STATISTICAL ANALYSIS OF STELLAR EVOLUTION: ONLINE SUPPLEMENT

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Color-Magnitude Diagrams (CMDs) are plots that compare the magnitudes (luminosities) of stars in different wavelengths of light (colors). High non-linear correlations among the mass, color and surface temperature of newly formed stars induce a long narrow curved point cloud in a CMD known as the main sequence. Aging stars form new CMD groups of red giants and white dwarfs. The physical processes that govern this evolution can be described with mathematical models and explored using complex computer models. These calculations are designed to predict the plotted magnitudes as a function of parameters of scientific interest such as stellar age, mass, and metallicity. Here, we describe how we use the computer models as a component of a complex likelihood function in a Bayesian analysis that requires sophisticated computing, corrects for contamination of the data by field stars, accounts for complications caused by unresolved binary-star systems, and aims to compare competing physics-based computer models of stellar evolution.

1. Introduction. This supplement contains four color figures and a description of the physics behind the computer-based stellar evolution models. This material was originally intended to be included in van Dyk *et al.* (2009), but was removed for editorial reasons. The images are visually impressive but not central to our statistical analysis. The section on the computer model provides details for readers interested in the inner workings of the likelihood function used in the main paper. This is not meant to be a standalone document and should be read in conjunction with the main paper.

2. Images of Planetary Nebula and Supernovae.   The end of the red giant phase in the evolution of a star results in some of the most visually spectacular objects in the sky. The loss of mass due to low gravity in the higher altitudes of a giant star as the core collapses lead to the formation of a very short lived *planetary nebula*. Figures 1 and 2 present images of two planetary nebulae, the Cat's Eye Nebula and the Eskimo Nebula.

Stars with initial mass greater than $8M_\odot$ eventually collapse into neutron stars. Matter falling into a newly formed neutron star sets off a shock wave that dramatically blows off the outer layers of the star in a supernova explosion. Figures 3 and 4 contain images of the Kepler's supernova and the Crab Nebula, both supernova remnants. These are two of the most recent supernovas in our galaxy and both explosions were observed on Earth in the past millennium.

3. Computer-Based Stellar Evolution Models.   Surprisingly, armed with only four basic principles we can create physical models that largely describe the energy generation, energy transport, density, and gravity of essentially any star as well as predict surface temperature and luminosity. The simplest of these principles is that the mass interior to any radius is the density-weighted volume from the star's center to that radius[1]. Second, stars must obey *conservation of energy*, which means that at each radius, the change in the *energy flux* (the rate of transfer of energy through a unit area) must equal the local rate of energy release. Third, stars are in *hydrostatic equilibrium* at each radius; that is the pressure from energy generation in the interior balances gravity so that the star neither expands nor contracts. Finally, the energy transport from the center to the surface takes the form of radiation, convection, and/or conduction. These four basic principles lead to four coupled differential equations.

While each of these equations is relatively simple, they contain terms that are dependent on other parameters in highly non-linear ways, and the resulting system of equations is difficult to solve. For instance, the rate of energy released by fusion in stellar cores depends sensitively on temperature, as high as $T_{\mathrm{core}}^{16.7}$ for some hydrogen-fusion reactions. Energy generation also depends on the ratios of elements present (e.g., metallicity and helium abundance). Energy transport is particularly difficult. Most stars transport the bulk of their energy by convection, for which no closed-form theory is available, and by radiation, for which detailed calculations of the *opacity* (a measure of the absorption of electromagnetic radiation) of each ion of each

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[1]Astronomers call this principle *conservation of mass*, but it is simply the computation of mass from density and volume.

FIG 1. *Planetary Nebula I: The Cat's Eye Nebula. The cat's eye nebula is one of the more structurally complex planetary nebulas known. This image is an X-ray optical composite constructed with images collected with the Hubble space telescope and the Chandra X-ray observatory.*

atom as a function of a broad range of wavelengths is necessary. Since the mix of ions for any atomic species depends on temperature, overall radiative opacity depends on wavelength, temperature, and composition. While the wavelength dependence of hydrogen's opacity is relatively simple and well-studied, some atoms, iron for instance, absorb light at millions of discrete wavelengths, many of which are still unknown.

In practice, computer models for stellar evolution iteratively solve the four coupled differential equations for a given stellar mass yielding a boundary condition (the stellar surface temperature) while referring to tabulated nuclear energy generation rates, opacities, and other pre-calculated physical

FIG 2. *Planetary Nebula II: The Eskimo Nebula. The image of the eskimo nebula was obtained with the Hubble space telescope.*

values. This iterative process creates a single static physical model of a star, which is how a star of a particular mass and a particular *radial abundance profile* would appear in terms of its luminosity and color. The radial abundance profile describes the composition of a star as a function of radius and is thus a more detailed version of metallicity and helium abundance. The computer models evolve a star by updating the abundance profile to account for the newly produced elements, for example helium, and then compute a new static stellar physical model. Thus, the computer models take initial mass and composition (i.e., metallicity and helium abundance) as inputs, iterate to a certain age of the star and yield the stellar surface temperature. Additional back end calculations compute absolute magnitudes from the surface temperature and composition. Finally interstellar absorption and distance
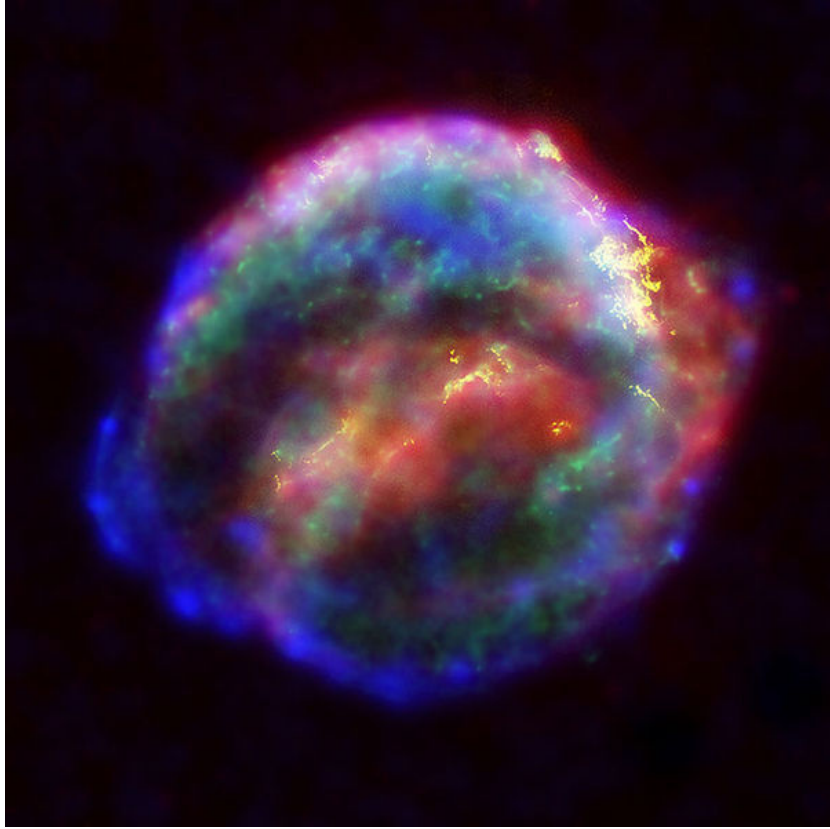
FIG 3. *Supernovae I: Kepler's Supernova. The image is the remnant of the most recently observed supernova in our Galaxy. It is known as Kepler's supernova or SN 1604 for Johannes Kepler who first observed it on October 9, 1604. This image was constructed using images collected with NASA's Spitzer Space Telescope, Hubble Space Telescope, and Chandra X-ray Observatory.*

can be used to convert absolute magnitudes into apparent magnitudes.

There are a number of different computational implementations of this stellar evolution model for main sequence and red giant stars. We use state-of-the-art models by Girardi *et al.* (2000), by the Yale-Yonsei group (Yi *et al.*, 2001), and of the Dartmouth Stellar Evolution Database (Dotter *et al.*, 2008). These models differ subtly in their use of different tabulated opacities, different equations of state (the relationship between temperature and pressure), and different mappings between stellar surface temperature, luminosity, and composition on the one hand, and the observed magnitudes on the other hand. These subtle differences in opacities and the equation of state cause little difference for the hotter more massive stars on the main
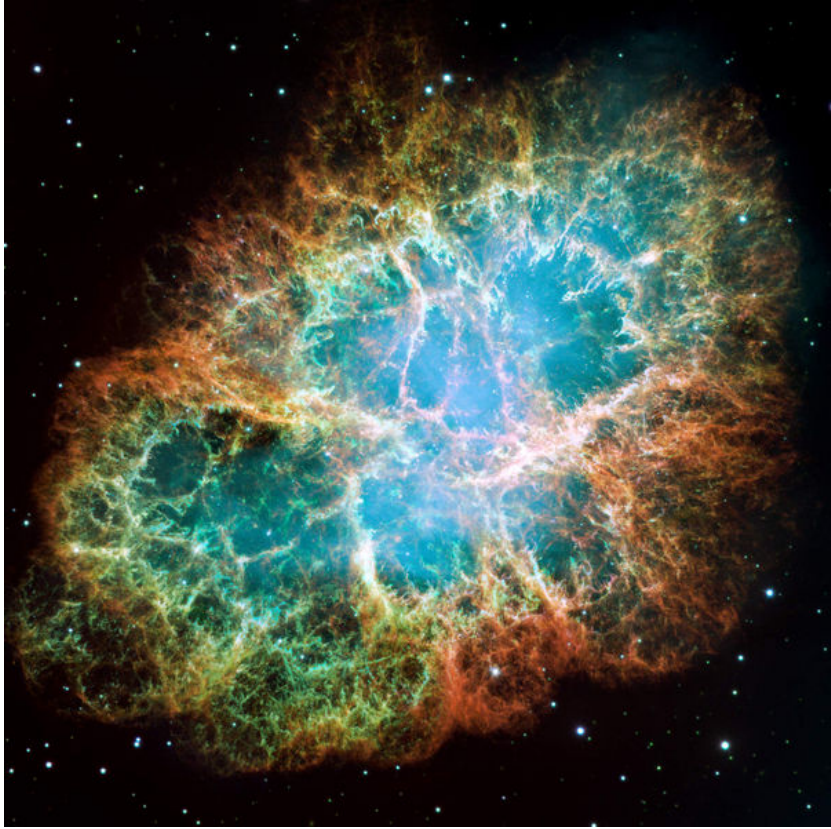
FIG 4. *Supernovae II: The Crab Nebula. The Crab Nebula is a six-light-year-wide expanding remnant of a star's supernova explosion. The supernova was observed by Japanese and Chinese astronomers in 1054. A neutron star at the center of the nebula continues to power the object ejecting two beams of radiation that appear as a pulse 30 times a second as the neutron star rotates. This image is a mosaic of Hubble Space Telescope images.*

sequence, but cause larger differences for cooler stars, where complex atmospheres with molecules and more substantial convection make the calculations both more difficult and more sensitive to these inputs. Note not all of the models allow for changes in the initial helium abundance. One of our primary goals is to compare these models empirically and to examine which if any of them adequately predict the observed magnitudes.

All of these models break down in the turbulent last stage of red giants as they fuse progressively heavier elements at different shells of their interior, begin to pulsate, contracting and expanding, finally lose their outer layers in planetary nebulae and form white dwarfs. This transition is physically very complex and dominated by chaotic terms. Other computer models are

used for white dwarfs. These models start with a newly formed white dwarf star and use the same coupled differential equations to model white dwarf evolution. For white dwarfs, however, energy transport is dominated by conduction which changes the implementation of the numerical solution. We use the white dwarf evolution models of Wood (1992) and the white dwarf atmosphere models of Bergeron *et al.* (1995) to convert the surface luminosity and temperature into magnitudes. Finally, to bridge the main sequence / red giant computer models with the white dwarf model we use an empirical mapping that links the initial mass of the main sequence star with the mass of the resulting white dwarf (Weidemann, 2000); this is the so-called *initial-final mass relation*. The combination of these various computer models for various stages of stellar evolution into one comprehensive stellar evolution model[2] was proposed by von Hippel *et al.* (2006).

In many ways, the white dwarf models are substantially simpler than the main sequence and red giant models. With no energy source at their centers, no conversion of one element to another and subsequent changes in their stellar structure, and with the bulk of their energy transport via conduction, white dwarfs maintain a nearly constant radius and their evolution is governed by the loss of their heat content into space. As far as we know, white dwarf evolution also does not depend on the metallicity or helium content of the precursor star. It does depends on the white dwarf mass, and on the relative abundances of hydrogen, helium, carbon, oxygen, and any heavier elements left over after evolution and mass loss during the earlier stages of evolution. The primary complications for white dwarfs are the imprecisely known amounts of remaining hydrogen and helium, the structures of their atmospheres when convective (at surface temperatures of approximately 12,000 Kelvin (K), corresponding to ages approximately 500 million years old), and whether oxygen gravitationally sinks to the center during carbon and oxygen crystallization. White dwarfs start out with surface temperatures of approximately 150,000 K, radiating most of their energy in ultraviolet light. As they age, they cool, with the coolest white dwarfs having temperatures of approximately 4000 K. White dwarfs are thus white for only a part of their lifetime, when their surface temperatures are approximately 10,000 K. Just as for models of main sequence stars and red giants, the white dwarf models (e.g., Wood, 1992) predict the surface temperature and luminosity as a function of age. Additional calculations that incorporate

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[2]Our use of "stellar evolution model" is somewhat different than is in common use in the Astronomy literature, where it generally refers to a model for the evolution of the main sequence and red giants. We use it to refer to a more comprehensive model that includes the transition to and evolution of white dwarfs.

the details of white dwarf atmospheres (e.g., Bergeron *et al.*, 1995) convert these theoretical quantities to the observed quantities of magnitudes.

We combine (i) the main sequence and red giant models with (ii) the initial-final mass relation, and (iii) white dwarf cooling models, to create a model we call the stellar evolution model, as it is meant to depict stars in all of the main phases of stellar evolution. (We ignore exotic objects such as neutron stars and black holes and short-lived objects such as supernovae, as the former objects do not radiate substantially in visible light and the latter objects are too short-lived to model sensibly from the color-magnitude diagram.)

Note: The images in Figures 1–4 are in the public domain because they were created by NASA and the European Space Agency. Hubble material is copyright-free. Material published by NASA may be freely used as in the public domain without restriction, although NASA requests credit and notification of further use. ESA requires that they be credited as the source of any of their material that is used. The material was created for NASA by STScI under Contract NAS5-26555 and for ESA by the Hubble European Space Agency Information Centre.

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