Introduction

The subject of complex variables appears in many areas of mathematics as it has been truly the ancestor of many subjects. It is employed in a wide range of topics, including, algebraic geometry, number theory, dynamical systems, and quantum field theory, to name a few. Basic examples and techniques in complex analysis have been developed over a century into sophistication methods in analysis. On the other hand, as the real and imaginary parts of any analytic function satisfy the Laplace equation, complex analysis is widely employed in the study of two-dimensional problems in physics, for instance in, hydrodynamics, thermodynamics, ferromagnetism, and percolations.

In complex analysis one often starts with a rather weak requirement (regularity) of differentiability. That is, a map $f : U \to \mathbb{C}$ is called *holomorphic* on Ω, if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists at every point in the open set $U \subseteq \mathbb{C}$. Then with little effort one concludes from the above property that f is infinity many times differentiable, and indeed it has a convergent power series. This is in a direct contrast with the notions of C^k regularities we have for real maps of Euclidean spaces. That is, there are C^k real maps that are not C^{k+1}, for any $k \geq 1$. Or, there are C^∞ real maps that have no convergent power series. The difference is rooted in the fact that here h tends to 0 in all directions, and there is a multiplication operation on the plane that interacts nicely with the addition. Due to this difference, complex analysis is not merely extending the calculus to complex-valued functions; rather it is a subject of mathematics on its own.

Let Ω be an open set in $\mathbb{C}$ that is bounded by a piece-wise smooth simple closed curve, and let $f : \omega \to \mathbb{C}$ be a holomorphic map. For any C^1 simple closed curve γ in Ω, if we know the values of f on γ, the Cauchy Integral Formula provides a simple formula for the values of f inside γ:

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz.$$

Also, there is a similar formula for the higher order derivatives of f at any point inside γ. On the other hand, if we know all derivatives of f at some point $z_0 \in \Omega$, then the infinite series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n,$$

is convergent for z close enough to z_0, and the value of the series is equal to $f(z)$.

When the domain Ω enjoys some form of symmetry, for example, when Ω is the unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

with rotational symmetry, the objects of interest in complex analysis often find simple algebraic forms. In Chapters 2 and 6 we prove some results of this nature, including,

Theorem 0.1. *Let $f : \mathbb{D} \to \mathbb{D}$ be a one-to-one and onto holomorphic mapping. Then, there are $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$ such that*

$$f(z) = e^{i\theta} \frac{a - z}{1 - \overline{a}z}.$$

Although the above type of results point to the restrictive nature of holomorphic property, there are also statements that go in the other direction. For example, in Chapter 4, we prove the Riemann mapping theorem, which, as a special case, implies the following.

Theorem 0.2. *Let $\Omega \subset \mathbb{C}$ be an open set that is bounded by a continuous simple closed curve, and let $z_0 \in \Omega$. Then, there is a one-to-one and onto holomorphic map $f : \mathbb{D} \to \Omega$ with $f(0) = z_0$.*

The domain Ω in the above theorem may have a very complicated shape (geometry), or may have a highly irregular boundary (analysis) obtained from a randomly generated curve. See Figure 1.

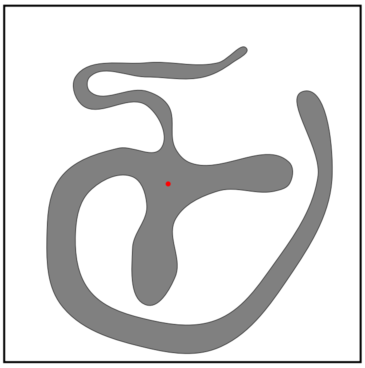

Figure 1: An arbitrary open set Ω bounded by a continuous simple closed curve.

The map f in the above theorem is called the *uniformlization* of the domain Ω. One aim of this course is to study the behavior of the uniformizations in connection with the geometric shape of Ω and its boundary. We also look for such geometric quantities that remain invariant under conformal mappings.

In applications, one often comes across domains Ω that have very complicated shapes, or very irregular boundaries. Although the above theorem provides us with a seemingly nice behaving map, there are little chances that we know the higher order derivatives of f at some $z_0 \in \Omega$ or the behavior of the map f on the boundary of $\mathbb{D}$ in order to use the Taylor series or the Cauchy Integral Formula to study the behavior of f. But, is it still possible to say something about the map f? As we shall see in Chapter 5 there are some *universal laws* that every one-to-one holomorphic map must obey. Let us give an example of this type. For an arbitrary $\theta \in [0, 2\pi)$, one map ask how fast the curve $r \mapsto f(re^{i\theta})$, for $r \in [0, 1)$, move away from 0, or how fast it may spiral about 0? In Chapter 5 we prove some results of the following type.

Theorem 0.3. *Let $f : \mathbb{D} \to \mathbb{C}$ be an arbitrary one-to-one and holomorphic map normalized with $f(0) = 0$ and $f'(0) = 1$. Then, for every $\theta \in [0, 2\pi]$ and $r \in (0, 1)$ we have*

$$|\arg f'(re^{i\theta})| \leq 2 \log \frac{1 + r}{1 - r}. \tag{1}$$

While Theorem 0.2 is strong and general, its proof is far from constructive. On rare occasions we are able to provide a formula for the map f (a list of such examples appear in Chapter 4). This is a rather general theme in holomorphic mappings that we often know that a holomorphic function with some prescribed conditions exists, but we don't have a constructive approach to it.

In Chapter 7 we introduce a generalization of conformal maps known as quasi-conformal maps. Roughly speaking, these are homeomorphisms whose first partial derivatives exist almost everywhere, and the Cauchy-Riemann condition is nearly satisfied (being small instead of 0). These maps naturally come up in complex analysis in several ways. It turns out that such maps still enjoy many properties of conformal maps, while having a more constructive nature. Many problems related to the behavior of conformal maps through quasi-conformal maps reduce to the study of a certain type of partial differential equation, where there are constructive approaches to the solutions.

Although the above method turns out to be unexpectedly powerful, we must remain humble. It is easy to pose simple looking open (and probably extremely hard) questions in complex analysis, for instance,

Brennans conjecture: For every one-to-one and onto holomorphic map $f : \mathbb{D} \to \Omega$, and every real p with $-2 < p < 2/3$, we have

$$\int_{\mathbb{D}} |f'(re^{i\theta})|^p \, dxdy < \infty.$$

We will see in this course that geometric complications and irregularities in the boundary of Ω results in large and small values for $|f'|$. The above conjecture suggests some bounds on the average values of $|f'|$. This is part of a set of conjectures knows as *universal integral means spectrum*, and encompasses some conjectures of Littlewood on the extremal growth rate of the length of the closed curves $f(r \cdot e^{i\theta})$, $\theta \in [0, 2\pi]$, as r tends to 1. These questions are motivated by important problems in statistical physics.

The actual prerequisite for this course is quite minimal. We assume that the students taking this class are familiar with the notions of holomorphic maps and their basic properties. This is a concise math course with ε-δ proofs, and so precise forms of definitions and statements appear in the notes. To rectify the challenge of where we start, we have summarized in Chapter 1 (in three pages) the basic results from complex analysis that we will rely on.

I prepared these notes for the course Geometric Complex Analysis, M3/4/5P60, for the autumn term of 2016 at Imperial College London. I am very pleased with the maths department for agreeing to offer this course for the first time. Complex analysis with its surprises is one of the most beautiful areas of mathematics. You may help me to improve these notes by emailing me any comments or corrections you have.