Explicit reduction modulo p of certain 2-dimensional crystalline representations.

Kevin Buzzard, Toby Gee

February 20, 2009

Abstract

We use the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ to explicitly compute the reduction modulo p of certain 2-dimensional crystalline representations of small slope, and give applications to modular forms.

1 Introduction.

Let $f = \sum_{n \geq 1} a_n q^n$ be a weight k cusp form for the group $\Gamma_1(N) \subseteq \mathrm{SL}_2(\mathbf{Z})$, and assume that f is normalised ($a_1 = 1$), is an eigenform for all the Hecke operators, and has character ψ (a Dirichlet character modulo N). The coefficients of f are complex numbers, but are well-known to be algebraic over $\mathbf{Q}$ and hence can be regarded (after some choices) as elements of $\overline{\mathbf{Q}}_p$, where p is a prime number. Deligne associated to f a p-adic Galois representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p).$$

Let us normalise the construction so that if f is associated to an elliptic curve over $\mathbf{Q}$ then ρ_f is the Tate module of the curve; this choice of normalisation is sometimes not ideal, but our main results appear cleaner with this choice.

A lot of explicit information is known about the "local" structure of ρ_f, by which we mean $\rho_f|_{D_\ell}$, where D_ℓ denotes a decomposition group at some prime number ℓ. For example if ℓ is a prime not dividing Np then ρ_f is unramified at ℓ and the characteristic polynomial of $\rho_f(\mathrm{Frob}_\ell)$ (with Frob_ℓ an arithmetic Frobenius) is $X^2 - a_\ell X + \ell^{k-1}\psi(\ell)$. In particular the local structure of ρ at ℓ is determined by the local structure of f at ℓ, which, because $\ell \nmid N$, can be interpreted as the triple $(a_\ell, k, \psi(\ell))$. This is visibly an explicit description of $\rho_f|_{D_\ell}$ (there is still a question regarding semisimplicity of $\rho_f(\mathrm{Frob}_\ell)$ in the case where the two eigenvalues coincide, but conjecturally this should never occur in weight $k \geq 2$, and in weight 1 $\rho_f(\mathrm{Frob}_\ell)$ is known to be semisimple anyway).

If $\ell | N$ but $\ell \neq p$ then the local Langlands conjecture for GL_2 (a bijection which can be explicitly written down in essentially all cases) and a local-global theorem of Carayol (following Deligne and Langlands) again gives us an explicit description of $\rho_f|_{D_\ell}$.

Let us now turn our attention to the local structure of ρ_f at p. Now one might argue that the theorems describing $\rho_f|_{D_p}$ in terms of the data attached to f are far from "explicit"—and indeed, how could one expect them to be explicit: a 2-dimensional p-adic representation of D_p is a very complicated object. However the *mod* p 2-dimensional representations of D_p are easily classified: the reducible ones are, up to semisimplification, the sum of two characters, and the irreducible ones are all induced from characters of the absolute Galois group of the unramified quadratic extension of $\mathbf{Q}_p$. So using a little local class field theory it is easy to explicitly list all these representations. The following "practical" question then arises:

Question 1.1. If $f = \sum a_n q^n$ is a normalised cuspidal level N eigenform and p is a prime, and if $\overline{\rho}_f$ is the associated semisimple representation

$$\overline{\rho}_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p),$$

then can one explicitly read off $(\overline{\rho}_f|_{D_p})^{\mathrm{ss}}$ from the weight, character and q-expansion of f?

In this generality, the answer to the question is in some sense "no". For example, if f is a newform of level $\Gamma_0(N)$ and p divides N exactly once, then to know $\rho_f|_{D_p}$ is (essentially) to know the value of the so-called $\mathcal{L}$-invariant of f at p, and this invariant is subtle: as far as anyone knows, it cannot be easily read off from the q-expansion of f, and is not a "local" invariant of the classical automorphic representation attached to f. Furthermore, the problem does not go away when reducing mod p: $\overline{\rho}_f|_{D_p}$ also depends heavily on the $\mathcal{L}$-invariant, even for small weight modular forms: see for example Théorème 4.2.4.7 of [BM02] for some examples of how the $\mathcal{L}$-invariant affects the local mod p representation. If furthermore p divides N more than once, then even less is known, and the explicit dictionary is no doubt even more complicated.

However, if $p \nmid N$ the situation is much better. The local data attached to f at p is the triple $(a_p, k, \psi(p))$ (in the sense that the local component π_p of the automorphic representation π associated to f is completely determined by this data), and in some cases this local data does determine a lot about the local representation. We now explain some of the results known in this situation. First we introduce some notation.

We identify D_p with the local Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Let μ_α denote the unramified character of D_p sending a geometric Frobenius to α. Let us normalise the isomorphisms of local class field theory by identifying uniformisers with geometric Frobenii. Let $\epsilon : D_p \to \mathbf{Z}_p^\times$ denote the cyclotomic character, and let $\omega : D_p \to \overline{\mathbf{F}}_p^\times$ denote the mod p reduction of ϵ. Let $\omega_2 : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_{p^2}) \to \overline{\mathbf{F}}_p^\times$ denote a character such that the induced map $\mathbf{Q}_{p^2}^\times \to \overline{\mathbf{F}}_p^\times$ sends p to 1 and such that the induced map $\mathbf{F}_{p^2}^\times \to \overline{\mathbf{F}}_p^\times$ is induced by a morphism of fields $\mathbf{F}_{p^2} \to \overline{\mathbf{F}}_p$. There are two such choices for ω_2; we fix one. We call ω_2 a "fundamental character of niveau 2". The other fundamental character is ω_2^p. Note that $\omega_2^{p+1} = \omega$ on I_p, the inertia subgroup of D_p. Abusing notation, let us also use ω_2 to mean $\omega_2|_{I_p}$. By local class field theory we can also consider ω and ω_2 as characters of $\mathbf{Q}_p^\times$ and $\mathbf{Q}_{p^2}^\times$ respectively.

The following theorems were proved by global methods:

Theorem 1.2. *(1) (Deligne-Serre) If $k = 1$ then ρ_f is unramified at p, and $\rho_f(\mathrm{Frob}_p)$ is semisimple with characteristic polynomial $X^2 - a_pX + \psi(p)$.*

(2) (Deligne) If $k \geq 2$ and a_p is a p-adic unit then $\rho_f|_{D_p}$ is reducible (and may or may not be semisimple), and the semisimplification of $\rho_f|_{D_p}$ is isomorphic to $\mu_{a_p^{-1}} \oplus \epsilon^{k-1}\mu_{a_p/\psi(p)}$.

(3) (Fontaine, Edixhoven) If $2 \leq k \leq p+1$ and a_p is not a p-adic unit then $\overline{\rho}_f|_{D_p}$ is irreducible, and $\overline{\rho}_f|_{I_p} \cong \omega_2^{k-1} \oplus \omega_2^{p(k-1)}$.

Remark 1.3. Part (1) of the theorem was proved by Deligne and Serre in [DS74]. Part (2) was proved in a 1974 letter from Deligne to Serre which apparently has never been published, although published proofs are now in the literature (see for example Theorem 2 of [Wil88]). Part (3) was proved (for $k \leq p$) in 1979 by Fontaine in two letters to Serre, but as far as we know the first published proof was given by Edixhoven in [Edi92] and this proof uses global methods. Note finally the relative strengths of the results: (1) and (2) describe the p-adic representation, and (2) has no restrictions on the weight, whereas (3) only describes the mod p representation and only for small weights—this is because parts (1) and (2) concern ordinary modular forms, and the situation is much more complicated in the non-ordinary case.

The problem with extending these global methods to the higher weight non-ordinary case is that they typically rely on the arithmetic of mod p modular forms and the geometry of modular curves over $\overline{\mathbf{F}}_p$, and hence find it very hard to distinguish between a_ps with positive valuations. However for $k > p + 1$ one can easily find examples on a computer of modular forms f_1 and f_2 of the same weight and level (prime to p) and character, with $v(a_p(f_1)) > 0$, $v(a_p(f_2)) > 0$ and $(\overline{\rho}_{f_1}|_{I_p})^{\mathrm{ss}} \not\cong (\overline{\rho}_{f_2}|_{I_p})^{\mathrm{ss}}$. In particular, vaguely speaking, the mod p Galois representation associated to a modular form, locally at p, depends on more than the mod p reduction of the local data attached to the form at p.

There is however a completely different and far more local way of approaching the problem, which relies on a coincidence in p-adic Hodge Theory which is very specific to 2-dimensional representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Let f be a normalised cuspidal eigenform of level N prime to p. Then $\rho_f|_{D_p}$ is a crystalline Galois representation, and we would like to say something concrete about $\overline{\rho}_f|_{D_p}$. Because of Theorem 1.2 above, we may now restrict our attention to the case $k \geq 2$ and $v(a_p) > 0$. Moreover, after an unramified twist we may assume $\psi(p) = 1$. The coincidence is that if we furthermore assume that the roots of $X^2 - a_pX + p^{k-1}$ are distinct (or equivalently that $a_p^2 \neq 4p^{k-1}$, an inequality which always holds if $k = 2$ and conjecturally holds for any $k \geq 2$) then there is up to isomorphism a *unique* $\overline{\mathbf{Q}}_p$-vector space D_{k,a_p} equipped with a semisimple linear Frobenius ϕ with characteristic polynomial $X^2 - a_pX + p^{k-1}$ and a weakly admissible filtration with jumps at 0 and $k-1$. In this case we have the following theorem which follows from Scholl's work on modular forms and the p-adic comparison isomorphism:

Theorem 1.4. *(Scholl, Faltings)*
If $k \geq 2$, if $v(a_p) > 0$, and furthermore if $a_p^2 \neq 4p^{k-1}$, then $D_{\mathrm{cris}}((\rho_f|_{D_p})^*) \cong D_{k,a_p}$.

Note that the omitted case $a_p^2 = 4p^{k-1}$ does not occur if $k = 2$ and should not occur if $k > 2$ (it would contradict a conjecture of Tate: see [CE98]). Note also that we need to take the $\overline{\mathbf{Q}}_p$-dual of ρ_f; this is because of our conventions.

Again under the assumptions $k \geq 2$, $a_p \in \overline{\mathbf{Q}}_p$ with $|a_p| < 1$ and $a_p^2 \neq 4p^{k-1}$, let V_{k,a_p} denote the crystalline representation V such that $D_{\mathrm{cris}}(V_{k,a_p}^*) \cong D_{k,a_p}$. Our conclusion is that (with notation as above) $\rho_f|_{D_p} \cong V_{k,a_p}$, and in particular the mod p reduction of $\rho_f|_{D_p}$ is determined by the local data (a_p, k) associated to f. But this reconstruction is *far* from explicit! It leads us to formulate the explicit purely local

Question 1.5. Say $k \geq 2$ and $a_p \in \overline{\mathbf{Q}}_p$ with $v(a_p) > 0$ and $a_p^2 \neq 4p^{k-1}$. What is the isomorphism class of $\overline{V}_{k,a_p}$ (the semisimplification of the reduction of the D_p-representation V_{k,a_p}) as an explicit function of k and a_p?

A few years ago this question seemed to be regarded as almost intractible for weights $k > p$: as far as we know, the only results for high weights were those of Berger, Li and Zhu ([BLZ04]) who showed that for $v(a_p)$ sufficiently large (an explicit bound depending on k) the answer was the same as for the case $a_p = 0$ (which was already known). Not only that, computational evidence collected by one of us (KB) seemed to indicate that the answer to the question was in general rather subtle.

However, recent work of Breuil, Berger and Colmez on the p-adic and mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ gives us a completely new approach for attacking this problem. We now summarise what we need to know about the p-adic and mod p Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ here (although we do not go into the details of extension classes, an important subtlety which we will not need here, and we shall only discuss the case of crystalline p-adic representations; much is now known in more general cases but we shall not need these results). Note also that from this point on, our notational conventions will be that a bar over a $\overline{\mathbf{Q}}_p$-vector space (a Galois representation, or a representation of $\mathrm{GL}_2(\mathbf{Q}_p)$) will mean (some sensible variant of) "take a stable lattice, reduce, and then semisimplify", whereas a bar over a lattice (for example the $\overline{\mathbf{Z}}_p$-module Θ_{k,a_p} that we shall see later on) will mean "reduce mod p and don't semisimplify".

Breuil in [Bre03a] has classified the irreducible smooth admissible representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ over $\overline{\mathbf{F}}_p$. Furthermore he also wrote down an explicit injective map $\overline{V} \mapsto LL(\overline{V})$ from the set of isomorphism classes of semisimple 2-dimensional $\overline{\mathbf{F}}_p$-representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ to the set of isomorphism classes of semisimple finite length admissible smooth representations of $\mathrm{GL}_2(\mathbf{Q}_p)$, the idea being that this map should be the "correct" version of the Local Langlands correspondence in this setting. Note that the image of the map LL does not contain every irreducible representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ and it does contain some reducible ones. In particular the situation is not quite as simple as the classical local Langlands correspondence.

Now let V denote an irreducible 2-dimensional crystalline representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ over $\overline{\mathbf{Q}}_p$. Berger and Breuil in [BB08] associate to V a p-adic Banach space $B(V)$ equipped with a unitary action of $\mathrm{GL}_2(\mathbf{Q}_p)$, such that $B(V)$ is topologically irreducible (see Corollaire 5.3.2 of [BB08]). The map B should be thought of as a p-adic local Langlands correspondence. If one chooses an open $\mathrm{GL}_2(\mathbf{Q}_p)$-stable lattice in this Banach space and then tensors the lattice with $\overline{\mathbf{F}}_p$, one obtains a finite length $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ whose semisimplification $\overline{B}(V)$ depends only on V. A crucial theorem of Berger (Théorème A of [Ber08]) is that $\overline{B}(V) \cong LL(\overline{V})$, where $\overline{V}$ denotes the semisimplification of the $\overline{\mathbf{F}}_p$-reduction of V. In words, the p-adic and mod p local Langlands dictionaries are compatible. In particular $\overline{B}(V)$ actually only depends on $\overline{V}$.

The final ingredient in our approach is the following. We have our data (a_p, k) with as usual $k \geq 2$, $v(a_p) > 0$ and $a_p^2 \neq 4p^{k-1}$. Given this data, Breuil constructs an algebraically irreducible $\overline{\mathbf{Q}}_p$-representation Π_{k,a_p} of the group $\mathrm{GL}_2(\mathbf{Q}_p)$, and this representation stabilises a lattice Θ_{k,a_p}, so $\overline{\Theta}_{k,a_p} := \Theta_{k,a_p} \otimes \overline{\mathbf{F}}_p$ is an $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ (we will see explicit definitions later; the constructions were made in [Bre03b] and the proofs of the assertions we are making here are in loc. cit. and in section 5 of [BB08]). Breuil and Berger have shown (again in section 5 of [BB08]) that $B(V_{k,a_p})$ is a certain completion of Π_{k,a_p} and it follows that $\overline{\Theta}_{k,a_p}^{\mathrm{ss}} \cong \overline{B}(V_{k,a_p})$. In particular, $\overline{\Theta}_{k,a_p}^{\mathrm{ss}}$ *is in the image of* LL. This opens up the possibility of computing it via a process of elimination: if we examine the image of LL and manage to rule out all but one element of it as a possibility for $\overline{\Theta}_{k,a_p}^{\mathrm{ss}}$ then we have computed $\overline{\Theta}_{k,a_p}^{\mathrm{ss}}$ and hence $\overline{V}_{k,a_p}$! Note that this approach relies heavily on both Breuil's classification and the explicit happy (ϕ, Γ)-module coincidences for $\mathrm{GL}_2(\mathbf{Q}_p)$ implicit in Berger's work, and hence seems to be restricted to the case of 2-dimensional representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, but it does enable us to give the first explicit computations of $\overline{V}_{k,a_p}$ valid in the non-ordinary case where $v(a_p)$ is small and k is unbounded. We remark that Paskunas independently proved similar results using similar ideas.

Note that if $1 \leq t \leq p$ then $\mathrm{ind}(\omega_2^t)$ (induction from $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_{p^2})$ to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$) is an irreducible 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ with determinant ω^t whose restriction to I_p is $\omega_2^t \oplus \omega_2^{pt}$. For $n \in \mathbf{Z}$ let $[n]$ denote the unique integer in $\{0, 1, 2, \ldots, p-2\}$ congruent to n mod $p-1$. We prove the following result in this paper, via the technique explained above:

Theorem 1.6. *If $k \geq 2$ and $0 < v(a_p) < 1$, and $t = [k-2]+1$, then either $\overline{V}_{k,a_p} \cong \mathrm{ind}(\omega_2^t)$ is irreducible, or $k \equiv 3$ mod $p-1$, and $\overline{V}_{k,a_p}|_{I_p} \cong \omega \oplus \omega$.*

Note that the theorem is true but vacuous if $p = 2$. Note also that for $p > 2$, work of Breuil and Berger completely determines $\overline{V}_{k,a_p}$ (for all a_p) in the case $k \leq 2p+1$ (see for example Théorème 3.2.1 of [Ber08]). Our contribution is hence the case $k \geq 2p+2$ and $p > 2$, where the assumption $v(a_p) < 1$ of the theorem implies that the roots of $X^2 - a_pX + p^{k-1}$ cannot have ratio 1 or p, which helps our exposition somewhat because these correspond to "special cases" in the theory that sometimes have to be dealt with separately.

If $k \not\equiv 3$ modulo $p-1$ then the theorem tells you $\overline{V}_{k,a_p}$, but if $k \equiv 3$ mod $p-1$ then there are two possibilities in the conclusion of the theorem and we know of no neat criterion to distinguish between them. This initially surprising special case actually has a simple global explanation. Take a weight 3 newform of level $\Gamma_1(N) \cap \Gamma_0(p)$; it will have slope $1/2$. Moreover the local mod p representation attached to this newform is "hard" to determine, because it requires knowledge of the $\mathcal{L}$-invariant of the form. Now any classical eigenform in a Coleman family sufficiently close to this newform will be old at p, have weight congruent to 3 mod $p-1$, and will still have slope $1/2$, and hence the mod p Galois representation attached to this eigenform is also "hard" to determine and in particular will depend on more than the slope of the form. On the positive side, if $p > 2$ then we at least know what is happening for k small: if $k = 3$ or $k = p+2$ then the reducible case can never occur, and if $k = 2p+1$ then a computation of Breuil explains exactly when the reducible case occurs: see Théorème 3.2.1 of [Ber08].

2 The mod p and p-adic Langlands correspondences for $\mathrm{GL}_2(\mathbf{Q}_p)$.

We recall some results on the mod p representations of $\mathrm{GL}_2(\mathbf{Q}_p)$. Nothing in this section is due to the authors. Let p be a prime, let $\overline{\mathbf{Z}}_p$ be the integers in $\overline{\mathbf{Q}}_p$ and let $\overline{\mathbf{F}}_p$ be the residue field of

$\overline{\mathbf{Z}}_p$. Say $r \in \mathbf{Z}_{\geq 0}$. Let K be the group $\mathrm{GL}_2(\mathbf{Z}_p)$, and for R a $\mathbf{Z}_p$-algebra let $\mathrm{Symm}^r(R^2)$ denote the space $\oplus_{i=0}^r Rx^{r-i}y^i$ of homogeneous polynomials in two variables x and y, with the action of K given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^{r-i}y^i = (ax+cy)^{r-i}(bx+dy)^i.$$

Set $G = \mathrm{GL}_2(\mathbf{Q}_p)$, and let Z be its centre. If V is an R-module with an action of K, then extend the action of K to the group KZ by letting $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ act trivially, and let $I(V)$ denote the representation $\mathrm{ind}_{KZ}^G(V)$ (compact induction). Explicitly, $I(V)$ is the space of functions $f : G \to V$ which have compact support modulo Z and which satisfy $f(kg) = k.(f(g))$ for all $k \in KZ$. This space has a natural action of G, defined by $(gf)(\gamma) = f(\gamma g)$. Note that §2.2 of [BL94] explains how an R-linear G-endomorphism of $I(V)$ can be interpreted as a certain function $G \to \mathrm{End}_R(V)$ (by Frobenius reciprocity).

If $V = \mathrm{Symm}^r(R^2)$ for some integer $r \geq 0$ and $\mathbf{Z}_p$-algebra R, then, using the dictionary mentioned above, Barthel and Livné identified a certain endomorphism T of $I(V)$ which corresponds to the function $G \to \mathrm{End}_R(V)$ which is supported on $KZ\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)KZ$ and sends $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to the endomorphism of $\mathrm{Symm}^r(R^2)$ sending $F(x,y)$ to $F(px,y)$. We refer to Lemme 2.1.4.1 of [Bre03b] and the remarks following this lemma for many basic facts about this endomorphism.

We now recall the classification of smooth irreducible mod p representations of $\mathrm{GL}_2(\mathbf{Q}_p)$, due to Breuil and Barthel-Livné. For $r, n \in \mathbf{Z}_{\geq 0}$ set $\sigma_r := \mathrm{Symm}^r(\overline{\mathbf{F}}_p^2)$ and set $\sigma_r(n) := \det{}^n \otimes \mathrm{Symm}^r(\overline{\mathbf{F}}_p^2)$. These are $\overline{\mathbf{F}}_p$-representations of K, irreducible if $0 \leq r \leq p-1$. If R is a ring and $t \in R^\times$ then let μ_t denote the map $\mathrm{GL}_1(\mathbf{Q}_p) \to R^\times$ which is trivial on $\mathbf{Z}_p^\times$ and which sends p to t. If $\chi : \mathbf{Q}_p^\times \to \overline{\mathbf{F}}_p^\times$ is a character, if $\lambda \in \overline{\mathbf{F}}_p$ and if $0 \leq r \leq p-1$ then define

$$\pi(r, \lambda, \chi) := (I(\sigma_r)/(T-\lambda)) \otimes (\chi \circ \det).$$

The classification is due to Barthel-Livne and Breuil, and is as follows (see for example Théorème 2.7.1 of [Bre03a], which summarises results in [BL94], and also Corollaire 4.1.1, Corollaire 4.1.4 and Corollaire 4.1.5 of loc. cit.):

(i) If $\lambda \neq 0$ and $(r, \lambda) \notin \{(0, \pm 1), (p-1, \pm 1)\}$ then $\pi(r, \lambda, \chi)$ is irreducible.

(ii) There is a certain infinite-dimensional representation St of G, called the Steinberg representation. For $r \in \{0, p-1\}$ and $\lambda = \pm 1$, $\pi(r, \lambda, \chi)$ has two Jordan-Hoelder factors, one 1-dimensional and isomorphic to $\chi\mu_\lambda \circ \det$ and the other equal to a twist of the Steinberg representation by this same character.

(iii) The representations $\pi(r, 0, \chi)$ are all irreducible, and are called the supersingular representations of G (we remark that this is the result due to Breuil and it is this part which does not generalise to local fields other than $\mathbf{Q}_p$). No Jordan-Hoelder factor of $\pi(r, \lambda, \chi)$ with $\lambda \neq 0$ is supersingular.

(iv) We have just seen that all the representations $\pi(r, \lambda, \chi)$ have finite length. Conversely, any smooth irreducible $\overline{\mathbf{F}}_p$-representation of G with a central character is a Jordan-Hoelder constituent of some $\pi(r, \lambda, \chi)$.

(v) The only isomorphisms between the $\pi(r, \lambda, \chi)$ are the following:

(a) If $\lambda \neq 0$ and $(r, \lambda) \notin \{(0, \pm 1), (p-1, \pm 1)\}$, then $\pi(r, \lambda, \chi) \cong \pi(r, -\lambda, \chi\mu_{-1})$,

(b) If $\lambda \neq 0$ and $\lambda \neq \pm 1$ then $\pi(0, \lambda, \chi) \cong \pi(p-1, \lambda, \chi)$ (and these are also isomorphic to $\pi(0, -\lambda, \mu_{-1}\chi)$ and $\pi(p-1, -\lambda, \mu_{-1}\chi)$ as already mentioned).

(c) $\pi(r, 0, \chi) \cong \pi(r, 0, \chi\mu_{-1}) \cong \pi(p-1-r, 0, \chi\omega^r) \cong \pi(p-1-r, 0, \chi\omega^r\mu_{-1})$.

Note that the Jordan-Hoelder factors of $\pi(0, \lambda, \chi)$ and $\pi(p-1, \lambda, \chi)$ coincide even if $\lambda = \pm 1$.

We now move on to the p-adic part of the story. Say $k \geq 2$ and $a_p \in \overline{\mathbf{Z}}_p$ with $|a_p| < 1$, as usual. We now furthermore make the assumption that the roots of $X^2 - a_pX + p^{k-1}$ do not have ratio 1 or p; in other words we assume $a_p^2 \neq 4p^{k-1}$ and $a_p \neq \pm(1+p)p^{(k-2)/2}$. These assumptions are not always necessary, but they make for a slightly cleaner exposition, and in our final application we have $k \geq p+2$ and $v(a_p) < 1$ and so they will be satisfied.

Definition 2.1. *Let $\Pi_{k,a_p} := \mathrm{ind}_{KZ}^G \mathrm{Symm}^{k-2}(\overline{\mathbf{Q}}_p^2)/(T-a_p)$ (compact induction, as before), and let Θ_{k,a_p} be the image of $\mathrm{ind}_{KZ}^G \mathrm{Symm}^{k-2}(\overline{\mathbf{Z}}_p^2)$ in Π_{k,a_p}.*

Alternatively, Θ_{k,a_p} is the quotient of $\mathrm{ind}_{KZ}^G \mathrm{Symm}^{k-2}(\overline{\mathbf{Z}}_p^2)/(T-a_p)$ by its torsion. Then Π_{k,a_p} is irreducible by Proposition 3.3(i) of [Bre03b] and Θ_{k,a_p} is a lattice in it, by Corollaire 5.3.4 of [BB08].

The p-adic Langlands correspondence associates a unitary p-adic Banach space representation $B(V)$ of $\mathrm{GL}_2(\mathbf{Q}_p)$ to an irreducible crystalline representation V, and $B(V_{k,a_p})$ (under our assumptions on k and a_p above) is isomorphic to the completion of Π_{k,a_p} with respect to the gauge of Θ_{k,a_p}. We deduce that $\overline{B}(V_{k,a_p})$ (the semisimplification of the reduction of an open $\mathrm{GL}_2(\mathbf{Q}_p)$-stable lattice in $B(V_{k,a_p})$) is isomorphic to $\overline{\Theta}_{k,a_p}^{\mathrm{ss}}$, where $\overline{\Theta}_{k,a_p} := \Theta_{k,a_p} \otimes \overline{\mathbf{F}}_p$. We now recall the explicit mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ formulated by Breuil and the compatibility of the mod p and p-adic Langlands correspondences for trianguline representations proved by Berger (see Théorème A of [Ber08]), and in particular what these things tell us about the situation in hand. Let $[x]$ be the unique integer in $[0, p-2]$ congruent to $x \in \mathbf{Z}$ modulo $p-1$.

Theorem 2.2. *We have*

$$\overline{V} \cong (\mathrm{ind}(\omega_2^{r+1})) \otimes \chi \iff \overline{B}(V) \cong \pi(r, 0, \chi)$$

and

$$\overline{V} \cong (\mu_\lambda \omega^{r+1} \oplus \mu_{\lambda^{-1}}) \otimes \chi \iff \overline{B}(V) \cong \pi(r, \lambda, \chi)^{ss} \oplus \pi([p-3-r], \lambda^{-1}, \chi\omega^{r+1})^{ss}.$$

In particular, if $k \geq 2$, $a_p \in \overline{\mathbf{Q}}_p$ with $v(a_p) > 0$, and if the roots of $X^2 - a_p X + p^{k-1}$ don't have ratio 1 or $p^{\pm 1}$, we have

$$\overline{V}_{k,a_p} \cong (\mathrm{ind}(\omega_2^{r+1})) \otimes \chi \iff (\overline{\Theta}_{k,a_p})^{\mathrm{ss}} \cong \pi(r, 0, \chi)$$

and

$$\overline{V}_{k,a_p} \cong (\mu_\lambda \omega^{r+1} \oplus \mu_{\lambda^{-1}}) \otimes \chi \iff (\overline{\Theta}_{k,a_p})^{\mathrm{ss}} \cong \pi(r, \lambda, \chi)^{\mathrm{ss}} \oplus \pi([p-3-r], \lambda^{-1}, \chi\omega^{r+1})^{\mathrm{ss}}.$$

3 Lemmas about mod p representations of $\mathrm{GL}_2(\mathbf{Q}_p)$.

Our strategy for proving Theorem 1.6 is to compute $\overline{V}_{k,a_p}$ when $0 < v(a_p) < 1$ by analysing $\overline{\Theta}_{k,a_p}$ and its possible Jordan-Hoelder factors. The following lemma follows directly from the explicit description of the Jordan-Hoelder factors of $\pi(r, \lambda, \chi)$ given in the previous section.

Lemma 3.1. *If $\lambda \neq 0$ and $\pi(r, \lambda, \chi)$ and $\pi(r', \lambda', \chi')$ have a common Jordan-Hoelder factor, then $\lambda' \neq 0$, $r \equiv r' \bmod p-1$, and χ/χ' is unramified.*

□

The next lemma is a straightforward strengthening of Proposition 32 of [BL94].

Lemma 3.2. *If $0 \leq r \leq p-1$ and F is a quotient of $I(\sigma_r)$ which has finite length as an $\overline{\mathbf{F}}_p[G]$-module, then every Jordan-Hoelder factor of F is a subquotient of $I(\sigma_r)/(T-\lambda) = \pi(r, \lambda, 1)$ for some λ (with λ possibly depending on the factor).*

Proof. Induction on the length of F. If F is irreducible then Théorème 2.7.1(i) of [Bre03a] shows that in fact F is a quotient of some $\pi(r, \lambda, 1)$, and in particular a subquotient. Assume now that F has length greater than one, so that F has an irreducible quotient J, and the kernel K has smaller length. Again by loc. cit., the composite map $I(\sigma_r) \to F \to J$ factors as $I(\sigma_r) \to \pi(r, \lambda, 1) \to J$ for some λ. Let γ denote this latter arrow.

Now consider the following commutative diagram:

$$\begin{tikzcd} & I(\sigma_r) \arrow[r, "T-\lambda"] \arrow[d, "\alpha"] & I(\sigma_r) \arrow[r] \arrow[d, "\beta"] & \pi(r,\lambda,1) \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & K \arrow[r] & F \arrow[r] & J \arrow[r] & 0 \end{tikzcd}$$

Since β is surjective, the snake lemma implies that $\mathrm{coker}(\alpha)$ is a subquotient of $\pi(r,\lambda,1)$ and hence each of its Jordan-Hoelder factors are too. Moreover, the inductive hypothesis tells us that every Jordan-Hoelder factor of $\mathrm{im}(\alpha)$ is a subquotient of $\pi(r,\lambda',1)$ for some λ' (depending on the factor), hence any Jordan-Hoelder factor of K and hence of F is a subquotient of $\pi(r,\lambda',1)$ for some λ' and we are done. □

The following proposition will be crucial for us. It uses the previous lemma to conclude that we can say a lot about a finite length $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ which is a quotient of $I(W)$ for some *irreducible* W, and whose semisimplification is in the image of LL. Note that any supersingular representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ is a quotient of $I(W)$ for some irreducible W; the point is that almost no other elements of the image of LL are. Recall that if W is an irreducible $\overline{\mathbf{F}}_p$-representation of K then $W=\sigma_s(n)$ for some $0\leq s\leq p-1$.

Proposition 3.3. *If V is an irreducible crystalline representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, if Θ is an open $\mathrm{GL}_2(\mathbf{Q}_p)$-stable lattice in $B(V)$ and if there is a surjection $I(\sigma_s(n))\to\overline{\Theta}$ for some s with $0\leq s\leq p-1$, then either $\overline{V}\cong\mathrm{ind}(\omega_2^{s+1+(p+1)n})$ is irreducible, or $\overline{V}$ is reducible and scalar on inertia. If $p>2$ then in the reducible case we must furthermore have $s=p-2$ and $\overline{V}|_{I_p}=\omega^n\oplus\omega^n$.*

Remark 3.4. If $p=2$ then this proposition is vacuous, as if $p=2$ then *every* semisimple 2-dimensional mod p representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is either irreducible or scalar on inertia.

Proof. Firstly observe that $I(\sigma_s(n))\cong I(\sigma_s)\otimes(\omega\circ\det)^n$ as $\mathrm{GL}_2(\mathbf{Q}_p)$-representations, so (twisting V by an appropriate power of the cyclotomic character) we may assume that $n=0$ (as LL is compatible with twists).

We know that $\overline{\Theta}^{\mathrm{ss}}$ must be in the image of LL. Hence if $\overline{\Theta}^{\mathrm{ss}}$ is irreducible then it is of the form $\pi(r,0,\chi)$ for some r,χ. However Lemma 3.2 tells us that any irreducible quotient of $I(\sigma_s)$ must be a Jordan-Hoelder factor of $\pi(s,\lambda,1)$ for some λ, and by the classification theorem we must have $\pi(r,0,\chi)\cong\pi(s,0,1)$. By Theorem 2.2 we deduce $\overline{V}\cong\mathrm{ind}(\omega_2^{s+1})$ in this case.

So now say $\overline{\Theta}^{\mathrm{ss}}$ is reducible (and hence $\overline{V}$ is reducible). The crucial point is that if $\lambda\in\overline{\mathbf{F}}_p^\times$ then usually the Jordan-Hoelder factors of $\pi(r,\lambda,\chi)^{\mathrm{ss}}\oplus\pi([p-3-r],\lambda^{-1},\chi\omega^{r+1})^{\mathrm{ss}}$ (a general reducible element of the image of LL) cannot *all* be subquotients of $I(\sigma_s)$ for the same s. Indeed, applying Lemma 3.2 to $\overline{\Theta}$ we see that if $\overline{V}$ is reducible then the Jordan-Hoelder factors of $\pi(r,\lambda,\chi)$ and $\pi([p-3-r],\lambda^{-1},\chi\omega^{s+1})$ *all* have to be subquotients of the $\pi(s,\lambda',1)$ for varying λ', and by Lemma 3.1 above we note that this forces both χ and $\chi\omega^{r+1}$ to be unramified, so $r\equiv s\equiv p-2$ modulo $p-1$, and in this case we see from Theorem 2.2 that $\overline{V}$ is unramified. □

4 The kernel of the map $I(\sigma_{k-2})\to\overline{\Theta}_{k,a_p}$.

Let us re-iterate our assumptions: $k\geq 2$, $v(a_p)>0$ and the roots of $X^2-a_pX+p^{k-1}$ do not have ratio 1 or $p^{\pm1}$. It is clear from the definition that $\overline{\Theta}_{k,a_p}$ admits a surjection from $I(\sigma_{k-2})/(T)$ and in particular from $I(\sigma_{k-2})$. Note however that Proposition 3.3 does not apply to this situation, because σ_{k-2} is not in general irreducible.

Let $X(k,a_p)$ denote the kernel of the surjection $I(\sigma_{k-2})\to\overline{\Theta}_{k,a_p}$. In this section we will analyse $X(k,a_p)$, and as a consequence find, in certain cases, that $\overline{\Theta}_{k,a_p}$ does admit a surjection from $I(W)$ for some *irreducible* $\overline{\mathbf{F}}_p$-representation W of K, enabling us to apply Proposition 3.3.

Firstly, we establish some notation, following [Bre03b]. Recall that for V a $\mathbf{Z}_p[K]$-module, the space $I(V)$ was defined previously to be a certain space of functions $G\to V$. We let $[g,v]$ denote the (unique) element of $I(V)$ which is supported on KZg^{-1}, and which satisfies $[g,v](g^{-1})=v$. Note that $g[h,v]=[gh,v]$ for $g,h\in G$, that $[gk,v]=[g,kv]$ for $k\in K$, and that the $[g,v]$ span $I(V)$ as an abelian group, as g and v vary. For $\lambda\in\mathbf{Q}_p$ write $g^0_{1,\lambda}:=\left(\begin{smallmatrix}p&\lambda\\0&1\end{smallmatrix}\right)$, and set $\alpha:=\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)$. Now let $V=\mathrm{Symm}^{k-2}(R^2)$ for some $\mathbf{Z}_p$-algebra R, thought of, as usual, as homogeneous polynomials in the variables x and y of degree $k-2$. An easy consequence of Lemma 2.2.1 of [Bre03b] is that

$T([\mathrm{Id}, v]) = T^+([\mathrm{Id}, v]) + T^-([\mathrm{Id}, v])$ where

$$T^+([\mathrm{Id}, v]) = \sum_{\lambda \in \mathbf{Z}_p : \lambda^p = \lambda} [g^0_{1,\lambda}, v(x, py - \lambda x)]$$

and

$$T^-([\mathrm{Id}, v]) = [\alpha, v(px, y)].$$

For simplicity now, write $r := k - 2 \in \mathbf{Z}_{\geq 0}$. Note that I is an exact functor, so if $W \subseteq V$ are K-representations then $I(W)$ is naturally a subset of $I(V)$.

Lemma 4.1. *If $r \geq p$ then $T.I(\sigma_r) = I(W_r)$ where W_r is the $\overline{\mathbf{F}}_p[K]$-submodule of σ_r generated by y^r.*

Remark 4.2. If $r \leq p - 1$ then the image of T is harder to describe. It is a minor miracle that one has such a succinct description of the image of T when $r \geq p$.

Proof. It is clear that $T.I(\sigma_r)$ is generated as an $\overline{\mathbf{F}}_p[G]$-module by elements of the form $T([\mathrm{Id}, v])$, for $v \in \sigma_r$. By our explicit formula for $T([\mathrm{Id}, v])$ above (which simplifies because $p = 0$ in this situation), we see that $T([\mathrm{Id}, v])$ may be written as a sum of terms $[h, w]$, all of which have the property that the ws are in $\overline{\mathbf{F}}_p x^r$ or $\overline{\mathbf{F}}_p y^r$. Because $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) y^r = x^r$, we see that $T.I(\sigma_r)$ is certainly contained in $I(W_r)$.

For the reverse inclusion, define

$$\overline{\theta} := xy^p - yx^p = x \prod_{\lambda : \lambda^p = \lambda} (y - \lambda x) \in \overline{\mathbf{F}}_p[x, y].$$

Then if $f := [\mathrm{Id}, y^{r-p}\overline{\theta}/x] \in I(\sigma_r)$ we have $T^+ f = 0$, and so $Tf = T^- f = [\alpha, y^r]$, which is easily checked to generate $I(W_r)$ as an $\overline{\mathbf{F}}_p[G]$-module. $\square$

Recall that $X(k, a_p)$ is the kernel of the map $I(\sigma_r) \to \overline{\Theta}_{k,a_p}$. Multiplication by $\overline{\theta}$ induces a map $\sigma_r \to \sigma_{r+p+1}$, which is K-equivariant when thought of as a map $\sigma_r(1) \to \sigma_{r+p+1}$.

Lemma 4.3. *If $v(a_p) < 1$ and $r \geq p + 1$ then $I(\overline{\theta}\sigma_{r-(p+1)}) \subseteq X(k, a_p)$.*

Proof. Set $\theta = x^p y - xy^p \in \overline{\mathbf{Q}}_p[x, y]$. For $g \in \overline{\mathbf{Z}}_p[x, y]$ of degree $r - (p + 1)$, set $f = [\mathrm{Id}, a_p^{-1}\theta g] \in \mathrm{ind}_{KZ}^G(\mathrm{Symm}^r(\overline{\mathbf{Q}}_p^2))$. Clearly the image of $Tf - a_p f$ in Π_{k,a_p} is zero. On the other hand, $\theta(x, py - \lambda x)$ (for $\lambda \in \mathbf{Z}_p$) and $\theta(px, y)$ are in $p\mathbf{Z}_p[x, y]$, and so by the explicit formula for T above one deduces that $Tf \in (p/a_p)\,\mathrm{ind}_{KZ}^G \mathrm{Symm}^r(\overline{\mathbf{Z}}_p^2))$. In particular Tf is zero in $\overline{\Theta}_{k,a_p}$. Yet $a_p f$ is $[\mathrm{Id}, \theta g]$, so $Tf - a_p f$ is in $\mathrm{ind}_{KZ}^G \mathrm{Symm}^r(\overline{\mathbf{Z}}_p^2))$, and $[\mathrm{Id}, \overline{\theta}\overline{g}]$ is in $X(k, a_p)$. $\square$

Remark 4.4. Although we will not use these results here (but they could be used if one were trying to formulate an analogue of Theorem 1.6 for larger $v(a_p)$), we note that similar tricks show that if $v(a_p) < t \in \mathbf{Z}_{\geq 1}$ and $r \geq t(p + 1)$ then $I(\overline{\theta}^t \sigma_{r-t(p+1)}) \subseteq X(k, a_p)$ (set $f = [\mathrm{Id}, a_p^{-1}\theta^t g]$) and that if $v(a_p) > n \in \mathbf{Z}_{\geq 0}$, if $0 \leq i \leq n$ and $r \geq i(p+1) + p$ then $I(\langle x^i y^{r-i} \rangle_K) \subseteq X(k, a_p)$, where $\langle x^i y^{r-i} \rangle_K$ denotes the sub-K-representation of σ_r generated by $x^i y^{r-i}$ (set $f = [\mathrm{Id}, (\theta/p)^i(\theta/x)y^{r-i(p+1)-p}]$).

5 Analysis of $0 < v < 1$.

If $k \geq p + 3$ and $0 < v(a_p) < 1$ then the roots of $X^2 - a_p X + p^{k-1}$ cannot have ratio 1 or $p^{\pm 1}$ so the results of the preceding sections give

Corollary 5.1. *If $0 < v(a_p) < 1$ and $k \geq p + 3$ then $X(k, a_p)$ contains $\mathrm{ind}_{KZ}^G(Y)$ where Y is the sub-K-representation of σ_r generated by $\overline{\theta}\sigma_{r-(p+1)}$ and y^r. In particular the map $\mathrm{ind}_{KZ}^G(\sigma_r) \to \overline{\Theta}_{k,a_p}$ factors through the induction $\mathrm{ind}_{KZ}^G(\sigma_r/Y)$ of the* irreducible *representation σ_r/Y. We have $\sigma_r/Y \cong \sigma_s(r)$ where $0 \leq s \leq p - 2$ and $s \equiv p - 1 - r$ mod $p - 1$.*

Proof. The first part follows from Lemmas 4.1 and 4.3. The rest of the Corollary follows once one knows the isomorphism $\sigma_r/Y \cong \sigma_s(r)$, which follows from Lemma 3.2(a) and (c) of [AS86]. □

Corollary 5.2. *Theorem 1.6 is true.*

Proof. By the comments following the statement of the theorem, we need only deal with the case $p > 2$ and $k \geq 2p+2$. In fact we prove the result for $p > 2$ and $k \geq p+3$. First fix s such that $0 \leq s \leq p-2$ and $s \equiv 2-k \bmod p-1$. By Corollary 5.1 we see $\overline{\Theta}_{k,a_p}$ is a quotient of $I(\sigma_s(k-2))$. The result now follows from Proposition 3.3 and some elementary arithmetic. □

Corollary 5.3. *If f is a modular form of weight $k \geq p+3$, level prime to p, and $0 < v(a_p) < 1$, and if $0 \leq g \leq p-2$ with $g \equiv k-2 \bmod p-1$, then either $(\overline{\rho}_f|_{D_p})^{ss} \cong \mathrm{ind}(\omega_2^{g+1})$, or $k \equiv 3 \bmod p-1$ and $(\overline{\rho}_f|_{I_p})^{ss} \cong \omega \oplus \omega$.*

Proof. This follows from the preceding corollary and Theorem 1.4. □

Remark 5.4. For $k \leq p+2$ slightly stronger results are known (for example if $k \leq p+2$ and $p > 2$ then the results of [Bre03b] show that the reducible case can't occur). On the other hand when $k = 2p+1$ and $p > 2$, unpublished calculations of Breuil, and numerical examples due to one of us (KB) show that when $v(a_p) = 1/2$ both reducible and irreducible possibilities can occur.

Corollary 5.5. *If $2 \leq p \leq 53$, then the slopes of level 1 modular forms of all weights are never in the range $(0, 1)$.*

Proof. If $p = 2$ this follows from the results of [Hat79]. If p is odd then the congruence $k = 3 \bmod p-1$ implies that k is odd, and there are no level 1 forms of odd weight. The corollary would follow if we knew that for f a modular form of level 1 and $p \leq 53$ then $\overline{\rho}_f|_{D_p}$ is always reducible. But Corollary 3.6 of [AS86] reduces this statement to a finite check (we only need to check level 1 eigenforms of weight at most $p+1 \leq 54$), and by Theorem 1.2 we see that what we must verify is that if $2 < p \leq 53$ then the T_p-eigenvalues on the space of cusp forms of level 1 and weight $k \leq p+1$ are all p-adic units, which is easily checked nowadays on a computer. □

Remark 5.6. If $p = 59$ and f is the normalised level 1 weight 16 cusp form then f is non-ordinary at p, the local mod p representation is irreducible, and there do exist level 1 modular forms with slope equal to 1/2 (for example, in weight 74).

References

[AS86] Avner Ash and Glenn Stevens, *Modular forms in characteristic l and special values of their L-functions*, Duke Math. J. **53** (1986), no. 3, 849–868. MR MR860675 (88h:11036)

[BB08] Laurent Berger and Christophe Breuil, *Sur quelques représentations potentiellement cristallines de* $\mathrm{GL}_2(\mathbb{Q}_p)$, Asterisque (to appear) (2008).

[Ber08] Laurent Berger, *Représentations modulaires de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et représentations galoisiennes de dimension 2*, Asterisque (to appear) (2008).

[BL94] L. Barthel and R. Livné, *Irreducible modular representations of* GL_2 *of a local field*, Duke Math. J. **75** (1994), no. 2, 261–292. MR MR1290194 (95g:22030)

[BLZ04] Laurent Berger, Hanfeng Li, and Hui June Zhu, *Construction of some families of 2-dimensional crystalline representations*, Math. Ann. **329** (2004), no. 2, 365–377. MR MR2060368 (2005k:11104)

[BM02] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en* $l = p$, Duke Math. J. **115** (2002), no. 2, 205–310, With an appendix by Guy Henniart. MR MR1944572 (2004i:11052)

[Bre03a] Christophe Breuil, *Sur quelques représentations modulaires et p-adiques de* $\mathrm{GL}_2(\mathbf{Q}_p)$*. I*, Compositio Math. **138** (2003), no. 2, 165–188. MR MR2018825 (2004k:11062)

[Bre03b] ———, *Sur quelques représentations modulaires et p-adiques de* $\mathrm{GL}_2(\mathbf{Q}_p)$*. II*, J. Inst. Math. Jussieu **2** (2003), no. 1, 23–58. MR MR1955206 (2005d:11079)

[CE98] Robert F. Coleman and Bas Edixhoven, *On the semi-simplicity of the U_p-operator on modular forms*, Math. Ann. **310** (1998), no. 1, 119–127. MR MR1600034 (99b:11043)

[DS74] Pierre Deligne and Jean-Pierre Serre, *Formes modulaires de poids* 1, Ann. Sci. École Norm. Sup. (4) **7** (1974), 507–530 (1975). MR MR0379379 (52 #284)

[Edi92] Bas Edixhoven, *The weight in Serre's conjectures on modular forms*, Invent. Math. **109** (1992), no. 3, 563–594. MR MR1176206 (93h:11124)

[Hat79] Kazuyuki Hatada, *Eigenvalues of Hecke operators on* $\mathrm{SL}(2, \mathbf{Z})$, Math. Ann. **239** (1979), no. 1, 75–96. MR MR516060 (80b:10037)

[Wil88] A. Wiles, *On ordinary λ-adic representations associated to modular forms*, Invent. Math. **94** (1988), no. 3, 529–573. MR MR969243 (89j:11051)