NUMERICAL SOLUTION OF RIEMANN–HILBERT PROBLEMS: PAINLEVÉ II

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- We present a method for computing solutions to matrix-valued Riemann–Hilbert problems:
  - Reduction to a linear equation via Cauchy transform (same as yesterdays talk by Dienstfrey)
    - Computing the Cauchy transform over a circle, real line and unit interval
    - Construction of a spectral method
  - Computing solutions to the homogeneous Painlevé II equation
  - Numerical nonlinear steepest descent
    - Negative x, Positive x and Hastings–McLeod solution

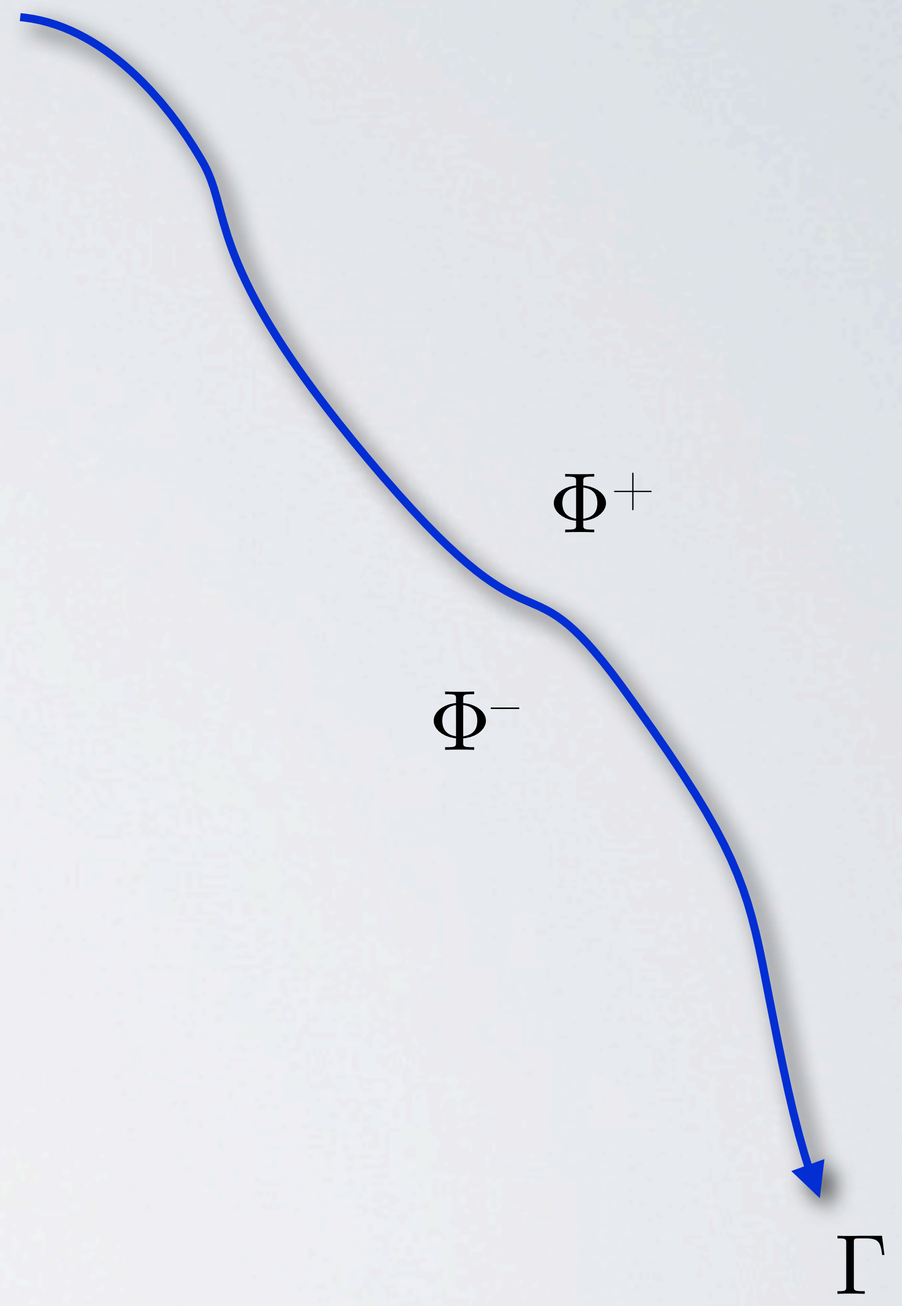
- We are concerned with computing the solution to the following *Riemann–Hilbert problem*:
  - Given an oriented curve Γ in the complex plane and a matrix-valued function G defined on Γ (in this talk, all functions on Γ are analytic along each piece of Γ);
  - Find a matrix-valued function Φ that is analytic everywhere in the complex plane off of Γ such that

$$\Phi^+(z) = \Phi^-(z)G(z) \quad \text{for} \quad z \in \Gamma \qquad \text{and} \qquad \Phi(\infty) = I$$

where

$$\Phi^+(z) = \lim_{\substack{x \to z \\ \text{where } x \text{ is left of } \Gamma}} \Phi(x)$$

$$\Phi^-(z) = \lim_{\substack{x \to z \\ \text{where } x \text{ is right of } \Gamma}} \Phi(x)$$



- Many linear differential equations have well-known integral representations
  - e.g., Airy equation, Bessel equation, Hypergeometric equation and Heat and wave equations (via Fourier transform)
- Matrix-valued RH problems can be (loosely) viewed as an analogue to these integral representations for *nonlinear equations*
- Importantly, RH problems can be used to determine asymptotics of solutions
  - This works similar to integral representations: the contour is deformed along the *path of steepest descent*
- The goal of this talk is to demonstrate that they can also be used as numerical tools
- RH problems have a major advantage over the originating differential equations: they are linear

SOME EXAMPLE RH PROBLEMS

KdV

$$u_t - 6uu_x + u_{xxx} = 0$$
$$u(x,0) = u_0(x)$$

$$\Phi^+ = \begin{pmatrix} 1 - |r(z)|^2 & -\bar{r}(z)\mathrm{e}^{-2\mathrm{i}(4tz^3+xz)} \\ r(z)\mathrm{e}^{2\mathrm{i}(4tz^3+xz)} & 1 \end{pmatrix} \Phi^-$$
$$\text{for } \Gamma = \mathbb{R}$$

Nonlinear Schrödinger equation

$$\mathrm{i}u_t + u_{xx} - 2\,|u|^2\,u = 0$$
$$u(x,0) = u_0(x)$$

$$\Phi^+ = \begin{pmatrix} 1 - |r(z)|^2 & -\bar{r}(z)\mathrm{e}^{-2\mathrm{i}(t2z^2+xz)} \\ r(z)\mathrm{e}^{2\mathrm{i}(2tz^2+xz)} & 1 \end{pmatrix} \Phi^-$$
$$\text{for } \Gamma = \mathbb{R}$$

Hilbert transform

$$\Phi^+(z) - \Phi^-(z) = f(z) \quad \text{and} \quad \Phi(\infty) = 0,$$
$$\Phi^+(z) + \Phi^-(z) = \frac{1}{\mathrm{i}\pi} ⨍_\Gamma \frac{f(t)}{t-z}\,\mathrm{d}t$$

Orthogonal polynomials

Random matrices

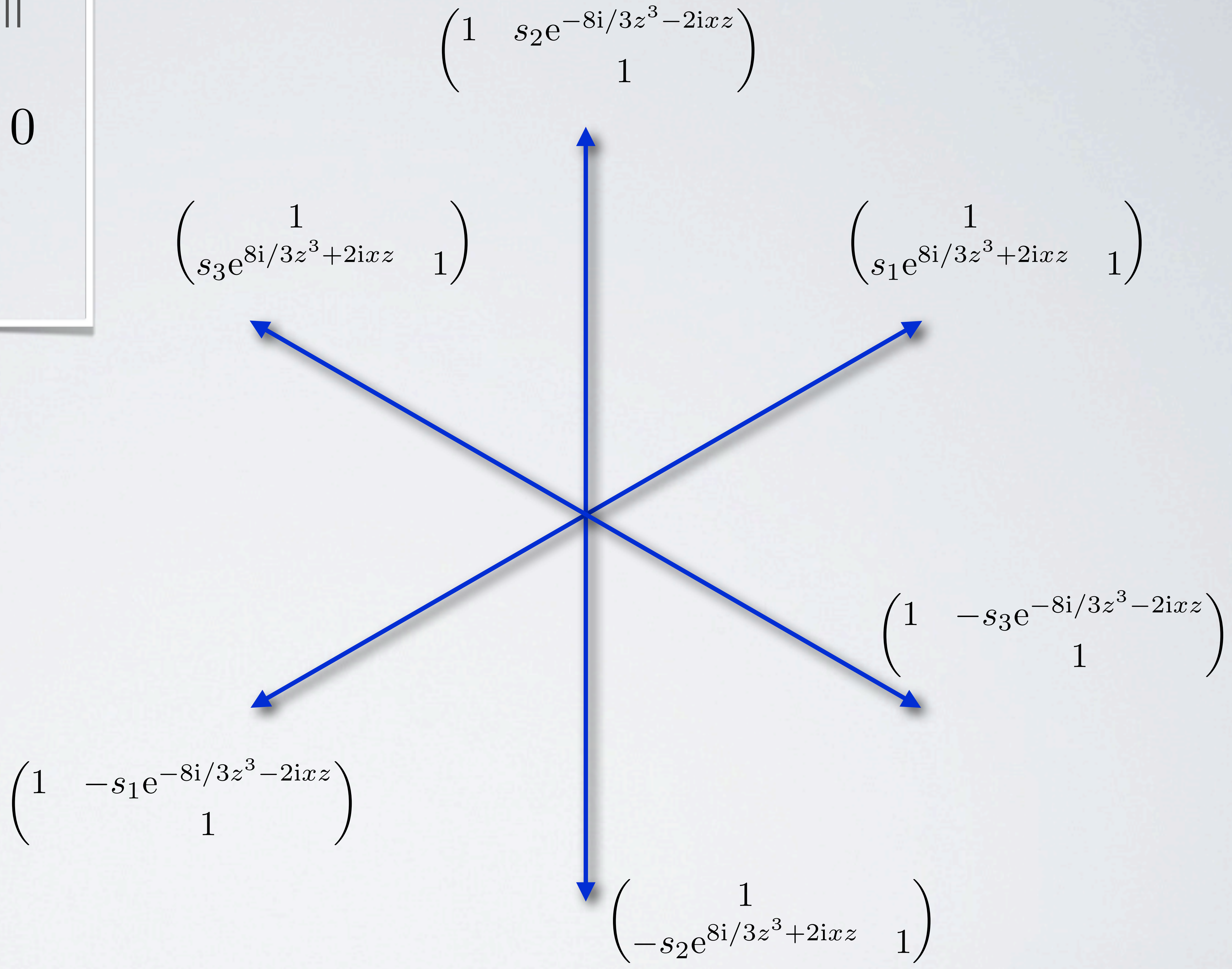
Homogeneous Painlevé II

$$u'' = xu + 2u^3$$

$$s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0$$

$$\Phi^+(z) = \Phi^-(z) G(z)$$

$$u(x) = 2 \lim_{z\to\infty} z\Phi_{12}(z)$$



(see eg. Fokas et al 2006)

• We are trying to solve

$$\Phi^+(z) = \Phi^-(z)G(z) \quad \text{for} \quad z \in \Gamma \qquad \text{and} \qquad \Phi(\infty) = I$$

• This is a linear operator, except for the condition at infinity. Therefore we change variables:

$$\begin{aligned} \Phi &= \Psi + I, \\ \mathcal{L}\Psi &= \Psi^+ - \Psi^- G = G - I \qquad \text{and} \qquad \Psi(\infty) = 0. \end{aligned}$$

Now $\mathcal{L}$ is a linear operator from the space of functions analytic off Γ which decay at infinity to the space of functions defined on Γ.

• We premultiply this operator by the Cauchy transform to obtain an operator from the space of functions defined on Γ to itself.

- Consider the *Cauchy transform*

$$\mathcal{C}_\Gamma f(z) = \frac{1}{2\mathrm{i}\pi} \int_\Gamma \frac{f(t)}{t - z} \, \mathrm{d}t.$$

This map defines a one-to-one correspondence between a function defined on Γ and a function which is analytic everywhere off Γ which decays at ∞

- The Cauchy transform also solves the simple Riemann–Hilbert problem

$$\mathcal{C}^+ f - \mathcal{C}^- f = f \qquad \text{and} \qquad \mathcal{C} f(\infty) = 0$$

- We thus need only solve the equation

$$\mathcal{L}\mathcal{C}V(z) = G(z) - I \qquad \text{for} \qquad z \in \Gamma$$

The operator $\mathcal{L}\mathcal{C}$ is a map from the set of functions defined on Γ to itself

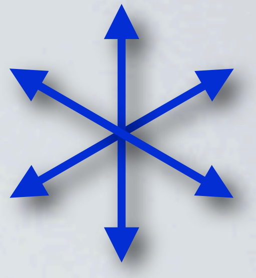
- Having a method to compute the Cauchy transform and its left and right limits allows us to compute

$$\mathcal{L}\mathcal{C}V = \mathcal{C}^+V - (\mathcal{C}^-V)G$$

where $\mathbf{V}$ is a matrix-valued function defined on Γ

- Since this is a linear operator, we can now construct a spectral/collocation method:
  - For some basis $\{\psi_1, \ldots, \psi_n\}$ of matrix-valued functions defined on Γ and set of nodes $\{x_1, \ldots, x_m\}$
  - Solve the linear system

$$c_1\mathcal{L}\mathcal{C}\psi_1(x_1) + \cdots + c_n\mathcal{L}\mathcal{C}\psi_n(x_1) = G(x_1) - I,$$
$$\vdots$$
$$c_1\mathcal{L}\mathcal{C}\psi_1(x_m) + \cdots + c_n\mathcal{L}\mathcal{C}\psi_n(x_m) = G(x_m) - I.$$

For homogeneous Painlevé II, we need to compute $\mathcal{C}$ over the domain 

- But we can decompose the transform to a sum over each of Γ's parts:

$$\mathcal{C}_{\ast} = \mathcal{C}_{\swarrow} + \mathcal{C}_{\downarrow} + \mathcal{C}_{\searrow} + \mathcal{C}_{\nearrow} + \mathcal{C}_{\uparrow} + \mathcal{C}_{\nwarrow}$$

- We will compute these by conformally mapping each ray to the unit interval
    - Thus we have reduced the construction of our spectral method to one problem: the computation of $\mathcal{C}_{(-1,1)}$

- This approach can be applied equally well to any Γ whose parts can individually be mapped to the unit interval. This includes all of the Painlevé Riemann–Hilbert problems.

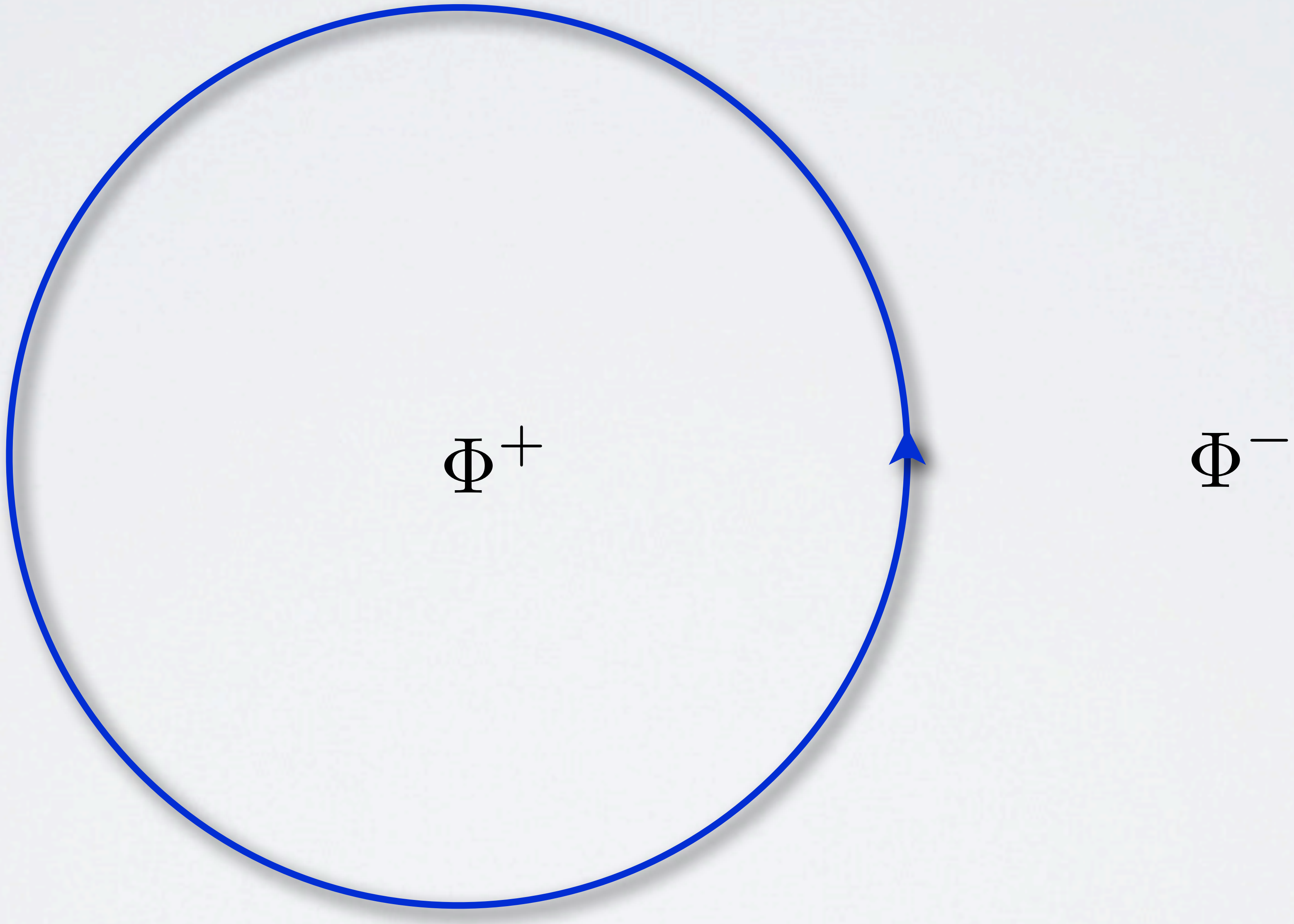
COMPUTATION OF CAUCHY TRANSFORMS

- Solvable over the unit circle using the FFT
- Then the real line by conformally mapping to the unit circle
- Then the interval by conformally mapping to the unit circle and using hypergeometric functions

CAUCHY TRANSFORM ON A CIRCLE: THE SIMPLEST RH PROBLEM

$$\Phi^+(z) - \Phi^-(z) = g(z) \qquad \text{and} \qquad \Phi(\infty) = 0$$

$$g(z) = \sum_{k=-\infty}^{\infty} \hat{g}_k z^k$$



(Used in Conjugation method, cf. Wegmann 1988)

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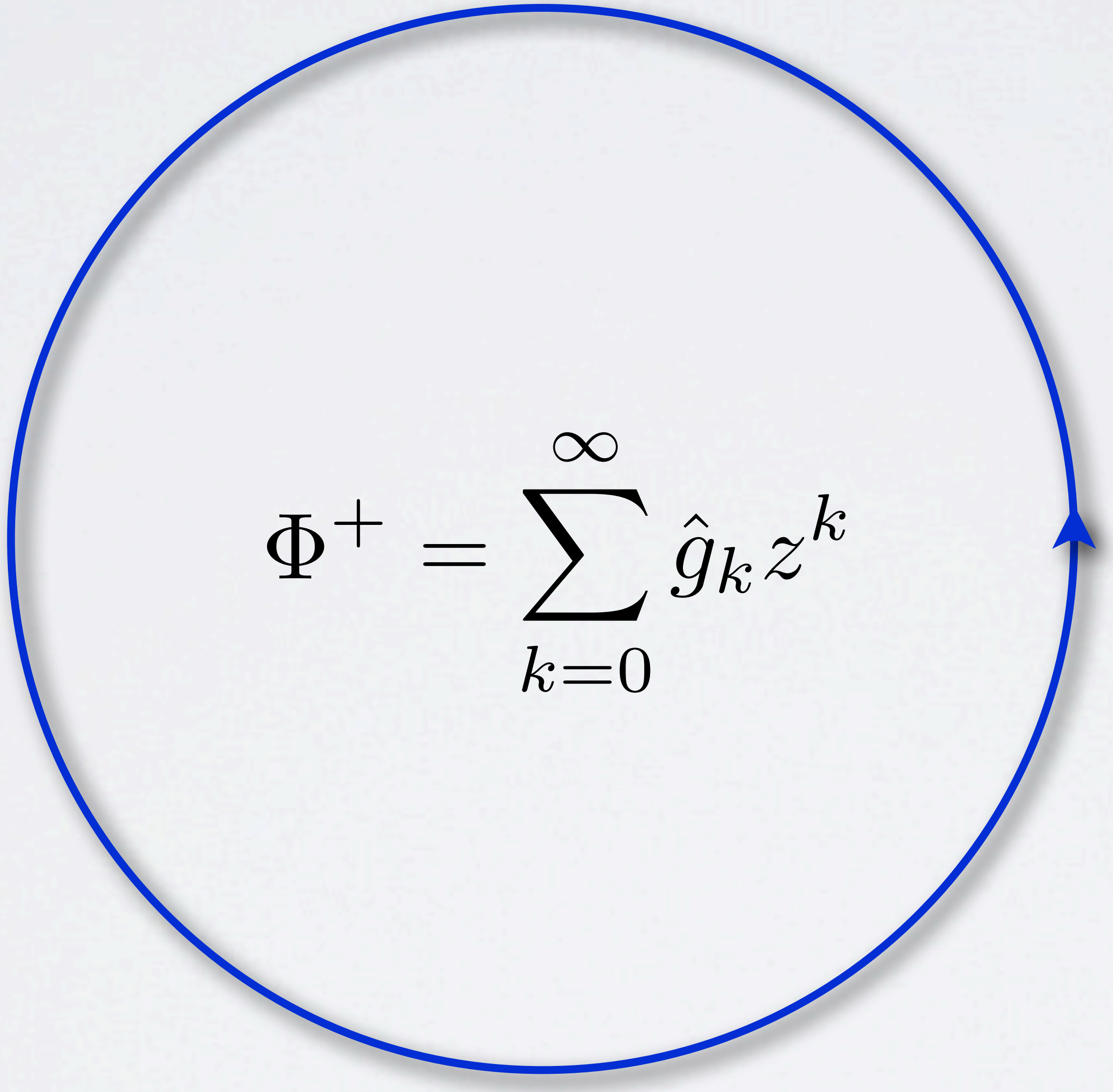
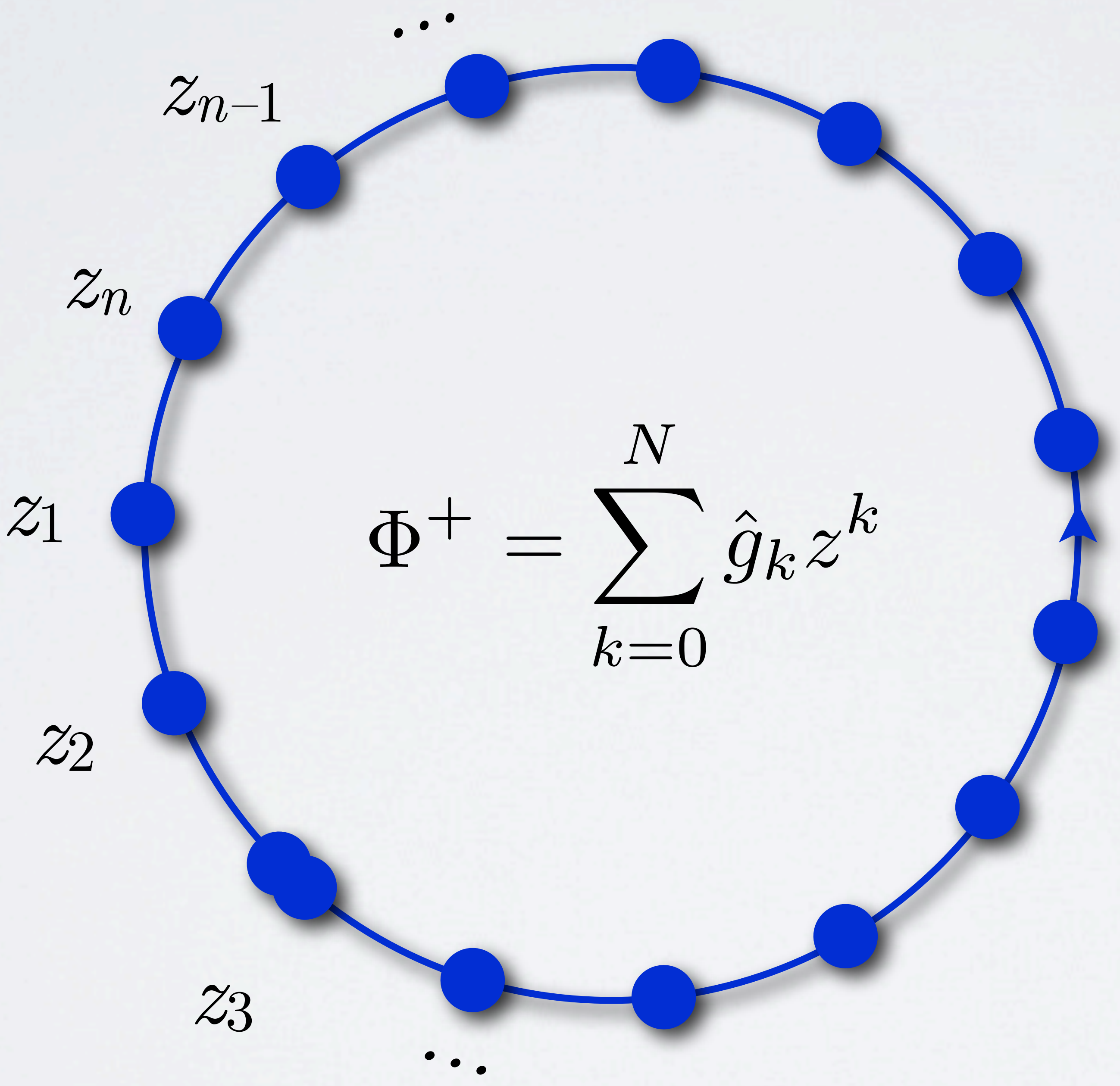


$$\Phi^+ = \sum_{k=0}^{\infty} \hat{g}_k z^k$$

$$\Phi^- = -\sum_{k=-\infty}^{-1} \hat{g}_k z^k$$

(Used in Conjugation method, cf. Wegmann 1988)

NUMERICAL SOLUTION: FFT

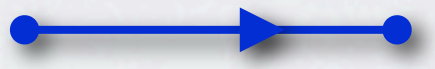


$$\Phi^+ = \sum_{k=0}^{N} \hat{g}_k z^k$$

$$\Phi^- = -\sum_{k=-N}^{-1} \hat{g}_k z^k$$

- We compute $\mathcal{C}$ over the following domains by mapping to the unit circle:

$\mathbb{R}$ 

$(-1,1)$ 

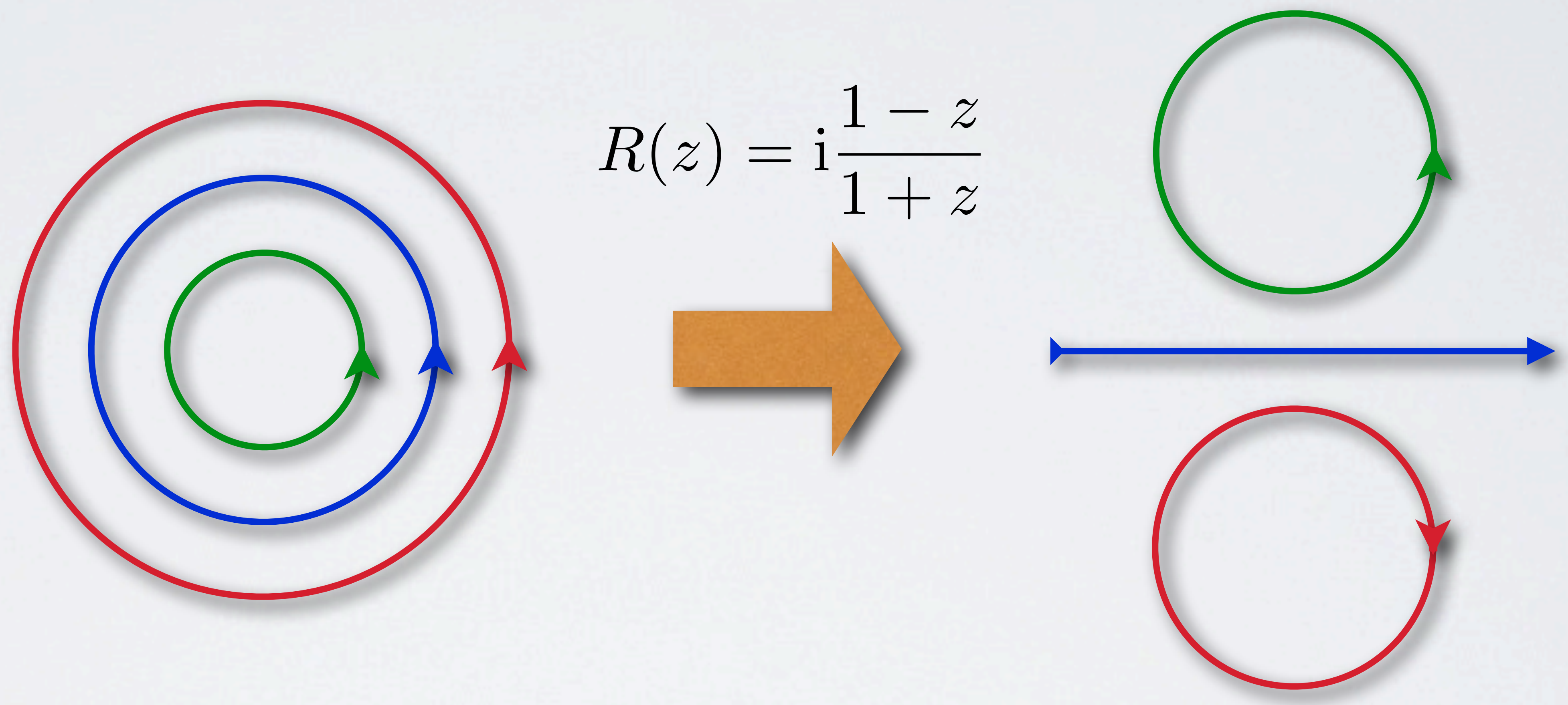
- This can be used in the computation of the Hilbert transform:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}\,\mathrm{d}t = \mathrm{i}(\mathcal{C}_{\mathbb{R}}^{+} f + \mathcal{C}_{\mathbb{R}}^{-} f)$$

- Over the real line and the unit interval, this is in some sense a rederivation of known results for computing Hilbert transforms and finite Hilbert transforms, in terms of the RH framework. What's new is that we obtain the Cauchy transform throughout the complex plane, as well as its left/right limits
    - Also, the known result on the interval is in terms of power series, where we obtain an expression in terms of Chebyshev series, computable globally in $n \log n$ time

$\mathbb{R}$

Consider the conformal map from the unit circle to the real line

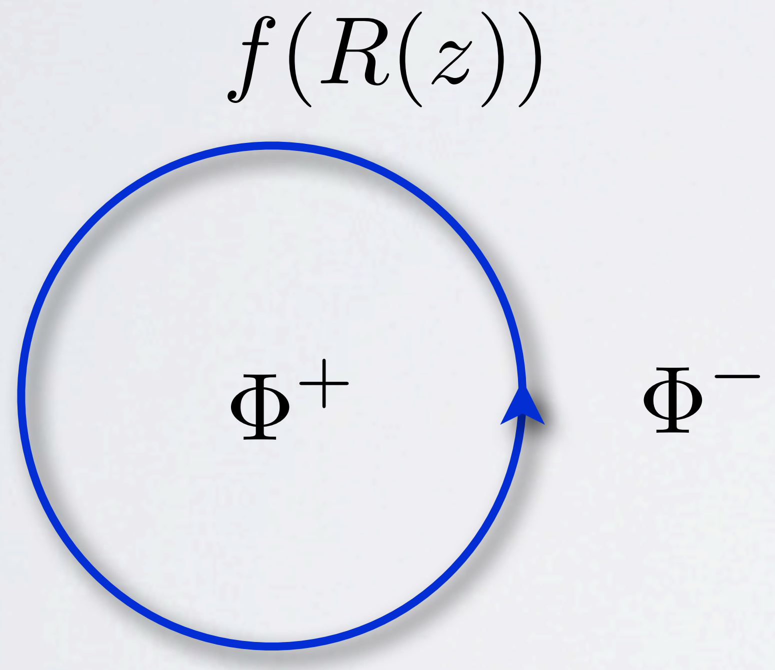


$$R(z) = \mathrm{i}\frac{1-z}{1+z}$$

Functions analytic inside the unit circle are mapped to functions analytic in the upper half plane, and functions analytic outside the unit circle are mapped to functions analytic in the lower half plane

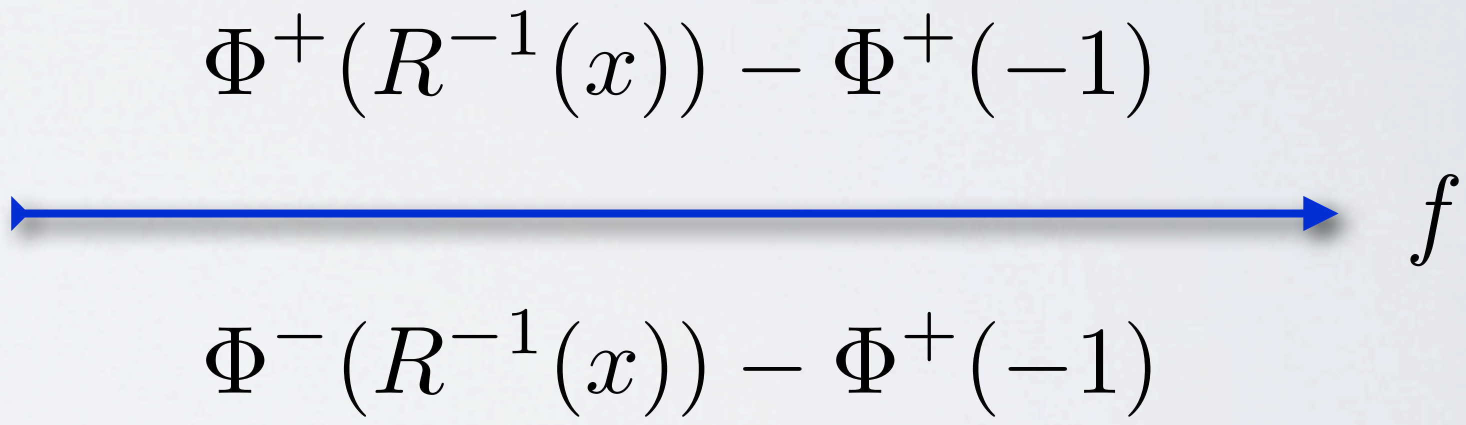
Suppose we compute the Cauchy transform of the mapped function on the unit circle

$$\Phi = \mathcal{C}_{\circ} f(R(z))$$



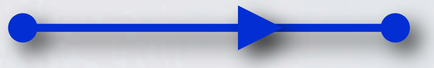
Then the Cauchy transform over the real line is

$$\begin{aligned}\mathcal{C}_{\mathbb{R}} f(x) &= \Phi(R^{-1}(x)) - \Phi(R^{-1}(\infty)) \\ &= \Phi(R^{-1}(x)) - \Phi^+(-1)\end{aligned}$$



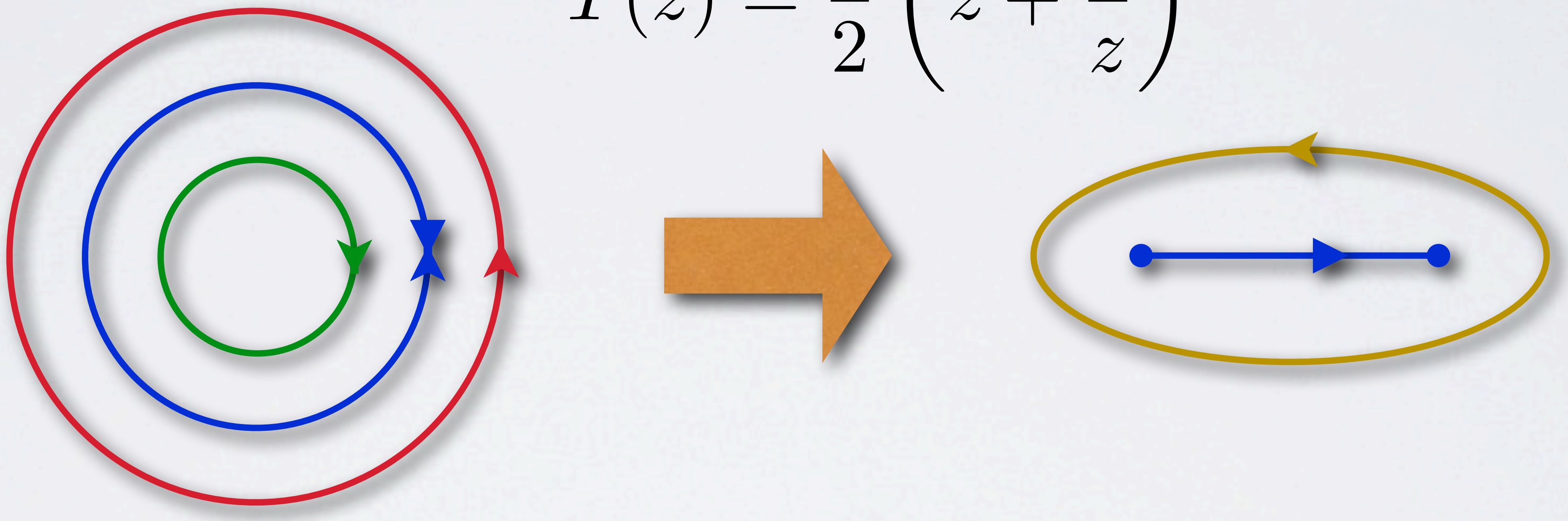
(Essentially a re-derivation of Weideman 1995)

$(-1, 1)$



Consider the Joukowski map from the unit circle to the unit interval

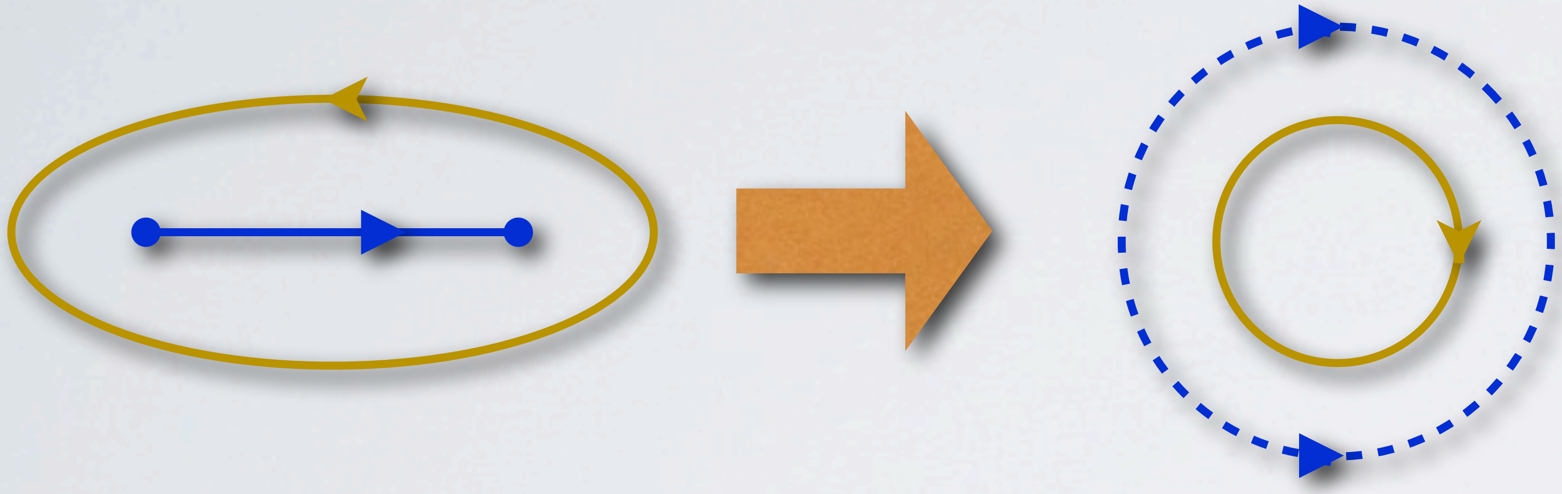
$$T(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$



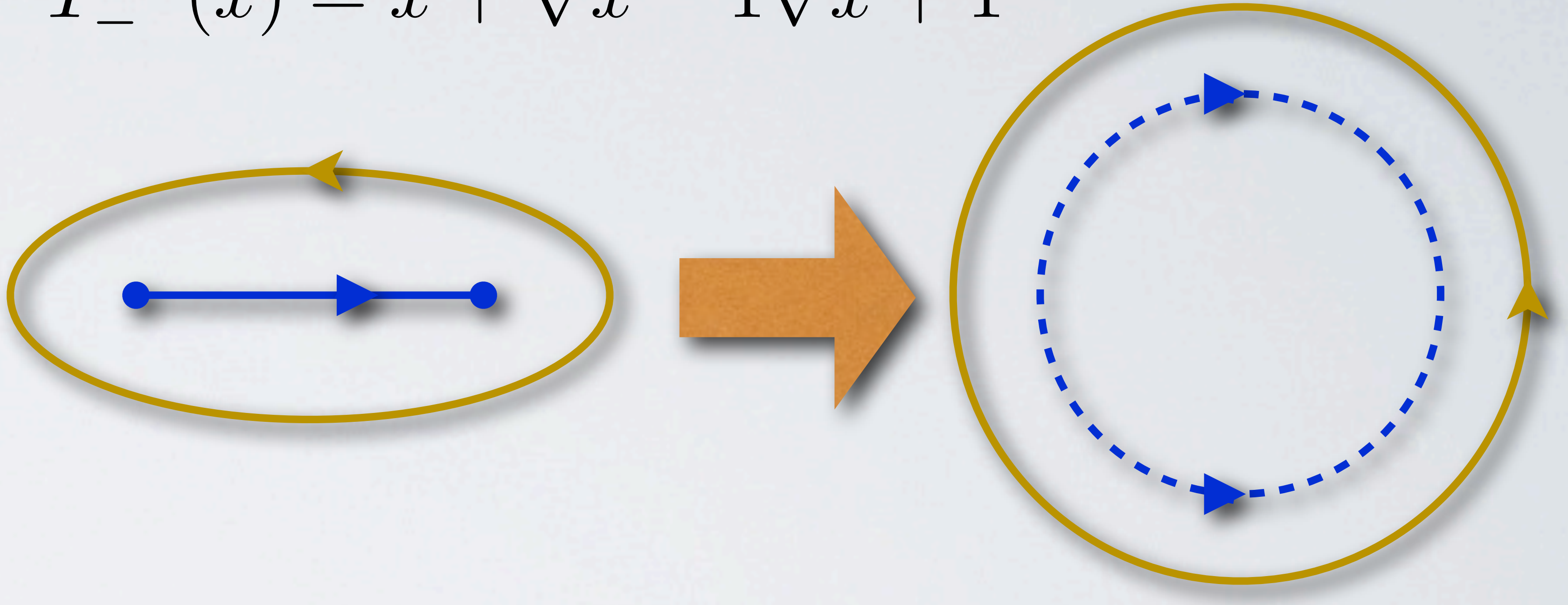
Functions analytic inside and outside the unit circle are mapped to functions analytic off the unit interval.

We define four inverses to the Joukowski map:

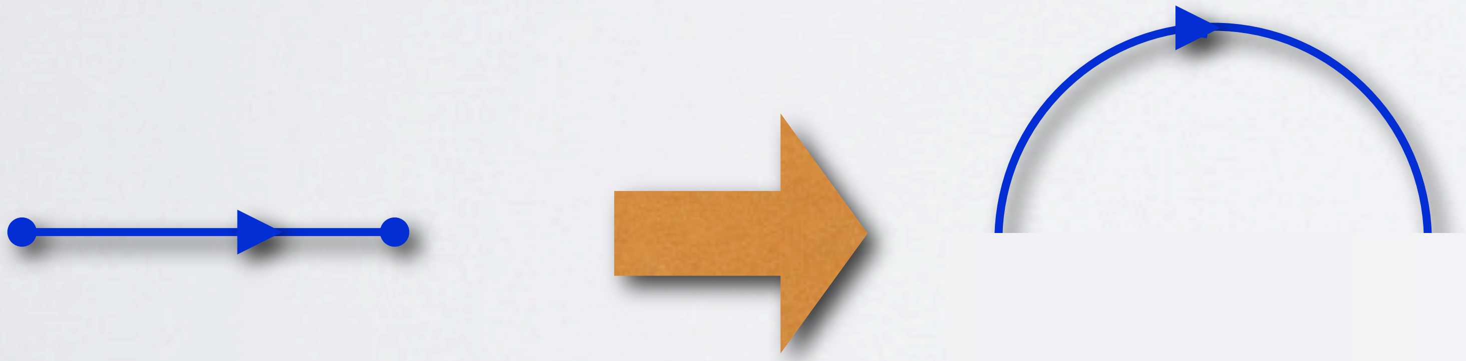
$$T_{+}^{-1}(x) = x - \sqrt{x-1}\sqrt{x+1}$$



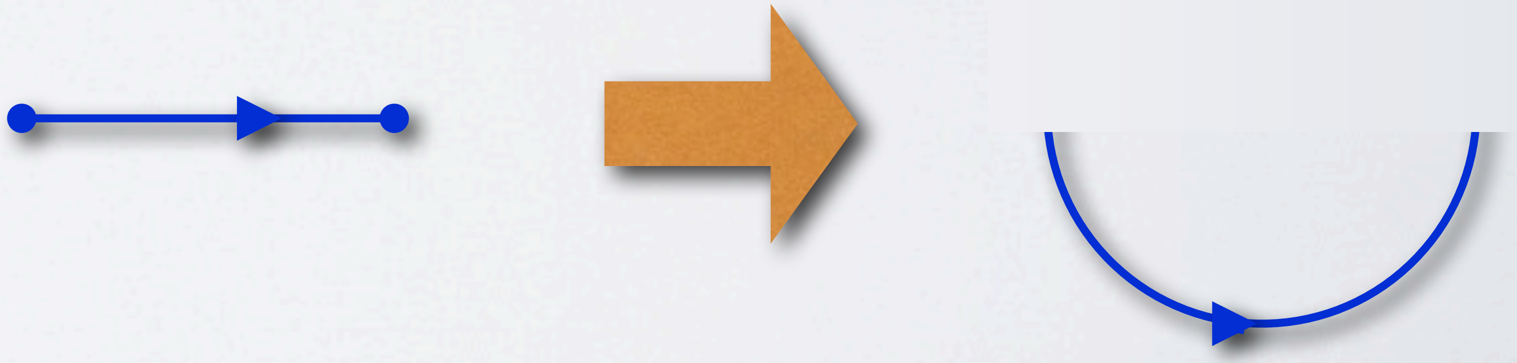
$$T_{-}^{-1}(x) = x + \sqrt{x-1}\sqrt{x+1}$$



$$T_{\uparrow}^{-1}(x) = x + \mathrm{i}\sqrt{1-x}\sqrt{1+x}$$

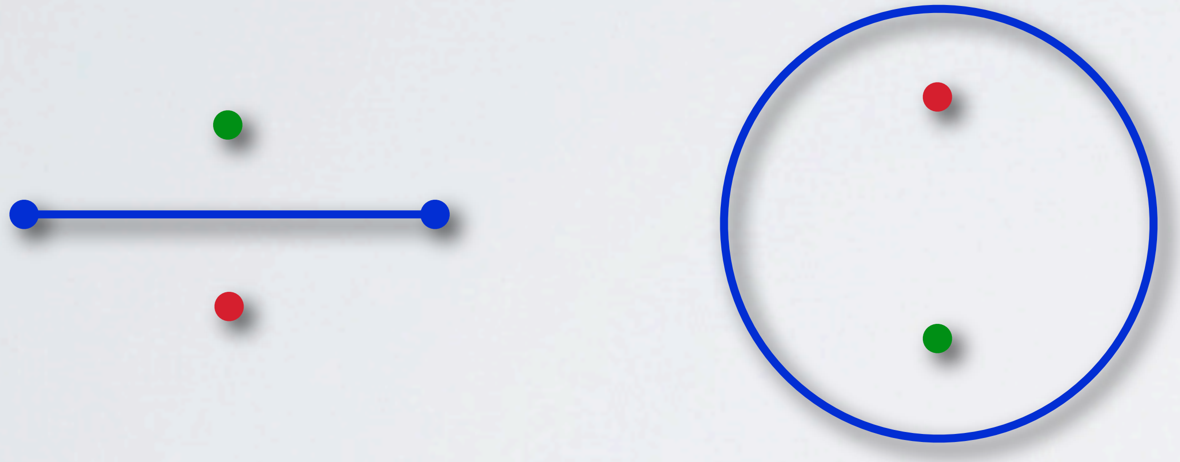


$$T_{\downarrow}^{-1}(x) = x - \mathrm{i}\sqrt{1-x}\sqrt{1+x}$$



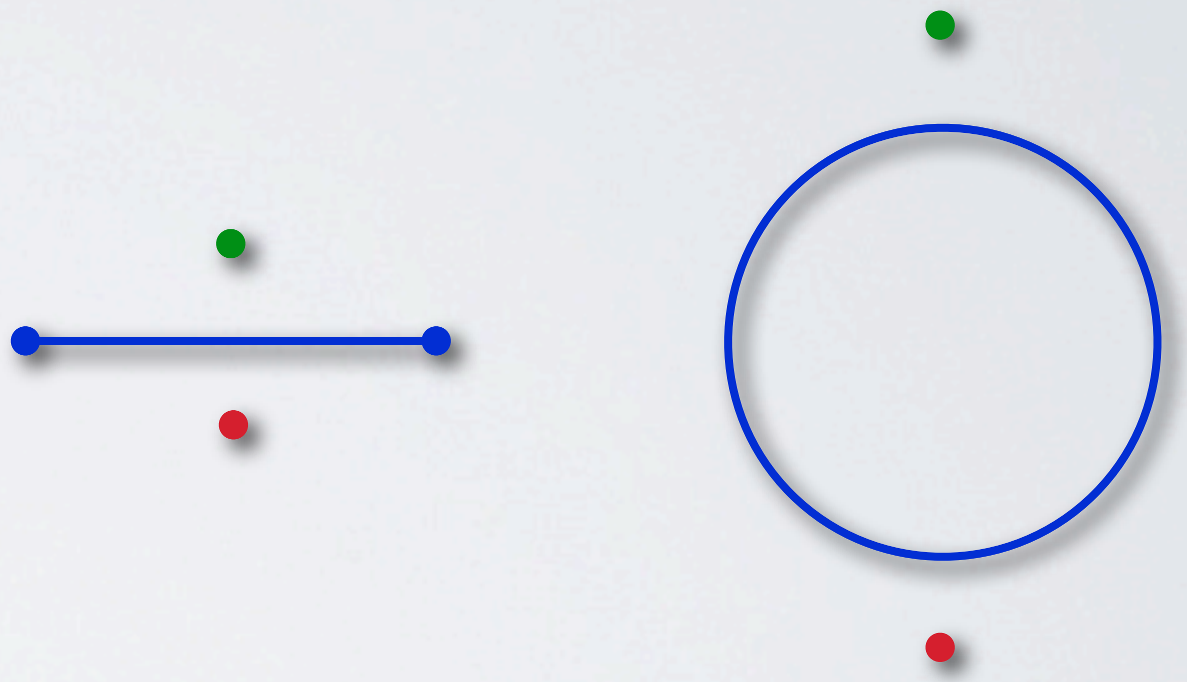
Relationship between inverses

$T_{+}^{-1}(x)$



$$T_{+}^{-1}(x^{+}) = T_{\downarrow}^{-1}(x)$$
$$T_{+}^{-1}(x^{-}) = T_{\uparrow}^{-1}(x)$$

$T_{-}^{-1}(x)$



$$T_{-}^{-1}(x^{+}) = T_{\uparrow}^{-1}(x)$$
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Relationship between inverses

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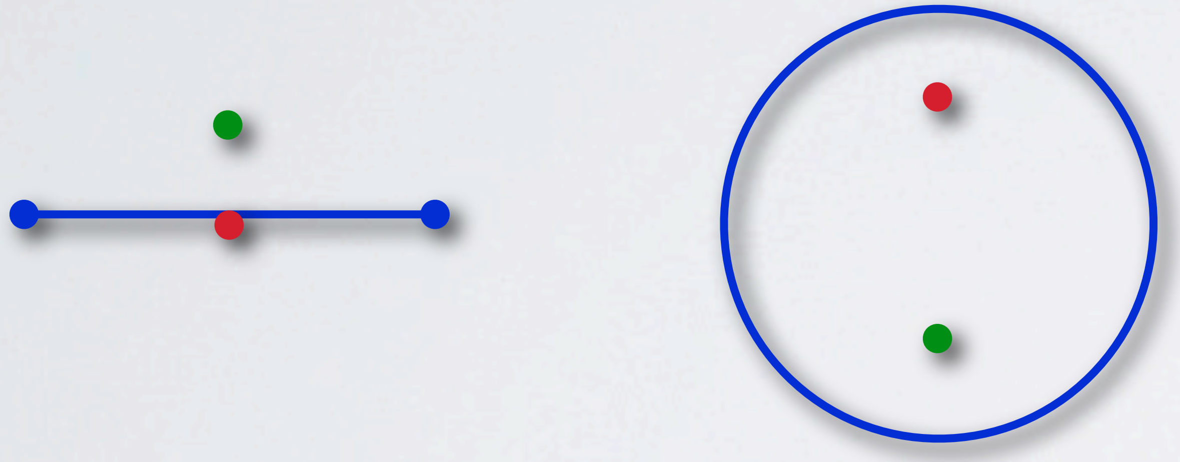

$$T_+^{-1}(x^+) = T_\downarrow^{-1}(x)$$
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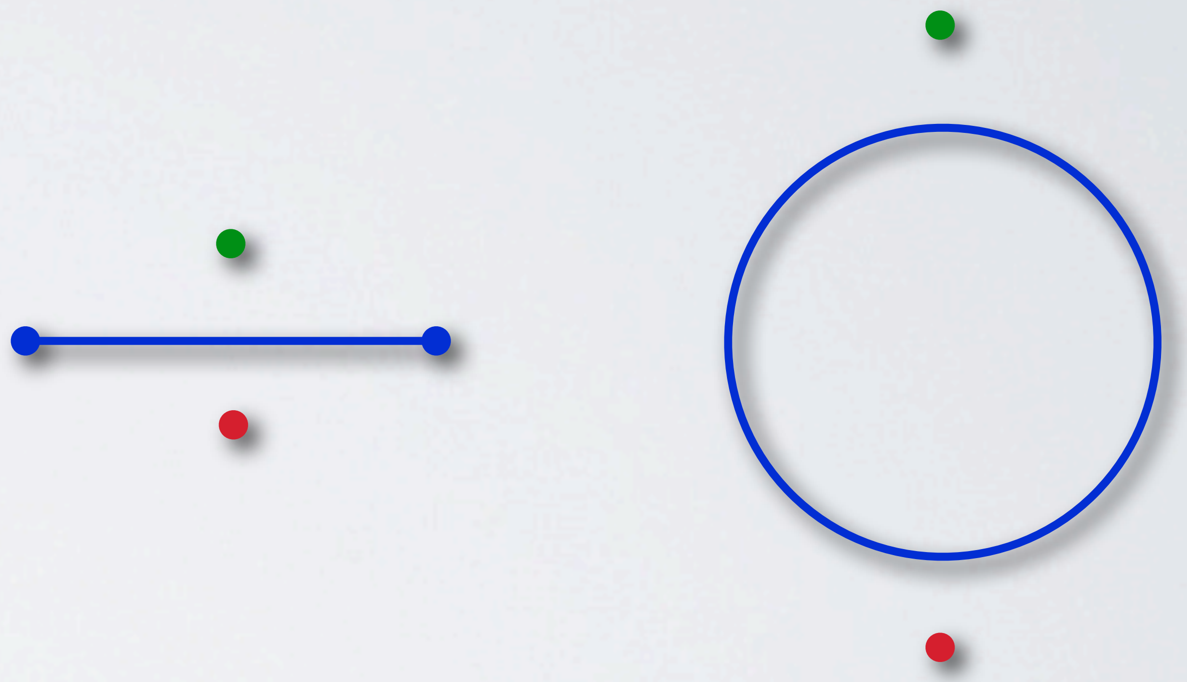
$T_-^{-1}(x)$



$$T_-^{-1}(x^+) = T_\uparrow^{-1}(x)$$
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Relationship between inverses

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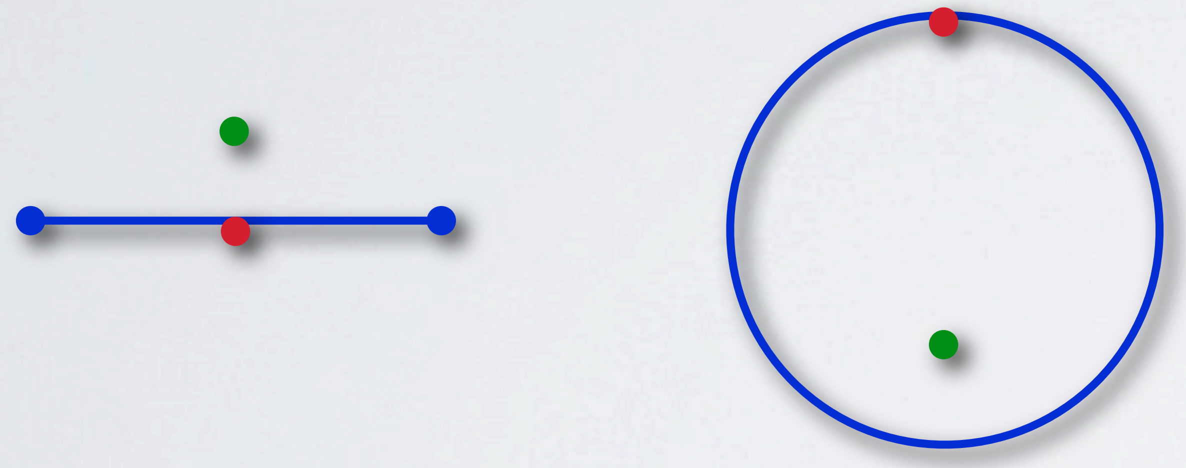
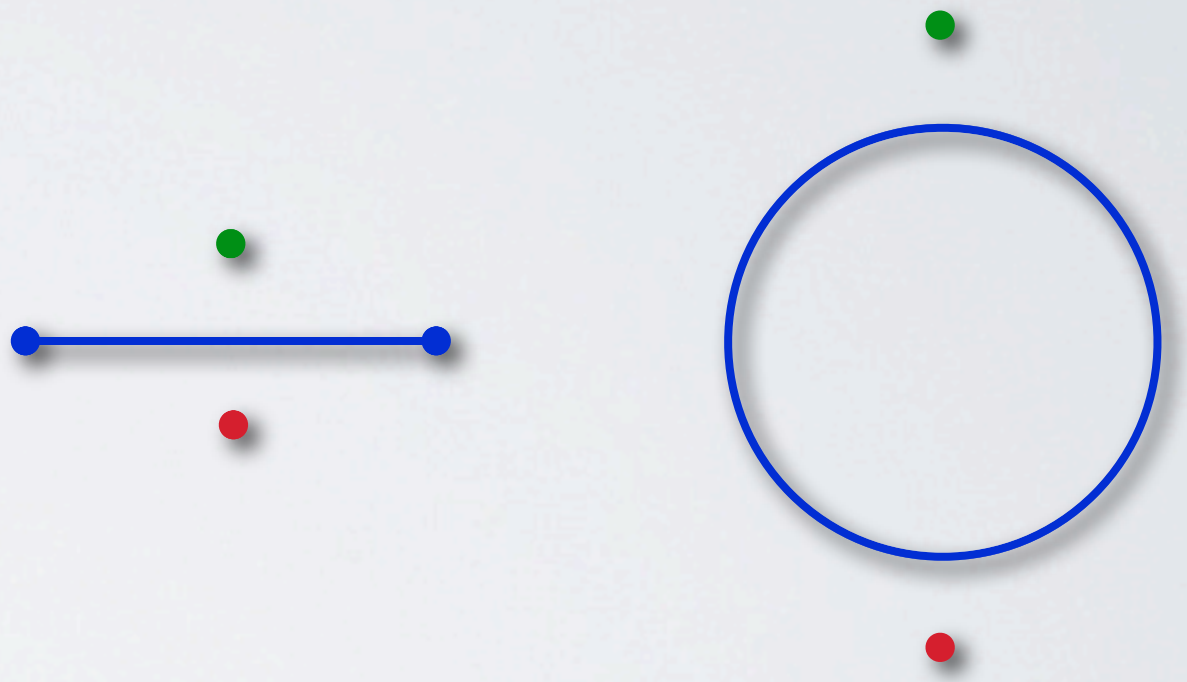


$T_-^{-1}(x)$



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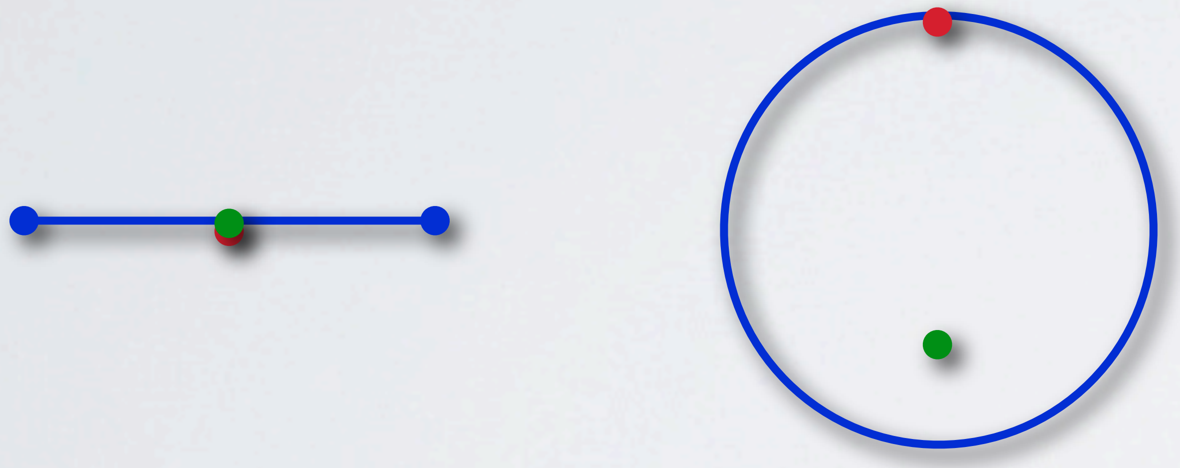

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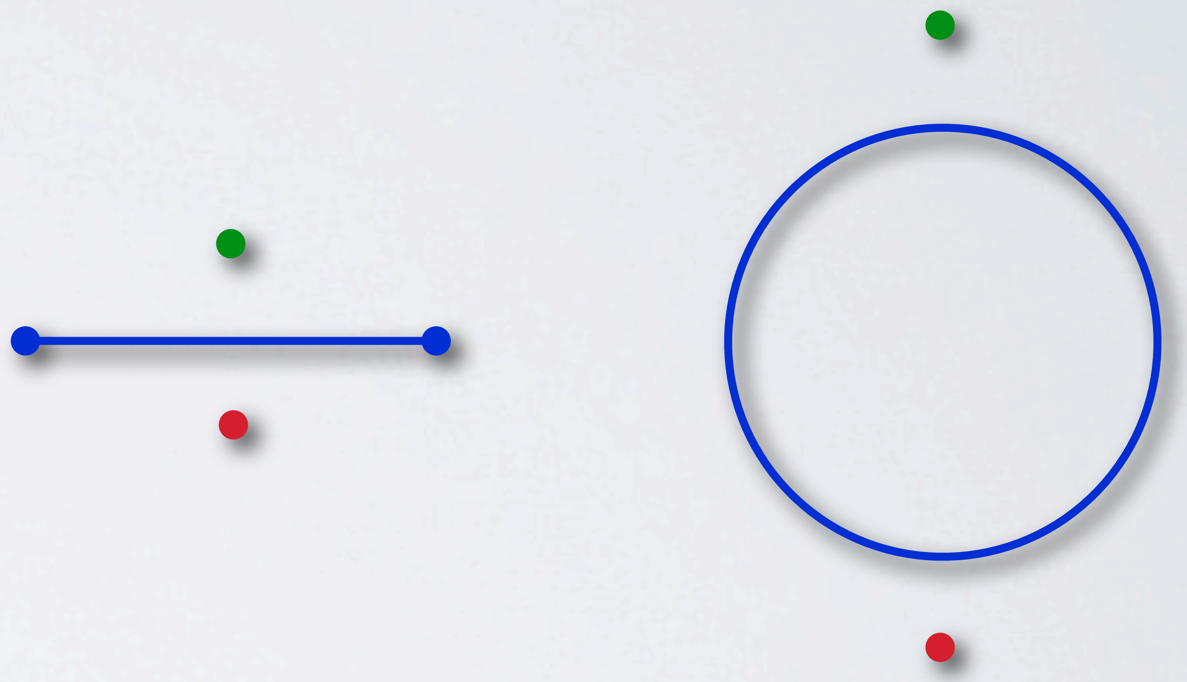

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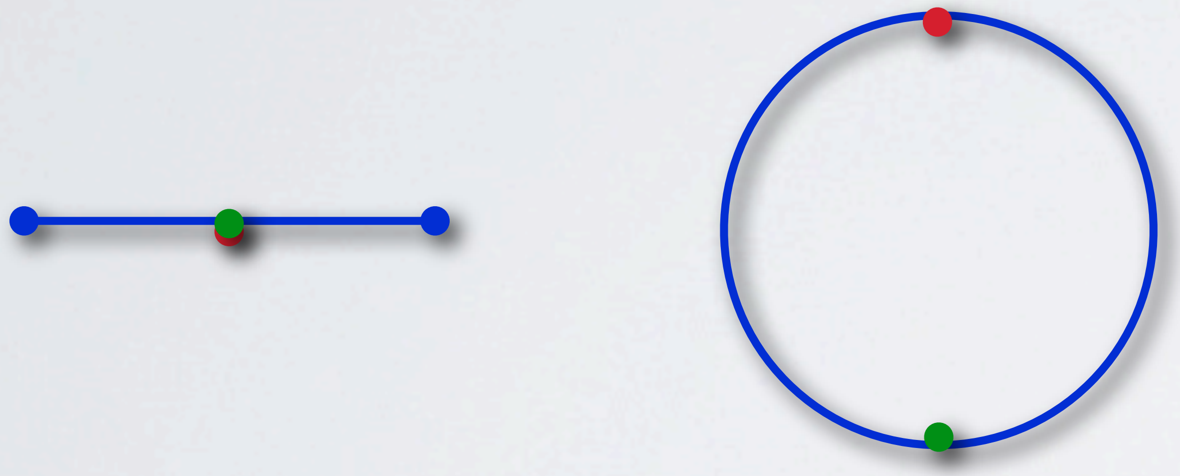

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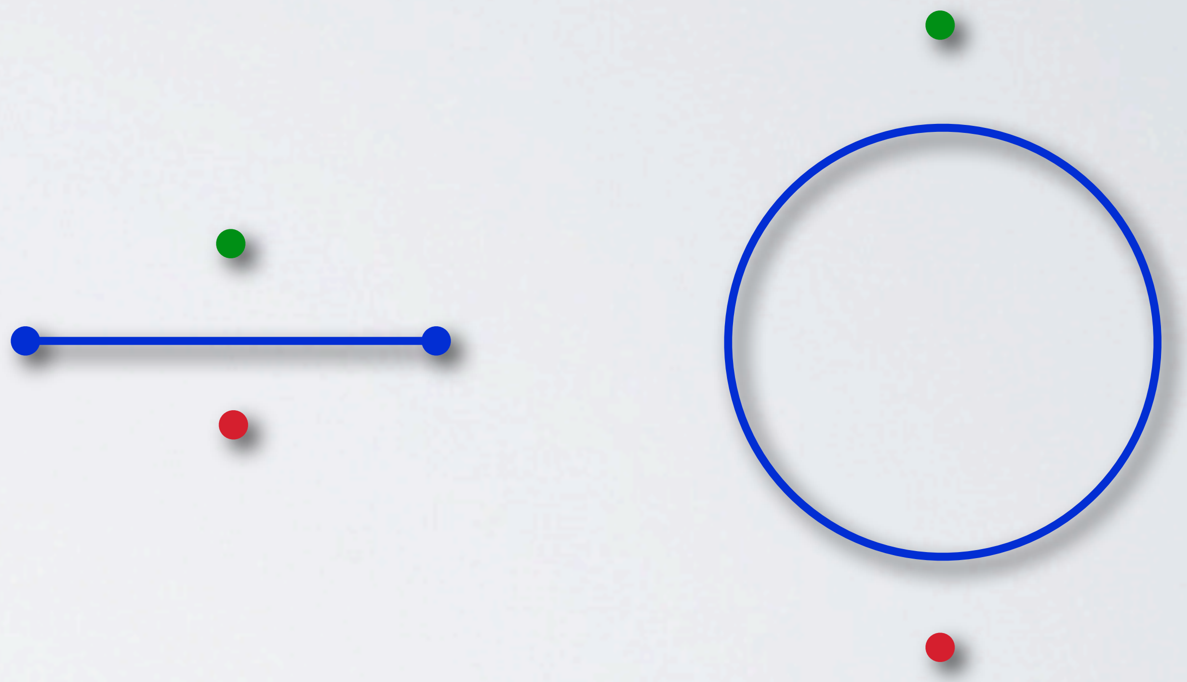

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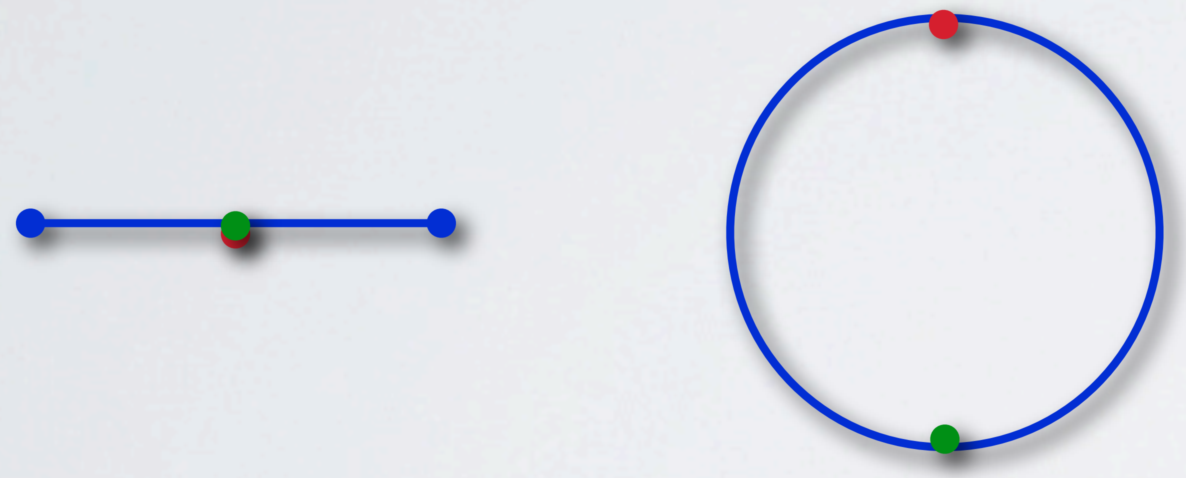

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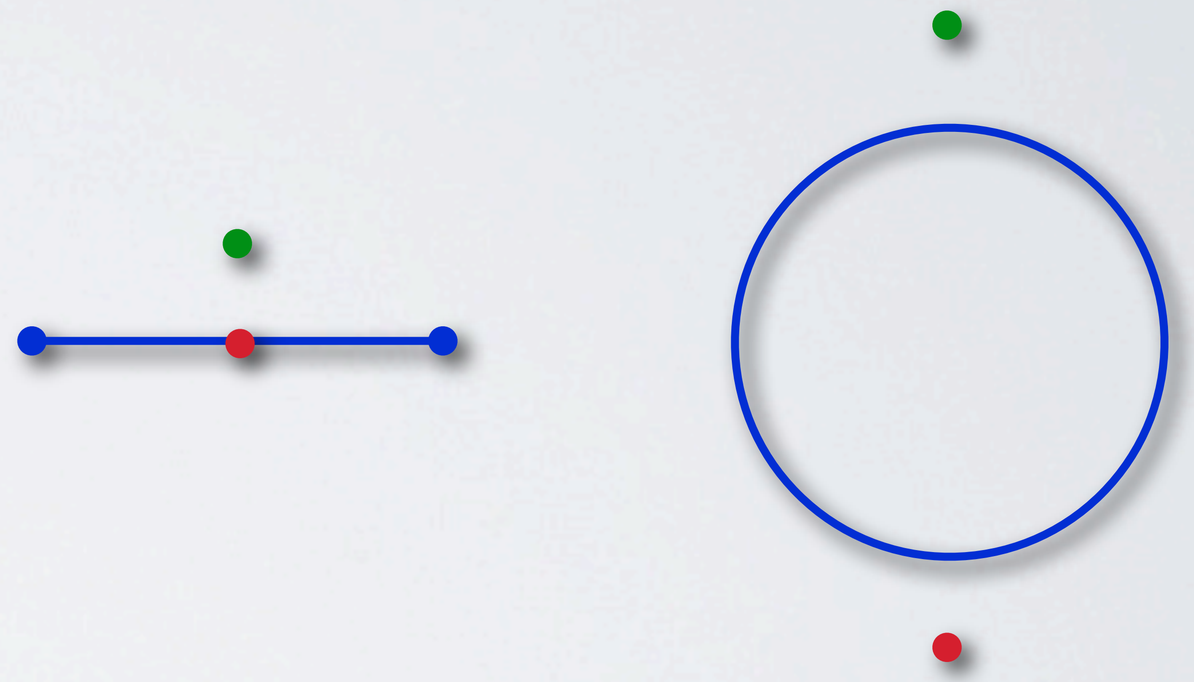

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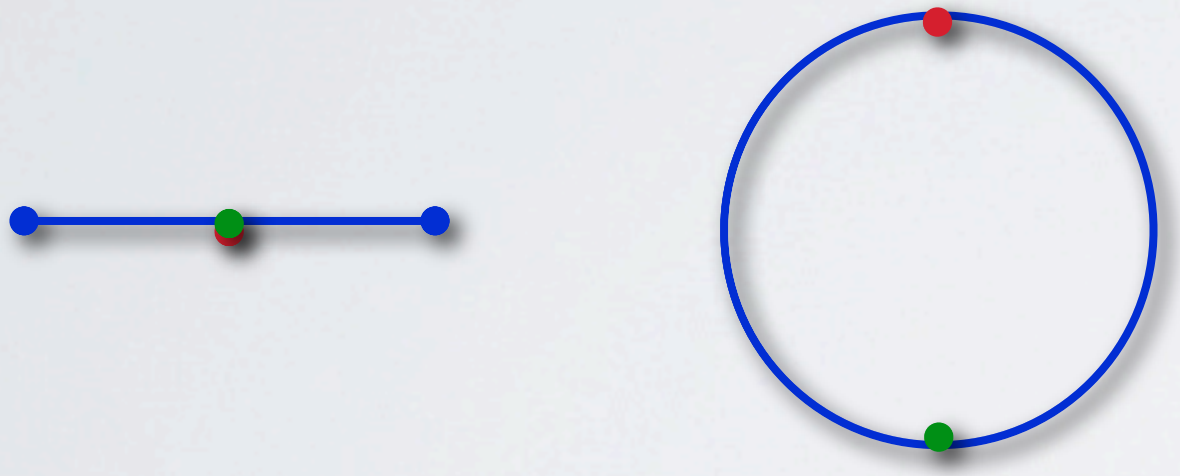
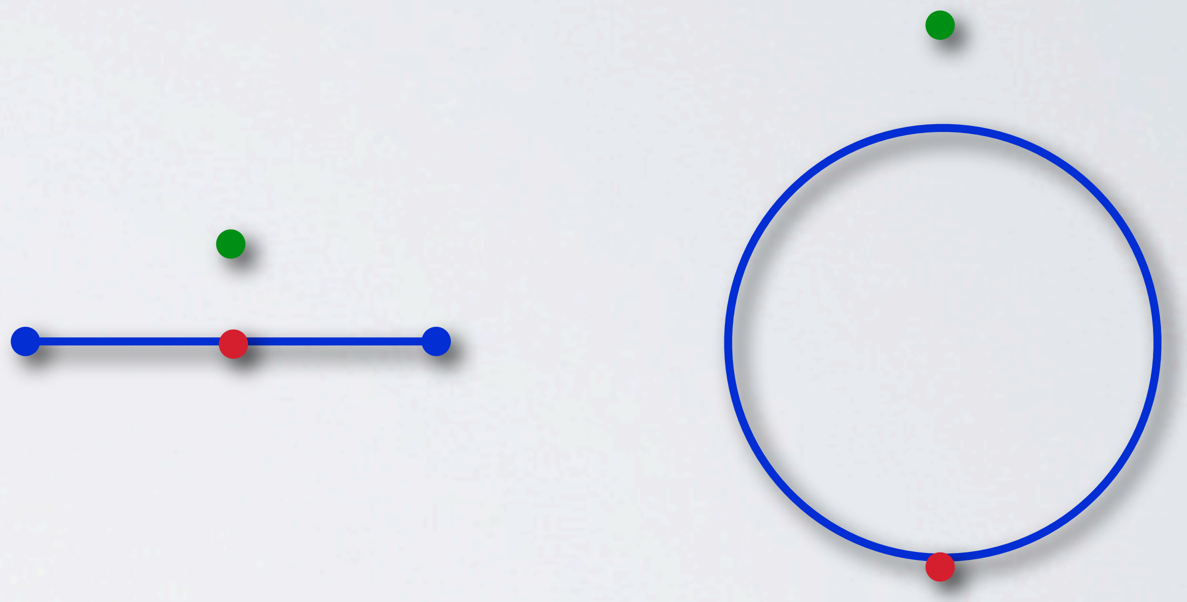


$T_-^{-1}(x)$



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Relationship between inverses

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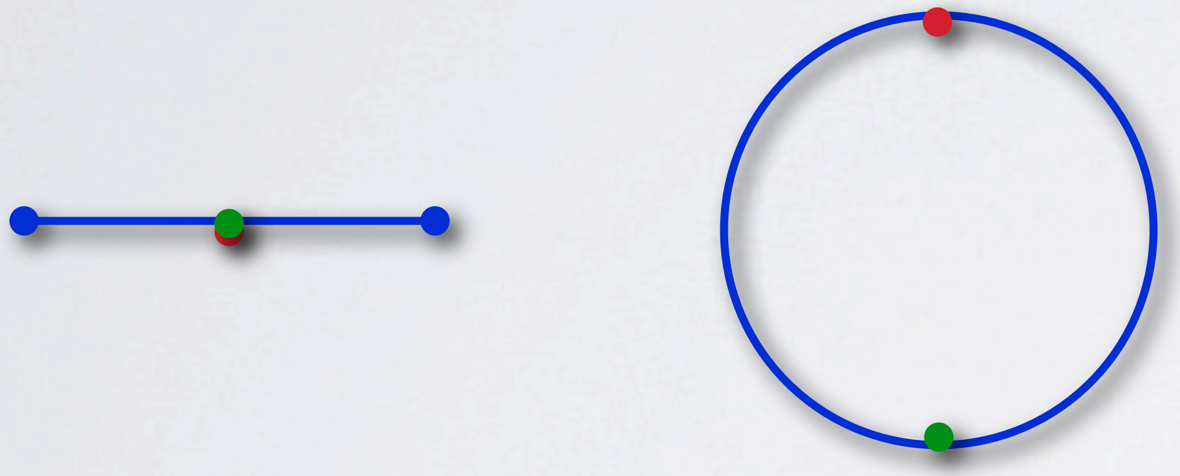

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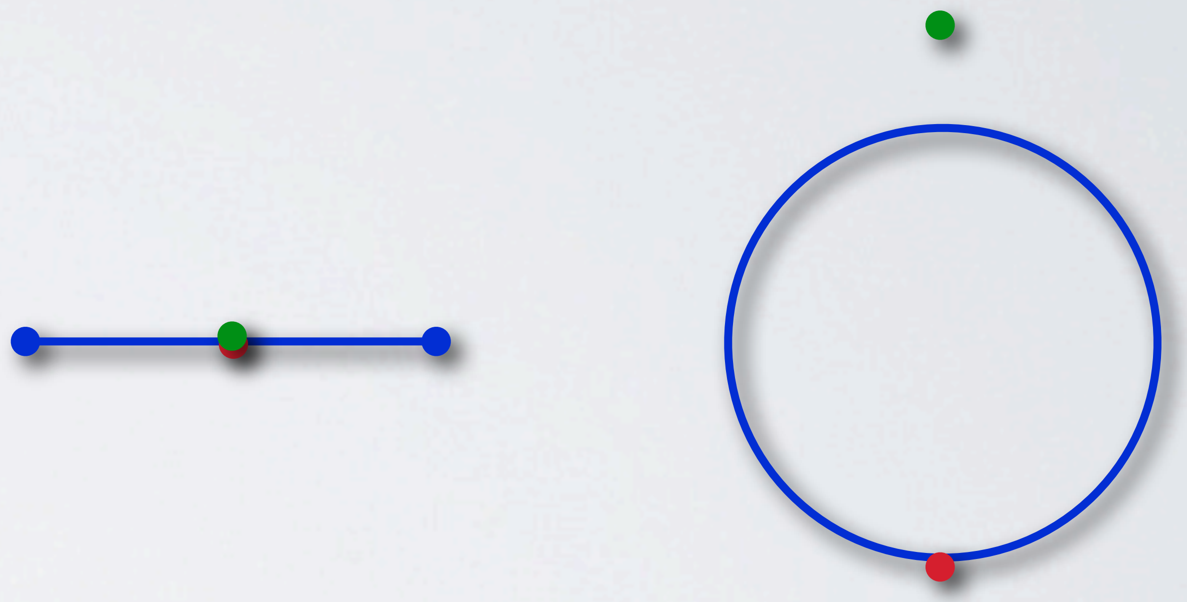

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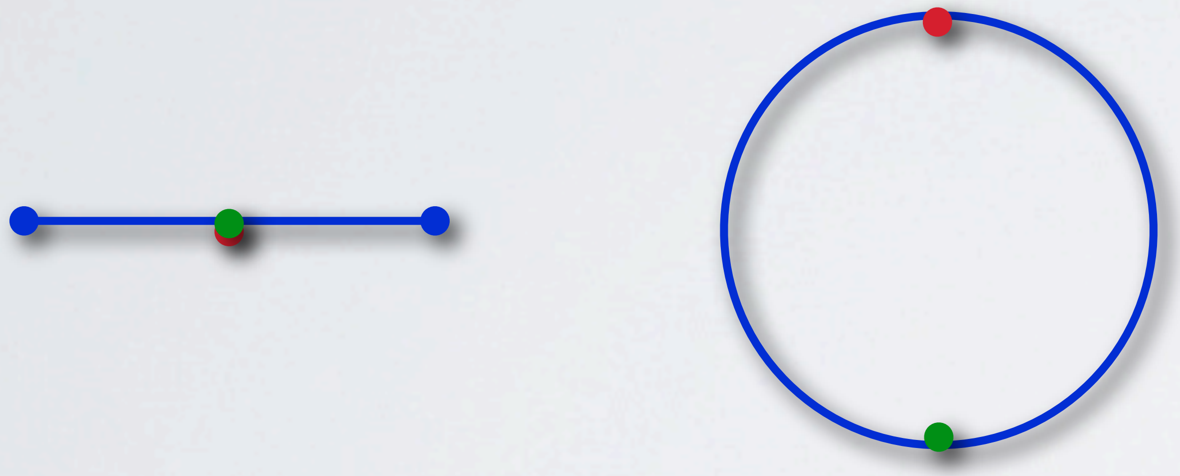

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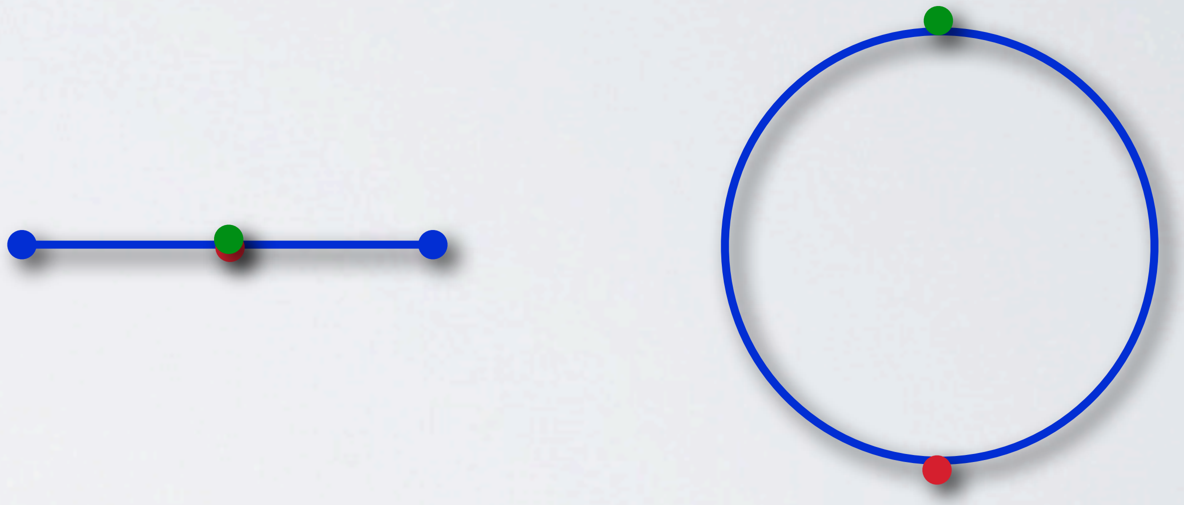


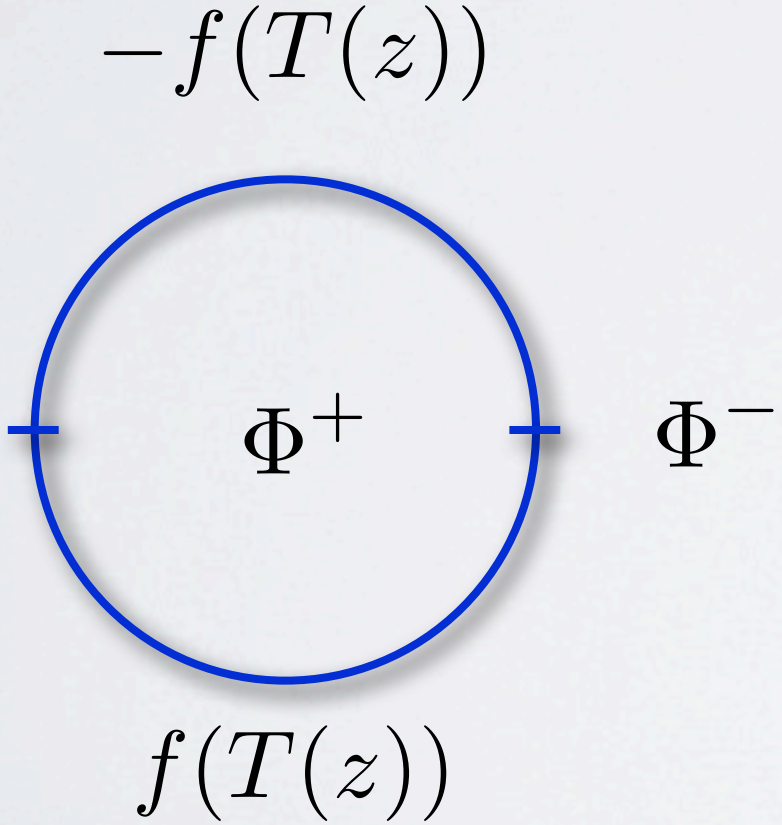
$$T_-^{-1}(x^+) = T_\uparrow^{-1}(x)$$
$$T_-^{-1}(x^-) = T_\downarrow^{-1}(x)$$

Suppose we compute the Cauchy transform of the mapped function on the unit circle

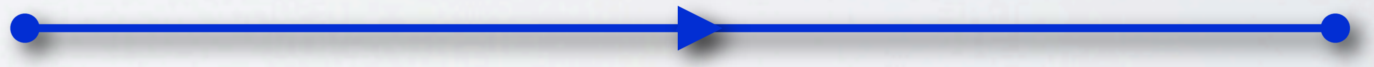
$$\Phi = -\mathcal{C}_{\circ} \operatorname{sgn} \arg z \, f(T(z))$$

Then the Cauchy transform over the interval is

$$\mathcal{C}_{(-1,1)} f(x) = \frac{1}{2}\left[\Phi(T_+^{-1}(x)) + \Phi(T_-^{-1}(x))\right]$$



$$\frac{1}{2}\left[\Phi^+(T_\downarrow^{-1}(x)) + \Phi^-(T_\uparrow^{-1}(x))\right]$$



$$\frac{1}{2}\left[\Phi^+(T_\uparrow^{-1}(x)) + \Phi^-(T_\downarrow^{-1}(x))\right]$$

Unless f decays to all orders at the endpoints, or has square root singularities,

has jumps at ± 1

But we can still represent it efficiently as

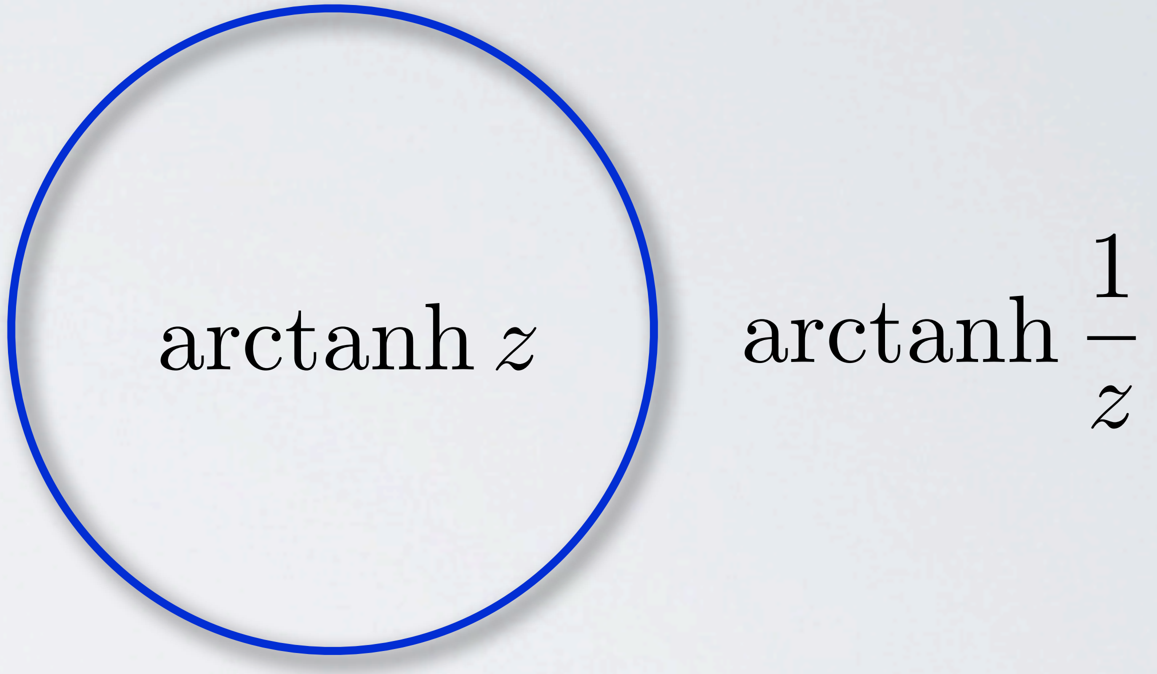
$$-\sum_{k=-\infty}^{\infty} \hat{f}_k z^k \operatorname{sgn} \arg z$$

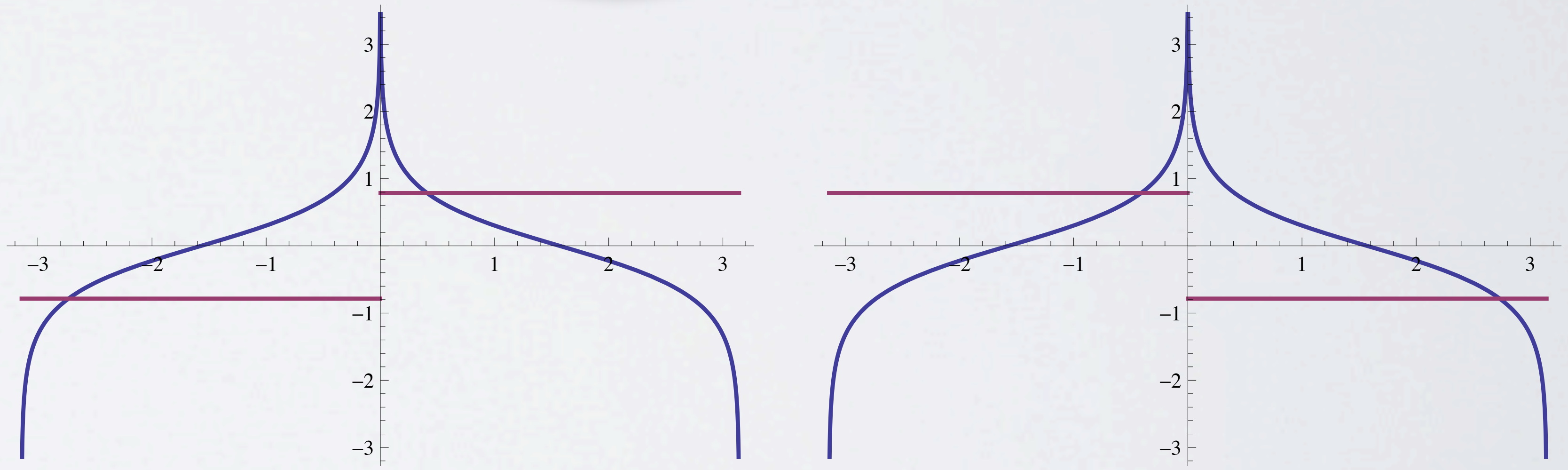
Thus we consider the moment problem

$$\mathcal{C} z^k \operatorname{sgn} \arg z$$

The first moment is straightforward:

$$\frac{i\pi}{2} \mathcal{C} \operatorname{sgn} \arg z =$$





$\operatorname{arctanh} z$ $\qquad$ $\operatorname{arctanh} \frac{1}{z}$

$$\frac{\mathrm{i}\pi}{2} u(z) = \qquad z \operatorname{arctanh} z \qquad z \operatorname{arctanh} \frac{1}{z}$$

Satisfies

$$u^+(z) - u^-(z) = z \operatorname{sgn} \arg z$$

However,

$$u(\infty) = \lim_{z \to 0} \frac{1}{z} \operatorname{arctanh} z = \lim_{z \to 0} \frac{1}{z} \left(z + \frac{z^3}{3} + \cdots \right) = 1 \neq 0$$

Thus we choose

$$\frac{\mathrm{i}\pi}{2}\mathcal{C} z \operatorname{sgn} \arg z = \qquad z \operatorname{arctanh} z - 1 \qquad z \operatorname{arctanh}\frac{1}{z} - 1$$

Higher order moments φ_k can be found in the same manner, by subtracting out the terms in the Taylor series of $\operatorname{arctanh}$.

A closed form solution which avoids cancellation can be written as Lerch transcendent functions, or alternatively (and more accurately in Mathematica) as Hypergeometric functions.

We obtain

$$\mathcal{C}_{(-1,1)} T_k(x) = -\frac{1}{4}\left[\varphi_k(T_+^{-1}(x)) + \varphi_k(T_-^{-1}(x)) + \varphi_{-k}(T_+^{-1}(x)) + \varphi_{-k}(T_-^{-1}(x))\right],$$

$$\mathcal{C}^+_{(-1,1)} T_k(x) = -\frac{1}{4}\left[\varphi_k(T_\downarrow^{-1}(x)) + \varphi_k(T_\uparrow^{-1}(x)) + \varphi_{-k}(T_\downarrow^{-1}(x)) + \varphi_{-k}(T_\uparrow^{-1}(x))\right],$$

$$\mathcal{C}^-_{(-1,1)} T_k(x) = -\frac{1}{4}\left[\varphi_k(T_\uparrow^{-1}(x)) + \varphi_k(T_\downarrow^{-1}(x)) + \varphi_{-k}(T_\uparrow^{-1}(x)) + \varphi_{-k}(T_\downarrow^{-1}(x))\right],$$

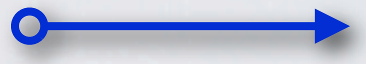
$$\mathcal{C}_{(-1,1)} T_k(x) \underset{x\to -1}{\sim} -\frac{1}{2\mathrm{i}\pi}(-1)^k\left[\log(-x-1) - \log 2\right] + \frac{1}{\mathrm{i}\pi}\phi_k(-1),$$

$$\mathcal{C}_{(-1,1)} f(x) \underset{x\to 1}{\sim} \frac{1}{2\mathrm{i}\pi}\left[\log(x-1) - \log 2\right] + \frac{1}{\mathrm{i}\pi}\phi_k(1),$$

where ϕ_k can be expressed in terms of φ_k

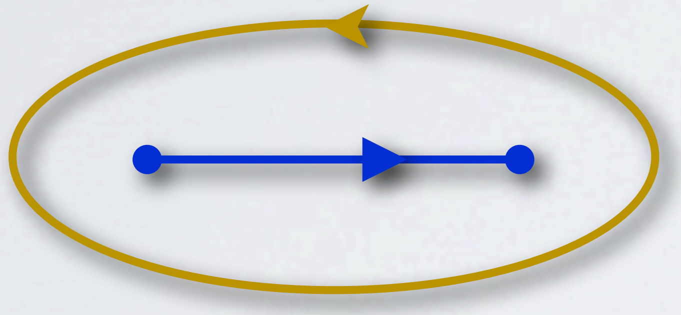
(Related formulæ for the Hilbert transform over the interval cf. King 2009)

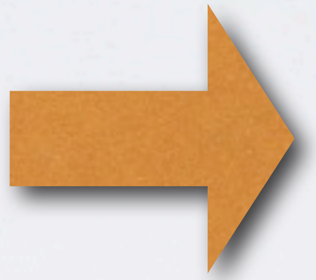
$(0, \infty)$

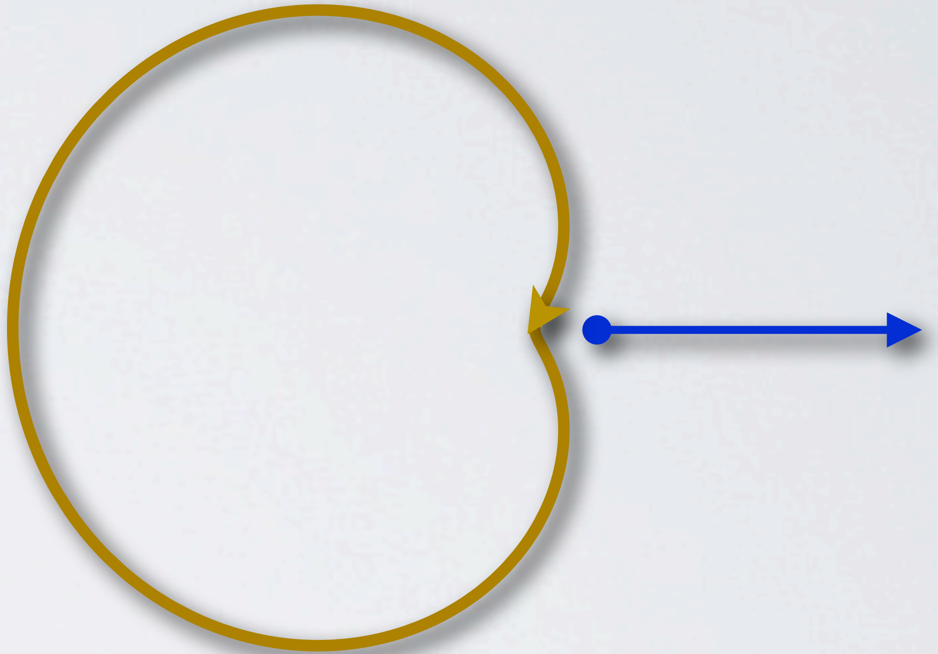


What we do now is map the unit interval to the half line

$$H(x) = \frac{1+x}{1-x}$$







We then obtain

$$\mathcal{C}r(y) = \Phi(H^{-1}(y)) - \Phi^{+}(1)$$

$$\Phi = \mathcal{C}f \qquad \text{for} \qquad f(x) = r(H(x))$$

Clearly, we can compute for other rays by rotation

COMPUTATION OF THE HILBERT TRANSFORM

- We can write

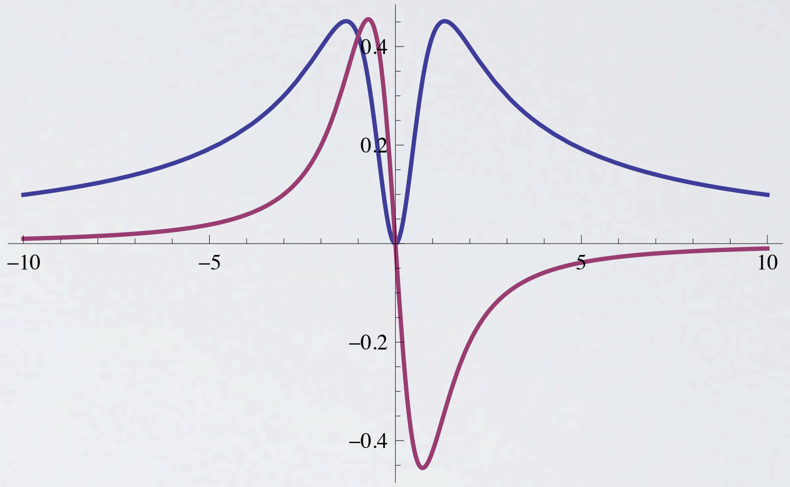
$$
\begin{aligned}
\mathcal{C}^{\pm}_{(-\infty,\infty)} r(y) &= \mathcal{C}^{\pm}_{(-\infty,0)} r(y) + \mathcal{C}_{(0,\infty)} r(y) \quad \text{for} \quad y \in (-\infty, 0), \\
\mathcal{C}^{\pm}_{(-\infty,\infty)} r(y) &= \mathcal{C}_{(-\infty,0)} r(y) + \mathcal{C}^{\pm}_{(0,\infty)} r(y) \quad \text{for} \quad y \in (0, \infty), \\
\mathcal{C}^{\pm}_{(-\infty,\infty)} r(0) &= \sum_{k=0}^{\infty} (\check{f}_{1,k} - \check{f}_{2,k})(\phi_k(1) - \phi_k(-1)) \\
f_1(x) &= r(H(x)),\, f_2(x) = r(-H(x))
\end{aligned}
$$

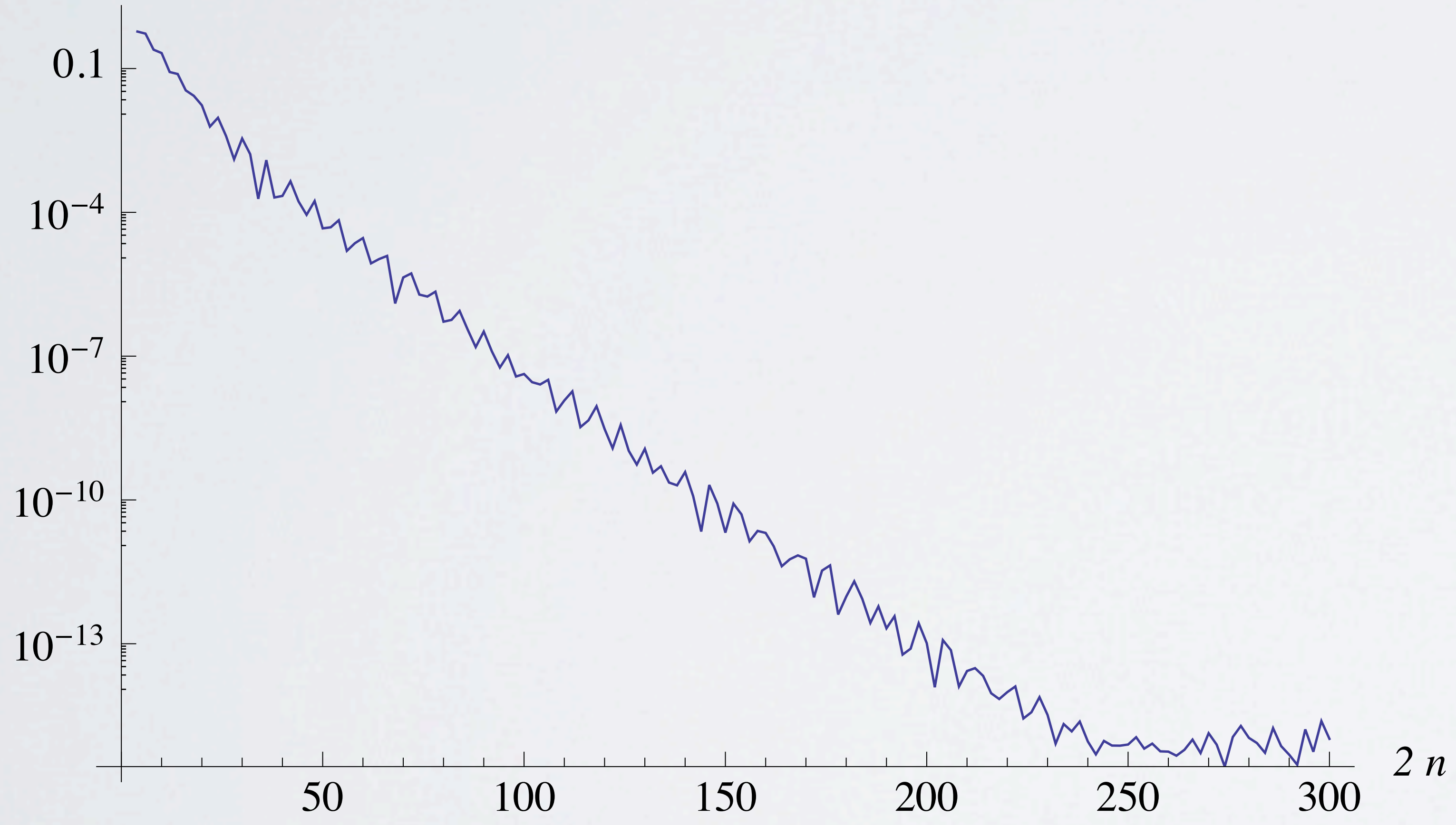
- Then

$$
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t - z} \, \mathrm{d}t = \mathrm{i}(\mathcal{C}^{+}_{\mathbb{R}} f + \mathcal{C}^{-}_{\mathbb{R}} f)
$$

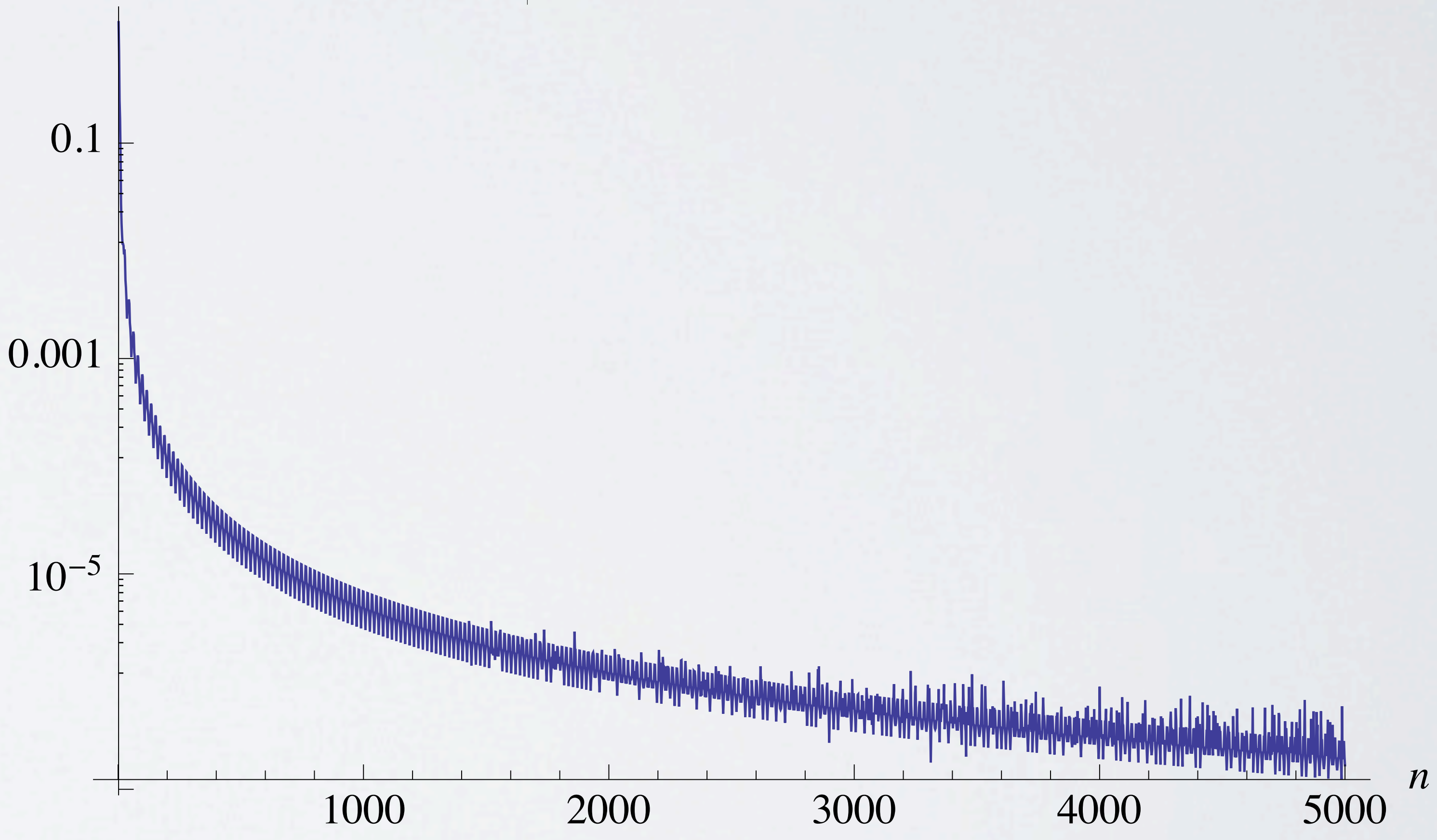
Applying this technique to compute the Hilbert transform, we obtain spectral accuracy for a wider class of functions than existing methods:

$$r(y) = \frac{\operatorname{erf} y}{y + \mathrm{i}} =$$





Two half lines



Real line mapped to circle

CONSTRUCTION OF THE SPECTRAL METHOD FOR RIEMANN–HILBERT PROBLEMS

- At the origin, the Cauchy transforms over the individual rays blow up:

$$\mathcal{C}_{\Gamma_k} f(z) \underset{z \to 0}{\sim} -\frac{f(0)}{2\mathrm{i}\pi} \log(-\mathrm{e}^{\mathrm{i}\theta_k} z) + C$$

We define the finite part along a curve at angle t as the circled part:

$$\mathcal{C}_{\Gamma_k} f(z) \sim \boxed{C - \frac{f(0)}{2\mathrm{i}\pi} \mathrm{i} \arg(-\mathrm{e}^{\mathrm{i}(\theta_k + t)})} - \frac{f(0)}{2\mathrm{i}\pi} \log|z|$$

- Whenever the limits of V along each ray sum to zero, this expression is an equality

$$\begin{aligned}
\mathcal{C}_{\Gamma} V(z) &= \mathcal{C}_{\Gamma_1} V_1(z) + \cdots + \mathcal{C}_{\Gamma_6} V_6(z) \\
&= -\frac{1}{2\mathrm{i}\pi} (V_1(0) + \cdots + V_6(0)) \log|z| + \text{bounded terms} \\
&\sim \text{bounded terms}
\end{aligned}$$

- Spectral method for the homogeneous Painlevé II equation:
  - Choose the basis of Chebyshev polynomials mapped to each ray
  - Using the Cauchy transform formulæ, construct the linear system

$$\mathcal{L}\mathcal{C}_{\Gamma_j}\boldsymbol{c}_1^{\Gamma_j}\psi_1^{\Gamma_j}(x_k)+\cdots+\mathcal{L}\mathcal{C}_{\Gamma_j}\boldsymbol{c}_n^{\Gamma_j}\psi_n^{\Gamma_j}(x_k)=G_j(x_k)-I,$$
$$j=1,\ldots,6 \text{ and } k=1,\ldots,4n.$$

where we take the finite part at zero

- This will be justified because the limits along each ray of the computed solution will always sum to zero whenever $s_1s_3 - s_1s_2 - s_2s_3 \neq 9$
- Otherwise, the linear system has an extra degree of freedom, and we can add the following condition:

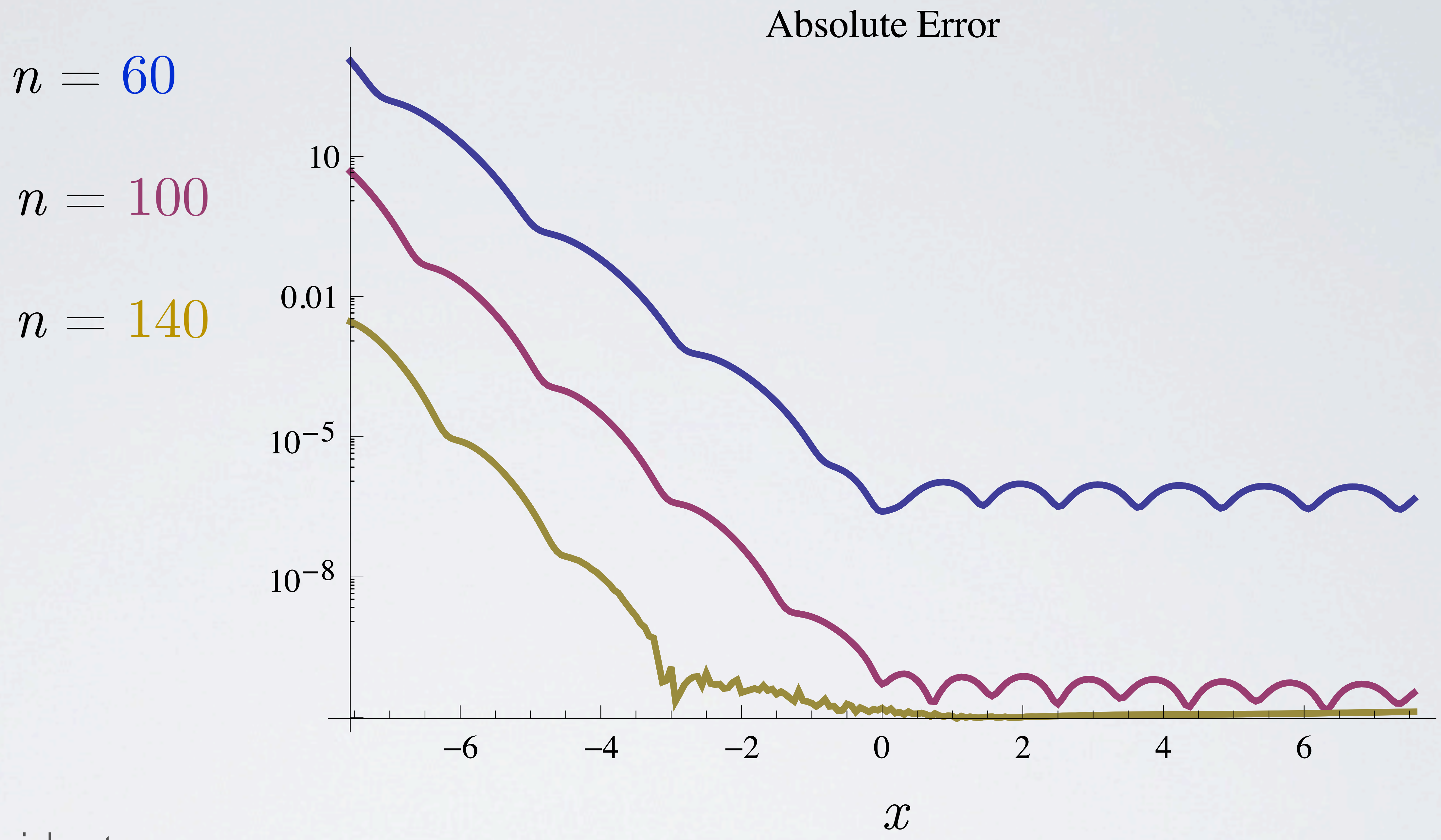
$$\sum_{k,j}\boldsymbol{c}_k^{\Gamma_j}\left[\text{Finite part of } \mathcal{L}\mathcal{C}\psi_k^{\Gamma_j} \text{ at zero}\right]=0$$

- We can easily find the asymptotic behaviour of $\mathcal{C}$ for each basis term at ∞, to compute

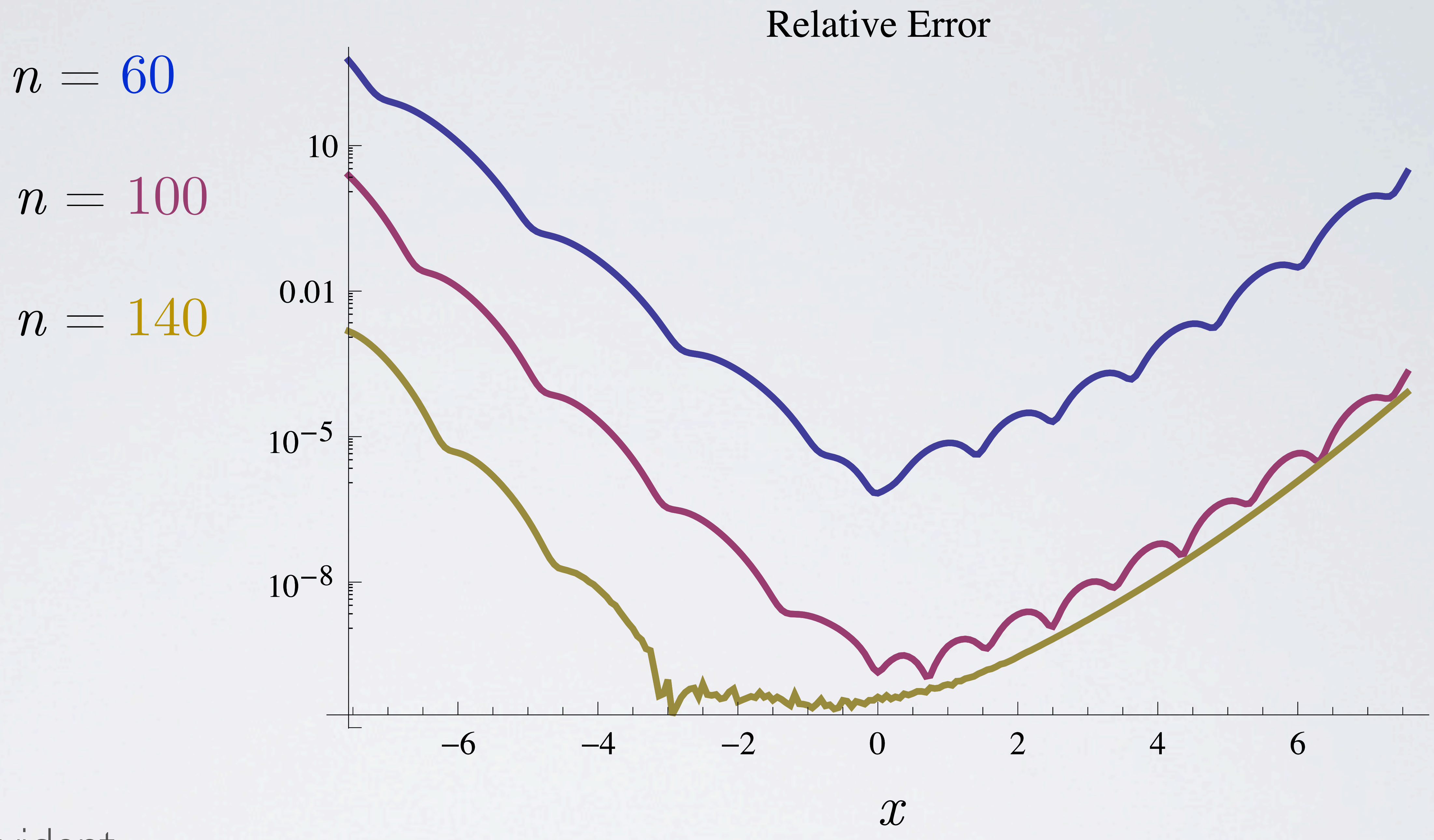
$$u(x) \approx 2 \lim_{z \to \infty} z \sum_{k,j} \boldsymbol{c}_{k,1,2} \mathcal{C}_{\Gamma_j} \psi_k^{\Gamma_j}(z)$$

- We can also apply this approach for computing the derivative of $u(x)$, reusing much of the computation
- This is possibly the first reliable numerical method for computing the initial conditions for given Stokes' constants
  - And asymptotics are determined from the Stokes' constants (though can very greatly with small errors)

- Consider the *Hastings–McLeod solution*, which is equivalent to the choice $(s_1, s_2, s_3) = (\mathrm{i}, 0, -\mathrm{i})$
- Numerical values of the Hastings–McLeod solution at a set of points are available (Prähofer and Spohn 2004)
  - Computed by using the known asymptotics to determine initial conditions for large x, then very high precision arithmetic to integrate to zero: a very inefficient method
  - This computation is particularly difficult because a small perturbation of initial conditions can introduce oscillations or poles



- Spectral convergence is evident
- When the Hypergeometric functions are precomputed, the method takes less than 4 seconds per point for $n = 140$
- (Bornemann 2010), where the related Tracy–Widom distribution is computed using its Fredholm determinant representation, is more efficient for this case (but doesn't generalize to other Stokes' constants)



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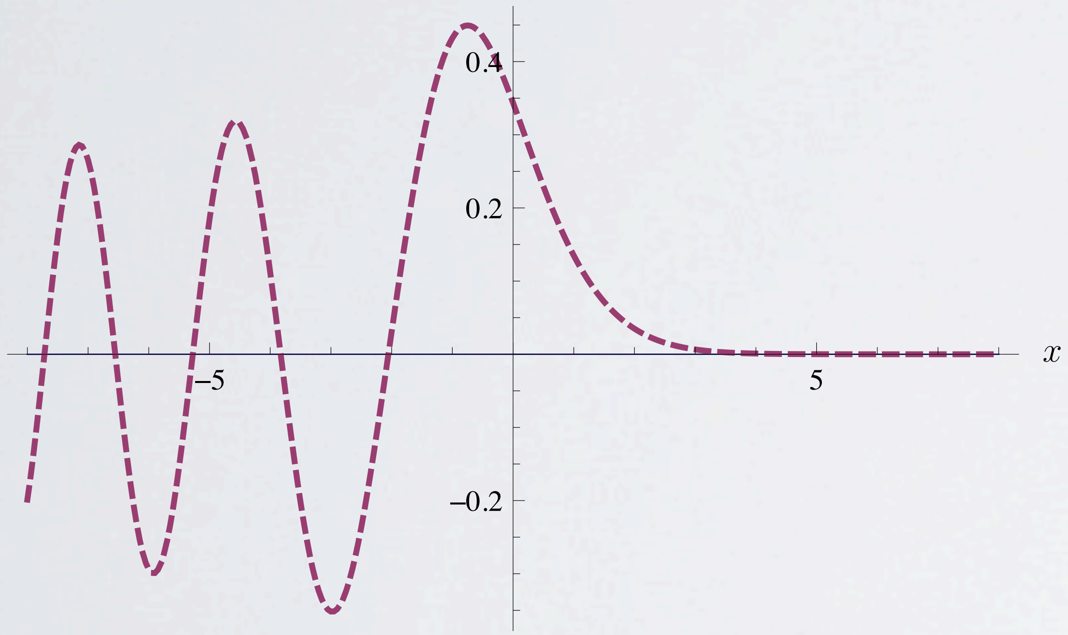
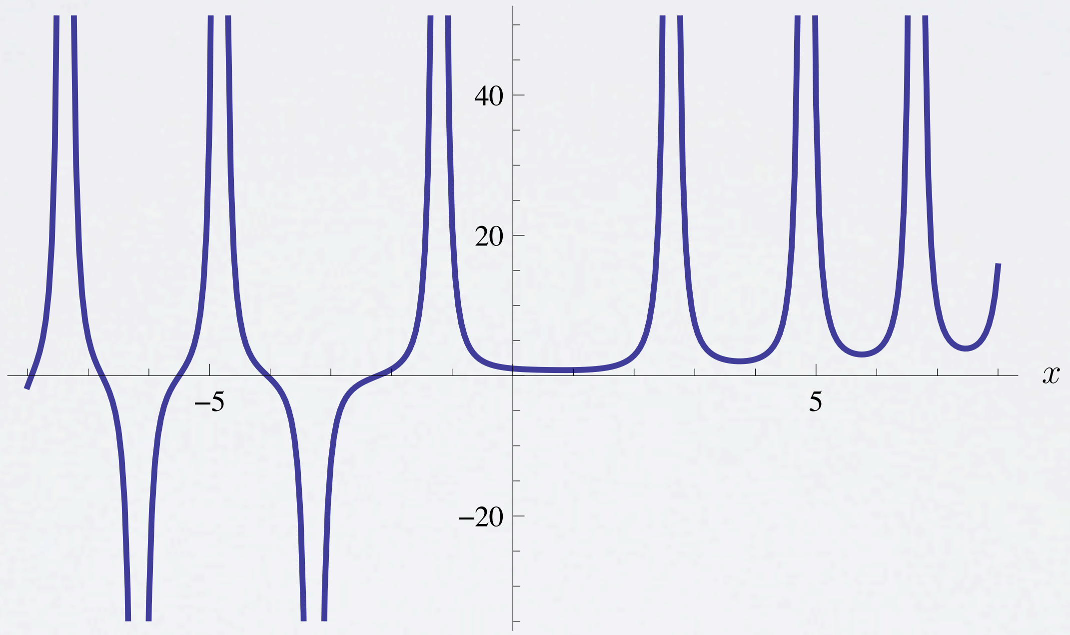
OTHER SOLUTIONS

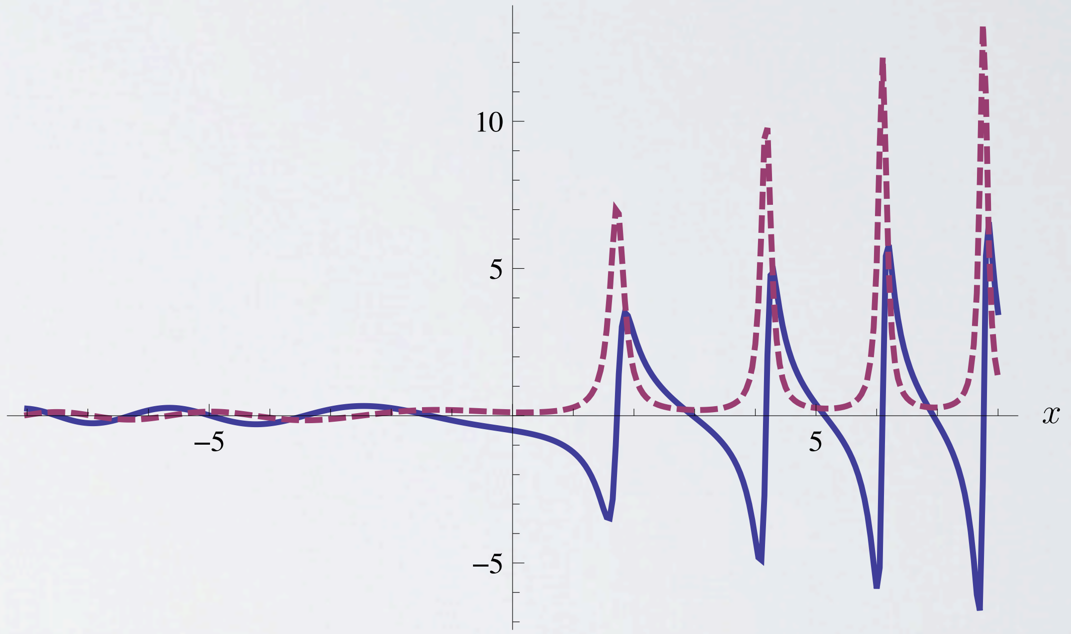
$(1,0,-1)$



$(1+\mathrm{i}, -2, 1-\mathrm{i})$



$(1,2,1/3)$

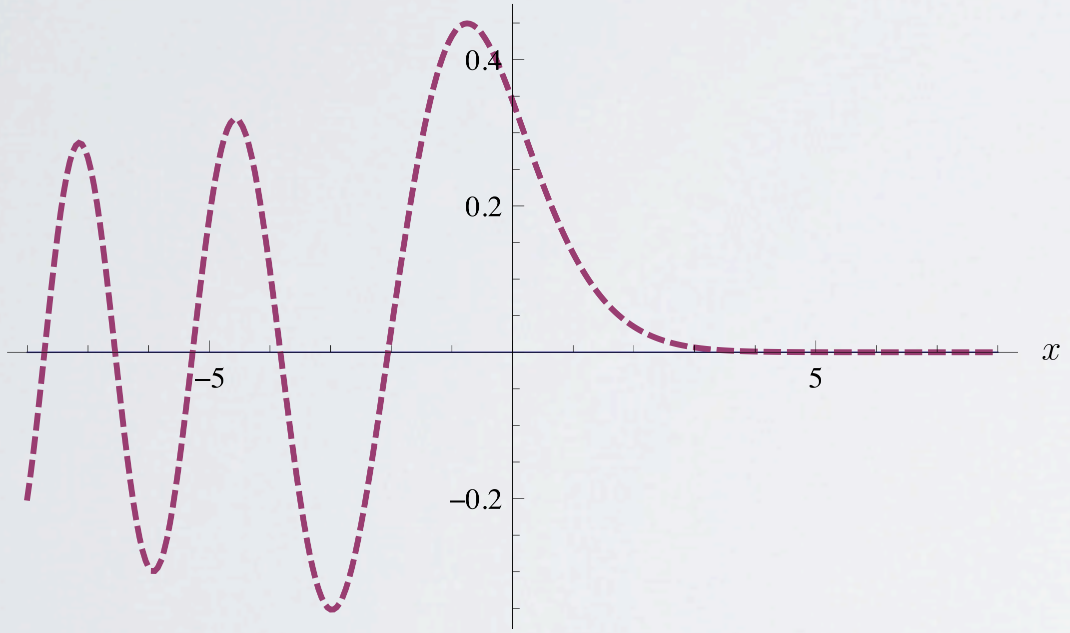


Real and imaginary parts

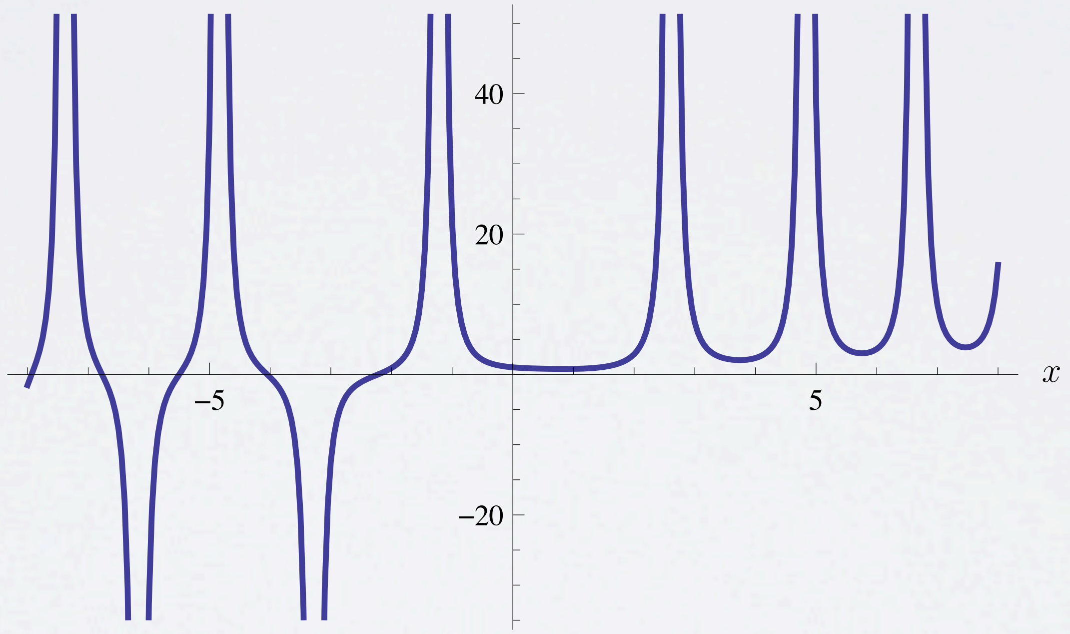
OTHER SOLUTIONS

Spectral system becomes badly conditioned at poles (can be used to compute location of poles)

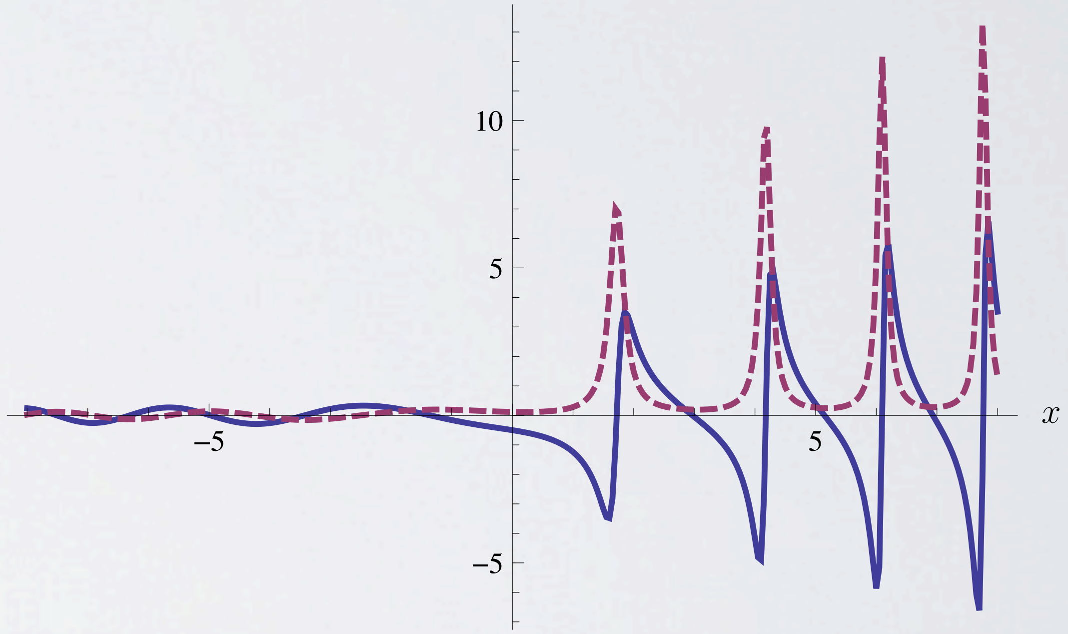
$(1,0,-1)$



$(1+\mathrm{i}, -2, 1-\mathrm{i})$



$(1,2,1/3)$



Real and imaginary parts

NONLINEAR STEEPEST DESCENT

- As x becomes large, the jump matrix G becomes increasingly oscillatory
  - Resolving oscillations requires more basis functions
  - When s_2 is nonzero, the method is also badly conditioned
- Both issues can be resolved by deforming Γ through the saddle points of the jump function and along the path of steepest descent
- This is how asymptotics of solutions to the Painlevé equation are determined (Deift & Zhou 1995); we instead use it as a computational tool
  - Obtaining higher order terms of the *asymptotic expansion* appears to be difficult, whereas our approach can easily achieve *high accuracy*

- The key to the numerical approach is that we can readily compute the Cauchy transform over domains conformally mappable to the unit interval
- We simply interpolate the path of steepest descent by a piecewise affine curve
- Motivated by (Huybrechs & Vandewalle 2006), which developed a numerical approach for steepest descent for oscillatory integrals, the collocation points (thus the break points of the linear interpolant) must coalesce at the saddle points in the asymptotic regime
- For negative x and $s_1 s_3 \neq 1$ this proceeds similar to deformation of oscillatory integrals, with the addition of requiring an LDU factorization and parametrix
- For positive x or $s_1 s_3 = 1$ (such as in Hastings–McLeod) we need to change the oscillator also to avoid exponential growth/cancellation

Numerical nonlinear steepest descent: negative x

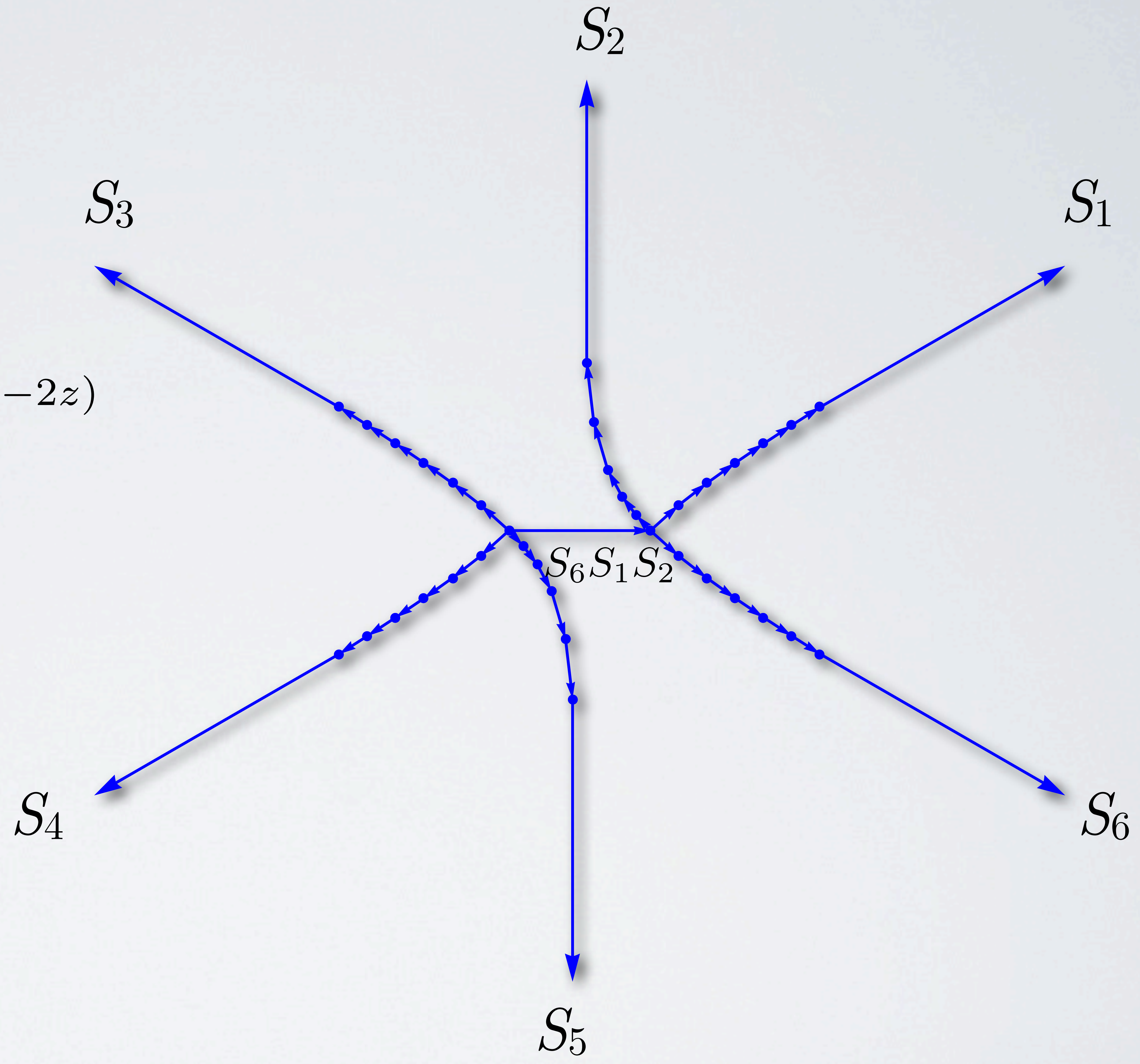
- We first do the transformation

$$z \mapsto \sqrt{-x}z$$

so that

$$\mathrm{e}^{\pm 8\mathrm{i}/3z^3 \pm 2\mathrm{i}x} \mapsto \mathrm{e}^{\pm\mathrm{i}(-x)^{3/2}(8/3z^3 - 2z)}$$

- This has two stationary points at $\pm 1/2$, thus we deform the contour to obtain the Riemann–Hilbert problem:



(Based on Deift & Zhou 1995 and Fokas et al 2006)

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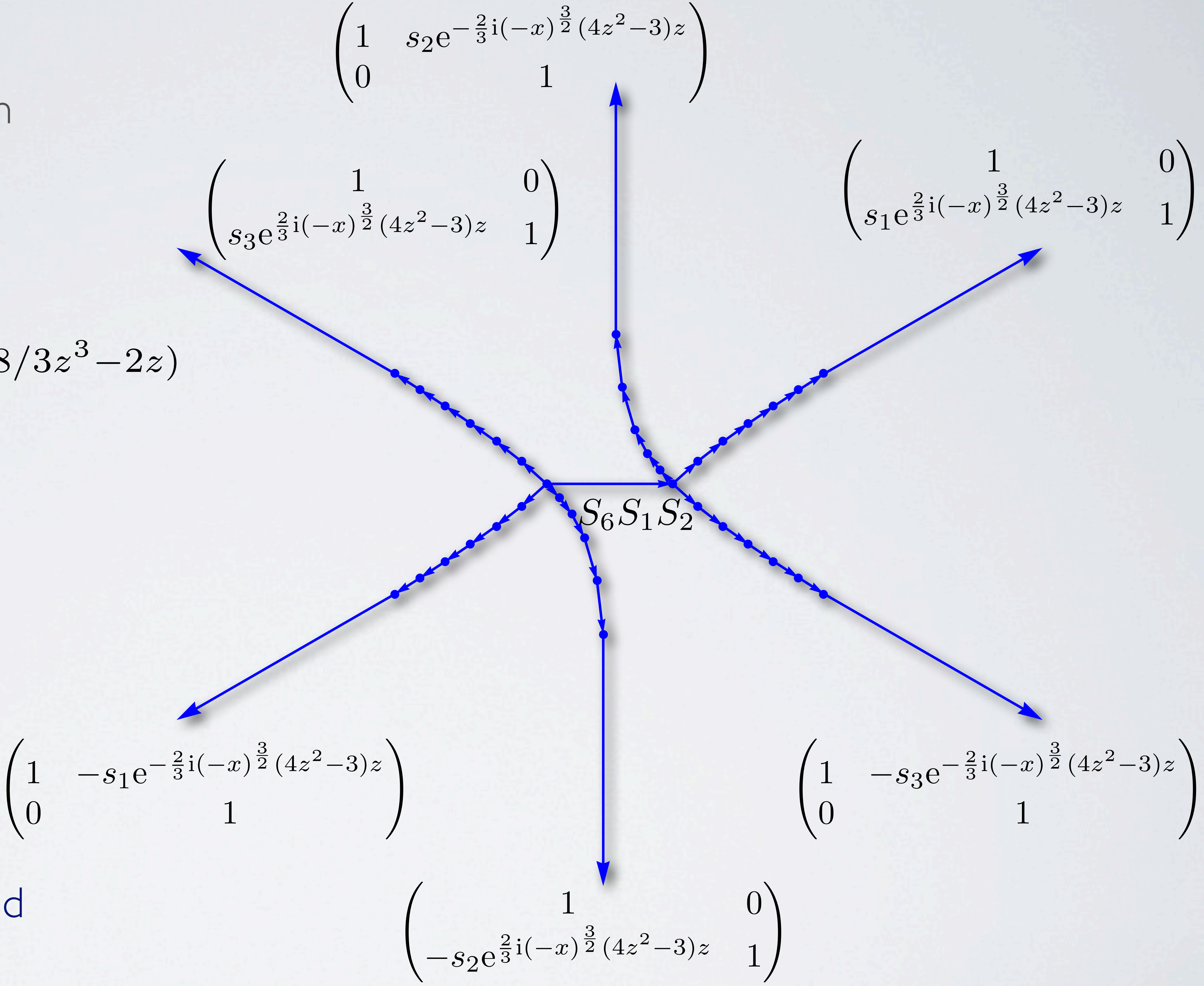


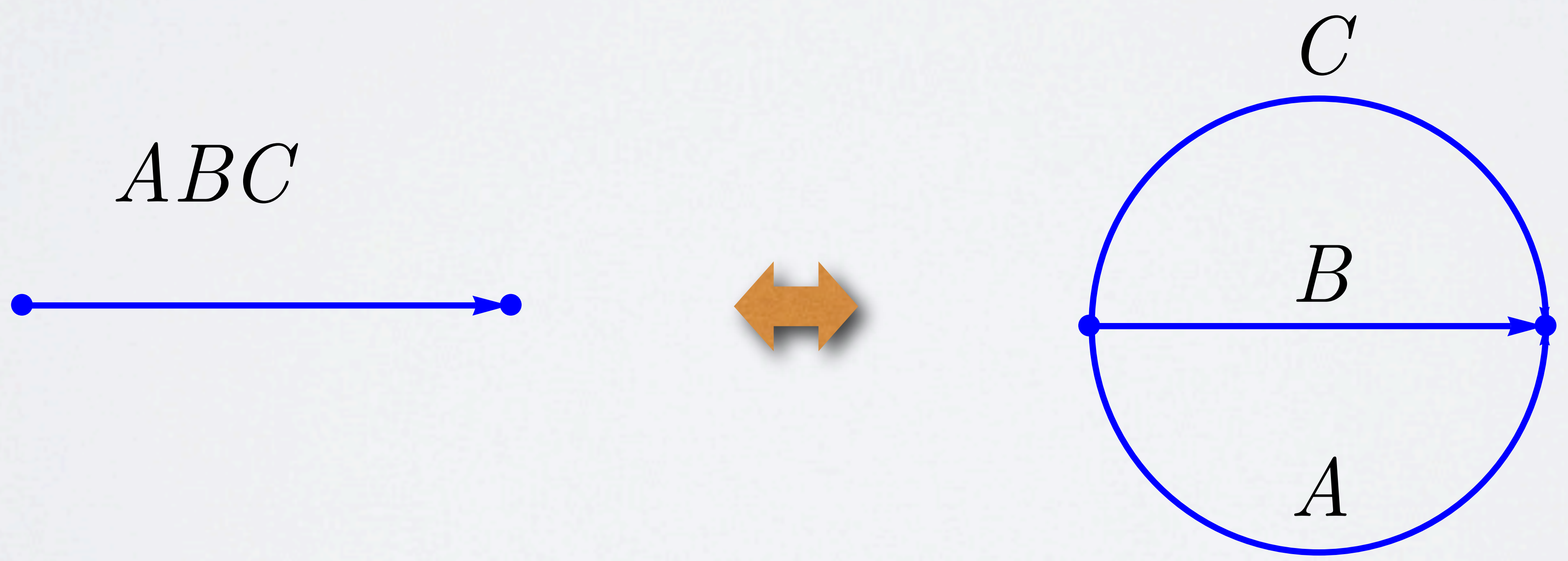
(Based on Deift & Zhou 1995 and Fokas et al 2006)

- Each of the paths to infinity have no oscillations and super-exponential decay
- But the path connecting $\pm 1/2$ is still oscillatory:

$$S_6 S_1 S_3 = \begin{pmatrix} 1 & -s_3 e^{-\frac{2}{3}\mathrm{i}(-x)^{\frac{3}{2}}(4z^2-3)z} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_1 e^{\frac{2}{3}\mathrm{i}(-x)^{\frac{3}{2}}(4z^2-3)z} & 1 \end{pmatrix} \begin{pmatrix} 1 & s_2 e^{-\frac{2}{3}\mathrm{i}(-x)^{\frac{3}{2}}(4z^2-3)z} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - s_1 s_3 & e^{-\frac{2}{3}\mathrm{i}(-x)^{\frac{3}{2}}(-3+4z^2)z} s_1 \\ e^{\frac{2}{3}\mathrm{i}(-x)^{\frac{3}{2}}(-3+4z^2)z} s_1 & 1 + s_1 s_2 \end{pmatrix}$$

- The key now is that we can split Riemann–Hilbert contours:

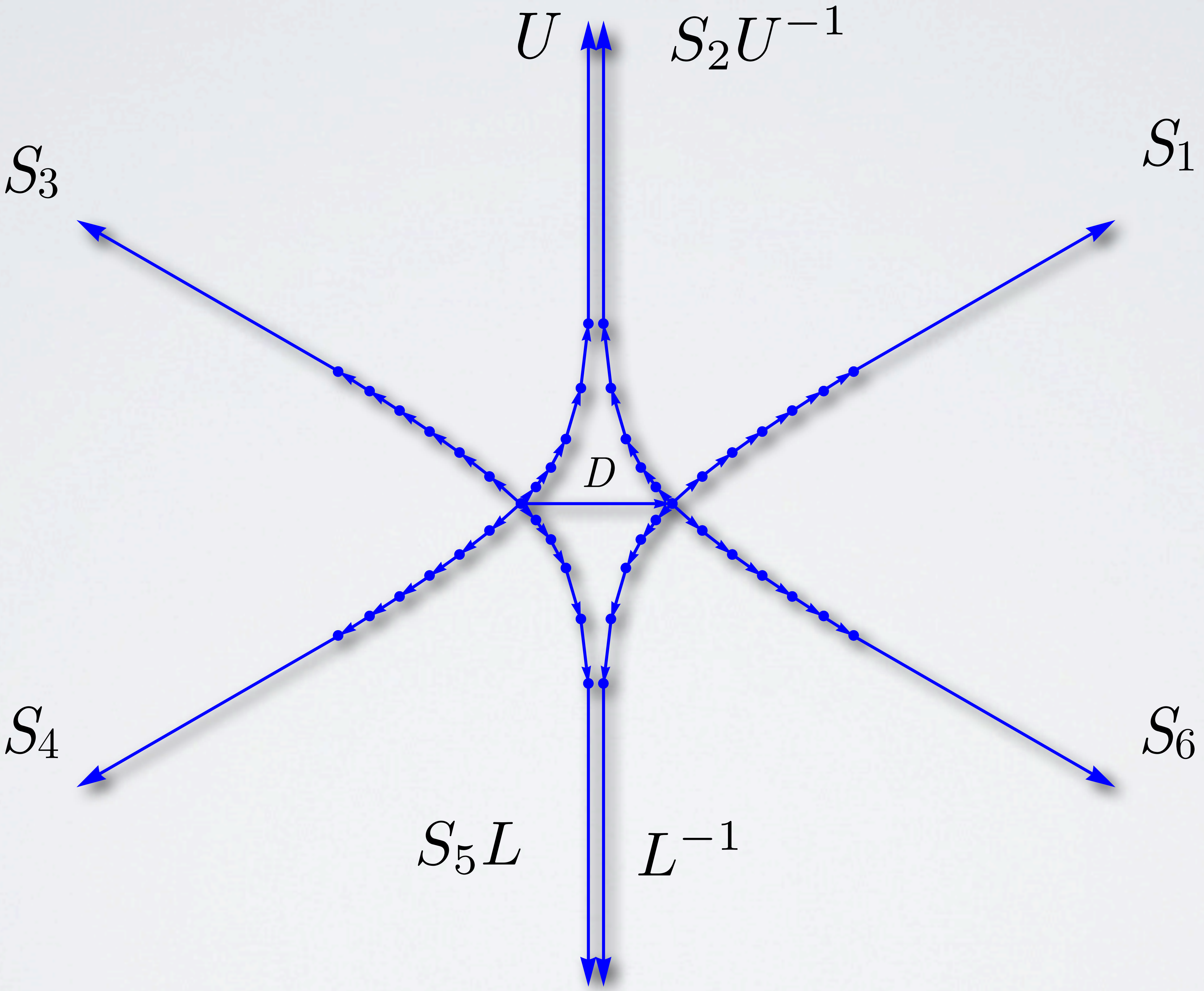


- We want to write $S_6 S_1 S_2$ as ABC where A goes to the identity matrix near the negative imaginary axis, B is nonoscillatory and C goes to the identity matrix near the positive imaginary axis
- This happens to be satisfied by the LDU factorization:

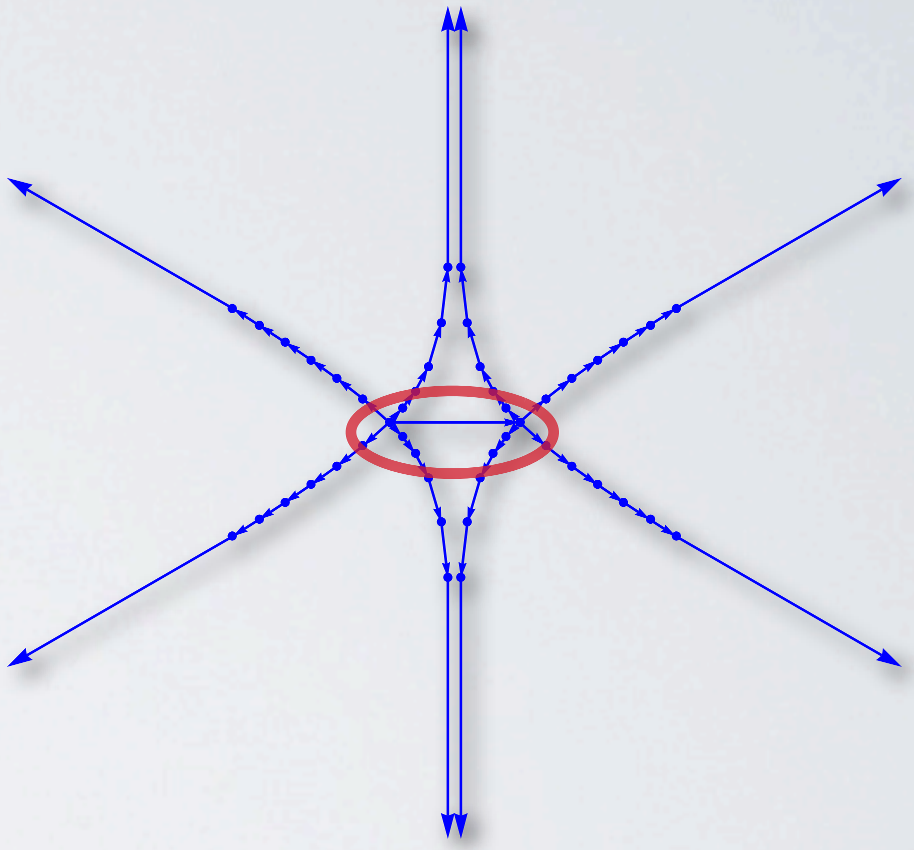
$$S_6 S_1 S_2 = LDU = \begin{pmatrix} 1 & 0 \\ \frac{s_1}{1-s_1 s_3} \mathrm{e}^{\frac{2}{3}\mathrm{i}(-x)^{\frac{3}{2}}(-3+4z^2)z} & 1 \end{pmatrix} \begin{pmatrix} 1 - s_1 s_3 & \\ & \frac{1}{1-s_1 s_3} \end{pmatrix} \begin{pmatrix} 1 & \frac{s_1}{1-s_1 s_3} \mathrm{e}^{-\frac{2}{3}\mathrm{i}(-x)^{\frac{3}{2}}(-3+4z^2)z} \\ 0 & 1 \end{pmatrix}$$

- Note that we must restrict our attention to the case where $s_1 s_3 \neq 1$
  - This excludes the Hastings–McLeod solution
  - Though a different factorization can be used in this case (will touch on later)

The RH problem for negative x and $s_1 s_3 \neq 1$



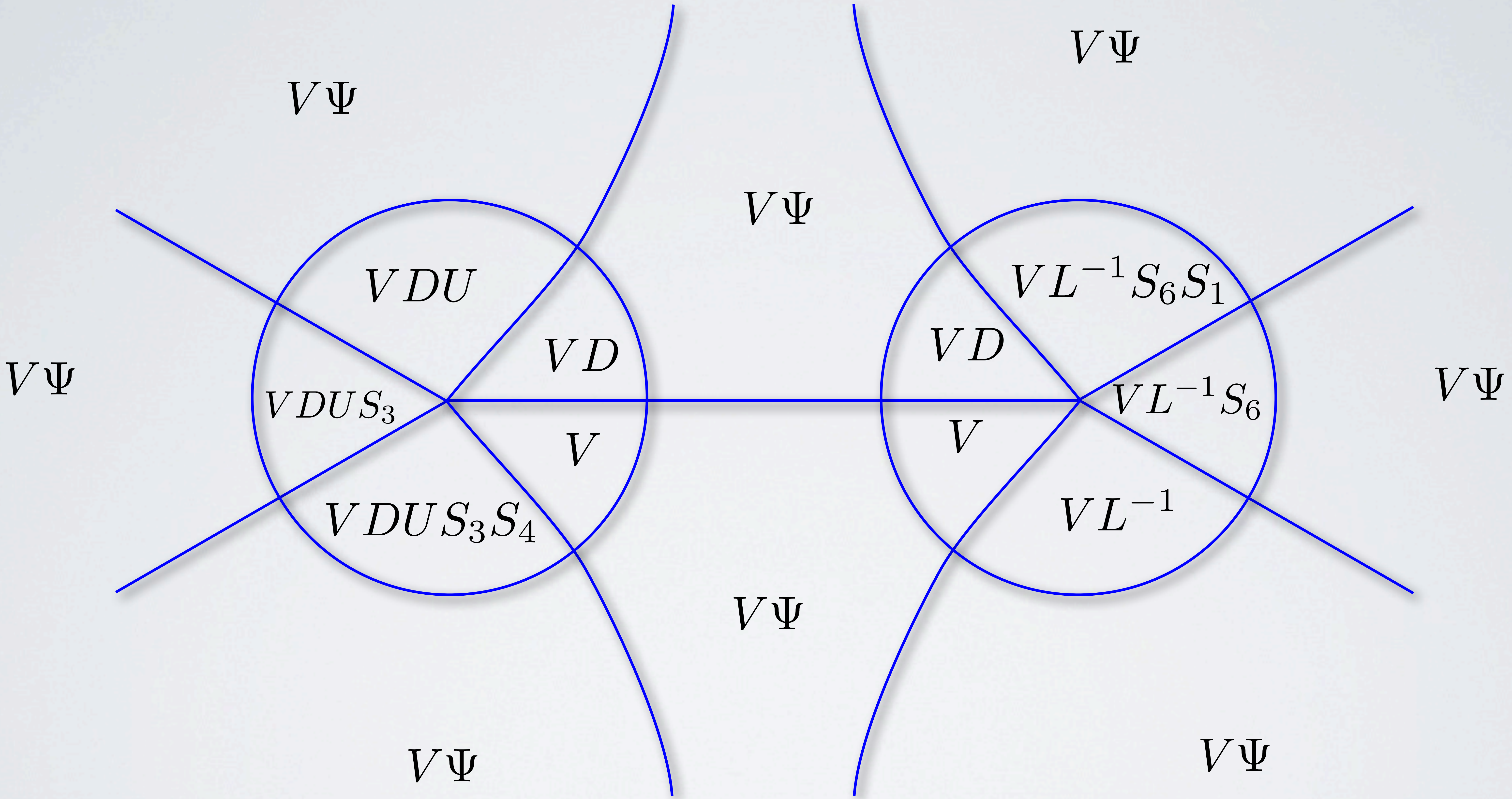
- We can implement a spectral method for this Riemann–Hilbert problem just as we did for the canonical six rays case
- The problem:
    - The solution is not analytic along circled connecting curve
    - Fortunately, we have a closed form solution (parametrix) for the contribution from that curve from the analytic development:



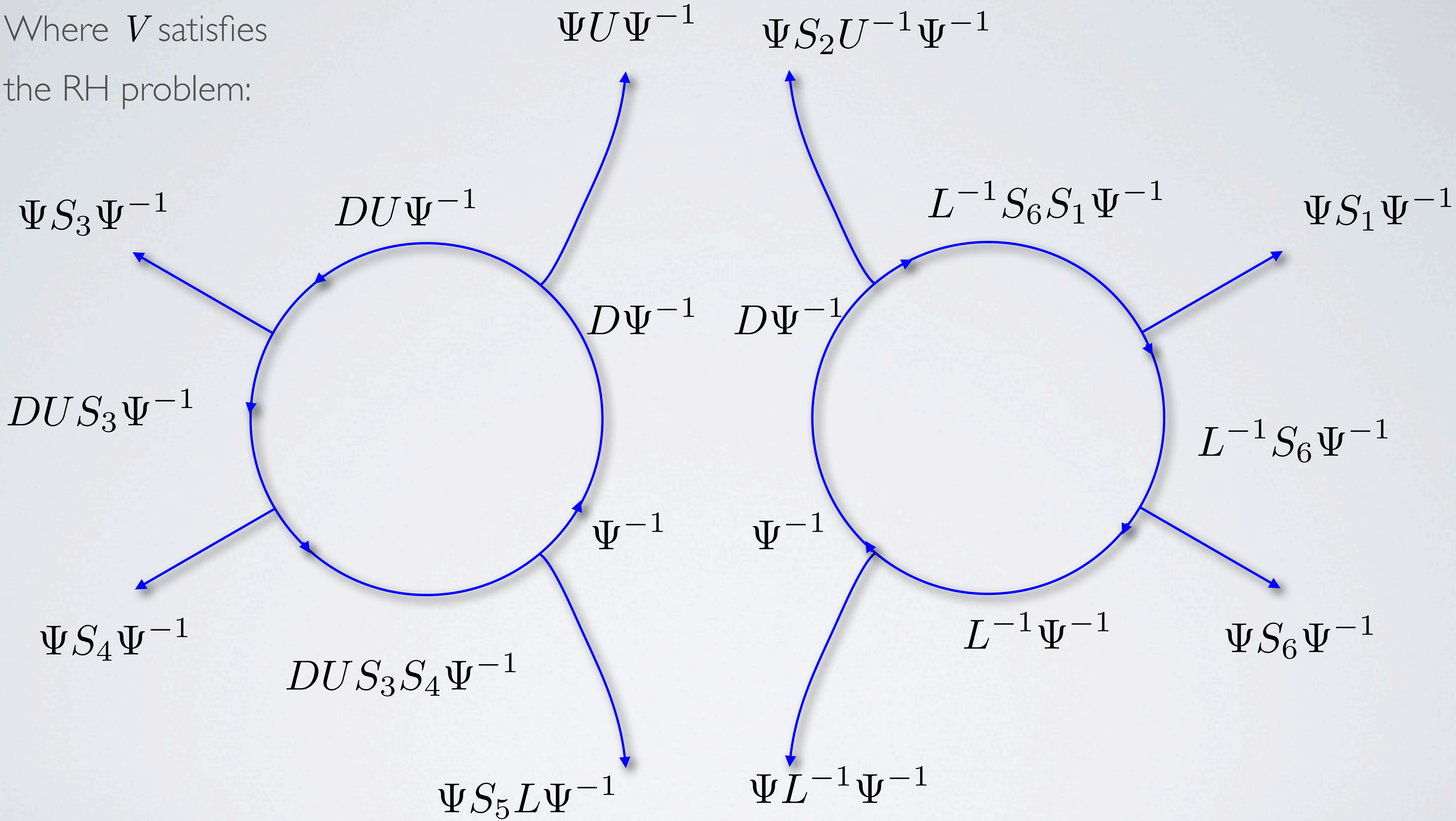
$$\Psi^+ = \Psi^- D$$

$$\Psi(z) = \begin{pmatrix} \left(\frac{1+2z}{2z-1}\right)^{\frac{\mathrm{i}}{2\pi}\log D_{11}} & \\ & \left(\frac{1+2z}{2z-1}\right)^{\frac{\mathrm{i}}{2\pi}\log D_{22}} \end{pmatrix}$$

We now write the solution as

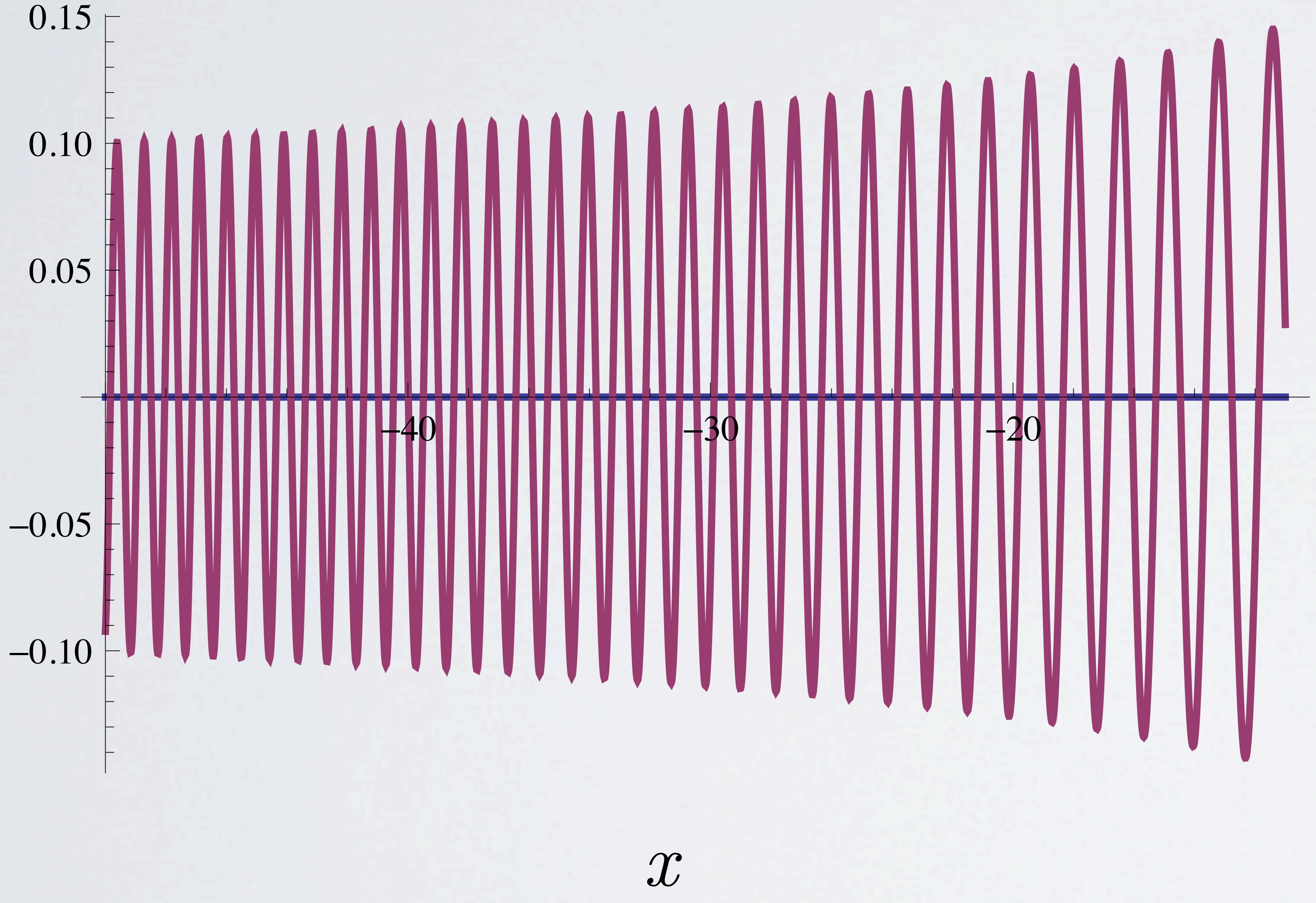


- Where V satisfies the RH problem:

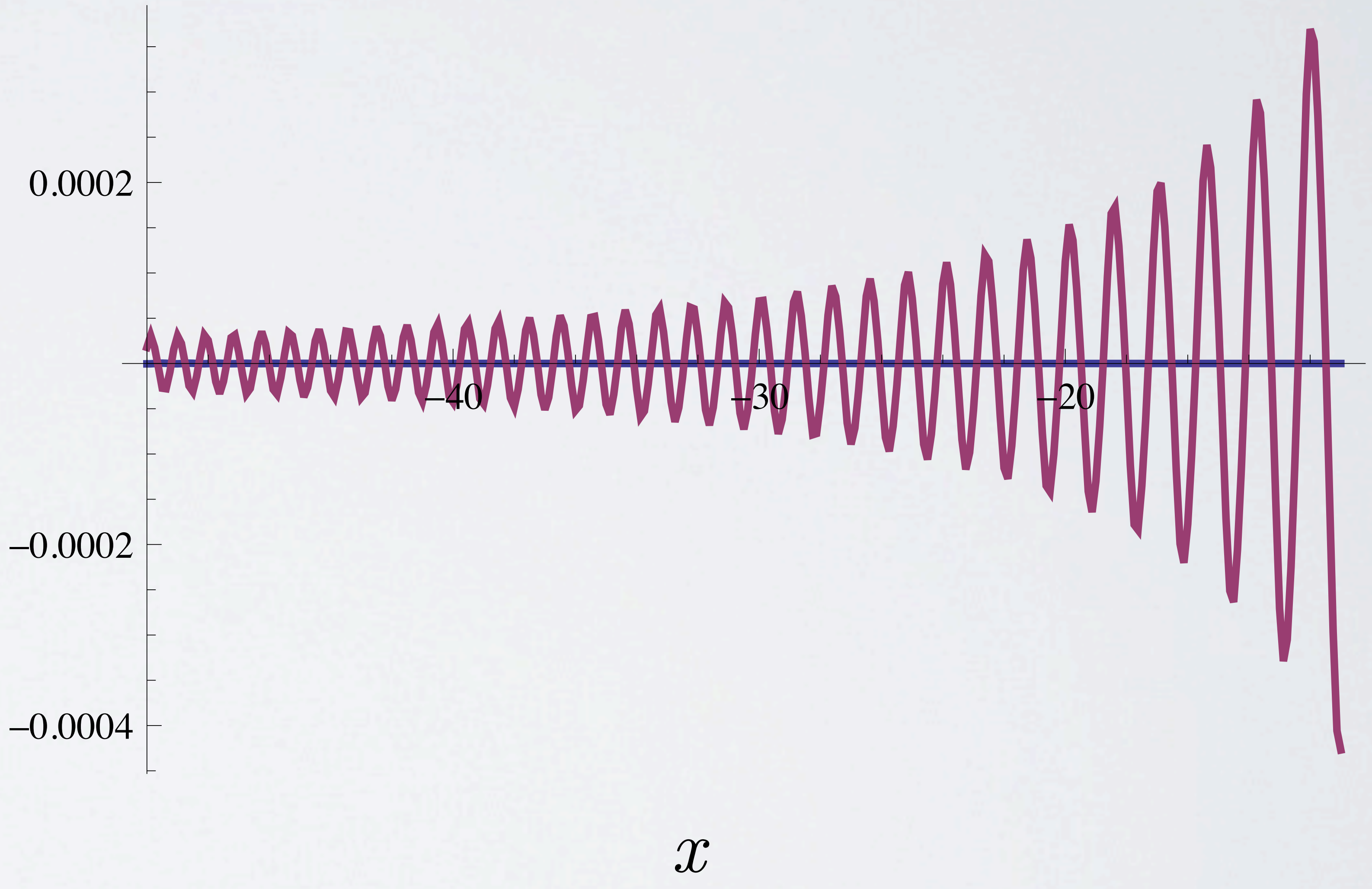


$$(s_1, s_2, s_3) = (0.5, 0, -0.5)$$

Plot of solution



Comparison
with known asymptotics



NUMERICAL STEEPEST DESCENT: POSITIVE x

- If we were to try the same approach as in the negative x case, we would have exponential growth at the stationary points which cancel, introducing large round-off errors
- Instead, as in the analytic development, a function g is introduced which has the same asymptotic behaviour as our oscillator, but is zero at the stationary points
  - This is possible by allowing it to have a branch cut along our RH graph
  - And is related to equilibrium measures
- A similar situation also applies when the LDU decomposition fails for negative x, such as the in the Hastings–McLeod solution

- We use the choice

$$g(z) = -\frac{4}{3}\left(\mathrm{i}z - \frac{1}{\sqrt{z}}\right)^{\frac{3}{2}}\left(\mathrm{i}z + \frac{1}{\sqrt{z}}\right)^{\frac{3}{2}}$$

- Note that this has three stationary points at

$$0, \pm\frac{\mathrm{i}}{\sqrt{2}} \text{ with } g(0) = \frac{\mathrm{i}\sqrt{2}}{3}, g\left(\pm\frac{\mathrm{i}}{\sqrt{2}}\right) = 0$$

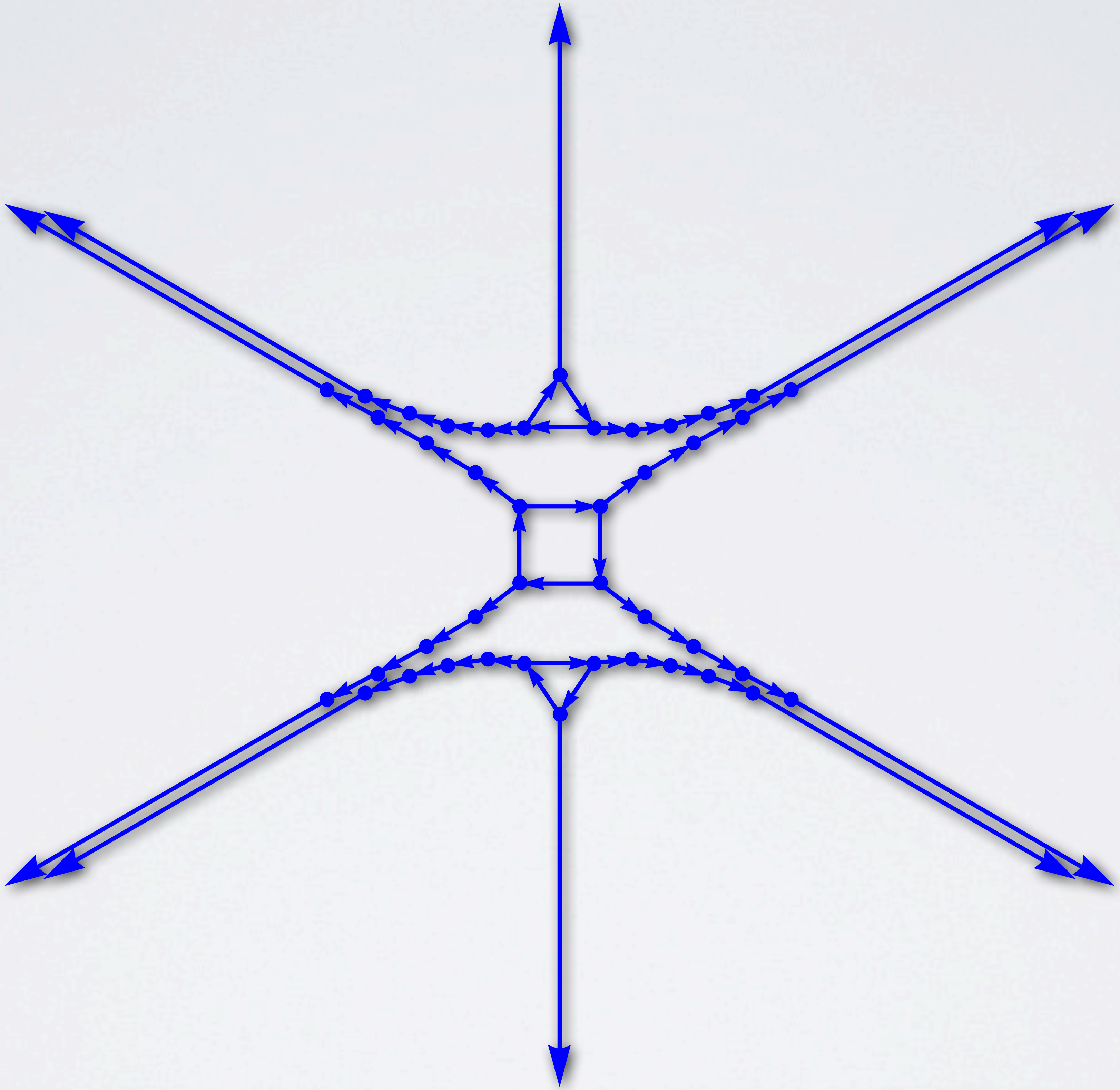
- Furthermore

$$g(z) = \frac{4}{3}\mathrm{i}z^3 + \mathrm{i}z + \mathcal{O}\left(\frac{1}{z}\right)$$

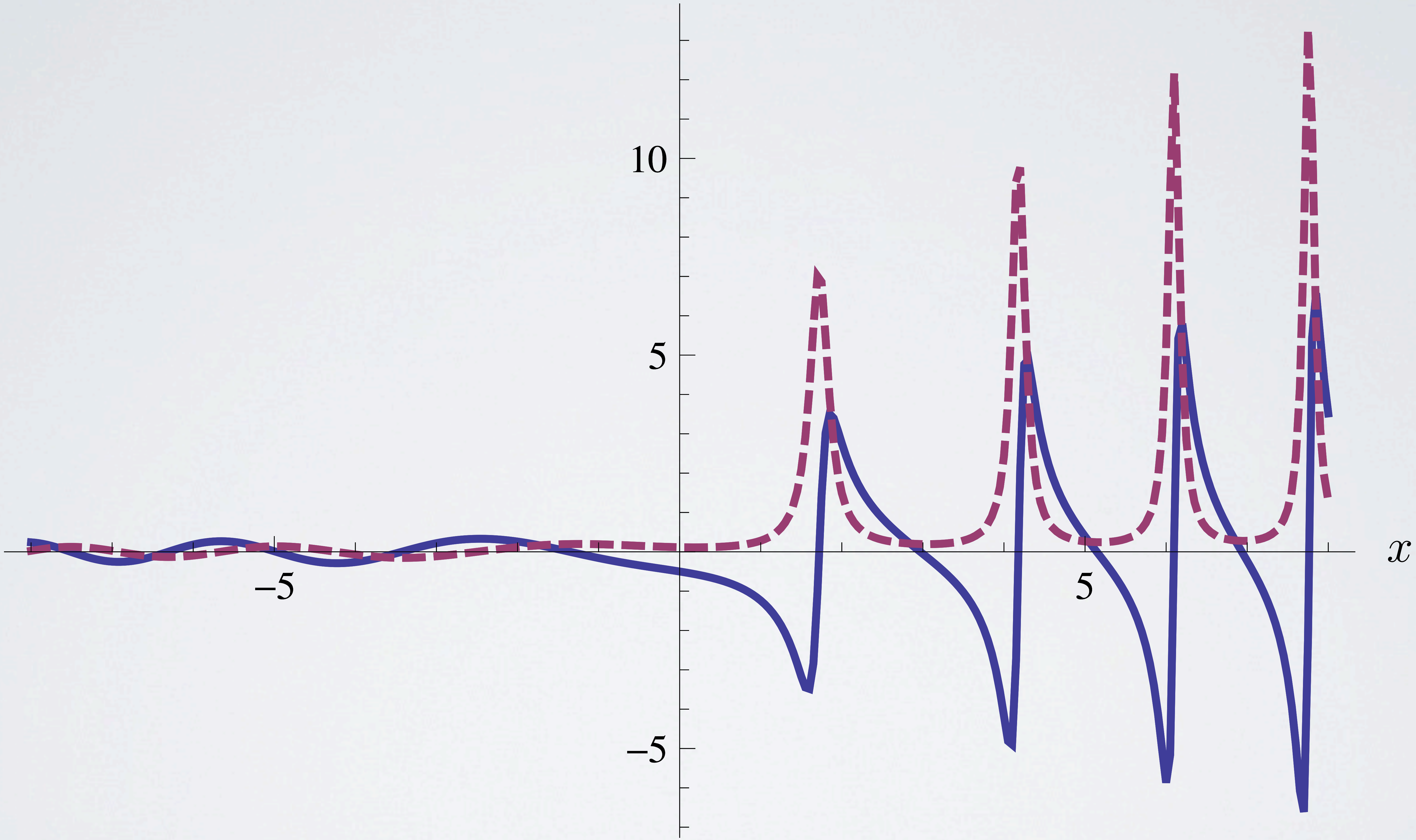
- We now need to do two separate decompositions, above and below zero

- As before, we also need to introduce a parametrix to avoid the singularities near the stationary points

Positive x RH problem



We can now extend the graph for $(s_1, s_2, s_3) = (1, 2, 1/3)$



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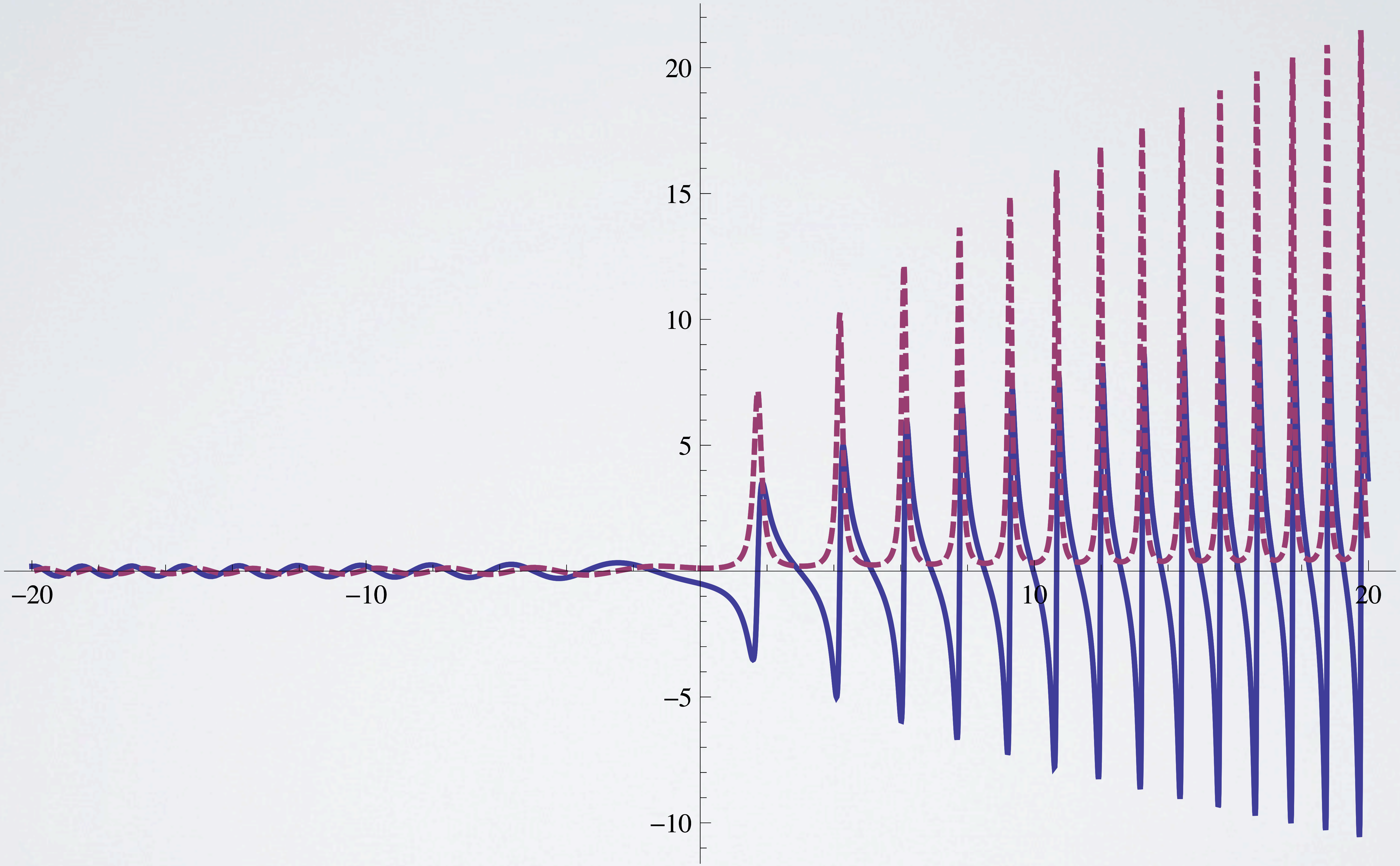

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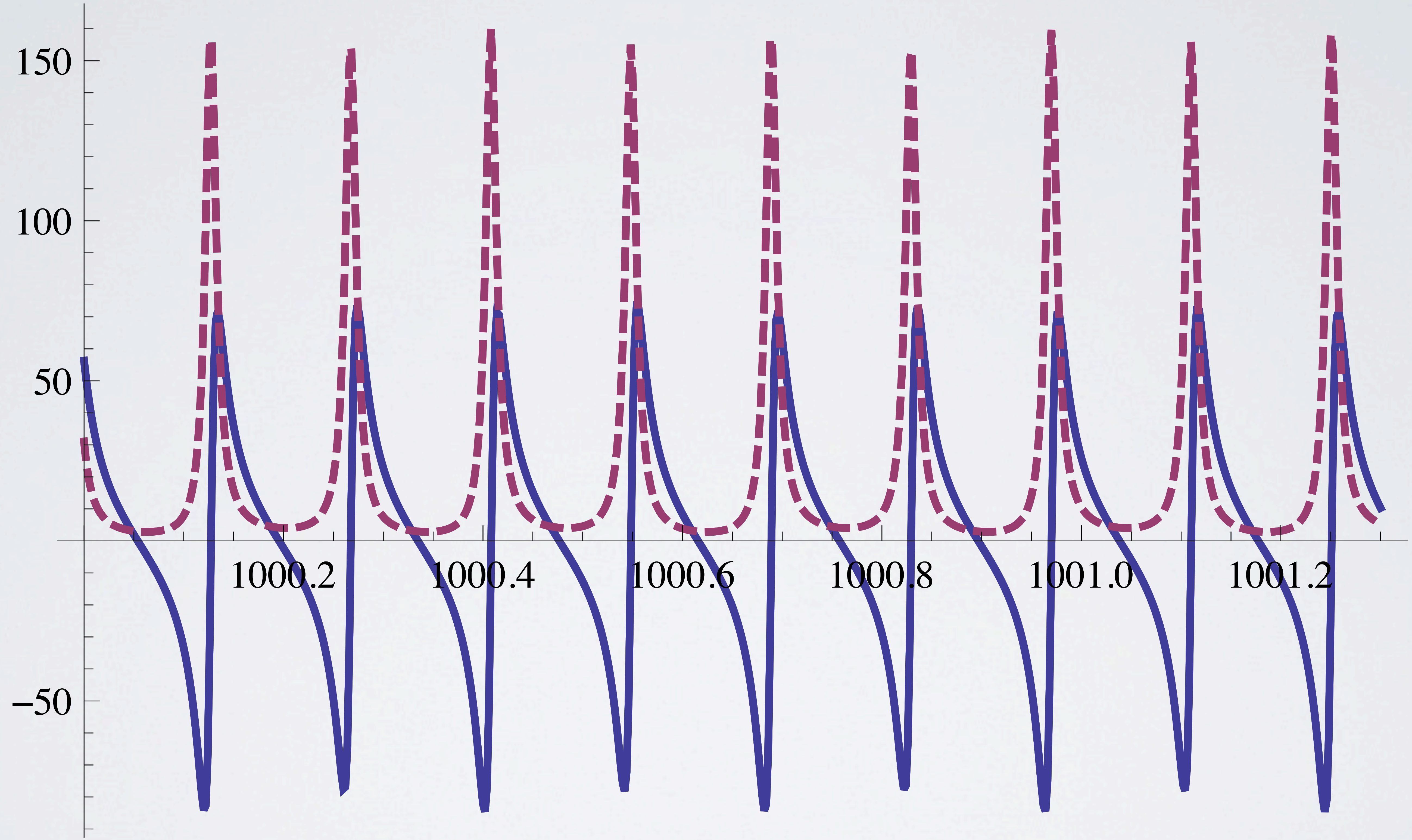


Hastings–McLeod $(s_1, s_2, s_3) = (\mathrm{i}, 0, -\mathrm{i})$

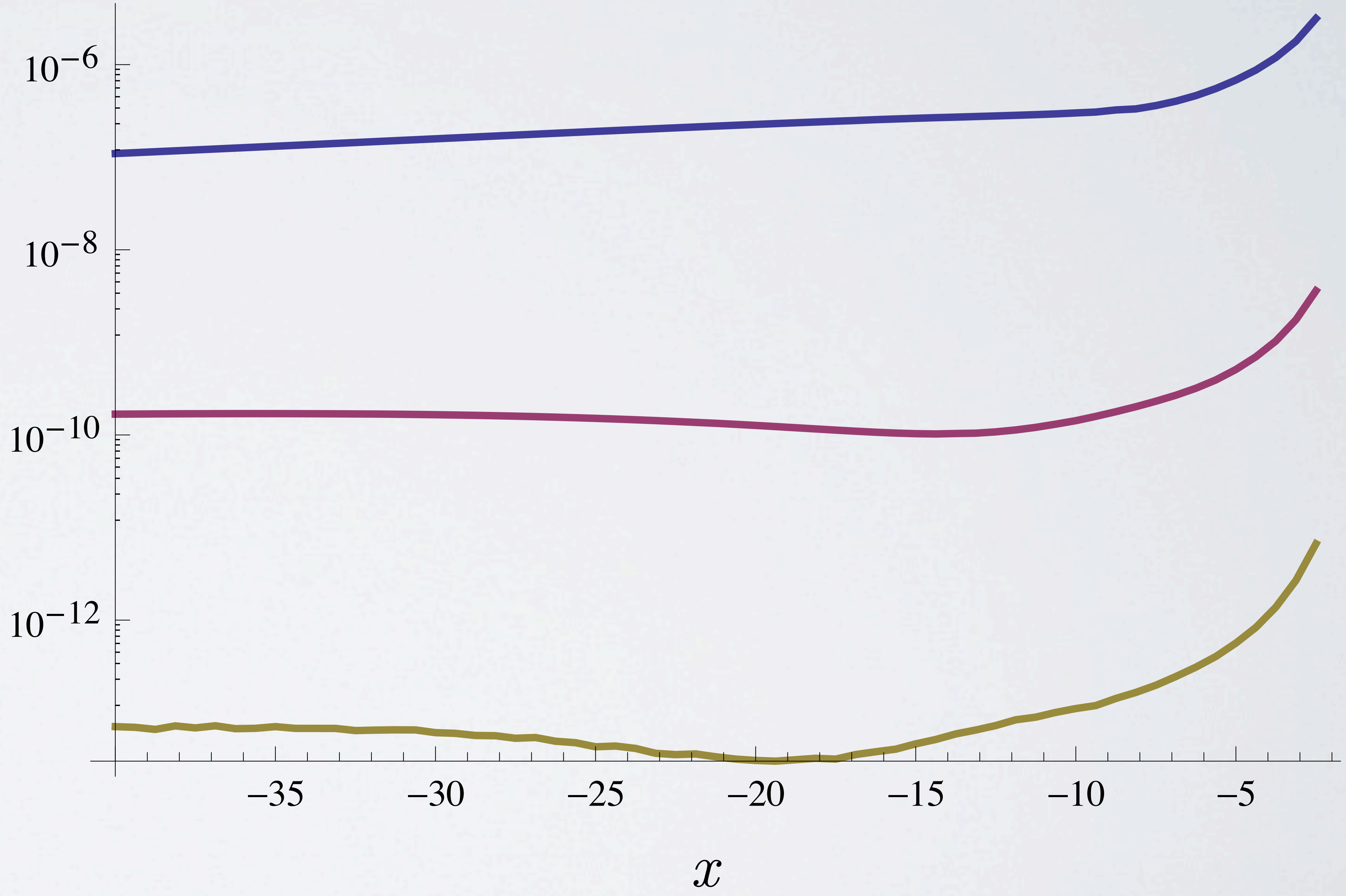
Relative error compared to (Prähofer and Spohn 2004)

Total # collocation points

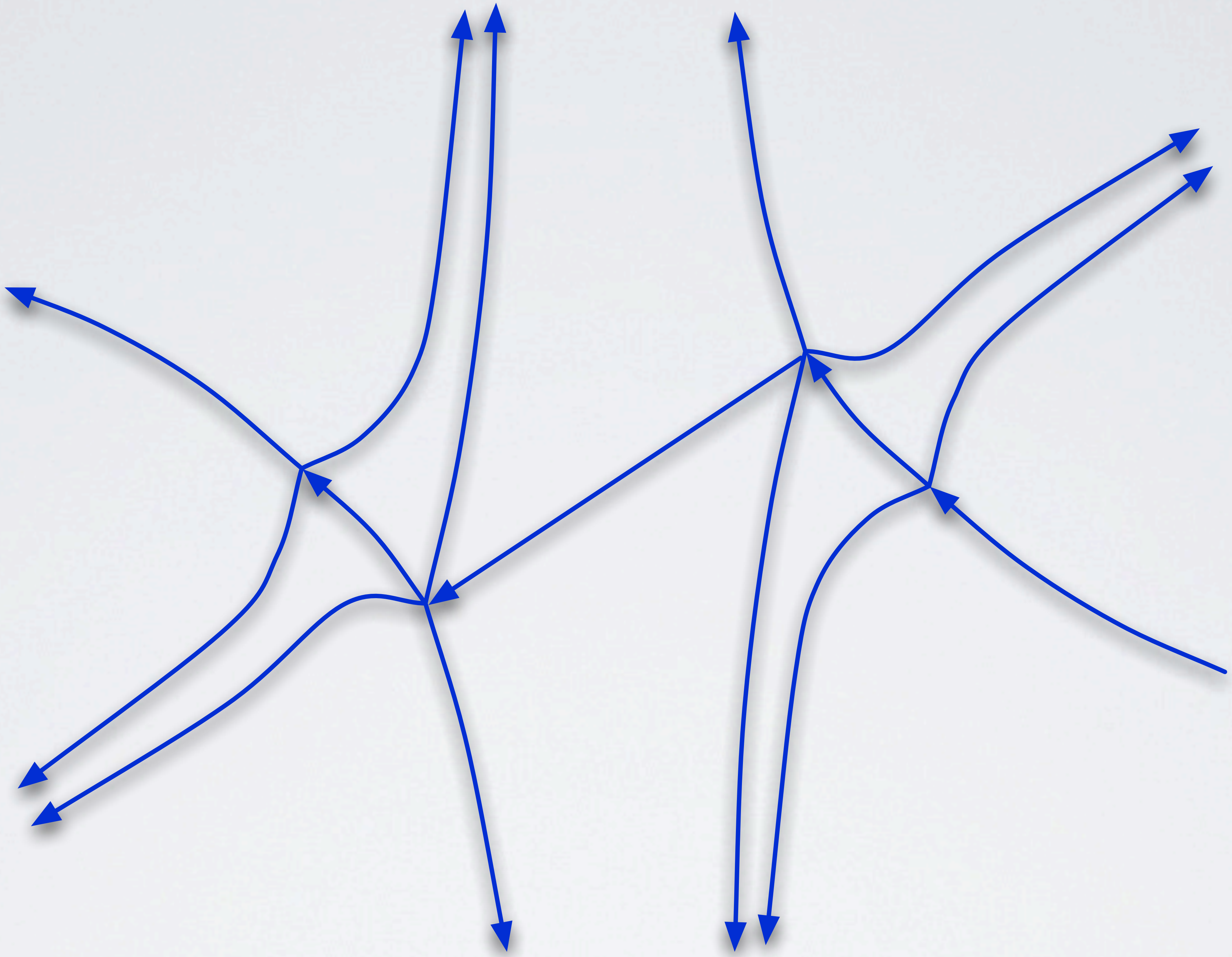
186

310

436



(Here (Bornemann 2010) has absolute accuracy for Tracy–Widom, so no accuracy for Hastings–McLeod)



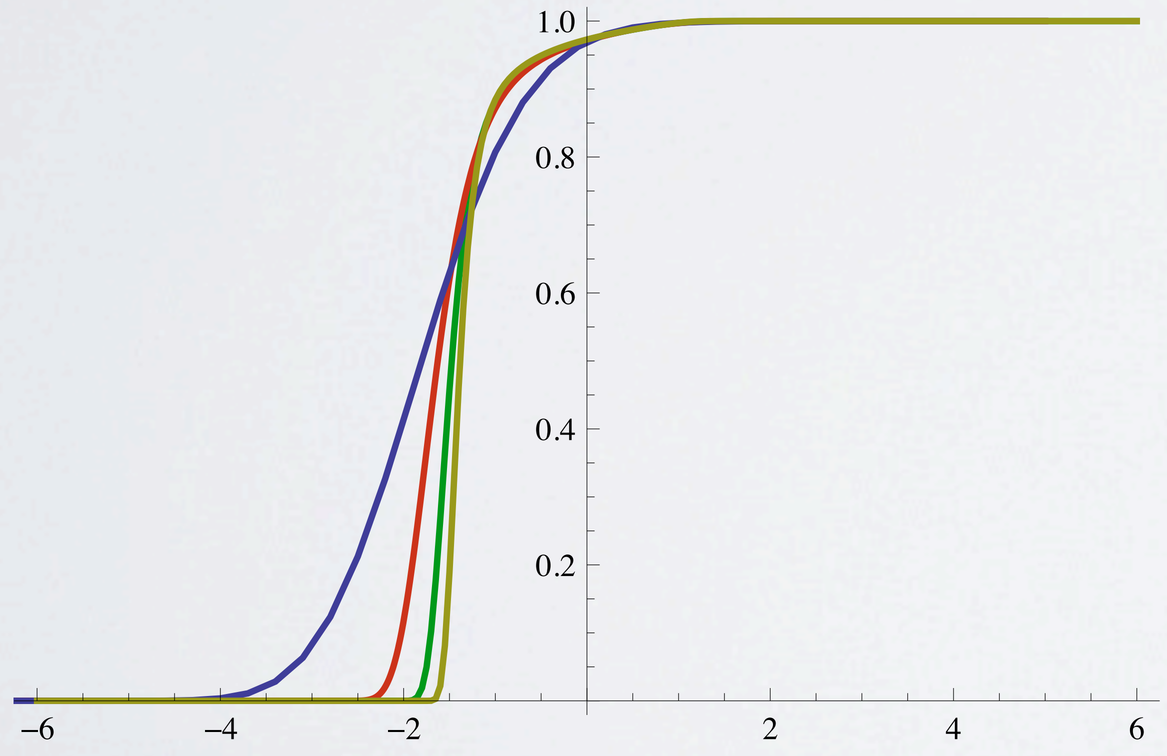
Sketch of Γ for complex x (not yet implemented)

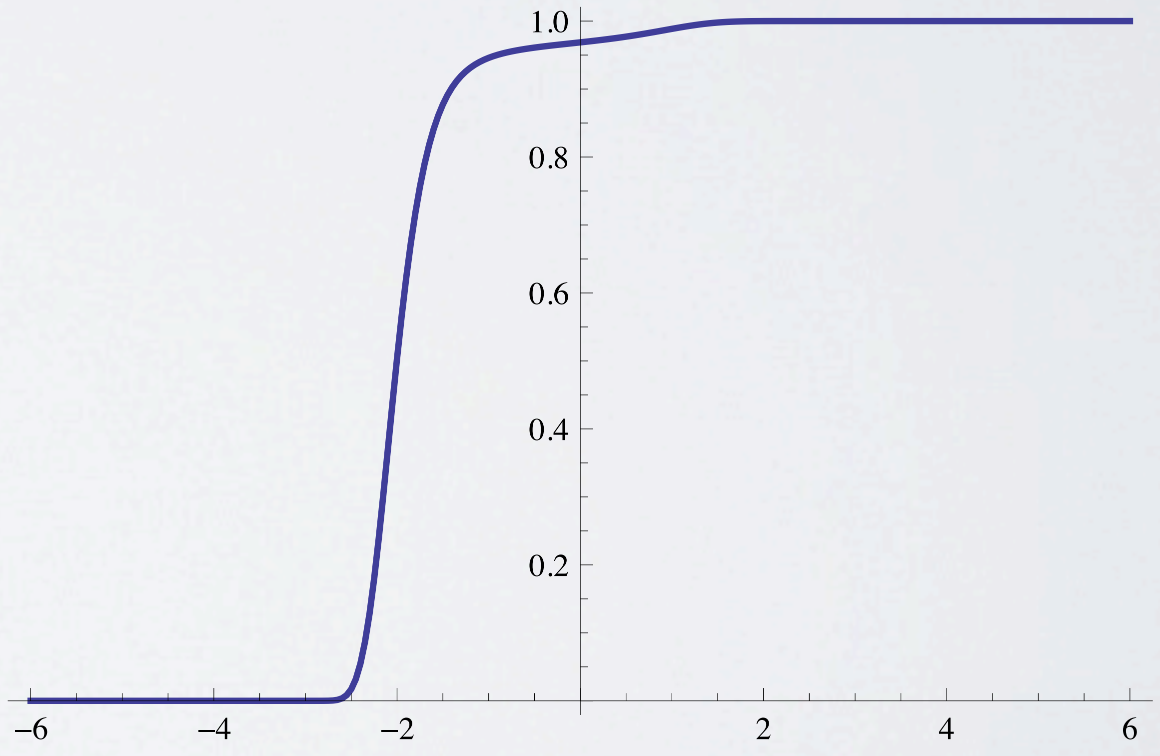
CONCLUSIONS

- Riemann–Hilbert problems can be numerically solved, efficiently and accurately
- This allows us to compute solutions to Painlevé equations
    - In short, we can connect initial conditions with asymptotic behaviour
    - This could potentially form the building block of a toolbox for computing Painlevé equations
- Conformally mapping the entire path would result in high asymptotic accuracy (and hence very few points needed, and a very rapid approximation)

OTHER APPLICATIONS

- Orthogonal polynomials
- Integrable PDEs (needs computation of reflection coefficients)
- Random matrix theory distributions (implemented for higher order Tracy–Widom with Kuijlaars and Claeys)





OTHER PAINLEVÉ RH PROBLEMS