# Chapter 6

# Growth and Distortion estimates

## 6.1 The classes of maps $\mathcal{S}$ and $\Sigma$

**Definition 6.1.** Let $U$ be an open subset of $\mathbb{C}$. A holomorphic map $f : U \to \mathbb{C}$ that is one to one is called a *univalent* map. These are also called *schlicht* maps.

In this section we are concerned with the class of maps

$$\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} : f \text{ is univalent on } \mathbb{D}, f(0) = 0, f'(0) = 1\}. \tag{6.1}$$

That is, holomorphic and univalent maps defined on $\mathbb{D}$ that are normalized by the condition $f(0) = 0$ and $f'(0) = 1$. Each member of $\mathcal{S}$ has a Taylor series expansion about 0

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \tag{6.2}$$

which is convergent for $|z| < 1$.

By virtue of the Riemann mapping theorem, elements of $\mathcal{S}$ correspond to simply connected regions in $\mathbb{C}$, distinct from $\mathbb{C}$ itself, modulo some translations and re-scaling. The translations and re-scalings allows us to imposed the two conditions $f(0) = 0$ and $f'(0) = 1$. Thus, theorems about elements of $\mathcal{S}$ often translate to geometric features of the simply connected domains obtain as the images of such elements. Before we discuss such results we give some simple, but key, examples of maps in $\mathcal{S}$.

(i) The identity map $f(z) = z$ is univalent on $\mathbb{D}$. Hence, $\mathcal{S}$ is not empty.

(ii) The Koebe function we discussed in Example 5.5

$$f(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \dots.$$

The map $f$ is univalent from $\mathbb{D}$ onto $\mathbb{C} \setminus (-\infty, -1/4]$. In many ways, as we shall see in this section, $f$ is a leading example in the class $\mathcal{S}$.

(iii) The map

$$f(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + z^7 + \dots$$

which maps $\mathbb{D}$ onto $\mathbb{C} \setminus (-\infty, -1/2] \cup [1/2, \infty)$. This is obtained from the map in Example 5.4 using the transformation $z \mapsto -if(iz)$.

(iv) The map

$$f(z) = \frac{1}{2} \log \frac{1+z}{1-z},$$

which maps $\mathbb{D}$ onto the strip $-\pi/4 < \operatorname{Im} w < \pi/4$.

(v) The map

$$f(z) = z - \frac{1}{2}z^2 = \frac{1}{2}(1 - (1-z)^2),$$

which maps $\mathbb{D}$ onto a cardioid.

Note that the class of maps $\mathcal{S}$ is not closed under addition and multiplication. For example, the maps $z \mapsto \frac{z}{1-z}$ and $z \mapsto \frac{z}{1+iz}$ are in class $\mathcal{S}$, but their sum is not univalent as it has a critical point at $(1+i)/2$.

However, the class of maps $\mathcal{S}$ is preserved under a number of transformations. We list these below.

Let $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ be an arbitrary element of $\mathcal{S}$.

(i) Conjugation: The map

$$g(z) = \overline{f(\overline{z})} = z + \overline{a_2} z^2 + \overline{a_3} z^3 + \dots$$

belongs to $\mathcal{S}$. This property implies that for every integer $k \geq 1$ the set

$$\{f^{(k)}(0) : f \in \mathcal{S}\}$$

is invariant under the complex conjugation. That is, symmetric with respect to the real axis.

(ii) Rotation: For every $\theta \in \mathbb{R}$, the map

$$g(z) = e^{-i\theta} f(e^{i\theta} z) = z + e^{i\theta} a_2 z^2 + e^{i2\theta} a_3 z^3 + \dots$$

belongs to $\mathcal{S}$. This property implies that for every integer $k \geq 1$ the set

$$\{f^{(k)}(0) : f \in \mathcal{S}\}$$

is invariant under the rotations about 0.

(iii) Dilation: For every $r \in (0, 1)$, the map

$$g(z) = r^{-1} f(rz) = z + ra_2 z^2 + r^2 a_3 z^3 + \dots$$

belongs to $\mathcal{S}$.

(iv) Disk automorphism: For every fixed $\alpha \in \mathbb{D}$, the map

$$g(z) = \frac{f\Big(\frac{z+\alpha}{1+\overline{\alpha}z}\Big) - f(\alpha)}{(1-|\alpha|^2) f'(\alpha)}$$

belongs to $\mathcal{S}$.

(v) Range transformation: If $\psi$ is a function that is analytic and univalent on the range of $f$ with $\psi(0) = 0$ and $\psi'(0) = 1$ then the map $g = \psi \circ f$ belongs to $\mathcal{S}$.

(vi) Omitted value transformation: If $w$ does not belong to the range of $f$ then the map

$$g(z) = \frac{w f(z)}{w - f(z)}$$

belongs to $\mathcal{S}$. This is a special case of the transformation in (v), where we have post composed the map $f$ with the transformation $z \mapsto (wz)/(w - z)$.

(vi) Square-root transformation: There is a well-defined and continuous branch of the map

$$g(z) = \sqrt{f(z^2)}$$

that belongs to $\mathcal{S}$. To see this first note that $f(z)$ has a unique zero at 0 which implies that $f(z^2)$ has a unique zero at 0 and this zero is of order 2. Thus, if we expand the map

$$f(z^2) = z^2 + a_2 z^4 + a_3 z^6 + a_4 z^8 + \dots = z^2(1 + a_2 z^2 + a_3 z^4 + a_4 z^6 + \dots).$$

In particular, the expression in the above parenthesis never becomes zero on $\mathbb{D}$. By Proposition 5.26, there is a continuous branch of the square root of $(1 + a_2 z^2 + a_3 z^4 + a_4 z^6 + \dots)$ defined on $\mathbb{D}$. There are two such branches, with values equal to $+1$ and $-1$ at 0. We choose the branch with value $+1$ at 0, and denote it by $h(z)$. Then

$$g(z) = \sqrt{f(z^2)} = z \cdot h(z).$$

We have $g(0) = 0$ and $g'(0) = 1 \cdot h(z) + z \cdot h'(z)|_{z=0} = 1$. It remains to show that $g$ is univalent on $\mathbb{D}$.

The map $h$ is an even functions, as $h(z) = h(-z)$. Hence, $g$ is an odd function, that is $g(-z) = -g(z)$, for all $z \in \mathbb{D}$. Let $z_1$ and $z_2$ be two points in $\mathbb{D}$ with $g(z_1) = g(z_2)$. Thus, $f(z_1^2) = f(z_2^2)$. As $f$ is one-to-one, we must have $z_1^2 = z_2^2$. This implies that $z_1 = \pm z_2$. However, if $z_1 = -z_2$, then $g(z_2) = g(-z_1) = -g(z_1)$. Hence, combining with $g(z_1) = g(z_2)$, we must have $g(z_1) = 0$, which is only possible if $z_1 = 0$.

Using $(1+x)^{1/2} = 1 + x/2 - x^2/4 + \dots$, we can see that

$$h(z) = 1 + \frac{a_2}{2} z^2 + \dots.$$

This implies that

$$g(z) = \sqrt{f(z^2)} = z + \frac{a_2}{2} z^3 + \dots.$$

The symmetrization of $f$ into $g$ leads to eliminating the second derivative at 0.

Define

$$\Delta = \{w \in \mathbb{C} : |w| > 1\}.$$

A closely related class of maps to $\mathcal{S}$ is the class of maps

$$\Sigma = \big\{g : \Delta \to \mathbb{C} : g \text{ is univalent on } \Delta, \lim_{z\to\infty} g(z) = \infty, g'(\infty) = 1\big\}.$$

Recall that the condition $\lim_{z\to\infty} g(z) = \infty$ implies that $g$ is holomorphic from a neighborhood of $\infty$ to a neighborhood of infinity. The derivative of $g$ at $\infty$ is calculated by looking at the derivative of the map $f(z) = 1/g(1/z)$ at 0. That is,

$$g'(\infty) = f'(0).$$

An element of $\Sigma$ has a series expansion

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots \tag{6.3}$$

that is convergent for $|z| > 1$. Each $g \in \Sigma$ maps $\Delta$ onto the complement of some compact and connected set in $\mathbb{C}$. It is useful to consider the subclass of maps

$$\Sigma' = \{f : \Delta \to \mathbb{C} : f \in \Sigma, 0 \notin f(\Delta)\}.$$

Note that every element of $\Sigma$ can be adjusted by adding a constant term to make it an element of $\Sigma'$. Such a transformation only translates the image of the element by a constant, and does not change the shape of the image.

There is a one-to-one correspondence between $\mathcal{S}$ and $\Sigma'$ obtained by inversion. That is, for each $f \in \mathcal{S}$ the map

$$g(z) = \frac{1}{f(1/z)}, |z| > 1,$$

belongs to $\Sigma'$. One can see that if $f$ has the series expansion given in Equation (6.2), then

$$g(z) = z - a_2 + \frac{a_2^2 - a_3}{z} + \dots.$$

In particular the class of maps $\Sigma'$ is invariant under the square-root transformation,

$$G(z) = \sqrt{g(z^2)} = z(1 + b_0 z^{-2} + b_1 z^{-4} + \dots)^{1/2}.$$

Note that the square-root transformation may not be applied to elements of $\Sigma$. That is because if $g(z^2)$ has a zero at some point in $\Delta$, then $G$ will necessary have a singularity at that point.

Recall that a set $E \subset \mathbb{C}$ is said to have Lebesgue measure zero, or of zero area, if for every $\varepsilon > 0$ there are $z_i \in \mathbb{C}$ and $r_i > 0$ such that $E \subseteq \cup B(z_i, r_i)$ and $\sum_i \pi r_i^2 \leq \varepsilon$.

A relevant subclass of $\Sigma$ is

$$\tilde{\Sigma} = \{f : \Delta \to \mathbb{C} : f \in \Sigma, \mathbb{C} \setminus f(\Delta) \text{ has zero Lebegue measure.}\}$$

The functions in the above class are sometimes referred to as *full mappings*.

## 6.2 Area theorem

Gronwall in 1914 discovered the following restriction on the coefficients of the functions in class $\Sigma$.

**Theorem 6.2** (Area theorem). *If*

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots$$

*belongs to $\Sigma$, then*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1, \tag{6.4}$$

*with the equality if and only if $g \in \tilde{\Sigma}$.*

The above theorem is the basis of a theory of univalent functions, parts of which we shall present in this section. The reason for the name area theorem comes from the proof.

*Proof.* For $r > 1$, let $C_r$ denote the image of the circle $|z| = r$ under $g$. Each $C_r$ is a simple, closed, and smooth curve. Let $E_r$ denote the bounded connected component of

$\mathbb{C} \setminus C_r$. Let $w = x + iy$ be the coordinate in the image of $g$. Then, by Green's theorem, for every $r > 1$,

$$\begin{aligned}
\operatorname{area}(E_r) = \int_{C_r} x\,dy = \frac{1}{2i}\int_{C_r} \overline{w}\,dw \\
&= \frac{1}{2i}\int_{|z|=r} \overline{g(z)}g'(z)\,dz \\
&= \frac{1}{2}\int_0^{2\pi} \Big(re^{-i\theta} + \sum_{n=0}^{\infty} \overline{b_n} r^{-n} e^{in\theta}\Big)\Big(1 - \sum_{\nu=1}^{\infty} \nu b_\nu r^{-\nu-1} e^{-i(\nu+1)\theta}\Big) re^{i\theta}\,d\theta \\
&= \pi\Big(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\Big).
\end{aligned}$$

Taking limits as $r$ tends to 1 from above in the above equation, we conclude that

$$\operatorname{area}(\mathbb{C} \setminus g(\Delta)) = \pi\Big(1 - \sum_{n=1}^{\infty} n|b_n|^2\Big).$$

(Note that we are allows to take limit of the infinite sum, since the infinite sum is a decreasing function of $r$ and is uniformly bounded from above. See the dominated convergence theorem.) Since the left-hand side of the above equation is $\geq 0$, we obtain the inequality in the theorem. $\square$

As each term in the sum in Equation (6.4) is positive, we conclude that for every $n \geq 1$ we must have

$$|b_n| \leq \frac{1}{\sqrt{n}}.$$

However, these bounds are not sharp for values of $n \geq 2$. For example, for $n \geq 2$, the function

$$g_n(z) = z + n^{-1/2} z^{-n}$$

is not univalent on $\Delta$. That is because,

$$g_n'(z) = 1 - n^{1/2} z^{-n-1}$$

vanishes at some points in $\Delta$. The inequality for $n = 1$ is sharp, as stated below.

**Corollary 6.3.** *If $g \in \Sigma$, then $|b_1| \leq 1$, with equality if and only if $g$ has the form*

$$g(z) = z + b_0 + b_1/z, |b_1| = 1.$$

*The above map $g$ is a conformal mapping of $\Delta$ onto the complement of a line segment of length 4.*

*Proof.* By Theorem 6.2, we must have $|b_1| \le 1$.

If the equality $|b_1| = 1$ occurs, we must have $b_n = 0$ for all $n \ge 2$. Thus, $g$ has the desired form in the corollary.

Indeed, we can show that for any $b_0$ and $b_1$ with $|b_1| = 1$, the map $g$ belongs to $\Sigma$. Given $b_0$ and $b_1$, let $a_1 = \sqrt{b_1}$, for some choice of the square root, and then let $a_2 = 1/a_1$. Define the maps $h_1(z) = a_1 z$ and $h_2(z) = a_2 z - a_2 b_0$. The maps $h_1$ and $h_2$ are automorphisms of $\mathbb{C}$. The map $f = h_2 \circ g \circ h_1$ is defined and univalent on $\Delta$. A simple calculation shows that $f(z) = z + 1/z$, for $z \in \Delta$. In Example 5.4 we have seen that $f$ is univalent on $\Delta$ with image equal to $\mathbb{C} \setminus [-2, 2]$. This implies that $g$ is univalent on $\Delta$ and its image is equal to some line segment of length 4.

It is also clear that $g(\infty) = \infty$, and $g'(\infty) = 1$. $\square$

As a consequence of Corollary 6.3, we obtain a short proof of the Bieberbach estimate on the second coefficient.

**Theorem 6.4** (Bieberbach's Theorem). *If $f \in \mathcal{S}$, then $|a_2| \le 2$, with equality if and only if $f$ is a rotation of the Koebe function.*

*Proof.* Let $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$. We apply the square root transformation to obtain

$$h(z) = \sqrt{f(z^2)} = z + \frac{a_2}{2} z^3 + \dots.$$

We saw in Section 6.1 that this is an element of $\mathcal{S}$. Applying an inversion to the map $h$ we obtain

$$g(z) = \frac{1}{h(1/z)} = \frac{1}{f(1/z^2)^{1/2}} = \frac{1}{1/z + \dfrac{a_2}{2z^3} + \dots} = z\Big(\frac{1}{1 + \frac{a_2}{2z^2} + \dots}\Big) = z - \frac{a_2}{2}\frac{1}{z} + \dots.$$

The map $g$ belongs to $\Sigma$. Thus by Corollary 6.3, $|a_2| \le 2$.

If $|a_2| = 2$, then $g$ reduces to the form

$$g(z) = z - e^{i\theta}/z,$$

which is equivalent to

$$f(1/z^2) = \frac{z^2}{z^4 - 2e^{i\theta} z^2 + e^{2i\theta}}.$$

Using the coordinate $w = 1/z^2$ on $\mathbb{D}$ we conclude that

$$f(w) = \frac{w}{(1 - e^{i\theta} w)^2} = e^{-i\theta} \frac{e^{i\theta} w}{(1 - e^{i\theta} w)^2} = e^{-i\theta} k(e^{i\theta} w),$$

where $k$ is the Koebe function. $\square$

Recall that any holomorphic map is an open mapping. That is, the image of every open set under a holomorphic map is open. In particular, this implies that for every $f \in \mathcal{S}$, $f(\mathbb{D})$ contains some disk of positive radius centered at 0. Around 1907, Koebe discovered that there is a uniform constant $\rho$ such that the image of every map in $\mathcal{S}$ contains the open disk $B(0, \rho)$. The Koebe map suggests that $\rho$ must be less than or equal to $1/4$. Koebe conjectured that $\rho = 1/4$. Bieberbach later established this conjecture.

**Theorem 6.5** (Koebe 1/4-Theorem)**.** *For every $f \in \mathcal{S}$, $f(\mathbb{D})$ contains the ball $|w| < 1/4$.*

*Proof.* Let $f(z) = z + a_2 z^2 + \dots$ be a function in $\mathcal{S}$ that omits a value $w \in \mathbb{C}$. Using the omitted value transformation, we build the map

$$h(z) = \frac{wf(z)}{w - f(z)} = z + \big(a_2 + \frac{1}{w}\big)z^2 + \dots$$

in class $\mathcal{S}$. By Theorem 6.4, we must have

$$\Big|a_2 + \frac{1}{w}\Big| \leq 2.$$

Combining with the estimate $|a_2| \leq 2$, we conclude that $|1/w| \leq 4$. That is, $|w| \geq 1/4$. This finishes the proof of the theorem. □

The above proof shows that the Koebe function, and its rotations, are the only functions omitting a $w$ with $|w| = 1/4$. Thus, any other function in $\mathcal{S}$ covers a larger disk.

## 6.3 Growth and Distortion theorems

Shapes in $\mathbb{D}$ are distorted under a map $f \in \mathcal{S}$ according to the changes in $f'(z)$. For instance, fast changes in the size of $|f'(z)|$ cause nearby curves of the same length to be mapped to curves of very different length, or fast changes in $\arg f'(z)$ make straight line segments to be mapped to curves with sharp bends. The upper bound on the size of the second derivative at 0, that is $|a_2| \leq 2$, leads to a collection of uniform bounds on the changes of $f'(z)$ as $z$ varies in $\mathbb{D}$. Here uniform means that estimates that are independent of the map in $\mathcal{S}$. The bounds we discuss in this section are known as the Koebe distortion theorems.

We first formulate a basic theorem that leads to the distortion estimates and related results.

**Theorem 6.6.** *For each $f \in \mathcal{S}$, we have*

$$\Big|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1 - r^2}\Big| \leq \frac{4r}{1 - r^2}, \quad r = |z| < 1. \tag{6.5}$$

*Proof.* Given $f \in \mathcal{S}$ and $z \in \mathbb{D}$, we use the disk automorphism transformation to build the map

$$F(w) = \frac{f\big(\frac{w+z}{1+\overline{z}w}\big) - f(z)}{(1-|z|^2)f'(z)} = w + \frac{1}{2}\Big((1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\Big)w^2 + \dots.$$

Since the map $F \in \mathcal{S}$, by Theorem 6.4, the absolute value of the coefficient of $w^2$ in the above expansion is bounded from above by 2. Thus,

$$\Big|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\Big| \leq 4,$$

which implies the desired inequality in the theorem. □

**Theorem 6.7** (Distortion Theorem)**.** *For each $f \in \mathcal{S}$, we have*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \quad r = |z| < 1. \tag{6.6}$$

*Moreover, one of the equalities hold at some $z \neq 0$, if and only if $f$ is a suitable rotation of the Koebe function.*

In order to prove the above theorem we need a lemma on calculating derivatives with respect to the polar coordinates.

**Lemma 6.8.** *There is a continuous branch of* $\log f'(z)$ *defined on* $\mathbb{D}$ *that maps* 0 *to* 0*. Moreover, for all $z = re^{i\theta}$ in $\mathbb{D}$ we have*

$$\frac{zf''(z)}{f'(z)} = r\frac{\partial}{\partial r}(\log|f'(z)|) + ir\frac{\partial}{\partial r}(\arg f'(z)).$$

*Proof.* Recall that $f'(0) = 1$, and since $f$ is univalent on $\mathbb{D}$, for all $z \in \mathbb{D}$, $f'(z) \neq 0$. Thus, by Proposition 5.26, there is a continuous branch of $\log f'(z)$ defined on $\mathbb{D}$ which maps 0 to 0.

Let $u(z) = u(re^{i\theta})$ be an arbitrary holomorphic function defined on some open set $U \subset \mathbb{C}$. Using the relation $z = r\cos\theta + ir\sin\theta$ we have

$$r\frac{\partial u}{\partial r} = r\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r} = r\frac{\partial u}{\partial z}\cdot(\cos\theta + i\sin\theta) = z\cdot\frac{\partial u}{\partial z}.$$

Applying the above formula to the function $\log f'(z)$, and using $\log z = \log|z| + i\arg z$, we obtain the desired relation

$$\frac{zf''(z)}{f'(z)} = z\cdot\frac{\partial}{\partial z}(\log f'(z)) = r\frac{\partial}{\partial r}(\log f'(z)) = r\frac{\partial}{\partial r}(\log|f'(z)|) + ir\frac{\partial}{\partial r}(\arg f'(z)).$$

□

*Proof of Theorem 6.7.* Note that inequality $|w-c| < R$ implies $c - R \leq \operatorname{Re} w \leq c + R$. In particular, by Equation (6.5), for $|z| = r$, we have

$$\frac{2r^2}{1-r^2} - \frac{4r}{1-r^2} \leq \operatorname{Re}\Big(\frac{zf''(z)}{f'(z)}\Big) \leq \frac{2r^2}{1-r^2} + \frac{4r}{1-r^2},$$

which simplifies to

$$\frac{2r^2-4r}{1-r^2} \leq \operatorname{Re}\Big(\frac{zf''(z)}{f'(z)}\Big) \leq \frac{2r^2+4r}{1-r^2}. \tag{6.7}$$

By Lemma 6.8, there is a continuous branch of $\log f'(z)$ defined on $\mathbb{D}$ that maps 0 to 0. Moreover, the relation in the lemma, and the above inequality implies that

$$\frac{2r-4}{1-r^2} \leq \frac{\partial}{\partial r} \log |f'(re^{i\theta})| \leq \frac{2r+4}{1-r^2}. \tag{6.8}$$

Now we fix $\theta$ and integrate the above equation from 0 to $R$ to obtain

$$\log \frac{1-R}{(1+R)^3} \leq \log |f'(Re^{i\theta})| \leq \log \frac{1+R}{(1-R)^3}. \tag{6.9}$$

Above we have used the explicit calculation

$$\int_0^R \frac{2r+4}{1-r^2}\, dr = \int_0^R \frac{3}{1-r} + \frac{2}{1+r}\, dr = -3\log(1-r) + \log(1+r)\Big|_{r=0}^{r=R} = \log \frac{1+R}{(1-R)^3}.$$

As the map $x \mapsto e^x$ is monotone, Equation (6.9) implies the desired inequality in the theorem.

Assume that for some $z = Re^{i\theta} \in \mathbb{D}$, $z \neq 0$, we have an equality in Equation 6.6. Then, we must have the corresponding equality in Equation (6.9) for $R$. The latter condition implies the corresponding equality in Equation (6.8) and then in Equation (6.7), for all $r \in (0, R)$. Now let $r$ tend to 0 from above, to obtain one of the equalities

$$\operatorname{Re}\Big(e^{i\theta} f''(0)\Big) = +4, \quad \text{or} \quad \operatorname{Re}\Big(e^{i\theta} f''(0)\Big) = -4.$$

Recall that since $f \in \mathcal{S}$, by Theorem 6.4, $|f''(0)| \leq 4$. Therefore, by the above equation we must have $|f''(0)| = 4$. By the same theorem, we conclude that $f$ must be a rotation of the Koebe function.

For the Koebe function $k(z) = z/(1-z)^2$, we have

$$k'(z) = \frac{1+z}{(1-z)^3},$$

so we have the right-hand equality at every $z = r \in (0,1)$.

On the other hand, for the function $h(z) = e^{i\pi} k(e^{-i\pi} z)$, where $k$ is the Koebe function we have

$$h'(z) = k'(e^{-i\pi} z) = \frac{1-z}{(1+z)^3},$$

so we have the left-hand equality at any $z \in (0,1)$. This finishes the proof of the if and only if statement. $\square$

**Theorem 6.9** (Growth Theorem)**.** *For each $f \in \mathcal{S}$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, |z| = r. \tag{6.10}$$

*Moreover, for each $z \in \mathbb{D}$ with $z \neq 0$, equality occurs if and only if $f$ is a suitable rotation of the Koebe function.*

*Proof.* An upper bound on $|f'(z)|$ as in Theorem 6.7 gives an upper bound on $|f(z)|$. That is, fix $z = re^{i\theta} \in \mathbb{D}$. Observe that

$$f(z) = \int_0^r f'(\rho e^{i\theta}) d\rho.$$

Then,

$$|f(z)| = \leq \int_0^r |f'(\rho e^{i\theta})| \, d\rho \leq \int_0^r \frac{1+\rho}{(1-\rho)^3} \, d\rho = \frac{r}{(1-r^2)}.$$

However, since we are working in dimension 2, a lower bound on $|f'|$ does not give a lower bound $|f|$. Let $z$ be an arbitrary point in $\mathbb{D}$. We consider two possibilities:

(i) $|f(z)| \geq 1/4$,

(ii) $|f(z)| < 1/4$.

Assume that (i) occurs. Since for all $r \in (0,1)$, $r/(1+r)^2 \leq 1/4$, we trivially have $r/(1+r^2) \leq |f(z)|$.

Now assume that (ii) occurs. By the Koebe 1/4-Theorem, the radial line $rz$, for $r \in [0,1]$ is contained in the image of of $f$. As $f$ is one-to-one, the pre-image of this radial line, is a simple smooth curve in $\mathbb{D}$ connecting 0 to $z$. Let $C$ denoted this curve. We have

$$f(z) = \int_C f'(w) \, dw.$$

By the definition of $C$, for any point $w$ on $C$, $f'(w)dw$ has the same argument as the argument of $z$. Thus,

$$|f(z)| = \Big| \int_C f'(w) \, dw \Big| = \int_C |f'(w)| \, |dw| \geq \int_0^r \frac{1-\rho}{(1+\rho)^3} \, d\rho = \frac{r}{(1+r)^2}.$$

It follows from the above arguments that an inequality in either side of Equation (6.10) implies the equality in the corresponding side of Equation (6.6), which by Theorem 6.7 implies that $f$ is a suitable rotation of the Koebe function.

Also, as in the proof of the previous theorem, suitable rotations of the Koebe function lead to the equality on either side of Equation (6.10). Thus, the bounds in the theorem are sharp. $\square$

It is possible to prove a distortion estimate involving both of $|f(z)|$ and $|f'(z)|$.

**Theorem 6.10** (combined growth-distortion Theorem)**.** *For each $f \in \mathcal{S}$,*

$$\frac{1-r}{1+r} \leq \Big|\frac{zf'(z)}{f(z)}\Big| \leq \frac{1+r}{1-r}, |z| = r. \tag{6.11}$$

*Moreover, for each $z \in \mathbb{D}$ with $z \neq 0$, equality occurs if and only if $f$ is a suitable rotation of the Koebe function.*

It is not possible to conclude the above theorem as a combination of the bounds in Theorems 6.7 and 6.9. But the proof is obtained from applying the Beiberbach Theorem 6.4 to a suitable disk automorphism applied to $f$. As we have already seen this technique we skip the proof of the above theorem.

**Theorem 6.11** (Radial distortion Theorem)**.** *For each $f \in \mathcal{S}$,*

$$|\arg f'(z)| \leq 2 \log \frac{1+r}{1-r}, |z| = r. \tag{6.12}$$

*Proof.* By considering the imaginary part of the inequality in Theorem 6.7, we obtain

$$-\frac{4r}{1-r^2} \leq \mathrm{Im}\Big(\frac{zf''(z)}{f'(z)}\Big) \leq \frac{4r}{1-r^2}.$$

By Lemma 6.8, this implies that

$$-\frac{4}{1-r^2} \leq \frac{\partial}{\partial r} \arg f'(re^{i\theta}) \leq \frac{4}{1-r^2}.$$

Integrating the above equation from $r = 0$ to $r = |z|$ we obtain

$$|\arg f'(z)| \leq \int_{r=0}^{r=|z|} \frac{4}{1-r^2} dr = 2 \log \frac{1+r}{1-r}.$$

This finishes the proof of the theorem. □

The quantity $\arg f'(z)$ has a geometric interpretation as the "local rotation" factor of $f$ at $z$. Unfortunately, in contrast to the other bounds we proved in this section, the upper bound in Theorem 6.11 is not optimal. The optimal bound is

$$|\arg f'(z)| \leq \begin{cases} 4 \sin^{-1} r & r \leq 1/\sqrt{2}, \\ \pi + \log \frac{r^2}{1-r^2} & r \geq 1/\sqrt{2}. \end{cases}$$

This lies much deeper than the arguments we have seen so far. The proof relies on a more powerful method known as Loewner evolution.

We have seen so far that for every $f \in \mathcal{S}$ we have $|a_2| \leq 2$. This naturally raises the question of finding the quantities

$$A_n = \sup_{f \in \mathcal{S}} |a_n|.$$

In Exercise 6.3 you will show that these are finite numbers. The Koebe function has coefficients

$$K(z) = \sum_{n=1}^{\infty} n z^n,$$

as the Koebe function is the extreme example in the distortion theorems, Bieberbach in 1916 conjectured that $A_n = n$, for all $n$. This conjecture motivated the development of many techniques in complex analysis and eventually settled by Louis de Branges in 1985.

## 6.4 Exercises

**Exercise 6.1.** Show that the class of maps $\mathcal{S}$ forms a normal family.

**Exercise 6.2.** Let $f : \mathbb{D} \to \mathbb{C} \setminus \{c\}$ be a one-to-one and holomorphic map. Prove that for every $z \in \mathbb{D}$ we have

$$|f(z)| \leq \frac{4|cz|}{(1-|z|)^2}.$$

**Exercise 6.3.** Let $k \geq 2$ be an integer and define

$$\Lambda_k = \{f^{(k)}(0) : f \in \mathcal{S}\}.$$

Prove that

(i) for every $k \geq 1$, there is $r_k > 0$ such that $\Lambda_k = \{w \in \mathbb{C} : |w| \leq r_k\}$;

(ii) there is a constant $C > 0$ such that for all $n \geq 1$ we have $r_n \leq Cn^2$.

**Exercise 6.4.** Show that for every integer $n \geq 1$, the function

$$h_n(z) = \frac{1}{n}(e^{nz} - 1),$$

satisfies $f_n(0) = 0$, and $f_n'(0) = 1$, but $f_n$ omits value $-1/n$.

**Exercise 6.5.** Let $\Omega$ be a non-empty simply connected subset of $\mathbb{C}$ that is not equal to $\mathbb{C}$. For $z \in \Omega$, the *conformal radius* of $\Omega$ at $z$ is defined as

$$\mathrm{rad}_{\mathrm{conf}}(\Omega, z) = |\varphi'(0)|,$$

where $\varphi : \mathbb{D} \to \Omega$ is the Riemann mapping with $\varphi(0) = z$.

(i) Prove that the quantity $\mathrm{rad}_{\mathrm{conf}}(\Omega, z)$ is independent of the choice of the Riemann map $\varphi$.

(ii) Define

$$r_z = \sup\{r > 0 : B(z, r) \subset \Omega\}.$$

Prove that

$$r_z \leq \mathrm{rad}_{\mathrm{conf}}(\Omega, z) \leq 4r_z.$$

(iii) Let $\Omega' \subset \Omega$ be a simply connected set that contains $z$. Prove that

$$\mathrm{rad}_{\mathrm{conf}}(\Omega', z) < \mathrm{rad}_{\mathrm{conf}}(\Omega, z).$$

**Exercise 6.6.** Prove that there is $r > 0$ such that for every one-to-one and holomorphic map $f : \mathbb{D} \to \mathbb{C}$, the set $f(B(0, r))$ is a convex subset of $\mathbb{C}$.