The Micro-Price

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High frequency traders (HFT)

- HFTs are good:
  - Optimal order splitting
  - Pairs trading / statistical arbitrage
  - Market making / liquidity provision
  - Latency arbitrage
  - Sentiment analysis of news

- HFTs are evil:
  - The flash crash
  - Front running
  - Market manipulation and spoofing
HFTs care about the imbalance

Figure: Buy and sell volume conditional on (pre-trade) Imbalance
The mid-price

- The mid-price $M = \frac{P^b + P^a}{2}$
- $P^b$ is the best bid price
- $P^a$ is the best ask price
- Not a martingale (Bid-ask bounce)
- Low frequency signal
- Doesn’t use volume at the best bid and ask prices.
The weighted mid-price

- The weighted mid-price $M^w = IP^a + (1 - I)P^b$
- The imbalance $I = \frac{Q^b}{Q^b + Q^a}$
- $Q^b$ is the bid size and $Q^a$ is the ask size.
- Gatheral and Oomen (2009)
- Not a martingale
- Noisy
- Counter-intuitive examples
The weighted mid-price example

- Assume $P^b = 32.17, \ Q^b = 9, \ P^a = 31.18, \ Q^a = 1$
- Assume the second best ask is $31.19$ and the second best ask size is 27
- $M^w = 32.179 = 0.1 \cdot 32.17 + 0.9 \cdot 32.18$
- Order of size 1 at $P^a = 31.18$ cancels
- New $M^w = 32.1725 = 0.25 \cdot 32.17 + 0.75 \cdot 32.19$
- The ‘fair’ price just moved down after an ask order canceled?
Features of the Micro-Price

- $P_{t}^{micro} = F(M_{t}, I_{t}, S_{t})$
- Markov
- Martingale
- Computationally fast
- Better short term price predictions
Outline

• General definition
• Toy models
  1. micro-price = mid price
  2. micro-price = weighted mid price
• A discrete Markov model
• Data analysis
• Conclusion
Micro-price definition

Define

\[ P_{t}^{\text{micro}} = \lim_{i \to \infty} P_{t}^{i} \]

where the approximating sequence of martingale prices is given by

\[ P_{t}^{i} = \mathbb{E} [M_{\tau_{i}} | \mathcal{F}_{t}] \]

- $\mathcal{F}_{t}$ is the information contained in the order book at time $t$.
- $\tau_{1}, ..., \tau_{n}$ are (random) times when the mid-price $M_{t}$ changes
Assumptions

Assumption

The information in the order book is given by the 3 dimensional Markov process $F_t = (M_t, I_t, S_t)$ where $M_t = \frac{1}{2}(P^b_t + P^a_t)$ is the mid-price $S_t = \frac{1}{2}(P^a_t - P^b_t)$ is the bid-ask spread $I_t = \frac{Q^b_t}{Q^b_t + Q^a_t}$ is the imbalance at the top of the order book.

Assumption

The dynamics of $(M_t, I_t, S_t)$ is independent of the level $M_t$, i.e.

$$\mathbb{E}[M_{\tau_1} - M_t | M_t, I_t, S_t] \triangleq g^1(I_t, S_t)$$
Main result

Theorem

Given Assumptions 1 and Assumption 2, the \(i\)-th approximation to the micro-price can be written as

\[ P_t^i = M_t + \sum_{k=1}^{i} g^k(I_t, S_t) \]

where

\[ g^1(I_t, S_t) = \mathbb{E} [M_{\tau_1} - M_t | I_t, S_t] \]

and

\[ g^{i+1}(I_t, S_t) = \mathbb{E} [g^i(I_{\tau_1}, S_{\tau_1}) | I_t, S_t], \forall j \geq 0 \]

can be computed recursively.
3 examples

1. Mid-price independent of imbalance
2. Brownian imbalance
3. Discrete-time, finite state space

Interesting questions:
- Does the micro-price converge?
- What does it converge to?
- Is the micro-price between the bid and the ask?
- Is it sensible for large tick and small tick stocks?
First example

If

- $M_s - M_t$ is independent of $I_t$ for all $s > t$
- $M_t$ is a continuous time random walk. The jumps are binomial and symmetric, i.e. $M_{\tau_{i+1}} - M_{\tau_i}$ takes values in $(-1, 1)$, have up and down probabilities of 0.5.
- The spread $S_t = 1$

then

$$P_t^{\text{micro}} = M_t$$
Second example

If

- The process $I_t$ is a Brownian motion on the interval $[0, 1]$.
- Let $\tau_{\text{down}} = \inf\{s > t : I_s = 0\}$ and $\tau_{\text{up}} = \inf\{s > t : I_s = 1\}$ and $\tau_1 = \min(\tau_{\text{up}}, \tau_{\text{down}})$
- When $I_t$ is absorbed to 1, the mid-price jumps up with probability 0.5 or bounces back with probability 0.5.
- When $I_t$ is absorbed to 0, the mid-price jumps down with probability 0.5 or bounces back with probability 0.5.
- The spread $S_t = 1$

then

$$P_t^{\text{micro}} = M_t + I_t - \frac{1}{2}$$
Assumptions

- The time step is now discrete with $t \in \mathbb{Z}^+$,
- The imbalance $I_t$ takes discrete values $1 \leq i_I \leq n$,
- The spread $S_t$ takes discrete values $1 \leq i_S \leq m$
- The mid-price changes $M_{t+1} - M_t$ takes integer values in $K = \{ k \mid 0 < |k| \leq 2m \}$.
- Define the state $X_t = (I_t, S_t)$ with discrete values $1 \leq i \leq nm$
Computing $g^1$

The first step approximation to the micro-price

$$g^1(i) = \mathbb{E}[M_{\tau_1} - M_t | X_t = i]$$

$$= \sum_{k \in K} k \cdot \mathbb{P}(M_{\tau_1} - M_t = k | X_t = i)$$

$$= \sum_{k \in K} \sum_s k \cdot \mathbb{P}(M_{\tau_1} - M_t = k \land \tau_1 - t = s | X_t = i)$$
The transition probability matrix $T_1$

Then we define an *absorbing* Markov chain completely identified by the transition probability matrix $T^1$ in canonical form:

$$T^1 = \begin{pmatrix} Q & R^1 \\ 0 & I \end{pmatrix}$$

- $Q$ is $nm \times nm$ matrix
- $R^1$ is $nm \times 4m$ matrix
- $I$ is the $4m \times 4m$ matrix
Computing $g^1$

Absorbing states

$$R_{ik}^1 := \mathbb{P}(M_{t+1} - M_t = k | X_t = i)$$

Transient states

$$Q_{ij} := \mathbb{P}(M_{t+1} - M_t = 0 \land X_{t+1} = j | X_t = i)$$

Note that $R^1$ is an $nm \times 4m$ matrix and $Q$ is an $nm \times nm$ matrix.

$$g^1(i) = \left( \sum_s Q^{s-1} R^1 \right) k = (1 - Q)^{-1} R^1 k$$

where $k = \left[ -2m, -2m + 1, \ldots, -1, 1, \ldots, 2m - 1, 2m \right]^T$
Computing $g^{i+1}$

Define a new matrix of absorbing states

$$R_{ik}^2 := \mathbb{P}(M_{t+1} - M_t \neq 0 \land I_{t+1} = k | I_t = i)$$

Once again applying standard techniques for discrete time Markov processes with absorbing states

$$g^{i+1}(i) = \left( \sum_s Q^{s-1} R^2 \right) g^i = (1 - Q)^{-1} R^2 g^i$$
Checking that the micro-price converges

Define $B := (1 - Q)^{-1} R^2$.

**Theorem**

*If $B$ has strictly positive entries and $\lim_{k \to \infty} B^k = W$ where $W$ is the unique stationary distribution and $W g_1 = 0$, then the limit $\lim_{i \to \infty} p_t^i = p_t^{micro}$ converges.*
A spectral decomposition for the micro-price

Perron-Frobenius decomposition

\[ p_t^{\text{micro}} = \lim_{i \to \infty} p_t^i = M_t + \sum_{i=2}^{nm} \exp(\lambda_i) B_i g^1 \]

where \( \lambda_i \) are the eigenvalues of \( B \) and \( B_i \) are matrices formed from normalized left and right eigenvectors of \( B \).
The data

Bid and ask quotes for Bank of America (BAC) and Chevron (CVX), for the month of March 2011.

Figure: Spread histograms for BAC and CVX. BAC is a typical large tick stock and CVX is a typical small tick stock.
The in-sample estimation

- Estimate transition probabilities $Q$, $R^1$ and $R^2$
- Compute $g^1 = (1 - Q)^{-1} R^1 k$. This function is symmetrized to ensure that $g^1(i_I, i_S) = 1 - g^1(n - i_I, i_S)$.
- Compute $B = (1 - Q)^{-1} R^2$. This function is symmetrized to ensure that $B(i_I, i_S), (j_I, j_S) = B(n - i_I, i_S), (n - j_I, j_S)$. Note that the symmetrizing procedure ensures that $B \dot{g}^1 = 0$ and that the micro-price converges as guaranteed by Theorem 2.
- Perform a spectral decomposition of $B$ in terms of eigenvalues $\lambda_i$ and matrices $B_i$
- Compute the micro-price adjustment:

$$G^* = p^{micro} - M = \sum_{i=2}^{nm} \exp(\lambda_i) B_i g^1$$
In-sample results

Figure: $G^* = p_t^{micro} - M_t$ as a function of $I$ and $S$
Out of sample validation part 1

- Compute averages of $M_{t+60} - M_t$ grouped by $I_t$ and $S_t$ for 3 out of sample days
- Compare to $G^*$ obtained from the first day or March.
Out of sample results part 1

Figure: $G^*$ vs 1 min price predictions on three consecutive days
Out of sample validation part 2

- Compute averages of $M_{t+60} - M_t$, $M_{t+300} - M_t$ and $M_{t+600} - M_t$ grouped by $I_t$ and $S_t$ for the entire month of March.
- Compare to $G^*$ obtained from the first day of March.
Out of sample results part 2

Figure: $G^*$ vs 1min, 5min and 10min price predictions for March 2011
Summary

1. Have defined the micro-price as the expected mid-price in the distant future
2. When fitting a Markov model, we have conditions that ensures this micro-price converges
3. Micro-price is a good predictor of future mid prices
4. Micro-price can fit very different microstructures
5. Micro-price needs less data to converge than averaging mid price changes over fixed horizons
6. Micro-price is horizon independent
7. Micro-price seems to live between the bid and the ask
Future work

1. Including other factors than imbalance and spread
2. Continuous models for the micro-price
3. Connections to quantities such as volatility, volume and tick size
4. High frequency volatility and correlation estimation
5. Applications to HFT strategies

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