CHAPTER XIX

LINEAR PROGRAMMING AND THE THEORY OF GAMES[1]

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The basic "scalar" problem of *linear programming* is to maximize (or minimize) a linear function of several variables constrained by a system of linear inequalities [Dantzig, II]. A more general "vector" problem calls for maximizing (in a sense of partial order) a system of linear functions of several variables subject to a system of linear inequalities and, perhaps, linear equations [Koopmans, III]. The purpose of this chapter is to establish theorems of duality and existence for general "matrix" problems of linear programming which contain the "scalar" and "vector" problems as special cases, and to relate these general problems to the theory of zero-sum two-person games.

1. NOTATION AND INTRODUCTORY LEMMAS

Capital letters, A, B, C, etc., denote rectangular matrices; lower-case letters, b, c, u, x, etc., denote vectors, regarded as one-column matrices; and Greek letters (lower case) δ, λ denote scalars—all quantities being real. A prime is used to denote transposition: thus A' denotes A transposed, and b' denotes a one-row matrix obtained by transposing the vector b. The number of components of a vector or the numbers of rows and columns of a matrix are not specified, but of course there are some implicit relations: thus the product Ax implies that the number of columns of A is the same as the number of components of x. Vector equations and inequalities are based on the following notation:

$u = 0$ means that all components of u are zero;
$u \geqq 0$ means that no components of u are negative;
$u \geq 0$ means $u \geqq 0$ with $u = 0$ excluded;
$u > 0$ means that all components of u are positive.

Other usages follow naturally: thus $u < 0$ means $-u > 0$, $u_1 \geqq u_2$ means $u_1 - u_2 \geqq 0$, etc. It should be noted that the inner product

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$b'u > 0$ if $b \geq 0$, $u > 0$; of course, $b'u \geqq 0$ if $b \geqq 0$, $u \geqq 0$. Matrix equations and inequalities use the same rules: thus $\Delta \geq D$ means $\Delta - D \geq 0$ (i.e., each element of $\Delta - D$ is nonnegative, and at least one element is positive).

The following lemmas provide the basis for the theorems in this chapter. Lemma 1 expresses a fundamental property of homogeneous linear inequalities observed by H. Minkowski [1896, p. 45]. Lemma 2 is an immediate consequence of Lemma 1, and Lemma 3 is a generalization of Lemma 2.

LEMMA 1: *In order that a homogeneous linear inequality* $b'u \geqq 0$ *hold for all* u *satisfying a system of homogeneous linear inequalities* $A'u \geqq 0$, *it is necessary and sufficient that* $b = Ax$ *for some* $x \geqq 0$.

For proofs the reader is referred to J. Farkas [1901, pp. 5–7], H. Weyl [1935 or 1950, Theorem 3], and in this volume David Gale [XVII, corollary to Theorem 2] and M. Gerstenhaber [XVIII, Theorem 11].

LEMMA 2: *In order that* $b'u < 0$ *for no* $u \geqq 0$ *such that* $A'u \geqq 0$, *it is necessary and sufficient that* $Ax \leqq b$ *for some* $x \geqq 0$.

PROOF: In Lemma 1 replace A by $[A \quad I]$ and x by $\begin{bmatrix} x \\ t \end{bmatrix}$, where I denotes an identity matrix. Then, in order that $b'u \geqq 0$ hold for all u satisfying $A'u \geqq 0$, $u \geqq 0$, it is necessary and sufficient that $b = Ax + t$ for some $x \geqq 0$, $t \geqq 0$. That is, in order that $b'u < 0$ for no $u \geqq 0$ such that $A'u \geqq 0$, it is necessary and sufficient that $Ax \leqq b$ for some $x \geqq 0$.

LEMMA 3: *In order that* $B'u \leq 0$ *for no* $u \geqq 0$ *such that* $A'u \geqq 0$, *it is necessary and sufficient that* $Ax \leqq By$ *for some* $x \geqq 0$, $y > 0$.

PROOF: To show that the x, y-condition is implied by the u-condition, we proceed as follows. Let b_k denote the kth column of the matrix B. Then the u-condition implies that $b_k'u < 0$ for no $u \geqq 0$ such that $A'u \geqq 0$, $-B'u \geqq 0$. Hence, substituting $[A \quad -B]$ for A and $\begin{bmatrix} x_k \\ y_k \end{bmatrix}$ for x in Lemma 2, we have $Ax_k - By_k \leqq b_k$ for some $x_k \geqq 0$, $y_k \geqq 0$. Then, summing for all columns of B, $A(\sum x_k) - B(\sum y_k) \leqq \sum b_k$. But $\sum b_k = Bj$, where j denotes a vector whose components are all 1's. So $A(\sum x_k) \leqq B(j + \sum y_k)$. That is, since $\sum x_k \geqq 0$ and $j + \sum y_k \geqq j > 0$, we have

$$Ax \leqq By \quad \text{for some} \quad x \geqq 0, y > 0.$$

This shows that the x, y-condition is implied by the u-condition.

To show that the x, y-condition implies the u-condition we assume, if possible, that

$$B'u_0 \leq 0 \quad \text{for some} \quad u_0 \geqq 0 \quad \text{such that} \quad A'u_0 \geqq 0.$$

Then

$$u_0'Ax \geqq 0 > u_0'By \quad \text{for all} \quad x \geqq 0, y > 0.$$

But, by the x, y-condition,

$$u_0'Ax \leqq u_0'By \quad \text{for some} \quad x \geqq 0, y > 0.$$

This contradiction shows that the denial of the u-condition implies the denial of the x, y-condition. Therefore the x, y-condition implies the u-condition. This completes the proof of Lemma 3.

2. Linear Programming Problems

Two general *dual* problems of linear programming are stated below. Each is based on the same given information—three matrices, A, B, C—and in each a matrix D is to be determined. A matrix D having a certain property is said to be *maximal* or *minimal* (under partial ordering by the rules of matrix inequalities explained in Section 1) if no other matrix Δ possessing the property is such that $\Delta \geq D$ or $\Delta \leq D$, respectively.

Problem 1: *To find a* maximal *matrix D having the property that*

(1) $Cx \geqq Dy$ for some $x \geqq 0, y > 0$ such that $Ax \leqq By$.

Problem 2: *To find a* minimal *matrix D having the property that*

(2) $B'u \leqq D'v$ for some $u \geqq 0, v > 0$ such that $A'u \geqq C'v$.

It will be shown (in Theorem 4) that there exists a matrix D providing solutions for both problems if the following existence conditions both hold:

(3) $Ax \leqq By$ for some $x \geqq 0, y > 0$,

(4) $A'u \geqq C'v$ for some $u \geqq 0, v > 0$.

It will also be shown (in Theorem 2) that Problem 1 admits a particular matrix D as solution if, and only if, Problem 2 also admits this D as a solution.

If the matrix B consists of a single column, b, and the matrix C consists of a single row, c', then D becomes a scalar, δ, and y and v become positive scalars that may be eliminated by dividing through by them. In this case the two general matrix problems reduce to the following two simple scalar problems: (a) to find the ordinary maximum, δ, of the linear function $c'x$ constrained by $Ax \leqq b$, $x \geqq 0$, and (b) to find the ordinary minimum, δ, of the linear function $b'u$ constrained by $A'u \geqq c$, $u \geqq 0$.

PROBLEM 1δ: *To find a* maximal *scalar* δ *having the property that*

$$c'x \geqq \delta \quad \text{for some} \quad x \geqq 0 \quad \text{such that} \quad Ax \leqq b.$$

PROBLEM 2δ: *To find a* minimal *scalar* δ *having the property that*

$$b'u \leqq \delta \quad \text{for some} \quad u \geqq 0 \quad \text{such that} \quad A'u \geqq c.$$

The "diet problem" of Cornfield and Stigler [1945] furnishes a typical example of Problem 2δ; another, more specialized, example occurs in the "transportation problem" of Hitchcock [1941] and Koopmans [XIV]. Fundamental methods for attacking such scalar problems have been developed by Dantzig [II, XXI, and XXIII]. The duality and existence theorems for Problems 1δ and 2δ are contained in the corollary to Theorem 2 (at the end of Section 3 of this chapter) and in the remark following the proof of Theorem 4 (in Section 4 of this chapter).

If the matrix B consists of a single column, b, but C consists of more than one row, then D becomes a vector, d, and y becomes a positive scalar that may be eliminated by division. In this case the two general matrix problems reduce to the following vector problems.

PROBLEM 1d: *To find a* maximal *vector* d *having the property that*

$$Cx \geqq d \quad \text{for some} \quad x \geqq 0 \quad \text{such that} \quad Ax \leqq b.$$

PROBLEM 2d: *To find a* minimal *vector* d *having the property that*

$$b'u \leqq d'v \quad \text{for some} \quad u \geqq 0, v > 0 \quad \text{such that} \quad A'u \geqq C'v.$$

A representative vector problem is the "efficient point" problem of Koopmans [III] from which the general matrix problems in this chapter have evolved. The following equations relate our notation to Koopmans' partitioning of his *technology* matrix A, *commodity* vector y, and *price* vector p, as regards *primary* and *final* commodities:

$$A = A_{\text{pri}}, \qquad b = -\eta_{\text{pri}}, \qquad u = p_{\text{pri}},$$

$$C = A_{\text{fin}}, \qquad d = y_{\text{fin}}, \qquad v = p_{\text{fin}}.$$

The extension to include *intermediate* commodities is indicated at the end of Section 6 of this chapter.

Of course, there are also vector problems, 1d′ and 2d′, that occur when the matrix C consists of a single row c' and v becomes a positive scalar that may be eliminated by division.

3. Duality

In preparation for the duality theorem (Theorem 2), we will now prove that the following new forms of Problems 1 and 2 are equivalent to the original forms.

PROBLEM 1 (new form): *To find a matrix D having both of the following properties:*

(1) $Cx \geqq Dy$ for some $x \geqq 0, y > 0$ such that $Ax \leqq By$,

(2*) $Cx \geq Dy$ for no $x \geqq 0, y \geqq 0$ such that $Ax \leqq By$.

PROBLEM 2 (new form): *To find a matrix D having both of the following properties:*

(2) $B'u \leqq D'v$ for some $u \geqq 0, v > 0$ such that $A'u \geqq C'v$,

(1*) $B'u \leq D'v$ for no $u \geqq 0, v \geqq 0$ such that $A'u \geqq C'v$.

Properties (1) and (2) occur also in the original statements of Problems 1 and 2. The new properties (2*) and (1*) are so denoted because they are equivalent to (2) and (1), respectively, as will be shown in the course of the proof of Theorem 2. It is to be remarked that a matrix D having both properties (1) and (2*) must produce equality, $Cx = Dy$, in property (1), and similarly that a matrix D having both properties (2) and (1*) must produce equality, $B'u = D'v$, in property (2).

THEOREM 1: *The new forms of Problems 1 and 2 are equivalent to the original forms.*

PROOF: To show that a solution D for the new Problem 1 is maximal as regards matrices having property (1), let us assume, if possible, that there is a matrix $\Delta \geq D$ having property (1). That is,

$$Cx \geqq \Delta y \quad \text{for some} \quad x \geqq 0, y > 0 \quad \text{such that} \quad Ax \leqq By.$$

Then $Cx \geqq \Delta y \geq Dy$ for the same x and y—thereby contradicting property (2*) possessed by D as a solution for the new Problem 1. Consequently, D is maximal as regards matrices having property (1). A similar argument shows that a solution D for the new Problem 2 is minimal as regards matrices having property (2).

To show that a solution D for the original Problem 1 possesses property (2*), let us assume, if possible, that

$$Cx_0 \geq Dy_0 \quad \text{for some} \quad x_0 \geqq 0, y_0 \geqq 0 \quad \text{such that} \quad Ax_0 \leqq By_0.$$

Adding this to (1), we get

$$C(x + x_0) \geq D(y + y_0) \quad \text{for some} \quad x + x_0 \geqq 0,\, y + y_0 > 0$$
$$\text{such that} \quad A(x + x_0) \leqq B(y + y_0).$$

In the system of inequalities $C(x + x_0) \geq D(y + y_0)$ there must be at least one individual inequality containing $>$, and so any element in the corresponding row of D may be increased slightly without disturbing the inequality. Then D is not maximal as regards matrices having property (1)—thereby contradicting the hypothesis that D is a solution for the original Problem 1. Hence D must possess property (2*). A similar argument shows that a solution, D, for the original Problem 1 possesses property (1*). This completes the proof of Theorem 1.

THEOREM 2 (duality theorem): *A matrix D is a solution for Problem 1 if, and only if, it is a solution for Problem 2.*

PROOF: It follows directly from Lemma 3, by substituting $\begin{bmatrix} A \\ -C \end{bmatrix}$ for A, $\begin{bmatrix} B \\ -D \end{bmatrix}$ for B, and $\begin{bmatrix} u \\ v \end{bmatrix}$ for u, that a matrix D has property (1) if, and only if, it has property (1*). Then, replacing A, B, C, D, x, y, u, v in (1) and (1*) by $-A', -C', -B', -D', u, v, x, y$, respectively, it follows that a matrix D has property (2) if, and only if, it has property (2*). In face of Theorem 1, this completes the proof of Theorem 2.

COROLLARY: *Problems 1δ and 2δ have a unique common solution, δ, or else no solution at all.*

PROOF: From Theorem 2 it follows that both problems have a common solution δ if either admits δ as a solution. Suppose that δ_1 provides another solution for either problem. Then, by Theorem 2, δ_1 provides also a solution for the dual problem. Clearly, δ_1 cannot exceed δ due to the maximal property of δ, nor can δ exceed δ_1 due to the maximal property of δ_1. So $\delta_1 = \delta$, which completes the corollary.

4. EXISTENCE

In preparation for the existence theorems (Theorems 4 and 5) we introduce a third problem based on the same data as Problems 1 and 2 and employing jointly the two properties involved in the original forms of Problems 1 and 2.

PROBLEM 3: *To find a matrix D that has both the following properties:*

(1) $Cx \geqq Dy$ for some $x \geqq 0,\, y > 0$ such that $Ax \leqq By$;

(2) $B'u \leqq D'v$ for some $u \geqq 0,\, v > 0$ such that $A'u \geqq C'v$.

A problem of this symmetric sort was formulated by von Neumann [1947] for the case in which D reduces to a scalar δ, corresponding to Problems 1δ and 2δ.

THEOREM 3: *A matrix D is a solution for Problem* 1 *or* 2 *if, and only if, it is a solution for Problem* 3.

PROOF: It is an immediate consequence of the equivalence of properties (1) and (1*), and of properties (2) and (2*), established in the proof of Theorem 2, that a matrix D has properties (1) and (2*) or (1*) and (2) if, and only if, it has properties (1) and (2); and of course, by Theorem 1, a matrix D has properties (1) and (2*) or (1*) and (2) if, and only if, it is a solution for the original Problem 1 or 2. This completes the obvious proof.

Remark: Problem 3 is not changed if the leading inequalities in properties (1) and (2) are made equalities: $Cx = Dy$ and $B'u = D'v$. This follows from the obvious facts (pointed out in sentences just preceding Theorem 1) that a matrix D having properties (1) and (2*) must give $Cx = Dy$ and that a matrix D having properties (2) and (1*) must give $B'u = D'v$.

THEOREM 4 (existence theorem): *There exists a solution, D, for Problem* 3, *and so for Problems* 1 *and* 2 *also, if, and only if, the following existence conditions are both satisfied:*

$$(3) \qquad Ax \leqq By \quad \text{for some} \quad x \geqq 0,\ y > 0,$$

$$(4) \qquad A'u \geqq C'v \quad \text{for some} \quad u \geqq 0,\ v > 0.$$

PROOF: Let $b = By_0$, and $c = C'v_0$, where y_0 and v_0 are the values of y and v in any particular set of x, y and u, v that satisfy the existence conditions (3) and (4). Then (3) and (4) imply that

$$(3\delta) \qquad Ax \leqq b \quad \text{for some} \quad x \geqq 0,$$

$$(4\delta) \qquad A'u \geqq c \quad \text{for some} \quad u \geqq 0.$$

[These two conditions are denoted by (3δ) and (4δ) because they are the counterparts of (3) and (4) for the scalar problems, 1δ and 2δ.]

By Lemma 2, (3δ) and (4δ) are equivalent to

$$(3\delta^*) \qquad b'u < 0 \quad \text{for no} \quad u \geqq 0 \quad \text{such that} \quad A'u \geqq 0,$$

$$(4\delta^*) \qquad c'x > 0 \quad \text{for no} \quad x \geqq 0 \quad \text{such that} \quad Ax \leqq 0,$$

where in the case of (4δ) and ($4\delta^*$) we must replace A, b, u, x in Lemma 2 by $-A'$, $-c$, x, u, respectively.

The inequality $b'u \geqq c'x$ holds for all $\lambda \geqq 0$, $u \geqq 0$, $x \geqq 0$ such that $Ax \leqq \lambda b$, $A'u \geqq \lambda c$. For, if $\lambda > 0$, we have

$$b'u \geqq \lambda^{-1}u'Ax \geqq c'x,$$

and, if $\lambda = 0$, we have

$$b'u \geqq 0 \geqq c'x,$$

by ($3\delta^*$) and ($4\delta^*$). Consequently,

$$\begin{bmatrix} 0 \\ b \\ -c \end{bmatrix}' \begin{bmatrix} \lambda \\ u \\ x \end{bmatrix} < 0 \quad \text{for no} \quad \begin{bmatrix} \lambda \\ u \\ x \end{bmatrix} \geqq 0$$

$$\text{such that} \quad \begin{bmatrix} b' & -c' \\ 0 & A \\ -A' & 0 \end{bmatrix}' \begin{bmatrix} \lambda \\ u \\ x \end{bmatrix} \geqq 0.$$

So, by Lemma 2,

$$\begin{bmatrix} b' & -c' \\ 0 & A \\ -A' & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ x_0 \end{bmatrix} \leqq \begin{bmatrix} 0 \\ b \\ -c \end{bmatrix} \quad \text{for some} \quad \begin{bmatrix} u_0 \\ x_0 \end{bmatrix} \geqq 0.$$

Multiplying these out, we get

$$b'u_0 \leqq c'x_0,\ Ax_0 \leqq b,\ A'u_0 \geqq c \quad \text{for some} \quad u_0 \geqq 0,\ x_0 \geqq 0.$$

But $b'u_0 \geqq u_0'Ax_0 \geqq c'x_0$, so

$$(\delta) \qquad b'u_0 = u_0'Ax_0 = c'x_0.$$

That is, replacing b and c by By_0 and $C'v_0$, we have

$$u_0'By_0 = u_0'Ax_0 = v_0'Cx_0.$$

Let

$$D = \frac{Cx_0u_0'B}{u_0'Ax_0} \quad \text{or} \quad \frac{hu_0'B}{v_0'h} + \frac{Cx_0j'}{j'y_0} \quad \text{according as} \quad u_0'Ax_0 \neq 0 \text{ or } = 0,$$

h and j denoting vectors all of whose components are 1's. Then, in either case,

$$Dy_0 = Cx_0, \quad \text{and} \quad v_0'D = u_0'B.$$

This means that our D has properties (1) and (2) for the y_0, v_0 taken initially and the x_0, u_0 arising in the course of the argument (see remark below). Consequently, D is a solution for Problem 3—and so, by Theorem 3, for Problems 1 and 2 also.

Conversely, it is obvious that (3) and (4) must hold if there exists a D having properties (1) and (2). This completes the proof of Theorem 4.

Remark: It is to be noted that the gist of the above proof—namely, the part from conditions (3δ) and (4δ) to equation (δ)—amounts to showing that Problems 1δ and 2δ have a common solution,

$$\delta = b'u_0 = u_0'Ax_0 = c'x_0,$$

when (3δ) and (4δ) both hold.

THEOREM 5 (existence theorem): *A solution, D, exists for Problem 1 if, and only if, the following existence conditions both hold:*

(3) $\quad Ax \leqq By \quad$ for some $\quad x \geqq 0,\ y > 0,$

(4*) $\quad Cx \geq 0 \quad$ for no $\quad x \geqq 0 \quad$ such that $\quad Ax \leqq 0.$

Similarly, a solution, D, exists for Problem 2 if, and only if, the following existence conditions both hold:

(4) $\quad A'u \geqq C'v \quad$ for some $\quad u \geqq 0,\ v > 0,$

(3*) $\quad B'u \leq 0 \quad$ for no $\quad u \geqq 0 \quad$ such that $\quad A'u \geqq 0.$

PROOF: By Lemma 3, conditions (3*) and (3) are equivalent. Likewise, replacing A, B, u, x, y in Lemma 3 by $-A'$, $-C'$, x, u, v, we see that (4*) and (4) are equivalent. Hence (3) and (4*) or (4) and (3*) hold if, and only if, (3) and (4) hold. And, by Theorem 4, a solution, D, exists for Problems 1 or 2 if, and only if, (3) and (4) hold. This completes the proof of Theorem 5.

Remarks: It is to be noted that each of the four existence conditions (3), (4), (3*), (4*) is necessary and sufficient that there exist a matrix D having the corresponding one of the four properties (1), (2), (1*), (2*). Thus (3) or (4) is implied by the existence of a matrix D having property (1) or (2); and conversely, if (3) or (4) holds, we can construct a matrix D having property (1) or (2) merely by taking large enough negative or positive elements, respectively. The equivalence of (1) to (1*), etc., then shows that (3*) or (4*) is necessary and sufficient for the existence of a matrix D having property (1*) or (2*), respectively.

It is to be noted also that the existence conditions (3), (3*), (4), (4*) can be interpreted in terms of special "null" problems, 1d′, 2d′ and 2d, 1d, in which $c' = 0$ and $b = 0$, respectively. For, with $C = c' = 0$, property (1) or (1*) is held by $D = d' = 0$ if, and only if, condition (3) or (3*) holds, while property (2*) or (1) is held trivially; and, with $B = b = 0$, property (2) or (2*) is held by $D = d = 0$ if, and only if, condition (4) or (4*) holds, while property (1*) or (2) is held trivially. Hence the special "null" problem, 1d′, 2d′, 2d, or 1d, admits a null solution ($d' = 0$ or $d = 0$) if, and only if, the corresponding existence condition (3), (3*), (4), or (4*) holds.

5. Programming and Games

Let A be the "payoff" matrix of a zero-sum two-person game [von Neumann and Morgenstern, 1944, Chapter III]. Then, to solve the game, we must find the *value*, λ, of the game and *optimal* (or good) *mixed strategies*, u and x, characterized by the following relations:

$$A'u \geqq \lambda i, \qquad u \geqq 0, \qquad g'u = 1,$$

$$Ax \leqq \lambda g, \qquad x \geqq 0, \qquad i'x = 1,$$

where g and i are vectors whose components are all 1's. The fact that such λ, u, x always exist—the main theorem for zero-sum two-person games—can be established as a by-product of Theorem 4. To this end, assume that $A > 0$—not an essential restriction, since the same arbitrary constant κ can be added to all the elements of a game matrix without affecting the game (except to increase the value of the game by κ). Then λ must be positive (if it exists), and the relations above can be divided throughout by λ. The divided relations may be rewritten in reverse order, as follows:

(1a) $\qquad i'x = \delta \quad \text{for some} \quad x \geqq 0 \quad \text{such that} \quad Ax \leqq g,$

(2a) $\qquad g'u = \delta \quad \text{for some} \quad u \geqq 0 \quad \text{such that} \quad A'u \geqq i;$

where now δ, x, u replace the previous $1/\lambda$, x/λ, u/λ. This amounts to Problem 3 for the special scalar case $A > 0$, $B = g$, $C = i'$, $D = \delta$. (See remark preceding Theorem 4 concerning the use of equations involving δ rather than inequalities.) By Theorem 4 this scalar problem has a solution, δ, because the existence conditions,

$$Ax \leqq g \quad \text{for some} \quad x \geqq 0; \quad A'u \geqq i \quad \text{for some} \quad u \geqq 0,$$

are easily satisfied by taking $x = 0$ and u sufficiently large. We carry the solution back to the initial game relations by dividing (1a) and (2a) throughout by δ, which is clearly positive—and unique, by the argument of the corollary to Theorem 2. Hence we conclude that the game with payoff matrix A has a unique value, $\lambda = 1/\delta$, and at least one pair of optimal mixed strategies, u and x. Such reduction of games to programming problems is treated in this volume by Dantzig [XX] and Dorfman [XXII].

It will now be shown that Problems 1 and 2, in full generality, are related through Problem 3 to a zero-sum two-person game.

THEOREM 6: *A matrix D is a solution for Problem 1 or 2 if, and only if, the game with the payoff matrix*

$$\begin{bmatrix} A & -B \\ -C & D \end{bmatrix}$$

has value zero and optimal mixed strategies

$$\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}$$

such that $v > 0$ and $y > 0$.

PROOF: Substituting

$$\begin{bmatrix} A & -B \\ -C & D \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} g \\ h \end{bmatrix}, \begin{bmatrix} i \\ j \end{bmatrix},$$

for A, u, x, g, i, respectively, in the basic relations for a zero-sum two-person game stated at the beginning of this section (g, h, i, j being vectors whose components are all 1's), and requiring $\lambda = 0$, $v > 0$, $y > 0$, we get

$$A'u \geqq C'v, \quad B'u \leqq D'v, \quad u \geqq 0, \quad v > 0, \quad g'u + h'v = 1;$$

$$Ax \leqq By, \quad Cx \geqq Dy, \quad x \geqq 0, \quad y > 0, \quad i'x + j'y = 1.$$

But these amount to properties (2) and (1) of Problem 3, coupled with the "normalizations" $g'u + h'v = 1$ and $i'x + j'y = 1$, which can always be achieved in Problem 3, because the inequalities $v > 0$ and $y > 0$ assure that (2) and (1) can be divided by $g'u + h'v$ and $i'x + j'y$, respectively. Therefore Theorem 6 is a direct consequence of Theorem 3. This completes the proof.

One further theorem relating linear programming to games is stated below. It follows out an ingenious idea of Dantzig [XX] and Brown [XXIV]. There does not seem to be any natural generalization for Problems 1 and 2.

THEOREM 7: *A solution, δ, exists for Problems 1δ or 2δ if, and only if, the* symmetric *game with the payoff matrix*

$$\begin{bmatrix} 0 & A & -b \\ -A' & 0 & c \\ b' & -c' & 0 \end{bmatrix}$$

has an optimal mixed strategy whose last component is positive.

PROOF: We will not give the proof explicitly, but it is contained in the proof of Theorem 4. (See the remark at the end of Theorem 4.)

Remark: Theorems 6 and 7 do not exclude necessarily the possibility that there also exist optimal mixed strategies lacking the specified positiveness. Thus the symmetric game above may also possess an optimal mixed strategy,

$$\begin{bmatrix} u \\ x \\ 0 \end{bmatrix},$$

even when Problems 1δ and 2δ have a solution, δ. In this particular event, $b'u = c'x = 0$ due to conditions (3δ*) and (4δ*).

6. Problems with Constraint Equations

The following dual problems present themselves when a system of *equations*,

$$Ex = Fy,$$

is added to the constraints $Ax \leqq By$, $x \geqq 0$, $y > 0$ in Problem 1.

PROBLEM 4: *To find a* maximal *matrix D having the property that $Cx \geqq Dy$ for some $x \geqq 0$, $y > 0$ such that $Ax \leqq By$, $Ex = Fy$.*

PROBLEM 5: *To find a* minimal *matrix D having the property that $B'u + F'w \leqq D'v$ for some $u \geqq 0$, $v > 0$, w, such that $A'u + E'w \leqq C'v$, the vector w being unrestricted in sign.*

These problems can be regarded as arising from Problems 1 and 2 by substituting $\begin{bmatrix} A \\ E \\ -E \end{bmatrix}$ for A, $\begin{bmatrix} B \\ F \\ -F \end{bmatrix}$ for B, and $\begin{bmatrix} u \\ w_1 \\ w_2 \end{bmatrix}$ for u. Then $w = w_1 - w_2$ is a vector whose components take all values, unrestricted in sign, as the vectors w_1 and w_2 vary subject to the constraints $w_1 \geqq 0$ and $w_2 \geqq 0$. Conversely, any vector w can be expressed as the difference $w_1 - w_2$ of two vectors $\geqq 0$, say, by taking $2w_1 = |w| + w$, and $2w_2 = |w| - w$, where $|w|$ is the vector whose components are the absolute values of the components of w.

There are exact analogues of Theorems 1–7 for these two problems, which the reader may easily formulate for himself.

If the matrices B and F consist of single columns, b and f, then D becomes a vector d, and y becomes a scalar that may be eliminated by division. In this case the general problems, 4 and 5, reduce to vector problems that bear on Koopmans' treatment of "efficient points" in the presence of *intermediate* commodities [III]. To cover this extension the following line should be added to the table of corresponding notations near the end of Section 2:

$$E = \pm A_{\text{int}}, \qquad f = 0, \qquad w = p_{\text{int}}.$$