Optimal portfolio choice with path dependent labor income

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Outline

1 Overview and motivation

2 Benchmark model (no path dependency)

3 Path-dependent wages

4 Conclusion
Lifecycle portfolio choice problem with borrowing (state) constraints where an agent receives labor income.

Novelty: path-dependency of the wage income process (“slow” adjustment to financial market shocks; “learning” your income) which leads to an infinite dimensional stochastic optimal control problem.

We solve completely the problem, and find explicitly the optimal controls in feedback form. Tool: explicit solution to the associated infinite dimensional Hamilton-Jacobi-Bellman (HJB) equation.

First step towards more general and interesting problems and more general solution methods.
Motivation: Portfolio choice

- **Merton (1971):** lifetime investment in risky stocks and riskless asset. Optimal for agents to allocate a *constant fraction of wealth in the risky asset* throughout their lives.

- Importance of labor income in shaping portfolio choice: e.g., Bodie et al. (1992), Campbell-Viceira (2002), Fahri-Panageas (2007), Dybvig-Liu (2010). The **total wealth** of an agent is given by both financial wealth and **human capital**, i.e., the market value of future labor income.

- **Key finding I:** investors should allocate a constant fraction of their **total wealth** to the risky asset.

- **Key finding II:** negative **hedging demand** for risky assets arises from the implicit holding of the risky assets in human capital.
Labour income dynamics

- ARMA processes commonly used to model the stochastic component of wages (e.g., MaCurdy, 1982; Abowd-Card, 1989; Meghir-Pistaferri, 2004; Storesletten et al., 2004).


Sticky wages

- Empirical evidence on wage rigidity suggests that labor income adjusts slowly to financial market shocks (e.g., Khan, 1997; Dickens et al., 2007; LeBihan et al., 2012).

- Delayed labor income dynamics as a tractable model to capture this feature.
Motivation: Human Capital II

Learning your income

- Shocks in labor income have modest persistency when heterogeneity in income growth rates is taken into account.
- Allowing agents to learn in (say) a Bayesian way about income growth can match several empirical features of consumption data (e.g., Guvenen, 2007, 2009).
- Bounded rationality and rational inattention can support the use of moving averages instead of optimal filters (e.g., Zhu and Zhou, 2009).
- Path dependent labor income retains tractability and delivers explicit solutions.
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Financial market of Black & Scholes type:

\[
dS_0(t) = rS_0(t)dt \\
\frac{dS_1(t)}{S_1(t)} = \mu dt + \sigma dZ(t),
\]

with \(0 < r < \mu, \sigma > 0\).

- \(Z\) is a Wiener process on a given filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\).

- We consider one risky asset for illustration only, the case of \(n > 1\) risky assets working in a similar way.
Consider the state equation (budget constraint and wage process)

\[
\begin{align*}
    dW(t) &= \left[ W(t)r + \theta(t)(\mu - r) - c(t) - \delta(B(t) - W(t)) \right] dt \\
    &\quad + (1 - R(t))y(t)dt + \theta(t)\sigma dZ(t), \quad W(0) = W_0 \\
    dy(t) &= y(t)(\mu_y dt + \sigma_y dZ(t)), \quad y(0) = y_0
\end{align*}
\]

- \( W(t) \) wealth process (state)
- \( y(t) \) labor income process (state)
- \( \theta(t) \) investment in the risky asset (control)
- \( c(t) \) consumption (control)
- \( B(t) \) bequest (control)
- \( R(t) := \mathbb{I}_{\{T \leq t\}} \) and \( T \) is the retirement date (control)
- \( \delta > 0 \) constant rate of mortality
- \( \mu_y, \sigma_y > 0. \)
The agent’s death time $\tau_\delta$ is modeled as a Poisson arrival time (with parameter $\delta > 0$) independent of the Wiener process $Z$.

We should consider as reference filtration the one generated by $\tau_\delta$ and $Z$, but we will actually work on $\{\tau_\delta > t\}$.

$B(t)$ is the bequest the agent targets for his/her beneficiaries:

- for $W(t) - B(t) < 0$, the agent purchases continuously life insurance with premium flow $\delta(B(t) - W(t))$;
- for $W(t) - B(t) > 0$, the agent is essentially receiving a life annuity flow $\delta(B(t) - W(t))$, as (s)he trades wealth in the event of death for a cash inflow while living.
Goal: maximize over \((c(\cdot), B(\cdot), \theta(\cdot), T)\) the objective

\[
\mathbb{E} \left\{ \int_0^{T(\delta)} e^{-\rho t} \left( (1 - R(t)) \frac{c(t)^{1-\gamma}}{1-\gamma} + R(t) \frac{(Kc(t))^{1-\gamma}}{1-\gamma} \right) dt 
\right. \\
+ e^{-\rho T(\delta)} \frac{(kB(T(\delta)))^{1-\gamma}}{1-\gamma} dt \right\},
\]

where \(K > 1\) allows the utility from consumption to differ before and after \(T\), and \(k > 0\) measures the intensity of preference for leaving a bequest.

The expectation above can be written as follows:

\[
J(W_0, y_0; c, B, \theta, T) := \mathbb{E} \left\{ \int_0^{+\infty} e^{-(\rho+\delta)t} \left( \frac{(K^{R(t)}c(t))^{1-\gamma}}{1-\gamma} 
\right. \\
+ \delta \frac{(kB(t))^{1-\gamma}}{1-\gamma} \left. \right) dt \right\}
\]
The state constraint

Dybvig-Liu (2010), Problem 1

For fixed retirement date $T \leq +\infty$, consider the following no-borrowing-without-repayment constraint:

$$ W(t) \geq -g(t)y(t), $$

with

$$ g(t) := \left( \frac{1 - e^{-\beta_1 (T-t)}}{\beta_1} \right)^+, $$

where we assume $\beta_1 > 0$, with $\beta_1 := r + \delta - \mu_y + \frac{(\mu-r)}{\sigma} \sigma_y$. 
Meaning of the constraint

Let $\xi(t)$ be the mortality risk adjusted state price density:

$$
\xi(t) := e^{-(r+\delta+\frac{1}{2} \frac{(\mu-r)^2}{\sigma^2})t - \frac{\mu-r}{\sigma} Z(t)},
$$

i.e., the solution of

$$
\begin{align*}
\left\{ 
\begin{array}{l}
    d\xi(t) = -\xi(t)(r + \delta)dt - \xi(t)\frac{\mu-r}{\sigma} dZ(t), \\
    \xi(0) = 1.
\end{array}
\right.
\end{align*}
$$

Then

$$
g(t)y(t) = \xi(t)^{-1} \mathbb{E} \left( \int_t^T y(s)\xi(s)ds \bigg| \mathcal{F}_t \right),
$$

which is nothing else than the human capital at time $t$. 
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Overview and motivation

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Our model

For simplicity we focus on the infinite horizon case \( T = +\infty \).

State equation:

\[
dW(t) = \left[ W(t)r + \theta(t)(\mu - r) - c(t) - \delta(B(t) - W(t)) \right] dt \\
+ y(t)dt + \theta(t)\sigma dZ(t), \quad W(0) = W_0
\]

\[
dy(t) = \left( y(t)\mu_y + \int_{-d}^{0} \alpha(\eta)y(t + \eta)d\eta \right) dt + y(t)\sigma_y dZ(t),
\]

\[
y(0) = y_0, \quad y(\eta) = y_1(\eta) \quad \forall \eta \in [-d, 0).
\]

\( W(t), y(t), \theta(t), c(t), B(t) \), as before.

\( \alpha(\cdot) \) square integrable function.
$J_1(W_0, y_0, y_1; c, B, \theta) :=$

$$
\mathbb{E} \left\{ \int_{0}^{+\infty} e^{-(\rho + \delta)t} \left( \frac{c(t)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB(t))^{1-\gamma}}{1-\gamma} \right) dt \right\}.
$$

(1)

**Problem**

Given $T = +\infty$, choose $c(\cdot)$, $\theta(\cdot)$, $B(\cdot)$ to maximize (1), with the following no-borrowing-without-repayment constraint:

$$W(t) \geq - \left( Gy(t) + \int_{-d}^{0} H(\eta)y(t + \eta)d\eta \right).$$
After some work we can write (Biffis-Prodocimi-Goldys, 2015):

\[
\xi(t)^{-1} E \left( \int_{t}^{+\infty} y(s) \xi(s) ds \bigg| \mathcal{F}_t \right) = G y(t) + \int_{-\infty}^{0} H(\eta) y(t + \eta) d\eta.
\]

The constant \( G \) and the function \( H \) are given by

\[
G := (\beta_1 - \beta_{\infty})^{-1},
\]

\[
H(\eta) := \int_{-\infty}^{\eta} e^{-(r + \delta)(\eta - s)} \alpha(s) ds,
\]

with \( \beta_{\infty} := \int_{-\infty}^{0} e^{-(r + \delta)s} \alpha(s) ds \).

For \( \alpha = 0 \) we have \( H = 0 \) and \( G \) coincides with \( g \).

The above shows that human capital is now shaped by two components:

- Current market value of the past trajectory of labor income,
  \( \int_{-\infty}^{0} H(\eta) y(t + \eta) d\eta \).
- Current market value of the future labor income stream, \( G y(t) \).
Stochastic control problem, infinite horizon I

- State space $H$, Hilbert space. Control space $C$ complete metric space.

- State equation

$$\begin{cases}
    dx(t) = b(x(t), c(t)) \, dt + \sigma(x(t), c(t)) \, dZ(t) \\
    x(s) = y, \quad s \geq 0, \; y \in H
\end{cases}$$

- Set of admissible controls (here when $C$ is bounded, if not integrability properties are needed)

$$\mathcal{U} := \{ c : [0, +\infty) \times \Omega \longrightarrow C \mid c \text{ is } \mathcal{F}_t\text{-adapted} \}.$$ 

- Objective functional

$$J(s, y; c(\cdot)) := \mathbb{E} \left\{ \int_s^{+\infty} e^{-\rho t} f(x^{(s,y)}(t), c(t)) \, dt \right\},$$
Stochastic control problem, infinite horizon 2

- value function

\[ V(s, y) := \sup_{c(\cdot) \in U^s} J(s, y; c(\cdot)), \text{ for any } (s, y) \in [0, +\infty) \times \mathbb{R} \]

we have

\[ V(s, y) = e^{-\rho s} V(0, y) = e^{-\rho s} V_0(y). \]

- Hamilton-Jacobi-Bellman equation for \( V_0 \)

\[ \rho v = \mathcal{H}(x, v_x, v_{xx}) \text{ for any } y \in \mathbb{R} \]

where

\[ \mathcal{H}(x, p, P) = \sup_{c \in C} \{ f(x, c) + b(x, c)p + \frac{1}{2}\sigma^2(x, c)P \} \]
Delay equations as ODEs in infinite dimensional spaces

- The state equation of $y(\cdot)$ is a stochastic delay differential equation.
- Classical theory works for Markovian state equations.
- We reformulate the problem in an infinite dimensional Hilbert space (e.g., Vinter, 1975; Chojnowska-Michalik, 1978; Da Prato-Zabczyk, 2014; Fabbri-Gozzi-Swiech, 2017).
- Consider the Hilbert space

$$\mathcal{H} := \mathbb{R} \times L^2([-d, 0]; \mathbb{R}),$$

with inner product for $x = (x_0, x_1), z = (z_0, z_1) \in \mathcal{H}$

$$\langle x, z \rangle_{\mathcal{H}} := x_0 z_0 + \int_{-d}^{0} x_1(\xi) z_1(\xi) d\xi$$

$$= x_0 z_0 + \langle x_1, z_1 \rangle_{L^2}$$
Set
\[ X(t) = (X_0(t), X_1(t)) := (y(t), y(t + \xi)|_{\xi \in [-d, 0]}), \]

- \( X(t) \) is an element of \( \mathcal{H} \) for all \( t \in [0, +\infty) \).
- Let \( X \) satisfy
\[
dX(t) = AX(t)dt + CX(t)dZ(t), \quad X(0) = (y_0, y_1) \in \mathcal{H}
\]

with
\[
A(x_0, x_1) := (\mu_y x_0 + \langle \alpha(\cdot), x_1(\cdot) \rangle_{L^2}, x_1'(\cdot)),
C(x_0, x_1) := (x_0 \sigma_y, 0)
\]

Then, the original problem is equivalent to the control problem with state \( X \) in the infinite dimensional space \( \mathcal{H} \) (e.g., Chojnowska 1989, Gozzi-Marinelli, 2004).
The value function $V_0$ is

$$V_0(W, x_0, x_1) := f_\gamma \frac{\Gamma^{1-\gamma}}{1-\gamma},$$

where

$$f_\infty := (1 + \delta k^{\frac{1}{1-\gamma}} - 1)\nu,$$

$$\nu := \frac{\gamma}{\rho + \delta - (1 - \gamma)(r + \delta + \frac{k^T \kappa}{2\gamma})} > 0,$$

$$\Gamma := W_0 + Gx_0 + \langle H, x_1 \rangle_{L^2} \geq 0,$$
The optimal strategies are given by:

\[ c^*(t) := f_{\infty}^{-1} \Gamma^*(t) \]

\[ B^*(t) := k^{-b} f_{\infty}^{-1} \Gamma^*(t) \]

\[ \theta^*(t) := \frac{(\mu - r) \Gamma^*(t)}{\gamma \sigma^2} - \frac{\sigma y}{\sigma} G y(t), \]

where \( \Gamma^*(t) := W^*(t) + G X_0(t) + \langle H, X_1(t, \cdot) \rangle_{L^2}. \)

We have

\[ \frac{d\Gamma^*(t)}{\Gamma^*(t)} = \left[ r + \delta + \frac{1}{\gamma} \left( \frac{\mu - r}{\sigma} \right)^2 - f_{\infty}^{-1} (1 + \delta k^{-b}) \right] dt \]

\[ + \frac{\mu - r}{\gamma \sigma} dZ(t). \]
Discussion

- With no labor income risk ($\sigma_y = 0$), the optimal ratios $\frac{\theta^*}{\Gamma^*}$ and $\frac{c^*}{\Gamma^*}$ are constant, as in the Merton model.

- Taking $\alpha = 0$, we recover the results of Dybvig-Liu.

- With $\alpha \neq 0$, the same logic as in Dybvig-Liu applies, but optimal total wealth (financial wealth + human capital) is now given by $\Gamma^*$:

  $$\Gamma^*(t) = W^*(t) + GX_0(t) + \langle H(t, \cdot), X_1(t, \cdot) \rangle_{L^2}.$$  

- The ratio $\frac{\theta^*}{\Gamma^*}$ is no longer constant and the negative hedging demand term $\frac{\sigma_y}{\sigma} G y(t)$ only takes into account the ‘future component’ of human capital.

- Richer empirical predictions than in the standard case: portfolio choice (e.g., stock market participation) depends on the relative importance of the past vs. future component of human capital.
Sketch of the proof

- Guess the value function to be
  \[ V(W_0, x_0, x_1) := f_\gamma \left( \frac{W_0 + Gx_0 + \langle H, x_1 \rangle_{L^2}}{1 - \gamma} \right)^{1-\gamma}. \]

- Putting \( V \) in the HJB equation, gives equations for \( f, G, H \).

- Solving these equations, we get that \( f, G, H \) are the constant as defined before.

- \( V \) is \( C^{1,2} \).

- Verification Theorem holds and the optimal feedback strategies are admissible.
Remarks I

Total wealth zero boundary:

- The **borrowing constraint** is not always slack.
- The case of binding constraint is reduced to a problem of viability.
- As opposed to Merton-type problems, the agent is not fully invested in the riskless asset along the boundary.
- At the zero boundary we have $c = 0$, $B = 0$, and $\theta = -\frac{\sigma_y}{\sigma} Gy(t)$.
- The agent is still invested in the risky asset, as (s)he needs to fully hedge his/her labor income risk.
Verification and preference parameter $\gamma > 0$:

- We cover in detail both the case of $\gamma \in (0, 1)$ and $\gamma > 1$.
- The first case is standard.
- The second case is not: it is at best neglected in the literature. We address this case and prove it explicitly.
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Conclusion and further/future research

Summary

- Extension of Merton’s problem to the case of realistic labor income dynamics and constraints.
- Explicit solutions can better match empirical data (e.g., hump shaped risky asset allocations, cross-sectional heterogeneity of portfolio choices, etc.).

Extensions

- The case with given retirement date (finite horizon) or with linear path dependent diffusion coefficient can be solved in a similar way.
- More general problems (e.g. non linear equation for \( y \)) call for new theoretical results on HJB equations or on the use of alternative methods (BSDEs through randomization, Maximum Principle, etc.).

[Lines of research: regularization of viscosity solutions using the classical definition (Fabbri-Gozzi-Swiech), or the PPDE definition (Ekren-Touzi-Zhang) in the finite dimensional case, and CossoFedericoGozziRosestolato-Touzi in the infinite dimensional case.]
THANK YOU