Static vs adaptive optimal trading and good execution

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Related papers: Journal of Financial Engineering, forthcoming [6], Quantitative Finance, 21:1, 143-163, 2020 [5], Mathematical Finance, forthcoming [4].

Agenda I

1. Optimal Trade Execution
   - Problem formulation
   - Static vs Adaptive solutions
   - Previous literature and adaptive/static mix
2. Optimal execution with trading signals
   - Price model, Cost and Risk
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   - Impact
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Optimal Trade Execution: Context

Partition a large trade into smaller trades so as to minimize the effect of market impact.

E.g. big sell order placed one-shot could alert market players that we wish to short a stock (we have a view the price will go down). Market players will then offer less for the stock and the price will move down.

Optimal Trade Execution: Context

Problem: Sell X shares by the time T by minimizing cost and risk in the execution. Costs are defined in terms of (instantaneous, transient, permanent) market impact. Risk may be defined in different ways.

P is "mid price", S is impacted price, X_t is the amount left to be traded at time t, we assume $X_0 = X, X_T = 0$. Note $S_t < P_t$ due to impact.

$$S_t = P_t - \kappa_{inst}\left(-\frac{dX_t}{dt}\right) - \kappa_{trans}\,\rho \int_{\{s<t\}} e^{-\rho(t-s)}(-dX_s) - \kappa_{perm}\,(X - X_t).$$

Trading the strategy $t \mapsto X_t$ will have a cost and a risk. Cost $C(X)$ is straightforward, risk $R(X)$ can be measured in different ways (below).

Find the optimal trading schedule $t \mapsto X_t^*$ that minimizes cost plus risk, or equivalently maximizes revenues minus risk.

Optimal Trade Execution: Context

In this talk, we consider static vs adaptive best solutions.

Q_{static} Static strategies $\subseteq$ Adaptive strategies $Q_{adapted}$

We compare minimization of cost plus risk

$$\text{minimize}_{X \in Q_{adapted}} \mathbb{E}[C(X) + \phi\, R(X)]$$

versus

$$\text{minimize}_{X \in Q_{static}} \mathbb{E}[C(X) + \phi\, R(X)].$$

How much worse is the second solution when compared with the first? Question complicated by the fact that popular papers jumped from one problem to the other one when using different models.

Bertsimas & Lo (1998) [7]:

- linearly increasing execution costs in the trading rate.
- “Mid” asset price as arithmetic Brownian motion (ABM).
- Cost minimization via dynamic programming (DP) over adaptive strategies gives a **static strategy**.
- Cost minimization with information (a AR1 process) gives an **adapted sol. that is not static**

Almgren and Chriss (1999-2000) [1] and [2]

- Assumes linearly increasing execution costs in the trading rate.
- Asset “mid price” follows ABM.
- combine expected execution cost and execution risk (taken as variance of the cost).
- *Solution sought in the class of* **static strategies**.

Gatheral & Schied (2011) [10]

- As above but asset follows geometric Brownian motion (GBM)
- VaR or ES risk added to cost to be minimized
- obtain a closed form **adapted (not static)** solution.

B. & Di Graziano (2014) [8]

- solve the problem using Gatheral and Schied result but with a displaced diffusion asset price model
- introduce new risk measures such as squared asset expectation
- again **adapted** solution is found, not static.

The above does not do justice to the literature, but shows an important point: **Sometimes the optimal adapted solution is sought, and it may turn out to be either static or adapted.**

Other times the optimal static solution is sought directly.

Question: how much worse is the optimal static solution compared with the optimal adapted one (in terms of revenues minus risk)?

B. and Piat [9] investigate this for the models of Bertsimas & Lo with info and Gatheral & Schied.

They find minor differences between the optimal cost $+$ risk using the static solution vs the adapted one for most realistic parameters configurations.

This means that there are not really important adaptive features in the adaptive solution in these simple models.

Question: can we find models where optimal static & optimal adaptive are relevantly different?

YES, but we may need to add TRADING SIGNALS [13]

Optimal Trade Execution with trading signals

Let us specify our problem more in detail

- The initial time is 0, the final time is T, and usually $t \in [0, T]$;
- P_t unaffected pre-impact price at t, S_t the impacted price;
- X_t: shares left to be traded at time t; Assumed absolutely continuous and adapted;
- $X_0 = X$ (sell X shares in total), $X_T = 0$ (all shares sold by T).
- Selling shares negatively impacts the shares price:

$$S_t = P_t + \underbrace{\kappa_{inst} \frac{dX_t}{dt}}_{negative} + \underbrace{\kappa_{trans}\ \rho \int_{\{s<t\}} e^{-\rho(t-s)} dX_s}_{negative}.$$

Optimal Trade Execution with trading signals

Cost and Risk of $t \mapsto X_t$:

- Cost $C(X) := \int_0^T S_t \, dX_t$; Revenue: $-C(X)$.
- Risk $R(X)$: several possible definitions (below). We use

$$R(X) = \phi \int_0^T X_t^2 \, dt$$

  see [3, 12, 16, 15].
- We are penalizing trading schedules that are far away from the target 0. Ideally we would jump to 0 immediately to minimize $R(X)$, but note that then the negative impact term $\kappa dX_t/dt$ contributing to $C(X)$ would be very large. Trade off.
- Find X that minimizes expected cost plus risk:

$$\text{minimize}_{X_\cdot \ \textit{adapted},\ X_T=0} \mathbb{E}[C(X) + \phi\, R(X)]$$

  with ϕ a leverage param

Optimal Trade Execution with trading signals

Our minimiz prob for $\mathbb{E}[C(X) + \phi R(X)]$ is almost completely specified:

- We need to postulate a Stochastic dynamics for P (and hence S).

For the dynamics, we use the model by Lehalle and Neumann [13]

$$dP_t = I_t dt + \sigma_P \, d\widetilde{W}_t, \quad P_0$$

$$dI_t = -\gamma I_t \, dt + \sigma \, dW_t, \quad I_0.$$

W and $\tilde{W}$ are independent. An example of the signal I is limit order book imbalance: best bid Q_B and best ask Q_A of the order book,

$$\mathrm{Imb}(\tau) = \frac{Q_B(\tau) - Q_A(\tau)}{Q_B(\tau) + Q_A(\tau)},$$

just before the occurrence of a transaction at time τ^+.

$$\text{Imb}(\tau) = \frac{Q_B(\tau) - Q_A(\tau)}{Q_B(\tau) + Q_A(\tau)},$$

If $Imb > 0$, more participants want to buy than sell, and the price will move up. The opposite if it is negative.

$$dP_t = I_t dt + \sigma_P \, d\widetilde{W}_t, \quad P_0, \quad dI_t = -\gamma I_t \, dt + \sigma \, dW_t, \quad I_0.$$

This explains why I is the correct drift for the price P.

Why is imbalance mean reverting?

If $Imb > 0$, more participants want to buy, but new participants who are keen to buy may post a limit order at a higher price than current best bid to avoid long queue. Price will go up and imbalance evens out.

I need not be precisely the imbalance as defined above. Related but different trading signals are associated for example to pair trading and operate on different time scales (see [13]).

Optimal Trade Execution: Dynamics and Risk

$$dP_t = I_t dt + \sigma_P \, d\widetilde{W}_t, \quad P_0$$
$$dI_t = -\gamma I_t \, dt + \sigma \, dW_t, \quad I_0.$$

Set the trading rate r equal to

$$r_t = -dX_t/dt.$$

The impacted price is $S < P$,

$$S_t = P_t - \kappa r_t.$$

In the full paper we also deal with *transient impact*.

Optimizing revenues minus risk: static vs adapted

$$dP_t = I_t dt + \sigma_P \, d\widetilde{W}_t, \quad P_0, \quad dI_t = -\gamma I_t \, dt + \sigma \, dW_t, \quad I_0.$$

$$\text{Impacted } \; S_t = P_t - \kappa r_t.$$

$$\text{Cost: } \; C(X) = \int_0^T S_t \, dX_t.$$

$$\text{Risk: } R(X) = \phi \int_0^T X_t^2 \, dt.$$

We now look at

$$\text{maximize}_{X_\cdot \ \mathit{adapted}, \ X_T=0} \mathbb{E}[-C(X) - \phi \, R(X)]$$

versus

$$\text{maximize}_{X_\cdot \ \mathit{static}, \ X_T=0} \mathbb{E}[-C(X) - \phi \, R(X)].$$

Optimizing revenues minus risk: adapted solution

$$\text{maximize}_{X_\cdot \ \textit{adapted},\ X_T=0} \mathbb{E}[-C(X) - \phi\, R(X)]$$

In [13] (see also [6]) the solution of the above problem is derived. One has ($r_t = -dX_t/dt$)

$$r_t^{*,adapted} = -\frac{1}{2\kappa}\left(2\bar{v}_2(t)X_t^{*,adapted} + I_t \int_t^T e^{-\gamma(s-t)+\frac{1}{\kappa}\int_t^s \bar{v}_2(u)du} ds\right),$$

where

$$\bar{v}_2(t) = \sqrt{\kappa\phi}\frac{1+e^{2\beta(T-t)}}{1-e^{2\beta(T-t)}}, \ \ \beta = \sqrt{\frac{\phi}{\kappa}}.$$

It is clear that this solution is adapted in general, and not just static, since I_t (but not S_t directly!) appears in the solution. If not for I_t, *the optimal adapted sol would be static* (theorem below).
The proof is based on HJB type analysis and verification theorems.

σ_P does not play a role since it is neutralized by the expectation. It can be included in case by slightly modifying the risk function.

Optimizing revenues minus risk: static solution

$$\text{maximize}_{X_{\cdot}\ \textit{static},\ X_T=0}\mathbb{E}[-C(X) - \phi\ R(X)]$$

In [6] the solution of the above problem is derived. One has

$$X_t^* = X\psi(t) + \varphi(t),$$

$$\psi(t) = \frac{\sinh(\beta(T-t))}{\sinh(\beta T)}, \quad \beta = \sqrt{\frac{\phi}{\kappa}}.$$

$$\varphi(t) = \frac{I_0}{2\kappa(\beta^2 - \gamma^2)}\left(1 - \frac{e^{-\gamma(T-t)}\sinh(\beta t) + e^{\gamma t}\sinh(\beta(T-t))}{\sinh(\beta T)}\right).$$

This is clearly a static solution, and we can now compare it with the adapted one. The proof is based on calculus of variations.

$\gamma = 0.1,\ \sigma = 0.1,\ T = 10,\ \kappa = 0.5,\ \phi = 0.1,\ X_0 = 10$

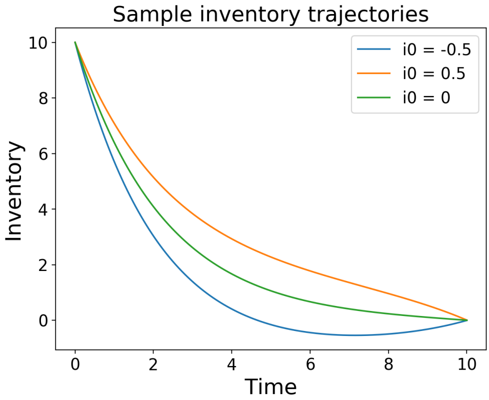


Figure: Plot of the optimal static inventory $X^{*,static}$ for the parameters above except for I_0. The optimal static strategy is presented for different initial values of the signal: $I_0 = 0.5$ (orange), $I_0 = 0$ (green) and $I_0 = -0.5$ (blue).

$\gamma = 0.1,\ \sigma = 0.1,\ T = 10,\ \kappa = 0.5,\ \phi = 0.1,\ X_0 = 10$

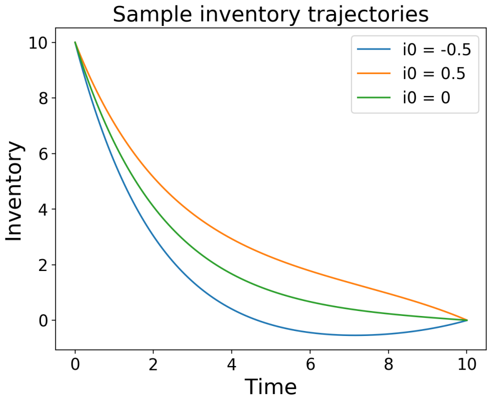


Figure: Note the negative blue plot. If the signal says the price goes down, that's not the right moment to start a liquidation. We haven't enforced a sign constraint (no exclusion of round trips)

$\gamma = 0.1,\ \sigma = 0.1,\ T = 10,\ \kappa = 0.5,\ \phi = 0.1,\ X_0 = 10$

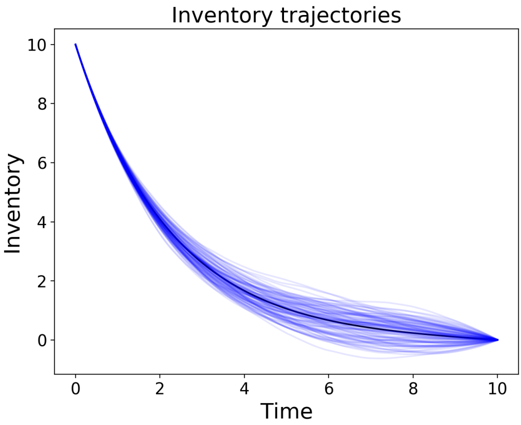
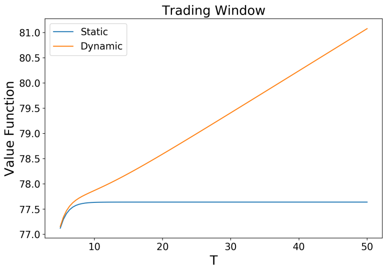


Figure: Simulation of the inventory $X^{*,adapted}$. The blue region is a plot of 1000 such trajectories of X^*. In the black curve we present the optimal static inventory. The parameters of the model are as above with $I_0 = 0.2$.

$\gamma = 0.1, \ \sigma = 0.1, \ T = 10, \ \kappa = 0.5, \ \phi = 0.1, \ X_0 = 10$



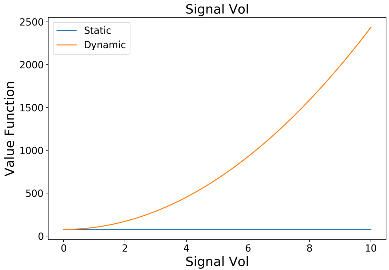


Figure: Left: comparison of $E[-C(X) - \phi R(X)]$ resulting from the optimal static strategy in blue, and the signal adaptive strategy in orange, for different values of trading windows T. The parameters of the model are as above plus $P_0 = 10$ and $I_0 = 0.2$. Right: the same comparison for different values of signal volatility σ. The model parameters (except form σ) are similar to the previous plot. *Large vols and long times make the difference relevant*

Good execution: Introduction

In the rest of the talk I would like to briefly introduce our “good execution” framework [5].

Let’s go back to static vs adaptive strategies in general.

- Static strategies
  - moderate model dependency, because only the expected quantity $E[S_t]$ enters the computations;
  - *does not* react to different realisations of S_t.
- adaptive strategies
  - heavy model dependency, because the full generator of the price dynamics is needed for HJB equation;
  - *react* to different realisations of S_t (e.g. Gatheral and Schied, in the model with signals we use here reacts to realizations of I_t).

We would like to find a middle ground between the adapted and static settings. Before proposing this middle ground, we introduce a further motivating consideration.

Introduction

Let us adopt a more general notation. Write cost plus risk as

$$J(X) = C(X) + \phi R(X) = \int_0^T F(t, S_t, X_t, \dot{X}_t)dt,$$

where $F = F(t, S, X, \dot{X})$ describes the cost of trading and the market impact that the execution itself of the trade has on the fundamental price S of the asset.

$$\text{E.g., previously } F(t, P, X, \dot{X}) = \dot{X}P + \underbrace{\kappa_{inst}\dot{X}^2 + \phi X^2}_{\tilde{F}(t, X, \dot{X})}$$

$$\int_0^T F(t, P, X, \dot{X})dt = \int_0^T (\dot{X}Pdt + \kappa_{inst}\dot{X}^2)dt + \int_0^T \phi X^2 dt =$$

$$\int_0^T \underbrace{(P_t + \kappa_{inst}\dot{X})}_{S_t} dX_t + \phi \int_0^T X^2 dt = C(X) + \phi R(X)$$

Degeneracy to static strategies

Recall: set of static strategies Q_{static} is a subset of $Q_{adaptive}$. It may happen that an optimal adaptive strategy turns out to be in particular static as in [7] (see also Cartea et al (2015) [Sec 6.3 & 6.4]).

Theorem (Optimal adaptive collapsing to static, e.g. $I_t = 0$ or det.).

$$\text{Assume that} \quad F(t, P, X, \dot{X}) = \underbrace{\dot{X} P}_{cost} + \underbrace{\tilde{F}(t, X, \dot{X})}_{impact\ plus\ risk}, \qquad (1)$$

for $\tilde{F}$ that does not depend on P. Let P be modelled

$$dP_t = \mu(t)dt + \sigma(t, P_t)dW_t, \qquad (2)$$

where the drift coefficient μ is taken to be a deterministic function of time only. Under (minor) technical assumptions, we have that the optimal adaptive solution is in fact static, namely it holds

$$\inf\left\{EJ(X) : X \in Q_{static}\right\} = \inf\left\{EJ(X) : X \in Q_{adaptive}\right\},$$

and the infimum is attained for some optimal deterministic X in Q_{static}

Good trade executions - motivation

Our aim in trying to find a setting that is in-between the static and adaptive frameworks is twofold:

- find a middle ground between moderate model dependency and ability to react to actual price realisations;
- avoid degeneracy to static strategy.

Our middle ground is “Good execution”.

Good trade executions - informal definition

We say that X is a good trade execution if for all η in two sufficiently large neighbourhoods (on different metrics) of X it holds

1. $EJ(X) \leq EJ(\eta)$ (for η in L^2 related neighborhoods of X);
2. $J(X) \leq J(\eta)$ (for η in pathwise-related neighborhoods of X);

We do not assume any specific dynamics for P_t, so that we are not bound by particular cases of HJB

The second requirement entails a pathwise assessment of the optimality of X. The fact that our concept of good trade executions avoids the degeneracy to static strategy hinges on this.

Moreover, in good execution we replace the fuel constraint $X_T = 0$ with the weakest requirement that $E[X_T] = 0$. There will be a liquidation error, for which we can introduce a penalization in case.

Good trade executions - example

$$\text{Let } \; F(t, P, X, \dot{X}) = \dot{X}P + \kappa_{inst}\dot{X}^2 + \phi X^2,$$

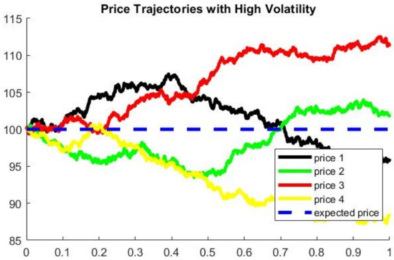
which is another way to write our previous example with linear instantaneous impact where κ is the market impact coeff. and ϕ is risk aversion. Then, for all stochastic models for P_t such that $t \mapsto E[P_t^2]$ is in $L^2[0, T]$, the following is a good trade execution

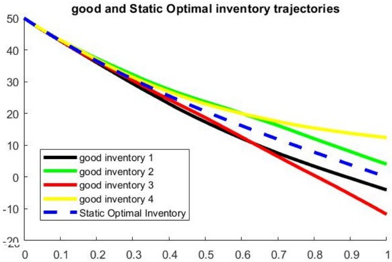
$$\begin{aligned} X(t) \quad := \quad & (1 - \alpha(t))X_0 + \alpha(t) \underbrace{X_T}_{=0 \text{ at least in } E} && \text{(convex combination)} \\ - \quad & \frac{1}{2\kappa}\int_0^t \cosh\left(\sqrt{\phi/\kappa}(t-u)\right) P_u du && \text{(adj based on realization)} \\ + \quad & K \sinh(\sqrt{\phi/\kappa}t), && \text{(ensures final expectation zero)} \end{aligned}$$

where $\alpha(t) = 1 - \sinh((T - t))/\sinh(\sqrt{\phi/\kappa}T)$, and K is a constant.

Still adapted but now P can be anything.

Good trade executions - example, cont'd





Good trade executions - remarks

1. A good trade execution commits in general an error of liquidation. Such an error depends on the volatility of the price. We can explicitly compute the variance of it and its financial interpretation is under investigation.
2. In the red trajectory the mid price is doing better than we expected (blue), so we liquidate fast as the negative impact will do less damage, and we complete liquidation around $t = 0.8$. In the yellow one the trade is unfinished and we'll have to trade a little longer.
3. Other lagrangians (other impact functions and other risk functions) than the one in the previous slide can be handled analytically within our framework of good trade execution.

Conclusions

- In Optimal execution one should seek the optimal adaptive solution, since traders monitor the market
- In some models this solution turns out to be static
- In other models it is truly adaptive
- Other authors seek the solution in the static class directly, giving up true (adaptive) optimality for tractability
- Can we compare optimal static vs adapted solutions?
- B. and Piat find that in classic models there is very little difference
- Here we showed that for models *with signals* the difference can be relevant
- Examples of signal come from book imbalance, pair trading
- Also, we introduced Good execution, a middle ground between the optimal static and fully optimal adaptive settings

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