Chapter 8

Appendix

8.1 Hints to exercises

In this appendix we provide hints to the exercises. Please note that these will be brief and do not suggest a suitable style of writing proofs in mathematics. The complete solutions to the difficult exercises are given. These have been indicated by [complete solution] at the beginning of the solution, and suggest a proper way of writing solutions.

Chapter 2

2.1: Let $\varphi_1(z) = rz + a$ and $\varphi_2(z) = sz + b$. Then, $\varphi_1 : \mathbb{D} \to B(a, r)$ and $\varphi_2 : \mathbb{D} \to B(b, s)$ are biholomorphisms. It follows that $\varphi_2^{-1} \circ f \circ \varphi_1 : \mathbb{D} \to \mathbb{D}$ is defined and holomorphic, and maps 0 to 0. By Lemma 2.1, we have $|(\varphi_2^{-1} \circ f \circ \varphi_1)'(0)| \leq 1$. This implies $|f'(a)| \leq s/r$.

2.2: We have seen that $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$ belongs to $\mathrm{Aut}(\mathbb{D})$. Recall that φ_a is the inverse of φ_a.

(i) Apply Lemma 2.1-(ii) to the map $\varphi_{f(a)}^{-1} \circ f \circ \varphi_a$, and explicitly calculate the derivatives of φ_a and $\varphi_{f(a)}$.

(ii) Apply Lemma 2.1-(i) to the map $\varphi_{f(a)}^{-1} \circ f \circ \varphi_a$ at $\varphi_a^{-1}(b)$.

2.3: The map $\varphi(z) = \frac{\operatorname{Im} a}{\operatorname{Im} h(a)} z + (\operatorname{Re} a - \frac{\operatorname{Im} a}{\operatorname{Im} h(a)} \operatorname{Re} a)$ is an automorphism of $\mathbb{H}$ that maps $h(a)$ to a. Let $\psi : \mathbb{D} \to \mathbb{H}$ be a biholomorphic map with $\psi(0) = a$. Then, apply Lemma 2.1-(ii) to the map $\psi^{-1} \circ \varphi \circ h \circ \psi$. Note that $(\psi^{-1})'(a) = 1/\psi'(0)$, so $|(\varphi \circ h)'(a)| \leq 1$. You need to calculate $\varphi'(h(a))$.

2.4: First note that it is enough to show that every point in $\mathbb{D}$ can be mapped to 0. Then compose such maps to obtain an automorphism that maps z to w.

Chapter 3

3.1: (i) Solve for A and B in $f(z) = Az + B$.

(ii) First note that it is enough to show that any three distinct points can be mapped to 0, 1, and ∞. Then, compose such maps and their inverses to get the desired map.

3.2: Apply the removable singularity theorem to the map $z \mapsto 1/f(1/z)$.

3.3: [Complete solution]

(i) Since Ω is an opens set, there is $r > 0$ such that $B(z_0, r) \subset \Omega$. Then, f has a convergent power series expansion on $B(z_0, r)$, say

$$f(z) = f(z_0) + a_1(z - z_0) + a_2(z - z_0)^2 + \dots.$$

As z_0 is a zero of order k, we must have $a_i = 0$ for all $1 \leq i \leq k-1$, and $a_k \neq 0$. Then,

$$f(z) = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \dots = (z - z_0)^k \big(a_k + a_{k+1}(z - z_0) + \dots\big).$$

The function $h(z) = a_k + a_{k+1}(z - z_0) + \dots$ is holomorphic on $B(z_0, r)$, and in particular it is continuous. Then, for $\varepsilon = |a_k|/2 > 0$ there is $\delta > 0$ such that if $|z - z_0| < \delta$ then $|h(z) - h(z_0)| < \varepsilon$. Here we may assume that $\delta < r$, as otherwise we may take $\min\{\delta, r\}$. The inequality means that h maps $B(z_0, \delta)$ into $B(a_k, |a_k|/2))$. On the other hand, since $B(a_k, |a_k|/2)$ does not meet the line segment $-a_k r$, for $r \in [0, \infty)$, there is a holomorphic branch of the k-th-root function defined on this ball. That is, $\sqrt[k]{h(z)}$ is defined and holomorphic on $B(z_0, \delta)$.

We have

$$f(z) = (z - z_0)^k \big(a_k + a_{k+1}(z - z_0) + \dots\big) = \big((z - z_0)\sqrt[k]{h(z)}\big)^k,$$

that is, $\psi(z) = (z - z_0)\sqrt[k]{h(z)}$.

(ii) For the map ψ obtained in part (i), we have $\psi(z_0) = 0$, and by the product rule, $\psi'(z_0) \neq 0$. By the inverse function theorem, ψ has an inverse defined on a neighborhood of $\phi(z_0) = 0$. Let g be this inverse map that is defined on $B(0, r)$, for some $r > 0$.

For every $w \in B(0, r^k)$, there are exactly k points $w_1, w_2, \dots, w_k$ in $B(0, r)$ such that $w_i^k = w$. Then the points $z_i = g(w_i)$ provide k solutions for the equation $f(z) = w$. To see that there are at most k solutions, assume that $f(z) = w$ for some $w \in B(0, r^k)$. Then $\psi(z)^k = w$, which implies that $\psi(z)$ is a k-th root of w. Thus, z_i are the only solutions.

3.4: (i) Assume that f is not constant. Let U be an open set in Ω. We need to show that $f(U)$ is open. Fix an arbitrary $w_0 \in f(U)$. There is $z_0 \in U$ with $f(z_0) = w_0$. Since, f is not constant, the function $z \mapsto f(z) - w_0$ has a zero of some finite order $k \geq 1$ at z_0. By the previous exercise, $f(z) - w_0$ is locally k to 1 near z_0. That is, for every w near w_0, there is z near z_0 such that $f(z) = w$. Since U is open, the points sufficiently close to z_0

are in U. This means that a neighborhood of w_0 is contained in $f(U)$. As $w_0 \in f(U)$ was arbitrary, we conclude that $f(U)$ is open.

(ii) Assume that f is not a constant map. As $f : \Omega \to \mathbb{C}$ is an open mapping, $f(\Omega)$ is an open set in $\mathbb{C}$. Fix an arbitrary $z \in \Omega$. As $f(\Omega)$ is open, there is $r > 0$ such that $B(f(z), r) \subset f(\Omega)$. Now, choose $w' \in B(f(z), r)$ with $|w'| > |f(z)|$. Since $B(f(z), r) \subset f(\Omega)$, there is $z' \in \Omega$ with $f(z') = w'$. Hence, $|f(z')| > |f(z)|$.

By the above argument, for every $z \in \Omega$, there is $z' \in \Omega$ such that $|f(z')| > |f(z)|$. This implies the maximum principle.

3.5: First show that the linear map $h(z) = az + b$ and the inversion $h(z) = 1/z$ map lines and circles to lines and circles. Then show that any Mobius transformation may be written as composition of these maps.

3.6: First show that there are at most finite number of points a_1, a_2, $\dots a_d$ in $\mathbb{D}$ with $g(a_i) = 0$.

Consider the function

$$h(z) = \prod_{j=1}^{d} \frac{z - a_j}{1 - \overline{a_j} z}$$

and show that it maps $\mathbb{D}$ into $\mathbb{D}$ and maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$.

Consider the rational function $\varphi(z) = g(z)/h(z)$. Show that there are no points in $\mathbb{D} \cup \partial\mathbb{D}$ that are mapped to 0.

Show that φ maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$. Conclude that for all $z \in \hat{\mathbb{C}}$ we have $\varphi(z) = 1/\overline{\varphi(1/\overline{z})}$, where $\overline{z}$ denotes the complex conjugate of z.

By the above two paragraphs, there are no points in $\hat{\mathbb{C}}$ which are mapped to 0 by φ. This implies that φ is a constant function, which must belong to $\partial\mathbb{D}$.

Chapter 4

4.1: You need to verify the three conditions for being a metric.

Property (i): This is obvious from the definition of the length of a curve. That is, the length of a curve is independent of the parametrization and the direction of the curve.

Property (ii): Since the length of any curve is non-negative, the infimum of a set of non-negative numbers is a non-negative number.

If $z = w$, then the constant curve from z to w has zero length with respect to ρ. Thus, $d_\rho(z, z) = 0$.

Now assume that $z \neq w$ and let $r = |z - w| > 0$. Since Ω is open there is $r_1 > 0$ such that $B(z, r_1) \subset \Omega$. Also, as the set of zero's of ρ is discrete, there is a positive

$\varepsilon < \min\{r, r_1\}$ such that ρ has at most one zero on $B(z, \varepsilon)$ (the only possible zero is z). Consider the compact set $A = \{\zeta \in \Omega \mid \varepsilon/4 \leq |z - \zeta| \leq \varepsilon/2\}$, which is contained in Ω. The function ρ is continuous and positive on A, and hence its minimum on A is strictly positive, say $m > 0$.

Let $\gamma : [a, b] \to \Omega$ be a piece-wise C^1 curve with $\gamma(a) = z$ and $\gamma(b) = w$. Then,

$$\begin{aligned}\ell_\rho(\gamma) = \int_{[a,b]} \rho(\gamma(s))|\gamma'(s)|ds \geq \int_{\{t\in[a,b];\gamma(t)\in A} \rho(\gamma(s))|\gamma'(s)|ds \\ \geq m \cdot \int_{\{t\in[a,b];\gamma(t)\in A} |\gamma'(s)|ds \geq m \cdot \frac{\varepsilon}{4}.\end{aligned}$$

As $m\varepsilon/4$ does not depend on γ, by the definition of infimum, $d_\rho(z, w) \geq m\varepsilon/4$. Hence, $d_\rho(z, w) > 0$.

By the above paragraphs, $d_\rho(z, w) = 0$ iff $z = w$.

Property (iii): Let η a piece-wise C^1 curve connecting x to z, and ξ a piece-wise C^1 curve connecting z to y. Then η followed by ξ is a piece-wise C^1 curve connecting x to y. By definition,

$$d_\rho(x, y) \leq \ell_\rho(\eta \cup \xi) = \ell_\rho(\eta) + \ell_\rho(\xi).$$

Now, take infimum over $\Gamma_{x,z}$, and then over $\Gamma_{z,y}$ to conclude the triangle inequality.

4.2: First note that $\rho \geq 1$, which implies that $d_\rho(z, w) \geq |z - w|$. In particular, if z_i converges to z w.r.t d_ρ, then, $|z_i - z| \to 0$.

On the other hand, if z_i converges to z w.r.t Euclidean distance, then, there is $r < 1$ such that $z_i \in B(0, r)$, for all i. Now, let M be the supremum of ρ on $B(0, r)$. M is a finite number. We have $d_\rho(z_i, z) \leq M|z_i - z|$. Hence, $d_\rho(z_i, z) \to 0$.

4.3: Let z_i be a Cauchy sequence in $(\mathbb{D}, \rho)$. First show that there is $r < 1$ such that for all $i \geq 1$, $z_i \in B(0, r)$. Then conclude that z_i is a Cauchy sequence w.r.t the Euclidean distance.

4.4: Use an isometry of the dist to map z to 0. Then use that the Mobius transformations map circles to circles in Exercise 3.5.

4.5: Show that there is a one-to-one correspondence between $\Gamma_{z,w}$ and $\Gamma_{f(z),f(w)}$.

4.6: By definition,

$$(F^*\rho)(w) = \rho(F(w)) \cdot |F'(w)| = \frac{1}{1 - |\frac{i-w}{i+w}|^2} \cdot \frac{2}{|i + w|^2} = \frac{2}{|i + w|^2 - |i - w|^2} = \frac{1}{2|\operatorname{Im} w|}.$$

Chapter 5

5.1: Write the circle of radius r as $re^{i\theta}$, and note that

$$f(re^{i\theta}) = (r + 1/r)\cos\theta + i(r - 1/r)\sin\theta,$$

and use the identity $\cos^2\theta + \sin^2\theta \equiv 1$.

5.2: From Example 5.6, replace sin and cos in terms of e^{iz} in $\tan z = \sin z/\cos z$.

5.3: On can do this by composition of a number of elementary transformations. First apply the biholomorphism $g_1(z) = i\frac{1-w}{1+w}$ (see Equation 2.1) to Ω to obtain $\mathbb{H} \setminus [0, 1/3]i$. Then apply $g_2(z) = -i\cdot z$ to get $B = \{w \in \mathbb{C} \mid \operatorname{Re} w > 0\} \setminus (0, 1/3)$. Next, apply $g_3(z) = z^2$, to obtain $\mathbb{C} \setminus (-\infty, 1/3)$, then apply $g_4(z) = z - 1/3$ to obtain $\mathbb{C} \setminus (-\infty, 0)$, and then apply $g_5(z) = \sqrt{z}$ to obtain the right half plane.

5.4: If a family of maps $\mathcal{F}$ is not uniformly bounded on compact sets, then there is a compact set $E \subset \Omega$ such that the family is not uniformly bounded on E. This means that for any $n \in \mathbb{N}$ there is $z_n \in E$ and $f_n \in \mathcal{F}$ such that $|f_n(z)| \geq n$. Since E is compact, $\{z_n\}$ has a sub-sequence, say n_k, converging to some $z \in E$. It follows that the sequence $\{f_{n_k}\}$ has no sub-sequence converging uniformly on compact subsets of Ω. That is because, if there is a sub-sequence of $\{f_{m_k}\}$ converging to some $g : \Omega \to \mathbb{C}$, then $g(z) = \lim f_{m_k}(z_{m_k}) = \infty$. This is a contradiction as g maps Ω to $\mathbb{C}$.

5.5: Properties (i) and (ii) are easy to see. For property (iii) introduce the function $h(r) = r/(1 + r)$, for $r \geq 0$. Prove $h(a + b) \leq h(a) + h(b)$ for all a and b in $(0, \infty)$.

5.6: First show that the functions

$$d_i''(f, g) = \frac{\sup_{E_i} |f(z) - g(z)|}{1 + \sup_{E_i} |f(z) - g(z)|}$$

satisfy the conditions for a metric on $C^0(E_i)$. Then prove that the sum of such metrics (multiplied by $1/2^i$ to make the sum convergent) is a metric on Ω, provided E_i form an exhaustion of Ω.

5.8: Use Proposition 5.10. That is, if $f_n \to f$ uniformly on compact subsets of Ω then $f_n' \to f'$ uniformly on compact subsets of Ω.

5.9: By Theorem 5.15, it is enough to show that the family is uniformly bounded on compact sets. Let E be a compact subsets of $\mathbb{D}$. There is $r < 1$ such that $E \subset B(0, r)$. Then, for all $z \in E$ we have

$$|f(z)| \leq r + \sum_{n=2}^{\infty} |a_n z^n| \leq r + \sum_{i=2}^{\infty} nr^n \leq \sum_{i=1}^{\infty} nr^n = M_r < \infty.$$

This means that the family is uniformly bounded from above on E.

5.10: See proof of Theorem 5.15

Chapter 6

6.1: [complete solution] By Montel's theorem from the lecture notes (Theorem 5.15) it is enough to show that the family $\mathcal{S}$ is uniformly bounded on compact subsets of $\mathbb{D}$. Let E be an arbitrary compact set in $\mathbb{D}$. There is $\delta < 1$ such that $E \subset B(0, \delta)$. By the growth theorem, 6.9, for every $z \in E$ we have

$$|f(z)| \leq \frac{|z|}{(1-|z|)^2} \leq \frac{\delta}{(1-\delta)^2} < \infty.$$

As the upper bound only depends on E, we conclude that the family is uniformly bounded on E.

6.2: First show that $|f'(0)| \leq 4c$. Then apply Theorem 6.9 to an appropriately normalized map.

6.3: [complete solution] (i) For every $k \geq 2$, the set Λ_k is uniformly bounded in $\mathbb{C}$. If this is not true, there is a sequence of maps f_n in $\mathcal{S}$ such that $f_n^{(k)}(0) \to \infty$. By Exercise 6.1, $\mathcal{S}$ is a normal family and there must be a sub-sequence of f_n that converges uniformly on compact sets to some holomorphic maps $g : \mathbb{D} \to \mathbb{C}$. In particular, $g^{(k)}(0)$ is defined and finite. This contradicts the convergence of $f_n^{(k)}(0) \to g^{(k)}(0)$ guaranteed in Theorem 5.10.

By the above paragraph, for every $k \geq 2$, the set

$$A_k = \{|w| \mid w \in \Lambda_k\}$$

is bounded from above. This set is also non-empty as it contains 0; the k-th derivative of the identity map in $\mathcal{S}$. It follows that the above set has a supremum which is finite. Let r_k denote the supremum of the above set. Therefore,

$$\Lambda_k \subseteq \{w \in \mathbb{C} \mid |w| \leq r_k\}.$$

Fix an arbitrary $k \geq 2$.

By the definition of supremum, either r_k belongs to A or there is a sequence of real numbers $a_i \in A_k$, for $i \geq 1$, such that $a_i \to r_k$. In the former case we conclude that there is $f \in S$ such that $|f^{(k)}(0)| = r_k$. In the latter case, let $f_i \in \mathcal{S}$ be such that $|f_i^{(k)}(0)| = a_i$. There is a sub-sequence of f_i that converges to some map g in $\mathcal{S}$. We must have $|g^{(k)}(0)| = r_k$. So, there is always an $f \in \mathcal{S}$ such that $|f^{(k)}(0)| = r_k$.

By the above paragraph there is a point on the circle $|w| = r_k$ that belongs to Λ_k. The operations of rotation and dilatation discussed in the lecture notes show that Λ_k is invariant under rotations about 0 and is invariant under multiplication by $r \in (0,1)$. We also showed earlier that 0 belongs to Λ_k. This proves that the above inclusion is equality.

(ii) By the Cauchy integral formula for the derivatives, for every $r \in (0,1)$ we have

$$f^{(k)}(0) = \frac{k!}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{k+1}}\, dz.$$

Then by the growth theorem, Theorem 6.9, we obtain

$$\begin{aligned} |f^{(k)}(0)| = \frac{k!}{2\pi} \Big| \int_{|z|=r} \frac{f(z)}{z^{k+1}}\, dz \Big| &\leq \frac{k!}{2\pi} \int_{|z|=r} \frac{|f(z)|}{r^{k+1}}\, |dz| \\ &\leq \frac{k!}{2\pi} \int_0^{2\pi} \frac{r}{(1-r)^2 r^{k+1}}\, r d\theta \leq \frac{k! r}{(1-r)^2 r^k}. \end{aligned}$$

The above bound holds for all $r \in (0,1)$. We may find the minimum of the function $\frac{k!r}{(1-r)^2 r^k}$ on $(0,1)$, by differentiating the function. The minimum occurs at $r = 1 - 1/k$, and the minimum value is

$$\frac{k! k^2}{(1-1/k)^{k-1}}.$$

The denominator of the above expression tends to the constant e as $k \to \infty$. Hence, the denominator is uniformly bounded away from 0, independent of k.

6.5: (i) Let φ_1 and φ_2 be two such maps. Apply the Schwarz lemma, 2.1, to the maps $\varphi_2^{-1} \circ \varphi_1$ and $\varphi_1^{-1} \circ \varphi_2$.

(ii) The upper bound follows from the 1/4-theorem, the lower bound follows from the Schwarz lemma.

6.6:[complete solution] Let $\gamma : [0,1] \to \mathbb{C}$ be a smooth simple closed curve. Then, γ bounds a convex region if the slope of the tangent to γ is increasing. This is equivalent to saying that

$$\frac{\partial}{\partial t}\big(\arg \gamma'(t)\big) > 0, \forall t \in [0,1]. \tag{8.1}$$

For instance, for the curve $\gamma_0(t) = e^{2\pi i t}$, for $t \in [0,1]$, we have $\gamma_0'(t) = 2\pi i e^{2\pi i t}$. Thus,

$$\frac{\partial}{\partial t}\big(\arg \gamma_0'(t)\big) = \frac{\partial}{\partial t}(\pi/2 + 2\pi t) = 2\pi > 0.$$

So, for the inequality in Equation (8.1) to hold, it is enough to have

$$\Big| \frac{\partial}{\partial t}\big(\arg \gamma'(t)\big) - \frac{\partial}{\partial t}\big(\arg \gamma_0'(t)\big) \Big| \leq \pi. \tag{8.2}$$

Next we note that $\frac{\partial}{\partial t}(\arg \gamma'(t)$ is given in terms of γ' and γ''. This implies that, there is $\delta > 0$ such that if for all $t \in [0, 1,$ if we have

$$|\gamma_0'(t) - \gamma'(t)| \leq \delta, \quad |\gamma_0''(t) - \gamma''(t)| \leq \delta, \tag{8.3}$$

then Equation (8.2) holds. (In other words, if a closed curve γ, is close enough to γ_0 in C^0, C^1, and C^2 metrics, then it bounds a convex region containing 0.)

For an arbitrary $f \in \mathcal{S}$ and $r \in (0, 1)$ let

$$f_r(z) = \frac{1}{r} \cdot f(r \cdot z), \forall z \in \mathbb{D}.$$

We have, $f(B(0, r)) = r \cdot f_r(\mathbb{D})$. In particular, $f(B(0, r))$ is a convex region, iff $f_r(\mathbb{D})$ is a convex region. We aim to show that for small enough r, independent of $f \in \mathcal{S}$, $f_r(\mathbb{D})$ is convex. As γ_0 is the image of the circle $|z| = 1$ under the identity map, by virtue of Equation 8.3, it is enough to show that for all $z \in \mathbb{D}$, we have

$$|f_r'(z) - 1| \leq \delta, \quad |f_r''(z) - 0| \leq \delta. \tag{8.4}$$

However, $f_r'(z) = f'(rz)$, and $f_r''(z) = f''(rz) \cdot r$. It follows from the distortion theorems 6.7 and 6.6, that for small enough r, independent of f, one may guarantee the above inequalities. This completes the proof.

Chapter 7

7.2: These may be reduced to the usual derivatives with respect to x and y using the formulas in Equation (7.5).

7.3: Define the map $h(z) = g^{-1} \circ f$ from $\mathbb{C}$ to $\mathbb{C}$. Show that $\partial h/\partial \overline{z} \equiv 0$, that is, h is 1-quasi-conformal. Then apply Corollary 7.9 to h to conclude that $h : \mathbb{C} \to \mathbb{C}$ is holomorphic and one-to-one. As $h(0) = 0$ and $h(1) = 1$, by 3.15, h must be the identity map.

7.4: Use the definition of absolute continuity with $\varepsilon = 1$ to obtain some δ. Then, $[a, b]$ is covered by at most $N = \lfloor |b - a|/\delta \rfloor + 1$ number of intervals of length bounded by δ.

7.5: Assume that a sequence $f^{\circ n_k}$ converges uniformly on compact subsets of U. By the open mapping property of f, $V = f(U)$ is open, and one can show that the sequence of functions $f^{\circ n_k - 1}$ converges uniformly on compact subsets of V. This shows that $R(\mathcal{F}(R)) \subseteq \mathcal{F}(R)$. The argument in the other direction is similar, and uses the $f^{-1}(U)$ is open.

7.6: [complete solution]

(i): Let us define the function $g(z) = R^{\circ n}(z)$. Let $\delta = |g'(z_0)| < 1$ and choose $\delta' \in (\delta, 1)$. By the continuity of $z \mapsto g'(z)$ there is $r > 0$ such that for all $z \in \hat{\mathbb{C}}$ with $d(z, z_0) < r$ we have $|g'(z)| \leq \delta'$. Let $U = \{z \in \hat{\mathbb{C}} : d(z, z_0) < r\}$. Now, for $z \in U$ we have

$$d(g(z), z_0) = d(g(z), g(z_0)) \leq \sup_{c \in U} |g'(c)| \cdot d(z, z_0) \leq \delta' r < r.$$

This implies that g maps U into U. In particular, for $z \in U$ the iterates $g^{\circ n}(z)$, for $n \geq 1$, are all defined and belong to U.

For $z \in U$ we have

$$d(g^{\circ k}(z), z_0) = d(g^{\circ k}(z), g^{\circ k}(z_0)) \leq \sup_{c \in U} |(g^{\circ k})'(c)| \cdot d(z, z_0) \leq (\delta')^k \cdot r.$$

Since $\delta' < 1$, $(\delta')^k \cdot r$ tends to zero as n tends to infinity. Hence, the iterates $g^{\circ k}$ converge uniformly on U to the constant map z_0. In particular, the iterates $g^{\circ k} = R^{\circ nk}$, for $k \geq 1$, converges uniformly on compact sets in U to the constant function z_0.

(ii): Let us in the contrary assume that there is an open neighborhood U of z_0 and a sequence of iterates $R^{\circ k_m}$, for $m \geq 1$, which converges on compact subsets of U to some holomorphic map $g : U \to \hat{\mathbb{C}}$. Consider the integers k_m modulo n, and observe that there must be a sub-sequence of k_m, denoted by j_m, that are the same modulo n. That is, there are integers $t_m \in \mathbb{N}$, and an integer $r \geq 0$ such that $j_m = t_m n + r$. Then,

$$(R^{\circ j_m})'(z_0) = (R^{\circ r} \circ (R^{\circ n})^{\circ t_m})'(z_0) = (R^{\circ r})'(z_0) \cdot \delta^{t_m}$$

As $\delta > 1$, we conclude that $(R^{\circ j_m})'(z_0)$ tends to infinity. But, by Theorem 5.10, we must have $g'(z_0) = \lim_{m \to \infty} (R^{\circ j_m})'(z_0) = \infty$. This contradiction shows that there is no convergent sub-sequence on any neighborhood of z_0.

(iii): Let $g = R^{\circ qn}(z)$. We have $g(z_0) = z_0$ and $g'(z_0) = 1$. There is a neighborhood of z_0 on which g has a convergent power series $g(z) = z_0 + (z - z_0) + a_d(z - z_0)^d + \dots$ with $a_d \neq 0$. A basic calculation shows that $g^{\circ k}(z) = z_0 + (z - z_0) + k a_d (z - z_0)^d + \dots$. This implies that the d-th derivatives $(g^{\circ k})^{(d)}(z_0)$ tend to ∞. As in part (ii), this implies that $g^{\circ k}$ has no sub-sequence that converges uniformly on compact sets on a neighborhood of z_0.

Assume that there is a sub-sequence $R^{\circ k_m}$ that converges on some open set U containing z_0. Let $k_m = (qn)t_m + r_m$ with integers t_m and $0 \leq r_m \leq qn - 1$. There is a further sub-sequence of k_m such that r_m are equal for different values of m. Let $r = r_m$ be this constant. It follows that $R^{\circ (qn) t_{m+1}} = R^{\circ (qn - r)} \circ R^{\circ k_m}$ converges uniformly on compact subsets of U. This contradicts the above paragraph.