PROBLEM CLASS: FINITE DIFFERENCES FOR AMERICAN OPTIONS

Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \(\mathbb{F} := (\mathcal{F}_t)_{t \in [0,T]}\) and a \(\mathbb{P}\)-Brownian motion \((W_t)_{t \in [0,T]}\). Let \((S_t)_{t \in [0,T]}\) be an asset price process \(S_t : [0,T] \times \Omega \to \mathbb{R}_+\). Denote by \(r\) a risk-free interest rate and consider an equivalent martingale measure \(Q \sim \mathbb{P}\) such that discounted price process is a \(Q\)-martingale. Under \(Q\) the dynamics of the stock price read

\[
dS_t = rS_t dt + \sigma S_t dW^Q_t,
\]

where \(W^Q\) is a \(Q\)-Brownian motion. Consider a European option on \(S\) with maturity \(T\) and payoff \(g(\cdot)\), and denote the \(t\)-time value of the option by \(v(t, s)\). We know that \(v(t, s)\) is the solution to the well-known Black-Scholes PDE

\[
L^r v(t, s) := L v(t, s) - rv(t, s) = 0,
\]

for all \((t, s) \in \mathbb{R}_+ \times [0, T)\) with terminal condition \(v(T, s) = g(s)\), where the differential operator \(L\) is defined as

\[
L v(t, s) := rs \partial_s v(t, s) + \frac{\sigma^2}{2} s^2 \partial_{ss} v(t, s) + \partial_t v(t, s).
\]

Now, consider an American option on \(S\), the pay-off \(A_T = g(s)\) where \(A_t\) is a random variable representing the pay-off of an American claim at time \(t\). Since an American option can be exercised at any time before the final maturity, the Snell envelope of the discounted pay-off \(v(t, s)\) satisfies

\[
v(t, s) := \sup_{\tau \in \mathcal{T}_{T,T}} \mathbb{E}^Q \{ e^{-r(T-\tau)} A_\tau | \mathcal{F}_t \},
\]

where \(\mathcal{T}_{T,T}\) is a set of stopping times valued in \([0, T]\). In fact \(v(t, s)\) is the value of an American option. The following lemma links the value of an American option to the Black-Scholes PDE via the variational inequality.

**Lemma 0.1.** The price of an American option with pay-off \(g(\cdot)\) is a function \(v : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+\) satisfying the following variational inequality

\[
\min \{ -L^r v(t, s), v(t, s) - g(s) \} = 0, \quad \text{for all } (t, s) \in [0, T) \times \mathbb{R}_+.
\]

In particular the lemma implies that \(L^r v(t, s) = 0\) for all \((t, s)\) in the continuation region \(C\) and \(v(t, s) = g(s)\) for all \((t, s)\) in the exercise region \(S\), where

\[
C := \{(t, s) \in [0, T) \times \mathbb{R}_+ : v(t, s) > g(s)\},
\]

\[
S := \{(t, s) \in [0, T] \times \mathbb{R}_+ : v(t, s) = g(s)\}.
\]

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Lemma 0.2. Suppose that there exists a function $w(t, s) \in C^{1,2}([0, T] \times \mathbb{R}_+)$ satisfying
\begin{align*}
\mathcal{L}^r w(t, s) &\leq 0, \\
g(s) &\leq w(t, s).
\end{align*}

Then $v(t, s) \leq w(t, s)$ for all $(t, s) \in [0, T] \times \mathbb{R}_+$.

The second lemma in particular implies that the price of the European call option on a non-dividend paying stock is equal to the price of an American option with the same characteristics. The usual Put-Call parity for European Options does not hold in general. However one can find a similar relationship for American Options.

Lemma 0.3. Let $P$ be the price of an American Put Option and $C$ the price of an American Call Option with strike $K$ and maturity $T$. Let the price of the underlying today is $S_0$ and $r$ be the risk-free interest rate. The following inequalities hold
\[ S_0 - K \leq C - P \leq S_0 - Ke^{-rT}. \]

Proof. Exercise. \qed