Spread options, Farkas's lemma and linear programming

How a joint distribution is constructed with given marginals is key to consistently pricing spread options. A classical result from probability ensures that can be done consistently with the principle of no arbitrage – and an easily implementable numerical approach can provide the opportunity for spotting arbitrage opportunities.

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OPTIONS ON INDIVIDUAL underlyings are very liquid in a variety of markets. In many markets, moreover, options on linear combinations of underlyings are also reasonably liquid. Of primary interest to us are markets with liquid spread options (that is, options on the difference of two underlyings), such as constant maturity swap (CMS) spread options in interest rate markets. Our discussion also naturally extends to other important examples such as foreign exchange markets with cross-rate options and equity markets with basket options, but we will concentrate on spread options from now on.

When spread options are available, it becomes important to be able to construct a joint distribution of two underlyings that is consistent with the spread option prices (and, naturally, with the marginal distributions of the underlyings themselves). This problem has attracted increased attention recently, with possible constructions proposed in Andersen & Piterbarg (2010) and Austing (2011). While it is very useful to be able to construct such distributions, these approaches suffer from at least two weaknesses. First, they are not guaranteed to produce a valid, in particular non-negative, probability density. And second, if these approaches fail, it does not necessarily mean that the joint distribution does not exist.

In this article, we derive necessary and sufficient conditions for the joint distribution with required properties to exist. The conditions turn out to have a deep financial meaning and can be formulated in terms of existence or absence of arbitrage among payouts of a certain type. From a practical perspective, we develop numerical methods to construct such joint distributions when they exist, and to find among all of them those that satisfy certain extremality criteria. Finally, we develop numerical methods to construct payouts that realise arbitrage in the case where a joint distribution does not exist.

It should be noted that our results rely on the existence of a liquid market in options on spread options of all strikes, and also assume no bid-ask spread. These assumptions clearly do not hold in practice. Nevertheless, such an idealised set-up is still valuable to develop a deeper understanding of the problem. Moreover, many trading desks mark spread options for a continuum of strikes for internal risk management, and the tools we develop could be used to test these marks for internal inconsistencies.

Existence

Let X and Y be two random variables representing two underlyings, and let S represent their spread. We assume their distributions are known from the options markets on X, Y and S.

Let $f(x)$, $g(y)$ be two functions; we think of them as defining payouts on X and Y correspondingly. In particular, the value of an option on X with payout $f(x)$ is $E(f(X))$ (here and everywhere discounting is ignored as we consider a fixed time horizon).

Let us define the spread envelope of f, g by:

$$\mathcal{E}_{f,g}(z) = \max_{x-y=z}\{f(x)+g(y)\}$$

The meaning of the spread envelope should be clear – it is the smallest function of $x - y$ only that dominates the function $f(x) + g(y)$ for all x, y:

$$\mathcal{E}_{f,g}(z) = \min\{h(z) : f(x) + g(y) \le h(x-y)\}$$

If (X, Y) have a joint distribution such that $X - Y$ has the same distribution as S, then we would clearly have:

$$E(f(X)) + E(g(Y)) \le E(\mathcal{E}_{f,g}(X-Y))$$

since by the no-arbitrage principle, if one payout is dominated by another, the value of the corresponding option is less. It turns out that the reverse is also true, and that is the main theoretical result of this article.

Theorem. Let three random variables X, Y, S be given. If for any continuous, bounded payouts $f(\cdot)$, $g(\cdot)$ we have that:

$$E(f(X)) + E(g(Y)) \le E(\mathcal{E}_{f,g}(S))$$

then, and only then, there exists a joint distribution of (X, Y) such that it has marginals X, Y and the distribution of $X - Y$ is the same as S.

As promised above, the result has a clear financial interpretation. It states that if we cannot construct an arbitrage that involves an arbitrary payout f of X only, g of Y only and their spread envelope (a function of S only), then a joint distribution exists. It can also be seen as a consequence of the fundamental theorem of asset pricing (see Duffie, 2001).

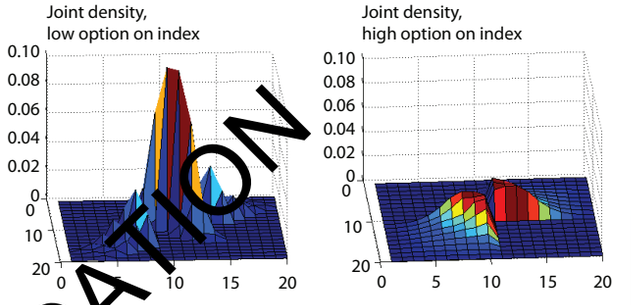
We provide a direct proof in a discrete setting in the Appendix. The proof is based on the so-called Farkas's lemma from convexity theory (see Farkas, 1902). The general statement of the theorem follows from the results in Gaffke & Rüschendorf (1984).

This result generalises ideas behind triangle arbitrage in spread options that is studied in McCloud (2011). Triangle arbitrage is based on the observation that

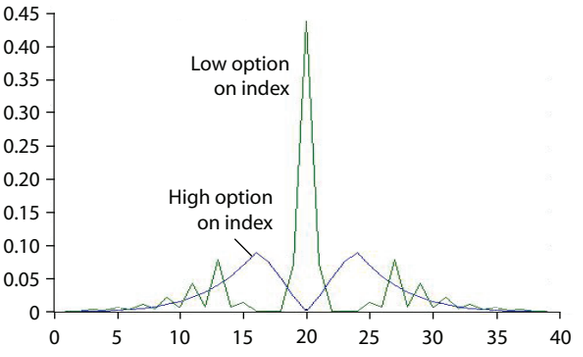
$$\begin{aligned}(X-K_x)^+ - (Y-K_y)^+ &\le (X-Y-(K_x-K_y))^+ \\ &\le (X-K_x)^+ + (K_y-Y)^+\end{aligned} \qquad (1)$$

for any strikes K_x, K_y, so if a joint distribution exists then the options on the spread must satisfy certain lower and upper bounds that depend on marginal distributions. Moreover, these bounds can be shown to be given by the Fréchet bounds on the copula that joins X and Y together (see McCloud, 2011, for details). The connection with our result is seen from the fact that the triangle inequality (1) is a special case of a spread envelope construction (for the lower bound we would take $f(x) = (x - K_x)^+$ and $g(y) = -(y - K_y)^+$ and similar for the upper bound). Absence of triangle arbitrage is, however, not sufficient for the existence of the joint distribution. In other words, we may have a situation where triangle arbitrage is absent but more general spread envelope arbitrage still exists[1] (see Piterbarg, 2011, and below). It is an open question whether it is sufficient to check the no-arbitrage conditions in our theorem for only a subset of functions f, g.

1 Joint density for Gaussian marginals and Gaussian distribution of the spread, optimised to have the lowest (left) and the highest (right) value of the option on the index



2 Density of the sum of two random variables, with the distributions of each variable and their difference Gaussian, optimised to have the highest and the lowest values of the option on the sum

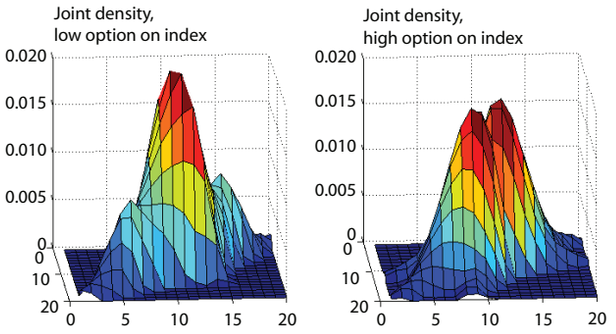


Spread options by linear programming

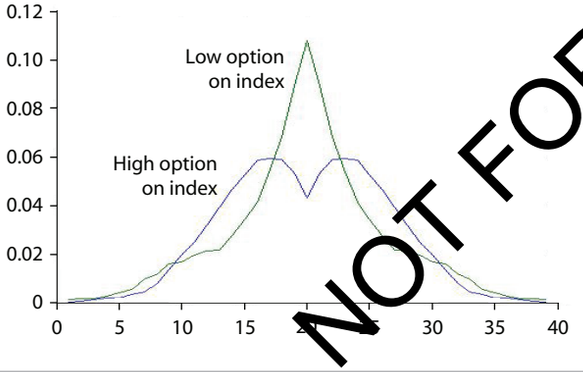
Our theorem gives us a nice theoretical result with a strong financial interpretation, but from a practical perspective it is not very useful as it would be impossible to check for spread envelope arbitrage for all the payouts. Here, we develop a practical approach to the existence problem and related questions. Given that in actual computer calculations we always work with discretised quantities, we assume that X, Y are

[1] *Of course more complicated strategies will incur higher costs (through bid-ask spreads) in practice, as more options need to be traded. We briefly discuss bid-ask spreads below*

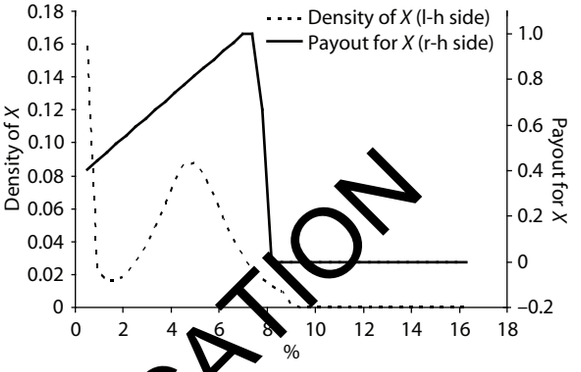
3 Joint density for Gaussian marginals and Gaussian distribution of the spread, optimised to be smooth and to have the lowest (left) and the highest (right) value of the option on the index



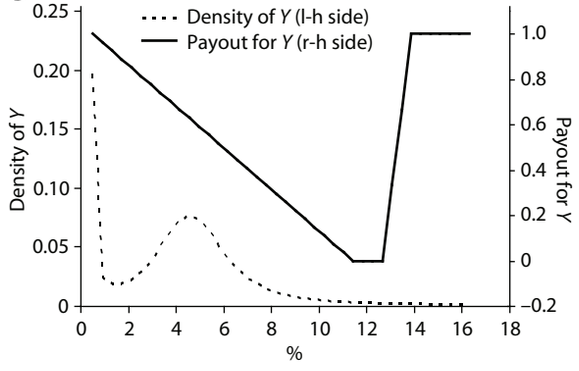
4 Density of the sum of two random variables, with the distributions of each variable and their difference Gaussian, optimised to be smooth and to have the highest and the lowest values of the option on the sum



5 Density and payout for the underlying X that realises spread envelope arbitrage



6 Density and payout for the underlying Y that realises spread envelope arbitrage



discrete, with the support on the common grid $\{i/N, i = 0, \ldots, N\}$. We assume that S is on the grid $(\{i - N)/N, i = 0, \ldots, 2N\}$. Let us denote the distributions of the three random variables by:

$$r_i = P\big(Y = (N - i)/N\big)$$
$$c_j = P\big(X = j/N\big)$$
$$d_k = P\big(S = (k - N)/N\big)$$

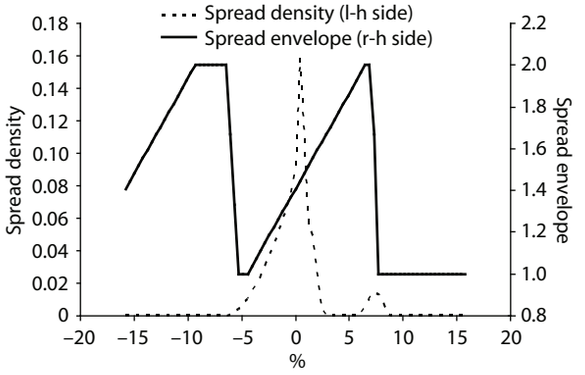
with $i, j = 0, \ldots, N$ and $k = 0, \ldots, 2N$. Vectors r, c, d are non-negative with the elements summing up to one. Note that we label rows r_i in reverse order (that is, r_0 corresponds to the maximum value of Y) for convenience and to reconcile the Cartesian view of the world (with the y-axis pointing upward) with the matrix indexing on computers where the row index increases in the downward direction.

Existence. We can reformulate the existence question in the discrete setting as follows. Let us see under what conditions on r, c, d there exists a matrix $p = \{p_{i,j}\}_{i,j=0}^{N}$ such that:

$$p_{i,j} \geq 0, \quad i, j = 0, \ldots, N \tag{2}$$

$$\sum_{j=0}^{N} p_{i,j} = r_i, \quad i = 0, \ldots, N \tag{3}$$

7 Spread envelope (function of $X - Y$) for the payouts in figures 5 and 6 that realises spread envelope arbitrage



$$\sum_{i=0}^{N} p_{i,j} = c_j, \qquad j = 0,...,N \tag{4}$$

$$\sum_{(i,j)\in D_k} p_{i,j} = d_k, \qquad k = 0,...,2N \tag{5}$$

where D_k is the kth diagonal:

$$D_k = \{(i,j) : i + j = k, 0 \le i, j \le N\}, \qquad k = 0,...,2N$$

While a theoretical answer is given by a discrete version of our theorem, ideas behind linear programming (LP) (see Gass, 2010) give us a way to find a practical solution. Recall that LP is concerned with optimising a linear function given a set of linear equality and inequality constraints. Since (2)–(5) are nothing but a set of linear equality and inequality constraints, the problem of finding p is just a problem of finding a feasible solution to an LP problem, that is, some point in the solution set that is not necessarily optimal. To find a feasible solution, another LP problem can be set up using so-called slack variables q_i, $i = 0, \ldots, 4N + 2$, where we have one for each row, column and diagonal constraint in (3)–(5). So we look for $\{p_{i,j}\}$, $\{q_i\}$ such that:

$$\sum_{i=0}^{4N+2} q_i \to \min$$
$$\text{subject to} \quad p_{i,j} \ge 0, \quad q_i \ge 0 \tag{6}$$
$$\sum_{j=0}^{N} p_{i,j} \underline{+q_i} = r_i, \ \sum_{i=0}^{N} p_{i,j} \underline{+q_{N+1+j}} = c_j, \ \sum_{(i,j)\in D_k} p_{i,j} \underline{+q_{2N+2+k}} = d_k$$

The slack variables are underlined. LP problems can be solved by the simplex method that, basically, goes vertex to vertex improving the objective function. A starting point to (6) is easy to find – we can just take $p_{i,j} = 0$ and:

Appendix: proof of theorem

We prove a discrete version of our theorem (see main text for notation). Let vectors c, r, d be given. We prove that for any vectors f, g:

$$\sum_j f_j c_j + \sum_i g_i r_i \le \sum_k d_k \max_{(i',j')\in D_k} \{f_{j'} + g_{i'}\} \tag{11}$$

then there exists a matrix p that satisfies (2)–(5).

The proof is by contradiction. Let $\mathcal{P}$ be the set of all matrices with non-negative entries, $\mathcal{P} = \mathbb{R}_+^{(N+1)\times(N+1)} \subset \mathbb{R}^{(N+1)\times(N+1)}$, and let $\mathcal{S}$ be the subset of $\mathcal{P}$ of matrices whose diagonal sums (that is, sums over D_k, $k = 0, \ldots, 2N$) are equal to d. Clearly $\mathcal{S}$ is not empty. Let R be the operator from $\mathbb{R}^{(N+1)\times(N+1)}$ into $\mathbb{R}^{N+1}$ that corresponds to summing matrix entries by row, and the same for C for columns. Let $\mathcal{D} \subset \mathcal{P}$ be the set of all matrices whose row and column sums are equal to vectors r, c correspondingly, that is, $\mathcal{D} = \{p \in \mathcal{P} : Rp = r, Cp = c\}$. Assume $\mathcal{D} \cap \mathcal{S} = \varnothing$ (that is, there is no matrix whose rows sum up to r, columns sum up to c and diagonals sum up to d). The set $\mathcal{S}$ is convex and so is $(R\mathcal{S}, C\mathcal{S}) \subset \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$. Hence there exists a hyperplane that separates $(C\mathcal{S}, R\mathcal{S})$ and (c, r). That is, there exist $f, g \in \mathbb{R}^{N+1}$ and $\alpha \in \mathbb{R}$ such that:

$$f^{\mathrm{T}} Cp + g^{\mathrm{T}} Rp < \alpha < f^{\mathrm{T}} c + g^{\mathrm{T}} r \text{ for any } p \in \mathcal{S}$$

or, in scalar notation:

$$\sum_j f_j \sum_i p_{i,j} + \sum_i g_i \sum_j p_{i,j} < \alpha < \sum_j f_j c_j + \sum_i g_i r_i \text{ for any } p \in \mathcal{S} \tag{12}$$

The left-hand side can be rewritten as:

$$\sum_j f_j \sum_i p_{i,j} + \sum_i g_i \sum_j p_{i,j} = \sum_{i,j} p_{i,j} (f_j + g_i) \tag{13}$$

Define p by the condition:

$$p_{i,j} = \begin{cases} d_k, & (i,j) \in D_k \text{ and } f_j + g_i = \max_{(i',j')\in D_k} \{f_{j'} + g_{i'}\} \\ 0, & \text{otherwise} \end{cases}$$

(if the max is not unique we just choose exactly one of the maximum points). Clearly p such defined has non-negative entries and each diagonal D_k has all zeros except one element that is equal to d_k. Hence $p \in \mathcal{S}$ and (12) holds for such p. Substituting, using (13) and the definition of p, we obtain:

$$\sum_k d_k \max_{(i',j')\in D_k} \{f_{j'} + g_{i'}\} < \alpha < \sum_j f_j c_j + \sum_i g_i r_i$$

which contradicts (11).

$$q_i = \begin{cases} r_i & i \le N \\ c_{i-N-1} & N+1 \le i \le 2N+1 \\ d_{i-2N-2} & 2N+2 \le i \end{cases}$$

Then running the simplex method we will find the optimal solution to (6). If it satisfies $q_i = 0$ for all i, then the original problem (2)–(5) has a solution given by $\{p_{i,j}\}$. If not, the solution to (2)–(5) does not exist. The simplex algorithm is very efficient and can easily handle the case of $N = 100$, say (implying about 10,000 variables to optimise over).

Extremal solutions. Having constructed a solution, the question then becomes how we can find all solutions or, failing that, solutions with special properties. Of particular interest in that regard are solutions that maximise or minimise the value of some options on X, Y that are not locked in by the construction, that is, options other than on X, Y or $X - Y$. For example, consider options on the index $X + Y$. An important practical question is what are the potential bounds on the value of such an option given that the marginals and all the spread options (options on $X - Y$) are fixed. Here, too, LP gives us a practical answer.

The value of an option with the payout $(X + Y - K_{idx})^+$ is a linear function in p:

$$T(p) = \sum_{i,j=0}^{N} \left(\frac{j}{N} + \frac{N-i}{N} - K_{idx} \right)^+ p_{i,j}$$

Hence, we can find the joint density p that satisfies constraints (2)–(5) and maximises the value of the index option (for a given strike K_{idx}) by solving the following LP problem:

$$T(p) \to \max, \quad \text{subject to constraints} (2)-(5) \tag{7}$$

Likewise, the density that gives the lowest value to the index option is found by replacing max with min in (7). In both cases the simplex algorithm can start from the feasible solution found above. The difference in values of the index option under the two densities gives a measure of model uncertainty for the case when only options on marginals and the spread are traded.

To consider an example, we look at a test case where both X and Y are Gaussian with zero mean and volatility 1%, and S is Gaussian with volatility 0.77% (implying correlation of 70% between X and Y). Figure 1 shows the two densities obtained. The values of the at-the-money index option in the two cases correspond to the implied Gaussian copula correlation of 9% (low option on index) and above 100% (high option on index). The densities of the index $X + Y$ in the two cases are shown in figure 2.

These figures also demonstrate an undesirable property of the simplex method in that it would set as many elements of p to zero as it can (normally number of variables minus number of constraints). The resulting density can be quite spiky and unrealistic from the financial perspective. To obtain smoother densities we can modify our method and instead of the linear optimisation use quadratic programming (QP), where the objective function is quadratic while the constraints are still linear. Efficient numerical methods for QP also exist. To put this into practice, we would replace the objective function in (7) with something like:

$$T(p) + w_{smooth} \sum_i \sum_j \left(p_{i,j} - p_{i,j+1} \right)^2 + \cdots \to \min$$

where the smoothing terms penalise high local variations in the density. Figure 3 shows two extreme densities with the smoothness constraints. Figure 4 shows the two corresponding densities of the index $X + Y$. The at-the-money index option value corresponds to the implied Gaussian copula correlation of 49% (low index option value) and 82% (high index option value).

Finding arbitrage. If no solution to (2)–(5) exists, our theorem implies that there is arbitrage that can be realised by trading in options on X, Y and $X - Y$. How do we find what positions we need to put on? Again, LP gives us a solution. We set up the following optimisation problem (a problem dual to (6)): find vectors f, g, h such that:

$$\sum_j f_j c_j + \sum_i g_i r_i - \sum_k h_k d_k \to \max \tag{8}$$

subject to:

$$h_k \ge f_{i'} + g_{j'} \text{ for all } (i', j') \in D_k \tag{9}$$

for all k and:

$$|f_i|, |g_j| \le 1 \text{ for all } i, j \tag{10}$$

The constraints (9) require the function $h(S)$ to dominate $f(X) + g(Y)$ for all values of the underlying random variables, and the set (10) is here to prevent infinite solutions. If the optimal value of the objective function in (8) is positive, then we have found arbitrage – we sell options with payouts $f(X)$ and $g(Y)$, buy the one with the payout $h(S)$, get positive cash upfront (from (8)), and then have a position that can never result in a negative payout (from (9)). Note that by the optimal-

ity property, the solution h will be the spread envelope of f and g.

In practice, the existence of bid-ask spreads will clearly complicate the construction of arbitrage strategies. While Peña, Vera & Zuluaga (2010) demonstrate how to add the consideration of bid-ask spreads to triangle arbitrage strategies (in the context of baskets on multiple underlyings), extending these ideas to arbitrary payouts that we consider in this article appears to be difficult.

An example solution is shown in figures 5, 6 and 7. This corresponds to CMS spread option prices observed in November 2010 in the euro. The spread density does not admit triangle arbitrage (or, equivalently, spread option prices satisfy lower/upper Fréchet bounds) but exhibit more general spread envelope arbitrage. The arbitrage profit from this strategy is of the order of a few basis points and in practice will not be realisable due to liquidity costs, yet the example is valuable as it highlights potential arbitrage issues when marking spread option smiles.

Conclusion

We demonstrate that the existence of a joint distribution that is consistent with the marginal distributions of two underlyings and the distribution of the spread between them is intimately linked to the presence or absence of arbitrage involving payouts on the underlyings and the spread – what we call the spread envelope arbitrage. We also propose practical numerical approaches based on LP to determine whether such a distribution exists. When it does exist, we show how to use LP methods to find densities satisfying certain optimality conditions, that allows us to quantify a measure of model uncertainty in non-spread two-dimensional payouts. Moreover, we show how the LP methods could be used to find payouts that realise arbitrage when the joint distribution does not exist. While we focus on spread options, straightforward extensions can be made to basket options and options on cross forex rates.

There remain a number of questions. In particular, can we meaningfully restrict the set of payouts for which we should check no-arbitrage conditions to guarantee the existence of a joint distribution? Or, more generally, is there a set of sufficient conditions for existence that is easier to check? Finally, is there a theoretical description, perhaps copula-like, of all distributions that match given marginals and the spread distribution? L&PR

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