Patching and $p$-adic local Langlands

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The $p$-adic local Langlands correspondence is an exciting, recent generalization of the classical Langlands program. For $GL_2(\mathbb{Q}_p)$ it consists of functors between two-dimensional, continuous $p$-adic representations of $\text{Gal}((\overline{\mathbb{Q}}_p)/\mathbb{Q}_p)$ and certain admissible unitary $p$-adic Banach space representations of $GL_2(\mathbb{Q}_p)$ [4, 9, 5]. The correspondence has several remarkable properties, namely it is compatible with deformations and reduction mod $p$, with the classical local Langlands correspondence via taking locally algebraic vectors, and with the global $p$-adic correspondence, i.e. with the completed cohomology of modular curves. These properties led to spectacular applications to the Fontaine-Mazur conjecture for $GL_2$ over $\mathbb{Q}$ [6, 8].

However, most techniques involved in the construction of the $p$-adic local Langlands correspondence seem to break down if one tries to move beyond $GL_2(\mathbb{Q}_p)$. For $GL_n(F)$, it is unclear even what the precise conjectures should be, though the best possible scenario would involve all three of the properties listed above. In this talk, I described the construction of a candidate for the $p$-adic local Langlands correspondence for $GL_n(F)$, where $F/\mathbb{Q}_p$ is a finite extension, using global techniques, specifically the Taylor-Wiles-Kisin patching method applied to completed cohomology [3].

More precisely, we associate to a continuous $n$-dimensional representation $r$ of $\text{Gal}((\overline{\mathbb{Q}}_p)/F)$ an admissible Banach space representation $V(r)$ of $GL_n(F)$, by $p$-adically interpolating completed cohomology for global definite unitary groups. The method involves working over an unrestricted local deformation ring of the residual $\overline{r}$, finding a global residual Galois representation which is automorphic and restricts to our chosen local representation $\overline{r}$, and gluing corresponding spaces of completed cohomology with varying tame level at so-called Taylor-Wiles primes. The output is a module $M_\infty$ over $R_{\overline{r}}$, which also has an action of $GL_n(F)$ and whose fibers over closed points are admissible, unitary $p$-adic Banach spaces. We define $V(r)$ to be the fiber of $M_\infty$ over the point of $R_{\overline{r}}$ corresponding to $r$.

We also show that, when $r$ is de Rham, we can recover the compatibility with classical local Langlands $r \mapsto \pi_{sm}(r)$ in many situations. More precisely, when $r$ lies on an automorphic component of a local deformation ring, we can compute the locally algebraic vectors in $V(r)$ and show that they have the expected form $\pi_{sm}(r) \otimes \pi_{alg}(r)$. This involves first establishing an inertial local Langlands correspondence via the theory of types. The next step is to construct a map from an appropriate Bernstein center to a local deformation ring for a specific inertial type, a map which interpolates classical local Langlands. Finally, we appeal to the automorphy lifting theorems of [1] to guarantee that the locally algebraic vectors we obtain are non-zero. Our control over locally algebraic vectors allows us to prove many new cases of an admissible refinement of the Breuil-Schneider conjecture [2], concerning the existence of certain unitary completions.
Theorem 1. Suppose that $p > 2$, that $r : G_F \to GL_n(\overline{\mathbb{Q}}_p)$ is de Rham of regular weight, and that $r$ is generic. Suppose further that either

1. $n = 2$, and $r$ is potentially Barsotti–Tate, or
2. $F/\mathbb{Q}_p$ is unramified and $r$ is crystalline with Hodge–Tate weights in the extended Fontaine–Laffaille range, and $n \neq p$.

Then $\pi_{\text{sm}}(r) \otimes \pi_{\text{alg}}(r)$ admits a nonzero unitary admissible Banach completion.

For example, when $F/\mathbb{Q}_p$ is unramified and $p$ is large, Theorem 1 applies to all unramified principal series representations. Note that this existence is a purely local result, even though it is proved using global, automorphic methods.

Unfortunately, the Taylor-Wiles patching method involves gluing spaces of automorphic forms with varying tame level in a non-canonical way, using a sort of diagonal argument to ensure that their compatibility can always be achieved. Therefore, it is not at all clear that $r \mapsto V(r)$ is a purely local correspondence: it depends on the choice of global residual representation as well as on the choice of a compatible system of Taylor-Wiles primes. If there was a purely local correspondence satisfying all three properties listed in the beginning, then our construction would necessarily recover it. This is the case for $GL_2(\mathbb{Q}_p)$ and, in fact, the six of us are in the process of writing a paper elaborating on this and reproving many properties of the $p$-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$, without making use of Colmez’s functors. Our arguments rely heavily instead on the ideas of [9], especially the use of projective envelopes. For $GL_n(F)$, the question of whether our construction is purely local seems quite hard.

However, there is forthcoming work of Scholze, who constructs a purely local functor in the opposite direction: from admissible unitary $p$-adic Banach space representations of $GL_n(F)$ to admissible representations of $D^\times \times W_F$, where $D$ is a division algebra with center $F$ and invariant $1/n$. His construction uses the cohomology of the Lubin-Tate tower, which is known to realize both classical local Langlands and the Jacquet-Langlands correspondence when $l \neq p$ [7]. This functor satisfies local-global compatibility, in the following sense: if the input is the completed cohomology for a definite unitary group $G$, split at $p$, then the output is the completed cohomology of a Shimura variety associated to an inner form $J$ of $G$ which is isomorphic to $D^\times$ at $p$. Just as one can patch completed cohomology for $G$, it is also possible to patch completed cohomology for $J$. Moreover, Scholze can even prove that if one uses our patched module $M_\infty$ as the input for his functor, then the output is the patched object for $J$. A consequence of this is that, at the very least, it should be possible to recover the Galois representation $r$ from the Banach space $V(r)$.

References