Chapter 1

Preliminaries from complex analysis

1.1 Holomorphic functions

In this section we recall the key concepts and results from complex analysis.

Let $\mathbb{R}$ denote the set of real numbers, and $\mathbb{C}$ denote the set of complex numbers. It is standard to write a point $z \in \mathbb{C}$ as $z = x + iy$, where x and y are real, and $i \cdot i = -1$. Here $x = \operatorname{Re} z$ is called the *real part* of z and $y = \operatorname{Im} z$ is called the *imaginary part* of z. With this correspondence $z \mapsto (x, y)$, $\mathbb{C}$ is heomeomorphic to $\mathbb{R}^2$.

Definition 1.1. Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$. Then f is called *differentiable* at a point $z \in \Omega$ if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is a finite complex number. This limit is denoted by $f'(z)$. The map f is called *holomorphic* (*analytic*) on Ω, if f is differentiable at every point in Ω.

It easily follows that if $f : \Omega \to \mathbb{C}$ is differentiable at $z \in \Omega$, then it is continuous at z.

It is important to note that in Definition 1.1 h tends to 0 in the complex plane. (This is rather an abuse of the terminology "differentiable", as we shall see in a moment!) In particular, h may tend to 0 in any direction. Let us write the map f in the real and imaginary coordinates as $f(x + iy) = u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ are real valued functions on Ω. When h tends to 0 in the horizontal direction, then

$$f'(z) = \lim_{x \to 0} \frac{f(z+x) - f(z)}{x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}. \tag{1.1}$$

On the other hand, if h tends to 0 in the vertical direction, that is, in the y direction, then

$$f'(z) = \lim_{y \to 0} \frac{f(z+iy) - f(z)}{iy} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial f}{\partial y} \tag{1.2}$$

Then, if $f'(z)$ exists, we must have

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \tag{1.3}$$

In terms of the coordinate functions u and v, we must have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.4}$$

The above equations are known as the *Cauchy-Riemann* equations. On the other hand, if u and v are real-valued functions on Ω that have continuous first partial derivatives satisfying Equation (1.4), then $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic on Ω.

Theorem 1.2 (Cauchy-Goursat theorem-first version)**.** *Let Ω be an open set in $\mathbb{C}$ that is bounded by a smooth simple closed curve, and let $f : \Omega \to \mathbb{C}$ be a holomorphic map. Then, for any piece-wise C^1 simple closed curve γ in Ω we have*

$$\int_\gamma f(z)\, dz = 0.$$

There is an important corollary of the above theorem, that we state as a separate statement for future reference.

Theorem 1.3 (Cauchy Integral Formula-first version)**.** *Let Ω be an open set in $\mathbb{C}$ that is bounded by a smooth simple closed curve, and let $f : \Omega \to \mathbb{C}$ be a holomorphic map. Then, for any C^1 simple closed curve γ in Ω and any point z_0 in the region bounded by γ we have*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz.$$

The condition Ω bounded by a smooth simple closed curve is not quite necessary in the above two theorem. Indeed, you may have only seen the above theorems when Ω is a disk or a rectangle. We shall see a more general form of these theorems later in this course, where a topological feature of the domain Ω comes into play.

Theorem 1.3 reveals a remarkable feature of holomorphic mappings. That is, if we know the values of a holomorphic function on a simple closed curve, then we know the values of the function in the region bounded by that curve, provided we *a priori* know that the function is holomorphic on the region bounded by the curve.

There is an analogous formula for the higher derivatives of holomorphic maps as well [1]. Under the assumption of Theorem 1.3, and every integer $n \geq 1$, the n-th derivative of f at z_0 is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz. \tag{1.5}$$

[1] Chauchy had proved Theorem 1.2 when the complex derivative $f'(z)$ exists and is a continuous function of z. Then, Édouard Goursat proved that Theorem 1.2 can be proven assuming only that the complex derivative $f'(z)$ exists everywhere in Ω. Then this implies Theorem 1.3 for these functions, and from that deduce these functions are in fact infinitely differentiable.

In Definition 1.1, we only assumed that the first derivative of f exists. It is remarkable that this seemingly weak condition leads to the existence of higher order derivatives. Indeed, an even stronger statement holds.

Theorem 1.4 (Taylor-series). *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on an open set $\Omega \subseteq \mathbb{C}$. For every $z_0 \in \Omega$, the infinite series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

is absolutely convergent for z close to z_0, with the value of the series equal to $f(z)$.

The above theorems are in direct contrast with the regularity properties we know for real maps on $\mathbb{R}$ or on $\mathbb{R}^n$. That is, we have distinct classes of differentiable functions, C^1 functions, C^2 functions, C^∞ functions, real analytic functions (C^ω). For any k, it is possible to have a function that is C^k but not C^{k+1} (Find an example if you already don't know this). There are C^∞ functions that are not real analytic. For example, the function defined as $f(x) = 0$ for $x \leq 0$ and $f(x) = e^{-1/x}$ for $x > 0$. But these scenarios don't exist for complex differentiable functions.

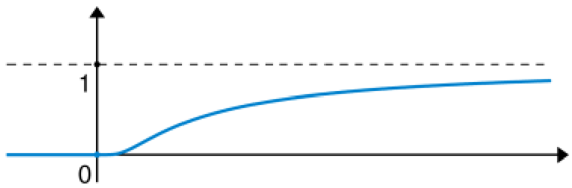

Figure 1.1: The graph of the function f.

Since a holomorphic function $f : \Omega \to \mathbb{C}$ is infinitely differentiable, higher order partial derivatives of u and v exist and are continuous. Differentiating Equations (1.4) with respect to x and y, and using $\partial_x \partial_y v = \partial_y \partial_x v$ and $\partial_x \partial_y u = \partial_y \partial_x u$, we conclude that the real functions $u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}$ are harmonic, that is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

hold on Ω. We state this as a separate theorem for future reference.

Theorem 1.5 (Harmonic real and imaginary parts). *Let $f(x + iy) = u(x, y) + iv(x, y)$ be a holomorphic function defined on an open set Ω in $\mathbb{C}$. Then, $u(x, y)$ and $v(x, y)$ are harmonic functions on Ω.*

A pair of harmonic functions u and v defined on the same domain $\Omega \subseteq \mathbb{C}$ are called *harmonic conjugates*, if they satisfy the Cauchy-Riemann equation, in other words, the function $f(x + iy) = u(x, y) + iv(x, y)$ is holomorphic.

Theorem 1.6 (maximum principle). *If $f : \Omega \to \mathbb{C}$ is a non-constant holomorphic function defined on an open set Ω, then its absolute value $|f(z)|$ has no maximum in Ω. That is, there is no $z_0 \in \Omega$ such that for all $z \in \Omega$ we have $|f(z)| \leq |f(z_0)|$.*

On the other hand, under the same conditions, either f has a zero on Ω or $|f(z)|$ has no minimum on Ω.

Let K be an open set in Ω such that the closure of K is contained in Ω. If $f : \Omega \to \mathbb{C}$ is an analytic function, $|f(z)|$ is continuous on K and by the extreme value theorem, $|f|$ has a maximum on the closure of K. But by the above theorem, $|f|$ has no maximum on K. This implies that the maximum of $|f|$ must be realized on the boundary of K. Similarly, the minimum of $|f|$ is also realized on the boundary of K.