

$$\sum_j \pi_j = 1, \quad \boldsymbol{\pi} Q = \mathbf{0}$$

Theorem For an irreducible process, the stationary distribution $\boldsymbol{\pi}$ is unique if it exists. If it exists the process is POSITIVE PERSISTENT and all rows of $P(t)$ converge to $\boldsymbol{\pi}$.

7.4 The Poisson Process

Recall Axioms of the Poisson process:

$$\begin{aligned}
\mathrm{P}(1 \text{ event in } [t, t+\delta t)) &= \lambda\delta t + o(\delta t) \\
\mathrm{P}(2 \text{ or more events in } [t, t+\delta t)) &= o(\delta t)
\end{aligned}$$

giving,

$$\mathrm{P}(0 \text{ event in } [t, t+\delta t)) = 1 - \lambda\delta t + o(\delta t).$$

Let $X(t)$ = number of events by time t, then we have

$$\begin{aligned}
\mathrm{P}(X(t+\delta t) = i+1 \,|\, X(t) = i) &= \lambda\delta t + o(\delta t) \\
\mathrm{P}(X(t+\delta t) = i \,|\, X(t) = i) &= 1 - \lambda\delta t + o(\delta t)
\end{aligned}$$

We have,

$$p_{ij}(\delta t) = \begin{cases} 1 - \lambda\delta t + o(\delta t) & j = i \\ \lambda\delta t + o(\delta t) & j = i+1 \\ o(\delta t) & \text{otherwise} \end{cases}$$

So, Q is given by

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & & & & \\ 0 & -\lambda & \lambda & 0 & & & \\ & 0 & -\lambda & \lambda & 0 & & \\ & & 0 & -\lambda & \lambda & 0 & \\ & & & & \ddots & \ddots & \\ & & & & & & \end{pmatrix}$$

We calculate $p_{0j}(t)$, $0 < j$, by solving

$$\frac{\mathrm{d}}{\mathrm{d}t} P(t) = P(t) Q$$

i.e.

$$\begin{pmatrix} & & \\ & \frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(t) & \\ & & \end{pmatrix} = \begin{pmatrix} & & \\ & p_{ij}(t) & \\ & & \end{pmatrix} \begin{pmatrix} -\lambda & \lambda & 0 & & & & \\ 0 & -\lambda & \lambda & 0 & & & \\ & 0 & -\lambda & \lambda & 0 & & \\ & & 0 & -\lambda & \lambda & 0 & \\ & & & & \ddots & \ddots & \\ & & & & & & \end{pmatrix}$$

Consider first row,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}p_{00}(t) &= -\lambda p_{00}(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}p_{0j}(t) &= -\lambda p_{0j}(t) + \lambda p_{0,j-1}(t) \qquad j \geq 1
\end{aligned}$$

Let $p_{0j}(t) = p_j(t)$, multiply by s^j and sum over j:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{j=0}^{\infty} s^j p_j(t)\right\} &= -\lambda \sum_{j=0}^{\infty} p_j(t)s^j + \lambda \sum_{j=1}^{\infty} p_{j-1}(t)s^j \\
&= \lambda(s-1)\sum_{j=0}^{\infty} p_j(t)s^j
\end{aligned}$$

so,

$$\begin{aligned}
\frac{\partial \Pi(s,t)}{\partial t} &= \lambda(s-1)\Pi(s,t) \\
\Rightarrow \log \Pi(s,t) &= \lambda(s-1)t + A(s) \\
\Pi(s,t) &= A^*(s)\exp(\lambda(s-1)t)
\end{aligned}$$

Initial conditions: $\mathrm{P}(X(0)=0) = 1 \Rightarrow \Pi(s,0) = 1 \Rightarrow A^*(s) = 1$. So,

$$\Pi(s,t) = \exp[-\lambda t(1-s)],$$

which is the pgf of $Poisson(\lambda t)$.

7.5 Birth and death process

$X(t) =$ size of population at t

$$\begin{aligned}
p_{n,n+1}(\delta t) &= \beta_n \delta t + o(\delta t) \qquad \beta_n = \text{overall birth rate at size } n \\
p_{n,n-1}(\delta t) &= \nu_n \delta t + o(\delta t) \qquad \nu_n = \text{overall death rate at size } n
\end{aligned}$$

$\nu_0 = 0$ by definition,

$$p_{n,n}(\delta t) = 1 - \beta_n\delta t - \nu_n \delta t + o(\delta t).$$

So,

$$Q = \begin{pmatrix}
-\beta_0 & \beta_0 & 0 & & & \\
\nu_1 & -(\nu_1+\beta_1) & \beta_1 & 0 & & \\
0 & \nu_2 & -(\nu_2+\beta_2) & \beta_2 & 0 & \\
 & & & & & \\
 & & & \nu_i & -(\nu_i+\beta_i) & \beta_i \\
 & & & & &
\end{pmatrix}$$

We calculate $p_{0j}(t)$, $0 < j$, by solving

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t) = P(t)Q$$

i.e.

$$\left(\quad \frac{\mathrm{d}}{\mathrm{d}t}p_{ij}(t) \quad \right) = \left(\quad p_{ij}(t) \quad \right) \begin{pmatrix} -\beta_0 & \beta_0 & 0 & & & \\ \nu_1 & -(\nu_1+\beta_1) & \beta_1 & 0 & & \\ 0 & \nu_2 & -(\nu_2+\beta_2) & \beta_2 & 0 & \\ & & & & & \\ & & & \nu_i & -(\nu_i+\beta_i) & \beta_i \\ & & & & & \end{pmatrix}$$

From first row and putting $p_{oj}(t) = p_j(t)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}p_0(t) &= -\beta_0 p_0(t) + \nu_1 p_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}p_j(t) &= \beta_{j-1}p_{j-1}(t) - (\beta_j + \nu_j)p_j(t) + \nu_{j+1}p_{j+1}(t) \qquad j \geq 1 \end{aligned}$$

i.e. differential difference equations for general birth and death process.

In general, $P(t)$ too complicated to derive (though we will study simplifications in due course). However, we may be able to find a stationary distribution.

7.5.1 Stationary distribution

We need to solve,

$$\boldsymbol{\pi} Q = \mathbf{0}, \qquad \sum_j \pi_j = 1$$

$$\begin{aligned} -\beta_0\pi_0 + \nu_1\pi_1 &= 0 \\ \beta_{i-1}\pi_{i-1} - (\nu_i + \beta_i)\pi_i + \nu_{i+1}\pi_{i+1} &= 0 \quad i \geq 1 \\ \Rightarrow \nu_{i+1}\pi_{i+1} - \beta_i\pi_i &= \nu_i\pi_i - \beta_{i-1}\pi_{i-1} \\ &= \nu_{i-1}\pi_{i-1} - \beta_{i-2}\pi_{i-2} \\ &= \nu_{i-2}\pi_{i-2} - \beta_{i-3}\pi_{i-3} \\ &\ \vdots \\ &= \nu_1\pi_1 - \beta_0\pi_0 = 0 \end{aligned}$$

Therefore,

$$\pi_{i+1} = \frac{\beta_i}{\nu_{i+1}}\pi_i \quad \forall\, i \geq 0$$

We have,

$$\pi_n = \frac{\beta_{n-1}}{\nu_n}\pi_{n-1} = \ldots = \frac{\beta_{n-1}\beta_{n-2}\ldots\beta_0}{\nu_n\ldots\nu_1}\pi_0$$

Also,

$$\sum_{n=0}^{\infty}\pi_n = \pi_0\left(1 + \sum_{n=1}^{\infty}\frac{\beta_{n-1}\beta_{n-2}\ldots\beta_0}{\nu_n\ldots\nu_1}\right)$$

The stationary distribution exists iff

$$\sum_{n=1}^{\infty}\frac{\beta_{n-1}\beta_{n-2}\ldots\beta_0}{\nu_n\ldots\nu_1} < \infty.$$

If this sum converges then

$$\begin{aligned}
\pi_0 &= \left(1 + \sum_{n=1}^{\infty}\frac{\beta_{n-1}\beta_{n-2}\ldots\beta_0}{\nu_n\ldots\nu_1}\right)^{-1} \\
\pi_n &= \frac{\beta_{n-1}\beta_{n-2}\ldots\beta_0}{\nu_n\ldots\nu_1}\pi_0 \quad n \geq 1
\end{aligned}$$

7.5.2 Linear birth and death process

If $\nu_n = n\nu$ and $\beta_n = n\beta$, we have a linear birth and death process.
i.e. the probability of each individual in the population giving birth in an interval of length δt is

$$\beta\delta t + o(\delta t)$$

similarly for a death.
So, we have,

$$\begin{aligned}
p_{n,n+1}(\delta t) &= \beta n\delta t + o(\delta t) \qquad \beta_n = \beta n = \text{overall birth rate at size } n \\
p_{n,n-1}(\delta t) &= \nu n\delta t + o(\delta t) \qquad \nu_n = \nu n = \text{overall death rate at size } n
\end{aligned}$$

$$p_{n,n}(\delta t) = 1 - \beta n\delta t - \nu n\delta t + o(\delta t).$$

Let $X_i(t)$ = number in population at time t given $X(0) = i$.

We have both $\nu_0 = \beta_0 = 0$, once $X_i(t)$ reaches zero, it stays there forever.

$$Q = \begin{pmatrix} 0 & 0 & 0 & \dots & & \\ \nu & -(\nu+\beta) & \beta & 0 & & \\ 0 & 2\nu & -(2\nu+2\beta) & 2\beta & 0 & \\ 0 & 0 & 3\nu & -(3\nu+3\beta) & 3\beta & \\ & & & \ddots & \ddots & \ddots \\ & & & & & \end{pmatrix}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t) = P(t)Q$$

Giving

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}p_{i0}(t) &= \nu p_{i1}(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}p_{ij}(t) &= (j-1)\beta p_{i,j-1}(t) - j(\nu+\beta)p_{ij}(t) + (j+1)\nu p_{i,j+1}(t) \quad j \geq 1 \end{aligned}$$

Apply general method:

1. multiply by s^j.

2. sum over $0 \leq j \leq \infty$.

3. find differential equation for pgf $\Pi_i(s,t)$ of $X_i(t)$:

$$\Pi_i(s,t) = \sum_{j=0}^{\infty} p_{ij}(t)s^j$$

To give,

$$\frac{\partial}{\partial t}\sum_{j=0}^{\infty} p_{ij}(t)s^j = \beta\sum_{j=1}^{\infty}(j-1)p_{i,j-1}(t)s^j - (\nu+\beta)\sum_{j=1}^{\infty} jp_{ij}(t)s^j + \nu\sum_{j=0}^{\infty}(j+1)p_{i,j+1}(t)s^j$$

Note that,

$$\frac{\partial}{\partial s}\Pi_i(s,t) = \frac{\partial}{\partial s}\sum_{j=0}^{\infty} p_{ij}(t)s^j = \sum_{j=1}^{\infty} jp_{ij}(t)s^{j-1}.$$

$$\begin{aligned} \frac{\partial}{\partial t}\sum_{j=0}^{\infty} p_{ij}(t)s^j &= \beta s^2\sum_{j=1}^{\infty} jp_{ij}(t)s^{j-1} - (\nu+\beta)s\sum_{j=1}^{\infty} jp_{ij}(t)s^{j-1} + \nu\sum_{j=1}^{\infty} jp_{ij}(t)s^{j-1} \\ \frac{\partial}{\partial t}\Pi_i(s,t) &= (\beta s^2 - (\nu+\beta)s + \nu)\frac{\partial}{\partial s}\Pi_i(s,t). \end{aligned}$$

A partial differential equation involving both $\frac{\partial}{\partial t}\Pi_i(s,t)$, and $\frac{\partial}{\partial s}\Pi_i(s,t)$.

Excursus on Lagrange's equation

$$f(s,t,\Pi)\frac{\partial \Pi}{\partial s} + g(s,t,\Pi)\frac{\partial \Pi}{\partial t} = h(s,t,\Pi)$$

Step 1: Write down the auxiliary equations

$$\frac{\mathrm{d}s}{f} = \frac{\mathrm{d}t}{g} = \frac{\mathrm{d}\Pi}{h}$$

Step 2: Solve these, writing solutions in the form

$$c_1 = \phi_1(s,t,\Pi) \quad c_2 = \phi_2(s,t,\Pi).$$

Step 3: Write down the arbitrary functional equation

$$c_2 = \Psi(c_1) \quad (\text{or } c_1 = \Psi(c_2))$$

and rewrite it to give $\Pi(s,t)$ on one side.

Step 4: Find a particular solution for given initial conditions by identifying Ψ using initial conditions and substituting for Ψ in general solution.

Recall, for the linear birth and death process:

$$\frac{\partial}{\partial t}\Pi_i(s,t) = (\beta s^2 - (\nu+\beta)s + \nu)\frac{\partial}{\partial s}\Pi_i(s,t),$$

We have,

$$f(s,t,\Pi_i) = -(\beta s^2 + \nu - (\beta+\nu)s) \quad g(s,t,\Pi) = 1 \quad h(s,t,\Pi_i) = 0$$

Auxiliary equations:

$$\frac{\mathrm{d}s}{-(\beta s^2 + \nu - (\beta+\nu)s)} = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}\Pi_i}{0}$$

Let's first suppose $\beta \neq \nu$.

From last pair: $c_1 = \Pi_i \quad [\int 0\,\mathrm{d}t = \int 1\,\mathrm{d}\Pi_i]$

From first pair:

$$\begin{aligned}
\int \frac{-1}{\beta s^2 + \nu - (\beta+\nu)s}\mathrm{d}s &= \int 1\,\mathrm{d}t \\
\int \frac{1}{\beta-\nu}\left(\frac{1}{1-s} - \frac{\beta}{\nu-\beta s}\right)\mathrm{d}s &= \int 1\,\mathrm{d}t \\
\Rightarrow \frac{1}{\beta-\nu}\left(-\ln(1-s) + \ln(\nu-\beta s)\right) &= t + \text{const} \\
\Rightarrow \ln\left(\frac{\nu-\beta s}{1-s}\right) &= (\beta-\nu)t + \text{const} \\
\Rightarrow c_2 &= \frac{\nu-\beta s}{1-s}e^{-(\beta-\nu)t}
\end{aligned}$$

Hence the general solution is

$$\Pi_i(s,t) = \Psi\left(\frac{\nu-\beta s}{1-s}e^{-(\beta-\nu)t}\right)$$

Question: If there is 1 individual in the population at time 0, and assuming $\beta \neq \nu$, what is $\Pi_1(s,t)$?

Answer: Putting $t=0$ in the general solution:

$$\Pi_1(s,0) = \Psi\left(\frac{\nu-\beta s}{1-s}\right)$$

also,

$$\Pi_1(s,0) = \sum_{j=1}^{\infty} p_{1j}(0)s^j = s$$

So,

$$s = \Psi\left(\frac{\nu - \beta s}{1 - s}\right)$$

Putting $x = (\nu - \beta s)/(1 - s) \;\Rightarrow\; s = (\nu - x)/(\beta - x)$

$$\Rightarrow \Psi(x) = \frac{\nu - x}{\beta - x}$$

So that,

$$\Pi_1(s,t) = \frac{\nu(1-s) - (\nu - \beta s)e^{(\nu-\beta)t}}{\beta(1-s) - (\nu - \beta s)e^{(\nu-\beta)t}}$$

Note, with $X(0) = i$, the initial condition is

$$\Pi_i(s,0) = \sum_{j=1}^{\infty} p_{ij}(0)s^j = s^i$$

so that,

$$\Pi_i(s,t) = \left(\frac{\nu(1-s) - (\nu - \beta s)e^{(\nu-\beta)t}}{\beta(1-s) - (\nu - \beta s)e^{(\nu-\beta)t}}\right)^i$$

When $\beta = \nu$, with i individuals at $t = 0$, we find

$$\Pi_i(s,t) = \left(\frac{\beta t - s\beta t + s}{\beta t - s\beta t + 1}\right)^i$$

7.5.3 Expected population size

Recall the pgf for $X_i(t)$ = number in population when we start with i ($X(0) = i$).

$$\Pi_i(s,t) = \sum_{j=0}^{\infty} p_{ij}(t)s^j$$

Expected size at t, given $X(0) = i$:

$$\left.\frac{\partial}{\partial s}\Pi(s,t)\right|_{s=1}$$

For the linear birth and death process this is:

$$\Pi_i'(1,t) = \begin{cases} ie^{(\beta-\nu)t} & \beta \neq \nu \\ i & \beta = \nu \end{cases}$$

7.5.4 Extinction probabilities

When $s = 0$

$$\Pi_i(0,t) = p_{i0}(t) = \mathrm{P}(X_i(t) = 0)$$

i.e. P(extinct by time t) $= \Pi_i(0,t)$.

For $\beta \neq \nu$, this is

$$\Pi_i(0,t) = \left(\frac{\nu - \nu e^{(\nu-\beta)t}}{\beta - \nu e^{(\nu-\beta)t}} \right)^i$$

Now let $t \to \infty$

$$\lim_{t\to\infty} \Pi_i(0,t) = \begin{cases} \left(\frac{\nu}{\beta}\right)^i & \beta > \nu \\ 1 & \beta < \nu \end{cases}$$

i.e. ultimate extinction is certain if death rate $>$ birth rate, but not if reverse holds.

For $\beta = \nu$

$$\Pi_i(0,t) = \left(\frac{\beta t}{\beta t + 1} \right)^i$$

and as $t \to \infty$

$$\lim_{t\to\infty} \Pi_i(0,t) = 1,$$

so that extinction is certain eventually.

7.5.5 Embedded process

Recall the irreducible Markov chain for the process observed at the jumps with transition matrix P with elements

$$p_{ij} = \begin{cases} 0 & i = j \\ -\frac{q_{ij}}{q_{ii}} & i \neq j \end{cases}$$

For the linear birth and death process this is

$$P = \begin{pmatrix} 1 & 0 & 0 & \dots & & \\ \frac{\nu}{\nu+\beta} & 0 & \frac{\beta}{\nu+\beta} & 0 & & \\ 0 & \frac{\nu}{\nu+\beta} & 0 & \frac{\beta}{\nu+\beta} & 0 & \\ 0 & 0 & \frac{\nu}{\nu+\beta} & 0 & \frac{\beta}{\nu+\beta} & 0 \\ & & & \ddots & \ddots & \ddots \\ & & & & & \end{pmatrix}$$

Recognise the form?

This is the transition matrix of a simple random walk with an absorbing barrier at zero with $p = \beta/(\beta+\nu)$ (gambler's ruin when playing a casino). To determine extinction probabilities for the birth and death process, we can use results for the embedded process:

Recall: $p =$ probability A wins each game

$q = 1 - p$

stake = £1 on each game

We found that the probability of ultimate ruin (q_j) if A starts with £j is

$$q_j = \begin{cases} 1 & \text{when } p \leq q \\ \left(\frac{q}{p}\right)^j & \text{when } p > q \end{cases}$$

So, putting $p = \beta/(\beta+\nu)$, $q = \nu/(\beta+\nu)$ and $j = i$,

$$\text{P(ultimate extinction)} = \begin{cases} 1 & \text{when } \beta \leq \nu \\ \left(\frac{\nu}{\beta}\right)^i & \text{when } \beta > \nu \end{cases}$$

But note: the embedded process tells us nothing about the development of the process in time.

7.6 Immigration

- births and deaths occur as before,
- in addition, new arrivals occur according to a Poisson process with rate λ.

Proceed as usual:

1. Find Q and use the forward or backward equations to obtain the differential difference equations for $\{p_{ij}(t)\}$.
2. Obtain a partial differential equation for $\Pi_i(s,t)$.
3. Solve this to find properties of $X_i(t)$.

7.7 Immigration-birth-death process

Consider a linear birth and death process with immigration (rate λ). We have

$$\mathrm{P}(X(t+\delta t) = n+1 \,|\, X(t) = n) = (\lambda + n\beta)\delta t + o(\delta t).$$

So,

$$\begin{aligned}
p_{n,n+1}(\delta t) &= (\lambda + n\beta)\delta t + o(\delta t) \\
p_{n,n-1}(\delta t) &= n\nu\delta t + o(\delta t) \\
p_{n,n}(\delta t) &= 1 - (\lambda + n\beta + n\nu)\delta t + o(\delta t).
\end{aligned}$$

So,

$$Q = \begin{pmatrix}
-\lambda & \lambda & 0 & & & \\
\nu & -(\lambda+\beta+\nu) & \lambda+\beta & 0 & & \\
0 & 2\nu & -(\lambda+2\beta+2\nu) & \lambda+2\beta & 0 & \\
 & & & & & \\
 & & & i\nu & -(\lambda+i\beta+i\nu) & \lambda+i\beta \\
 & & & & &
\end{pmatrix}$$

i.e. general birth and death process with $\beta_n = n\beta + \lambda$ and $\nu_n = n\nu$.

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t) = P(t)Q$$

Giving

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}p_{i0}(t) &= -\lambda p_{i0}(t) + \nu p_{i1}(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}p_{ij}(t) &= (\lambda + (j-1)\beta)p_{i,j-1}(t) - (\lambda + j(\beta+\nu))p_{ij}(t) + (j+1)\nu p_{i,j+1}(t) \quad j \geq 1
\end{aligned}$$

Apply general method:

1. multiply by s^j.

2. sum over $0 \leq j \leq \infty$.

3. find differential equation for pgf $\Pi_i(s,t)$ of $X_i(t)$

Find,

$$(1-s)(\nu-\beta s)\frac{\partial\Pi_i(s,t)}{\partial s}-\frac{\partial\Pi_i(s,t)}{\partial t}=\lambda(1-s)\Pi_i(s,t)$$

Note: this includes, as special cases all processes involving any combination of immigration-birth-death (just set β, ν or $\lambda = 0$).

Case 1: $\beta \neq \nu$ Lagrange form, auxiliary equations

$$\begin{aligned}\Rightarrow c_1 &= \frac{\nu-\beta s}{1-s}e^{(\nu-\beta)t}\\ c_2 &= \Pi_i(s,t)(\nu-\beta s)^{(\lambda/\beta)}\end{aligned}$$

If $X(0) = 0$ then

$$\Pi_0(s,t)=\left(\frac{(\nu-\beta)e^{(\nu-\beta)t}}{(\nu-\beta s)e^{(\nu-\beta)t}-\beta(1-s)}\right)^{\lambda/\beta}$$

Note: λ only occurs as index, if $\lambda = 0$ (i.e. no immigration),

then $\Pi_0(s,t) = 1 = 1\cdot s^0 + 0\cdot s^1 + 0\cdot s^2 + \ldots$

(so $p_{0j}(t) = 0\ \forall\ j$ – that is the population size remains at 0).

Now, setting $p = (\nu-\beta)e^{(\nu-\beta)t}/(\nu e^{(\nu-\beta)t}-\beta)$, we have

$$\Pi_0(s,t)=\left(\frac{p}{1-qs}\right)^{\lambda/\beta}$$

which is the pgf of a negative binomial distribution,with

$$\begin{aligned}\mathrm{E}(X_0(t)) &= \frac{\lambda q}{\beta p}\\ \mathrm{var}(X_0(t)) &= \frac{\lambda q}{\beta p^2}\end{aligned}$$

Could also get these directly from pgf.

Case 2: $\beta = \nu$

$$\begin{aligned}\Pi_0(s,t) &= (1+\beta t-s\beta t)^{-\lambda\beta}\\ &= \left(\frac{1/(1+\beta t)}{1-s\beta t/(1+\beta t)}\right)^{\lambda/\beta}\\ &= \left(\frac{p}{1-qs}\right)^{\lambda/\beta}\end{aligned}$$

with $p = 1/(1+\beta t)$ – again the pgf of a negative binomial distribution.