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THE WEIGHT PART OF SERRE'S CONJECTURE FOR GL(2)

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Abstract

Let $p > 2$ be prime. We use purely local methods to determine the possible reductions of certain two-dimensional crystalline representations, which we call "pseudo-Barsotti–Tate representations", over arbitrary finite extensions of $\mathbb{Q}_p$. As a consequence, we establish (under the usual Taylor–Wiles hypothesis) the weight part of Serre's conjecture for GL(2) over arbitrary totally real fields.
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Overview

Let p be a prime number. Given an irreducible modular representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, the weight part of Serre's conjecture predicts the set of weights k such that $\overline{\rho}$ is isomorphic to the mod p Galois representation $\overline{\rho}_{f,p}$ associated to some eigenform of weight k and level prime to p. The conjectural set of weights is determined by the local representation $\overline{\rho}|_{G_{\mathbb{Q}_p}}$. In recent years, beginning with the work of [BDJ10], generalisations of the weight part of Serre's conjecture have become increasingly important, in particular because of their importance in formulating a p-adic Langlands correspondence (*cf.* [BP12]).

The weight part of Serre's original conjecture was settled in the early 1990s (at least if $p > 2$; see [CV92, Edi92, Gro90]). The paper [BDJ10] explored the generalisation of the weight part of Serre's original conjecture [Ser87] to the setting of Hilbert modular forms over a totally real field F in which p is unramified. Already in this case even formulating the conjecture is far more difficult; there are many more weights, and the conjectural description of them involves subtle questions in integral p-adic Hodge theory. The conjecture was subsequently extended to arbitrary totally real fields in [Sch08, Gee11a, BLGG13]; see [BLGG13, §4] for an extensive discussion. The present paper completes the proof of this general Serre weight conjecture for GL(2) (under the assumption that $p > 2$ and that a standard Taylor–Wiles hypothesis holds).

Work of [BLGG13] and [GK, New14] has reduced the weight part of Serre's conjecture for GL(2) to a certain statement about local Galois representations: namely, two sets of Serre weights associated to a mod p local Galois representation, each defined in p-adic Hodge-theoretic terms, must be seen to be equal. It is this problem that we resolve in this paper. In our earlier paper [GLS14] we proved the same result for totally real fields in which p is unramified, by establishing a structure theorem for Kisin modules which slightly extended Fontaine–Laffaille theory in this case. In contrast, in this paper we must work over an arbitrarily ramified base field, and the proof requires a delicate analysis of the Kisin modules associated to a certain class of crystalline representations, which we call pseudo-Barsotti–Tate representations. To the best of our knowledge, these are the first general results about the reductions of crystalline representations in any situation where arbitrary ramification is permitted.

1. Introduction

We begin by tracing the history of the work on the weight part of Serre's conjecture over the last decade. The first breakthrough towards proving the conjecture of [BDJ10] was the paper [Gee11b], which proved the conjecture under a mild global hypothesis as well as a genericity hypothesis on the local mod p representations. The global hypothesis comes from the use of the Taylor–Wiles–Kisin method, which is used to prove modularity lifting theorems; the modularity lifting theorems used in [Gee11b] are those of [Kis09, Gee06] for potentially Barsotti–Tate representations. The genericity hypothesis was needed for a complicated combinatorial argument relating Serre weights to the reduction modulo p of the types associated to certain potentially Barsotti–Tate representations. The natural output of the argument was a description of the Serre weights (in this generic setting) in terms of potentially Barsotti–Tate representations, while the conjecture of [BDJ10] is in terms of crystalline representations, and the comparison between the two descriptions involved a delicate calculation in integral p-adic Hodge theory. It was clear that the combinatorial arguments would not extend to cover the non-generic case, or to settings in which p is allowed to ramify.

All subsequent results on the problem have followed [Gee11b] in making use of modularity lifting theorems, and have assumed that $p > 2$, which we do for the remainder of this introduction. The next progress (other than special cases such as [GS11]) was due to the work of [BLGGT14], which proved automorphy lifting theorems for unitary groups of arbitrary rank, in which the weight of the automorphic forms is allowed to vary. As the conjectural description of the set of weights is purely local, there is a natural analogue of the weight part of Serre's conjecture for inner forms of U(2), and the paper [BLGG13] used the results of [BLGGT14] to prove a general result on these conjectures. Specifically, given a global modular representation $\overline{\rho}$ associated to some form of U(2), the main results of [BLGG13] show that $\overline{\rho}$ is modular of all the weights predicted by the generalisations of the weight part of Serre's conjecture. The problem is then to prove that $\overline{\rho}$ cannot be modular of any other weight.

This remaining problem is easily reduced to a local problem. To describe this, let $K/\mathbb{Q}_p$ be a finite extension with residue field k, and $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ a continuous representation. In this local context a Serre weight is an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_2(k)$ (*cf.* Definition 4.1.1). Associated to $\bar{r}$ there are two sets of Serre weights $W^{\mathrm{explicit}}(\bar{r}) \subseteq W^{\mathrm{cris}}(\bar{r})$, defined in terms of p-adic Hodge theory. The set $W^{\mathrm{cris}}(\bar{r})$ is defined in terms of the existence of crystalline lifts with certain Hodge–Tate weights, and $W^{\mathrm{explicit}}(\bar{r})$ is defined as a subset of $W^{\mathrm{cris}}(\bar{r})$ by explicitly writing down examples of these crystalline lifts (as inductions and extensions of crystalline characters); see Definitions 4.1.3 and 4.1.4 below for precise definitions. These definitions are then extended to global representations $\overline{\rho}$ by taking the tensor products of the sets of weights for the restrictions of $\overline{\rho}$ to decomposition groups at places dividing p. In the case that p is unramified or $\bar{r}$ is semisimple, $W^{\mathrm{explicit}}(\bar{r})$ is the set of weights predicted by the conjectures of [BDJ10, Sch08].

If $\overline{\rho}$ is a global modular representation for U(2), then it is almost immediate from the definition that the set of weights in which $\overline{\rho}$ is modular is contained in $W^{\mathrm{cris}}(\overline{\rho})$. The main result of [BLGG13] shows that the set of weights contains $W^{\mathrm{explicit}}(\overline{\rho})$. Therefore, to complete the proof of the weight part of Serre's conjecture for U(2) (i.e. to prove that $W^{\mathrm{explicit}}(\overline{\rho})$ is the set of weights in which $\overline{\rho}$ is modular) it is only necessary to show in the local setting that $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$.

In the case that $K/\mathbb{Q}_p$ is unramified (that is, the setting of the original conjecture of [BDJ10]), this was proved by purely local means in our earlier paper [GLS14]. That work uses the second author's theory of $(\varphi, \hat{G})$-modules to prove a structure theorem for the Kisin modules associated to crystalline representations (over an unramified base) with Hodge–Tate weights just beyond the Fontaine–Laffaille range. A careful analysis of this structure theorem, and of the ways in which these Kisin modules can be extended to $(\varphi, \hat{G})$-modules, allowed us to compute the possible reductions of the crystalline representations, and explicitly check that they were all of the required form.

Until the present paper, the equality $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$ was not known in any greater generality. However, in the paper [GLS12] we were able to show that if p is totally ramified, then the set of modular weights for U(2) is exactly $W^{\mathrm{explicit}}(\overline{\rho})$, by making a rather baroque global argument using the results of [BLGG13] and a comparison to certain potentially Barsotti–Tate representations, motivated by the approach of [Gee11b]. This approach did not show that $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$; as in [Gee11b], this approach relies on combinatorial results that do not extend to the general case, although generalisations subject to a genericity hypothesis (and other related results) were proved using these techniques in [DS].

The results of [BLGG13, GLS14, GLS12] only concerned the analogues for U(2) of the conjecture of [BDJ10] and its generalisations, which were formulated using modular forms on quaternion algebras over totally real fields. The quaternion algebra setting of [BDJ10] is a more natural generalisation of the original conjectures of [Ser87], but it is harder, because of a parity obstruction coming from the units of the totally real field: algebraic Hilbert modular forms necessarily have paritious weight. This means, for example, that there are mod p Hilbert modular forms of level prime to p and some weight which cannot be lifted to characteristic zero forms of the same weight, and is one of the reasons for studying potentially Barsotti–Tate lifts instead (which correspond to Hilbert modular forms of parallel weight two).

The papers [GK, New14] independently succeeded (by rather different means) in transferring the results for U(2) described above to the setting of quaternion algebras over totally real fields. In particular, [GK] defines a set of weights $W^{\mathrm{BT}}(\bar{r})$, and shows (by global methods, using the results of [BLGG13]) that there are inclusions

$$W^{\mathrm{explicit}}(\bar{r}) \subseteq W^{\mathrm{BT}}(\bar{r}) \subseteq W^{\mathrm{cris}}(\bar{r}).$$

The definition of $W^{\mathrm{BT}}(\bar{r})$ can again be extended to global representations in exactly the same manner as before, and (in either the U(2) or quaternion algebra settings) the set of weights in which $\overline{\rho}$ is modular is always the set $W^{\mathrm{BT}}(\overline{\rho})$. In particular, this shows that the set of weights is determined purely locally, and in order to complete the proof of the weight part of Serre's conjecture, it would be enough to solve the purely local problem of showing that $W^{\mathrm{BT}}(\bar{r}) = W^{\mathrm{explicit}}(\bar{r})$.

Unfortunately, the definition of $W^{\mathrm{BT}}(\bar{r})$ is rather indirect, being defined as a linear combination of the Hilbert–Samuel multiplicities of certain potentially Barsotti–Tate deformation rings. These rings have only been computed when

$K = \mathbb{Q}_p$ ([Sav05]) or when $K/\mathbb{Q}_p$ is unramified and $\bar{r}$ is generic ([Bre14, BM12]), and they appear to be extremely difficult to determine in any generality.

The main local result of this paper (see Theorem 6.1.8) is that $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$ in complete generality (provided that $p > 2$). The proof is purely local, and will be described below. As a consequence, we deduce that $W^{\mathrm{BT}}(\bar{r}) = W^{\mathrm{explicit}}(\bar{r})$ when $p > 2$, and thus we obtain the following theorem (see Theorem 6.2.1).

A Theorem. *Let $p > 2$ be prime, let F be a totally real field, and let $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Assume that $\overline{\rho}$ is modular, that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible, and if $p = 5$ assume further that the projective image of $\overline{\rho}|_{G_{F(\zeta_p)}}$ is not isomorphic to A_5.*

For each place $v|p$ of F with residue field k_v, let σ_v be a Serre weight of $\mathrm{GL}_2(k_v)$. Then $\overline{\rho}$ is modular of weight $\otimes_{v|p}\sigma_v$ if and only if $\sigma_v \in W^{\mathrm{explicit}}(\overline{\rho}|_{G_{F_v}})$ for all v.

In particular, this immediately implies the generalisations of the weight conjecture of [BDJ10] proposed in [Sch08, Gee11a].

We remark that Newton and Yoshida ([NY14], forthcoming) have also established Theorem A for many weights $\otimes_{v|p}\sigma_v$ by methods that are rather different from ours, namely via novel arguments involving the study of the special fibres of integral models for Shimura curves.

We now turn to an outline of our approach. To prove that $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$, it is necessary to study the possible reductions of certain 2-dimensional crystalline representations. The Hodge–Tate weights of these representations (which were first considered in [Gee11a]) have a particular form, and we call these representations *pseudo-Barsotti–Tate* representations, because most of their Hodge–Tate weights (in a sense that we make precise in Definition 2.3.1) agree with those of Barsotti–Tate representations. In the case that $K/\mathbb{Q}_p$ is unramified, these are exactly the representations considered in [GLS14], and our techniques are a wide-ranging extension of those of that paper to a setting where p may be arbitrarily ramified.

The equality $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$ is by definition equivalent to the statement that the possible reductions of pseudo-Barsotti–Tate representations with given Hodge–Tate weights are in an explicit list, namely the list of representations arising as the reductions of pseudo-Barsotti-Tate representations (of the same Hodge–Tate weights) which are extensions or inductions of crystalline characters. The most obvious way to try to establish this would be to classify (the lattices in) pseudo-Barsotti–Tate representations (perhaps in terms of their associated $(\varphi, \hat{G})$-modules), and then to compute all of their reductions modulo p. Experience suggests that this is likely to be a very difficult problem, and this is not the approach that we take. Instead, we proceed more indirectly, guided by the particular form of $W^{\text{explicit}}(\bar{r})$.

Our first step is to make a detailed study of the filtrations on the various objects in integral p-adic Hodge theory which are attached to lattices in pseudo-Barsotti–Tate representations. This study, which is a considerable generalisation of much of the work carried out in [GLS14, §4] in the unramified case, culminates in Theorem 2.4.1, which classifies the possible structures of their underlying Kisin modules in terms of their Hodge–Tate weights. It is then relatively straightforward to determine the possible characters that can occur in the reduction modulo p of such a representation, and comparing this to the form of $W^{\text{explicit}}(\bar{r})$, we prove the conjecture in the case that $\bar{r}$ is a direct sum of two characters (see Theorem 3.1.4).

We next turn to the case that $\bar{r}$ is irreducible. In this case, we know that $\bar{r}$ becomes a direct sum of two characters after restriction to the absolute Galois group of the unramified quadratic extension, and applying the previous result in this case gives a constraint on the form of $\bar{r}$. It is not at all obvious that this constraint is sufficient to prove the conjecture in this case, but we are able to establish this by a somewhat involved combinatorial argument (see Theorem 3.1.5).

At this point we have established the conjectures of [Sch08], which treat the case that $\bar{r}$ is semisimple. There is, however, still a considerable amount of work to be done to deal with the case that $\bar{r}$ is an extension of characters. It is not surprising that more work should be needed in the non-semisimple case. For example, if the ramification degree of K is at least p and $\bar{r}$ is semisimple, then the inclusion $W^{\text{cris}}(\bar{r}) \subseteq W^{\text{explicit}}(\bar{r})$ is essentially trivial, because the set $W^{\text{explicit}}(\bar{r})$ consists of all the weights satisfying a simple determinant condition. On the other hand when $\bar{r}$ is not semisimple, the definition of $W^{\text{explicit}}(\bar{r})$, which is given in terms of certain crystalline Ext^1 groups, only becomes more complicated as the ramification index increases.

In the remaining case that $\bar{r}$ is an extension of characters, the result that we proved in the semisimple case shows that the two characters are of the predicted form, and it remains to show that the extension class is one of the predicted extension classes. We again proceed indirectly. From the definition of $W^{\text{explicit}}(\bar{r})$, we need to show that $\bar{r}$ has a pseudo-Barsotti–Tate lift of the given weight and which is an extension of crystalline characters. An approach to this problem naturally suggests itself: we could compute the dimension of the space of extensions in characteristic p that arise from the reductions of extensions of crystalline characters, and try to use our structure theorem to prove that the set of extensions that can arise as the reductions of possibly irreducible pseudo-Barsotti–Tate representations is contained in a space of this same dimension.

This is in effect what we do, but there are a number of serious complications that arise when we try to compute our upper bound on the set of extension classes coming from pseudo-Barsotti–Tate representations. It is natural to return to our structural result Theorem 2.4.1, and this gives us non-trivial information on the possible Kisin modules, which we assemble with some effort. We would then like to compare these Kisin modules with those arising from extensions of crystalline characters. It is at this point that two difficulties arise. One is that the functor from lattices in Galois representations to Kisin modules is not exact (see Example 5.2.1 and the discussion that follows it), which means that the Kisin module corresponding to the reduction of such an extension need not correspond to an extension of the corresponding rank-one Kisin modules. The other related difficulty is that when the ramification degree is large, there are many different crystalline characters with different Hodge–Tate weights to consider, and it is necessary to relate their reductions. We are able to overcome these difficulties by showing that there is a "maximal" pair of rank-one Kisin modules for the representation $\bar{r}$ and the Hodge–Tate weights under consideration, and reducing to the problem of studying their extensions. This is done in Sections 5.2 to 5.4.

The second complication is that Kisin modules do not completely determine the corresponding G_K-representations, but rather their restrictions to a certain subgroup G_{K_∞}. If $\bar{r}$ is an extension of $\overline{\chi}_2$ by $\overline{\chi}_1$ with $\overline{\chi}_1\overline{\chi}_2^{-1}$ not equal to the mod p cyclotomic character, it turns out that the natural restriction map from extensions of G_K-representations to extensions of G_{K_∞}-representations is injective (Lemma 5.4.2), and we have done enough to complete the proof. However, in the remaining case that $\overline{\chi}_1\overline{\chi}_2^{-1}$ is the mod p cyclotomic character, we still have more work to do; we need to study the uniqueness or otherwise of the extensions of the Kisin modules to $(\varphi, \hat{G})$-modules. We are able to do this with the aid of [GLS14, Cor. 4.10], which constrains the possible $\hat{G}$-actions coming from the reductions of crystalline representations.

1.1. Notation

1.1.1. Galois theory If M is a field, we let G_M denote its absolute Galois group. If M is a global field and v is a place of M, let M_v denote the completion of M at v. If M is a finite extension of $\mathbb{Q}_\ell$ for some ℓ, we let M_0 denote the maximal unramified extension of $\mathbb{Q}_\ell$ contained in M, and we write I_M for the inertia subgroup of G_M. If R is a local ring we write $\mathfrak{m}_R$ for the maximal ideal of R.

Let p be a prime number. Let K be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$ and residue field k. Fix a uniformiser π of K, let $E(u)$ denote the minimal polynomial of π over K_0, and set $e = \deg E(u)$. We also fix an algebraic closure $\overline{K}$ of K. The ring of Witt vectors $W(k)$ is the ring of integers in K_0.

Our representations of G_K will have coefficients in subfields of $\overline{\mathbb{Q}}_p$, another fixed algebraic closure of $\mathbb{Q}_p$, whose residue field we denote $\overline{\mathbb{F}}_p$. Let E be a finite extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ and containing the image of every embedding into $\overline{\mathbb{Q}}_p$ of the unramified quadratic extension of K; let $\mathcal{O}_E$ be the ring of integers in E, with uniformiser ϖ and residue field $k_E \subset \overline{\mathbb{F}}_p$.

We write $\mathrm{Art}_K\colon K^\times \to W_K^{\mathrm{ab}}$ for the isomorphism of local class field theory, normalised so that uniformisers correspond to geometric Frobenius elements. For each $\lambda \in \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$ we define the fundamental character ω_λ corresponding to λ to be the composite

$$I_K \longrightarrow W_K^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_K^{-1}} \mathcal{O}_K^\times \longrightarrow k^\times \xrightarrow{\lambda} \overline{\mathbb{F}}_p^\times.$$

We fix a compatible system of p^nth roots of π: that is, we set $\pi_0 = \pi$ and for all $n > 0$ we fix a choice of π_n satisfying $\pi_n^p = \pi_{n-1}$. Define $K_\infty = \cup_{n=0}^\infty K(\pi_n)$.

1.1.2. Hodge–Tate weights If W is a de Rham representation of G_K over $\overline{\mathbb{Q}}_p$ and κ is an embedding $K \hookrightarrow \overline{\mathbb{Q}}_p$ then the multiset $\mathrm{HT}_\kappa(W)$ of Hodge–Tate weights of W with respect to κ is defined to contain the integer i with multiplicity

$$\dim_{\overline{\mathbb{Q}}_p}(W \otimes_{\kappa,K} \widehat{\overline{K}}(-i))^{G_K},$$

with the usual notation for Tate twists. (Here $\widehat{\overline{K}}$ is the completion of $\overline{K}$.) Thus for example $\mathrm{HT}_\kappa(\varepsilon) = \{1\}$, where ε is the cyclotomic character. We will refer to the elements of $\mathrm{HT}_\kappa(W)$ as the "κ-labeled Hodge–Tate weights of W", or simply as the "κ-Hodge–Tate weights of W".

1.1.3. p-adic period rings Define $\mathfrak{S} = W(k)[[u]]$. The ring $\mathfrak{S}$ is equipped with a Frobenius endomorphism φ via $u \mapsto u^p$ along with the natural Frobenius on $W(k)$.

We denote by S the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. Let $\mathrm{Fil}^r S$ be the closure in S of the ideal generated by $E(u)^i/i!$ for $i \geq r$. Write $S_{K_0} = S[1/p]$ and $\mathrm{Fil}^r S_{K_0} = (\mathrm{Fil}^r S)[1/p]$. There is a unique Frobenius map $\varphi\colon S \to S$ which extends the Frobenius on $\mathfrak{S}$. We write N_S for the K_0-linear derivation on S_{K_0} such that $N_S(u) = -u$.

2. Kisin modules attached to pseudo-Barsotti–Tate representations

2.1. Finer filtrations on Breuil modules Let V be a d-dimensional E-vector space with continuous E-linear G_K-action which makes V into a crystalline representation of G_K. Let $D := D_{\mathrm{cris}}(V)$ be the filtered φ-module associated to V by Fontaine [Fon94]. Recall that we have assumed that the coefficient field E contains the image of every embedding of K into $\overline{\mathbb{Q}}_p$. Since V has an E-linear structure, it turns out that the filtration on the Breuil module $\mathcal{D}$ attached to D in [Bre97, §6] can be endowed with finer layers that encode the information of the κ-labeled Hodge–Tate weights of D. As we will see in later subsections, these finer filtrations play a crucial role in understanding the structure of Kisin modules ([Kis06]) associated to pseudo-Barsotti–Tate representations (Theorem 2.4.1). In this subsection, we construct such finer filtrations and study their basic properties.

2.1.1. Let $f = [K_0 : \mathbb{Q}_p]$ and recall that $e = [K : K_0]$. Fix an element $\kappa_0 \in \mathrm{Hom}_{\mathbb{Q}_p}(K_0, E)$, and recursively define $\kappa_i \in \mathrm{Hom}_{\mathbb{Q}_p}(K_0, E)$ for $i \in \mathbb{Z}$ so that $\kappa_{i+1}^p \equiv \kappa_i \pmod p$. Then the distinct elements of $\mathrm{Hom}_{\mathbb{Q}_p}(K_0, E)$ are $\kappa_0, \dots, \kappa_{f-1}$, while $\kappa_f = \kappa_0$. Now we label the elements of $\mathrm{Hom}_{\mathbb{Q}_p}(K, E)$ as

$$\mathrm{Hom}_{\mathbb{Q}_p}(K, E) = \{\kappa_{ij} \,:\, i = 0, \dots, f-1,\ j = 0, \dots, e-1\}$$

in any manner so that $\kappa_{ij}|_{K_0} = \kappa_i$.

Let $\varepsilon_i \in W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ be the unique idempotent element such that $(x \otimes 1)\varepsilon_i = (1 \otimes \kappa_i(x))\varepsilon_i$ for all $x \in W(k)$. Then we have $\varepsilon_i(W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E) \simeq W(k) \otimes_{W(k),\kappa_i} \mathcal{O}_E$.

Write $K_{0,E} = K_0 \otimes_{\mathbb{Q}_p} E$ and $K_E = K \otimes_{\mathbb{Q}_p} E$. We have a natural decomposition of rings $K_{0,E} = \prod_{i=0}^{f-1} \varepsilon_i(K_{0,E}) \simeq \prod_{i=0}^{f-1} E_i$ where $E_i = \varepsilon_i(K_{0,E}) \simeq K_0 \otimes_{K_0,\kappa_i} E$. Similarly we have $K_E = \prod_{i,j} E_{ij}$ with $E_{ij} = K \otimes_{K,\kappa_{ij}} E$. We will sometimes identify an element $x \in E$ with an element of E_i via the map $x \mapsto 1 \otimes x$, and similarly for E_{ij}.

2.1.2. Recall that D is a finite free $K_{0,E}$-module of rank d (see [BM02, §3.1], for example), so that $D_K := K \otimes_{K_0} D$ is a finite free K_E-module of rank d. In particular we have natural decompositions $D = \oplus_{i=0}^{f-1} D_i$ with $D_i = \varepsilon_i(D) \simeq D \otimes_{K_{0,E}} E_i$, and $D_K = \oplus_{i,j} D_{K,ij}$ with $D_{K,ij} = D \otimes_{K_E} E_{ij}$. Note that $\mathrm{Fil}^m D_K$ and $\mathrm{gr}^m D_K$ have similar decompositions, though of course they need not be free. Given a multi-set of integers $\{m_{ij}\}$ with $0 \le i \le f-1$ and $0 \le j \le e-1$, we define

$$\mathrm{Fil}^{\{m_{ij}\}} D_K := \bigoplus_{i=0}^{f-1} \bigoplus_{j=0}^{e-1} \mathrm{Fil}^{m_{ij}} D_{K,ij} \subset D_K.$$

If $m_{ij} = m$ for all i, j then of course $\mathrm{Fil}^{\{m_{ij}\}} D_K = \mathrm{Fil}^m D_K$.

2.1.3. Let A be an $\mathfrak{S}$-algebra. We write $A_{\mathcal{O}_E} := A \otimes_{\mathbb{Z}_p} \mathcal{O}_E$. This is a $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$-algebra, so that $A_{\mathcal{O}_E} \simeq \prod_{i=0}^{f-1} A_{\mathcal{O}_E,i}$ with $A_{\mathcal{O}_E,i} = \varepsilon_i(A_{\mathcal{O}_E}) \simeq A \otimes_{W(k),\kappa_i} \mathcal{O}_E$. Similarly, we write $A_E := A \otimes_{\mathbb{Z}_p} E$, so that we have $A_E \simeq \prod_{i=0}^{f-1} A_{E,i}$ with $A_{E,i} := A \otimes_{W(k),\kappa_i} E$.

Let us write ι for the isomorphism $\mathfrak{S}_{\mathcal{O}_E} \simeq \prod_{i=0}^{f-1} \mathfrak{S}_{\mathcal{O}_E,i}$ and ι_i for the projection $\mathfrak{S}_{\mathcal{O}_E} \twoheadrightarrow \mathfrak{S}_{\mathcal{O}_E,i}$. Note that $\mathfrak{S}_{\mathcal{O}_E,i} = \mathfrak{S} \otimes_{W(k),\kappa_i} \mathcal{O}_E \simeq \mathcal{O}_{E_i}[\![u]\!]$, where $\mathcal{O}_{E_i}$ denotes the ring of integers in E_i.

Recall that π is a fixed uniformiser of K, with minimal polynomial $E(u)$ over K_0. Write $\pi_{ij} = \kappa_{ij}(\pi) \in E$. For each κ_i, we define $E^{\kappa_i}(u) = \prod_{j=0}^{e-1}(u - \pi_{ij})$ in $E[u]$, so that $E^{\kappa_i}(u)$ is just the polynomial obtained by acting by κ_i on the coefficients of $E(u)$; note that identifying E_i with E will identify $\iota_i(E(u))$ with $E^{\kappa_i}(u)$.

Let f_π be the map $S \to \mathcal{O}_K$ induced by $u \mapsto \pi$. We will also write f_π for the map $f_\pi \otimes_{\mathbb{Z}_p} E : S_E \to (\mathcal{O}_K)_E$. We have surjections $\iota_{ij} : (\mathcal{O}_K)_E \to K_E \to E_{ij}$ for all i, j, and composing with f_π gives E-linear maps $f_{ij} := \iota_{ij} \circ f_\pi : S_E \to E_{ij}$. Restricting f_{ij} to $\mathfrak{S}_{\mathcal{O}_E}$ gives an $\mathcal{O}_E$-linear surjection $\mathfrak{S}_{\mathcal{O}_E} \to \mathcal{O}_{E_{ij}}$ that we also denote f_{ij}. (Here $\mathcal{O}_{E_{ij}}$ denotes the ring of integers in E_{ij}.) Set $\mathrm{Fil}^1_{ij} S_E := \ker(f_{ij})$ and $\mathrm{Fil}^1 \mathfrak{S}_{\mathcal{O}_E} := \mathfrak{S}_{\mathcal{O}_E} \cap \mathrm{Fil}^1_{ij} S_E$. Let $E_{ij}(u)$ be the unique element in $\mathfrak{S}_{\mathcal{O}_E}$ such that $\iota_\ell(E_{ij}(u)) = u - \pi_{ij}$ if $\ell = i$ and $\iota_\ell(E_{ij}(u)) = 1$ if $\ell \neq i$. We see that $E(u) \otimes 1 = \prod_{i,j} E_{ij}(u)$ in $\mathfrak{S}_{\mathcal{O}_E}$.

2.1.4 Lemma.

1. $E_{ij}(u) \in \mathrm{Fil}^1_{ij} S_E$.

2. $\bigcap_{ij} \mathrm{Fil}^1_{ij} S_E = \mathrm{Fil}^1 S \otimes_{\mathbb{Z}_p} E$.

3. $\mathrm{Fil}^1_{ij} \mathfrak{S}_{\mathcal{O}_E} = E_{ij}(u)\mathfrak{S}_{\mathcal{O}_E}$.

Proof. Note that $S_{E,i} \subset K_0[\![u]\!] \otimes_{K_0,\kappa_i} E \simeq E_i[\![u]\!]$, so that elements in $S_{E,i}$ can be regarded as power series with E_i-coefficients.

Unwinding the definitions, we see that $f_{ij} : S_E \to E_{ij}$ is the E-linear map sending u to π_{ij}. In fact the map f_{ij} can be factored as

$$f_{ij} : S_E \simeq \prod_{i'=0}^{f-1} S_{E,i'} \to \prod_{i'=0}^{f-1} (\mathcal{O}_K)_{E,i'} \to E_{ij}$$

where the second map is the product of the maps sending u to $\pi \otimes 1$, while the third map is the map $\mathcal{O}_K \otimes_{W(k),\kappa_i} E \twoheadrightarrow \mathcal{O}_K \otimes_{\mathcal{O}_K,\kappa_{ij}} E$ on the ith factor and is zero on the remaining factors. Now (1) is clear, and moreover an element $h \in S_{E,i}$ lies in $\mathrm{Fil}^1_{ij} S_E$ if and only if $h = (u - \pi_{ij})h'$ for some $h' \in E_i[\![u]\!]$. (We caution the reader that h' need not be in $S_{E,i}$.) Then (2) and (3) follow easily. □

For $0 \leq \ell \leq f-1$, define $\mathrm{Fil}^1_{ij} S_{E,\ell} := \mathrm{Fil}^1_{ij} S_E \cap S_{E,\ell}$ and $\mathrm{Fil}^1_{ij} \mathfrak{S}_{E,\ell} := \mathrm{Fil}^1_{ij} \mathfrak{S}_{\mathcal{O}_E} \cap \mathfrak{S}_{\mathcal{O}_{E,\ell}}$. Note that unless $\ell = i$ we have $\mathrm{Fil}^1_{ij} S_{E,\ell} = S_{E,\ell}$, and similarly for $\mathfrak{S}_{E,\ell}$.

2.1.5. Let N be the $W(k)$-linear differential operator on $\mathfrak{S}$ such that $N(u) = -u$, and extend N to $\mathfrak{S}_{\mathcal{O}_E}$ in the unique E-linear way. It is easy to check that N is compatible with ι in the following sense: if N is the $\mathcal{O}_E$-linear differential operator on $\mathcal{O}_{E,i}[\![u]\!]$ such that $N(u) = -u$, then $\iota_i(N(x)) = N(\iota_i(x))$ for any $x \in \mathfrak{S}_{\mathcal{O}_E}$.

Let $\mathcal{D} := S \otimes_{W(k)} D$ be the Breuil module attached to D (see e.g. [Bre97, §6]). We have a natural isomorphism $D_K \simeq \mathcal{D} \otimes_{S,f_\pi} \mathcal{O}_K$; therefore, we also have a natural isomorphism $D_{K,ij} \simeq \mathcal{D} \otimes_{S_E,f_{ij}} E_{ij}$. We again denote the projection $\mathcal{D} \twoheadrightarrow D_K$ by f_π, and the projection $\mathcal{D} \twoheadrightarrow D_K \twoheadrightarrow D_{K,ij}$ by f_{ij}.

2.1.6 Remark. Since $D_K = \oplus_{i,j} D_{K,ij}$, it is easy to check that $f_\pi = \oplus_{i,j} f_{ij}$. This is a useful fact in the $\mathbb{Q}_p$-rational theory. Unfortunately, this fails in general in the integral theory (unless $e = 1$), essentially because the idempotents in $K \otimes_{\mathbb{Q}_p} E$ may not be contained in $\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}_E$. While it is not hard to see that $f_{ij}(\mathfrak{S}_{\mathcal{O}_E}) = \mathcal{O}_{E_{ij}}$, the map $f_\pi : \mathfrak{S}_{\mathcal{O}_E} \to \oplus_{i,j} \mathcal{O}_{E_{ij}}$ in general is not surjective: indeed $f_\pi(\mathfrak{S}_{\mathcal{O}_E}) = \mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}_E$, and this need not equal $\oplus_{i,j} \mathcal{O}_{E_{ij}}$ unless $e = 1$.

2.1.7. Let $\{m_{ij}\}$ be a collection of integers indexed by $i = 0, \dots f-1$ and $j = 0, \dots, e-1$. We recursively define a filtration $\mathrm{Fil}^{\{m_{ij}\}} \mathcal{D} \subseteq \mathcal{D}$. We first set $\mathrm{Fil}^{\{m_{ij}\}} \mathcal{D} = \mathcal{D}$ if $m_{ij} \le 0$ for all i, j. Then define

$$\mathrm{Fil}^{\{m_{ij}\}} \mathcal{D} = \{x \in \mathcal{D} \ : \ f_{ij}(x) \in \mathrm{Fil}^{m_{ij}} D_{K,ij} \text{ for all } i, j, \text{ and } N(x) \in \mathrm{Fil}^{\{m_{ij}-1\}} \mathcal{D}\}.$$

This is a direct generalization of the usual filtration $\mathrm{Fil}^m \mathcal{D}$ on $\mathcal{D}$ defined in [Bre97, §6], and evidently $\mathrm{Fil}^m \mathcal{D} = \mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}$ when $m_{ij} = m$ for all i, j.

2.1.8. Let us discuss a slight variation of the filtration $\mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}$ defined in 2.1.7. Recall that $S_E \simeq \prod_{\ell=0}^{f-1} S_{E,\ell}$. Then $\mathcal{D} = \oplus_{\ell=0}^{f-1} \mathcal{D}_\ell$ with $\mathcal{D}_\ell = \mathcal{D} \otimes_{S_E} S_{E,\ell}$. We also have $\mathrm{Fil}^{\{m_{ij}\}} \mathcal{D} = \oplus_{\ell=0}^{f-1} \mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}_\ell$ with $\mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}_\ell = \mathrm{Fil}^{\{m_{ij}\}} \mathcal{D} \otimes_{S_E} S_{E,\ell}$. Note that $N(\mathcal{D}_\ell) \subset \mathcal{D}_\ell$ (because N is E-linear on D). We see easily that $\mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}_\ell$ depends only on the $m_{\ell j}$ for $0 \le j \le e-1$, and we can define $\mathrm{Fil}^{\{m_{\ell,0},\dots,m_{\ell,e-1}\}} \mathcal{D}_\ell := \mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}_\ell$. Note that $\mathrm{Fil}^{\{m_{i,0},\dots,m_{i,e-1}\}} \mathcal{D}_i$ also has the recursive description

$$\{x \in \mathcal{D}_i \ : \ f_{ij}(x) \in \mathrm{Fil}^{m_{ij}} D_{K,ij} \text{ for all } j, \text{ and } N(x) \in \mathrm{Fil}^{\{m_{i,0}-1,\dots,m_{i,e-1}-1\}} \mathcal{D}_i\}.$$

The following proposition summarizes some useful properties of $\mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}$ and $\mathrm{Fil}^{\{m_{i,0},\dots,m_{i,e-1}\}} \mathcal{D}_i$.

2.1.9 Proposition. *With notation as in 2.1.7, the filtration* $\mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}$ *has the following properties.*

1. *If* $m_{ij} \ge m'_{ij}$ *for all* i, j *then* $\mathrm{Fil}^{\{m_{ij}\}} \mathcal{D} \subseteq \mathrm{Fil}^{\{m'_{ij}\}} \mathcal{D}$.

2. $N(\mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}) \subseteq \mathrm{Fil}^{\{m_{ij}-1\}} \mathcal{D}$.

3. $\mathrm{Fil}^{\{m_{ij}\}}\mathcal{D}$ *is an* S_E*-submodule of* $\mathcal{D}$.

4. $f_\pi(\mathrm{Fil}^{\{m_{ij}\}}\mathcal{D}) = \oplus_{i,j}\mathrm{Fil}^{m_{ij}} D_{K,ij}$.

5. $(\mathrm{Fil}^r S)\,\mathrm{Fil}^{\{m_{ij}-r\}}\mathcal{D} \subseteq \mathrm{Fil}^{\{m_{ij}\}}\mathcal{D}$ *for all* $r \geq 0$.

6. *We have* $(\mathrm{Fil}^1_{i'j'} S_E)\,\mathrm{Fil}^{\{m_{ij}\}}\mathcal{D} \subseteq \mathrm{Fil}^{\{m'_{ij}\}}\mathcal{D}$ *with* $m'_{i'j'} = m_{i'j'}+1$ *and* $m'_{ij} = m_{ij}$ *if* $(i,j) \neq (i',j')$.

7. *Suppose that* $E(u)x \in \mathrm{Fil}^{\{m_{ij}\}}\mathcal{D}$ *with* $x \in \mathcal{D}$*. Then* $N^\ell(x) \in \mathrm{Fil}^{\{m_{ij}-1-\ell\}}\mathcal{D}$ *for all* $\ell \geq 0$.

8. *Suppose that* $E_{i'j'}(u)x \in \mathrm{Fil}^{\{m_{ij}\}}\mathcal{D}$ *with* $x \in \mathcal{D}$*. Then* $x \in \mathrm{Fil}^{\{m'_{ij}\}}\mathcal{D}$ *with* $m'_{i'j'} = m_{i'j'} - 1$ *and* $m'_{ij} = m_{ij}$ *if* $(i,j) \neq (i',j')$.

The analogous statements hold for the filtration $\mathrm{Fil}^{\{m_{i,0},\dots,m_{i,e-1}\}}\mathcal{D}_i$ *of 2.1.8, replacing* $\mathcal{D}$ *by* $\mathcal{D}_i$ *and* S_E *by* $S_{E,i}$*. (Note that we have defined* $\mathrm{Fil}^1_{i'j'} S_{E,i}$ *below Lemma 2.1.4.)*

Proof. The proofs for $\mathrm{Fil}^{\{m_{i,0},\dots,m_{i,e-1}\}}\mathcal{D}_i$ are essentially the same as those for $\mathrm{Fil}^{m_{ij}}\mathcal{D}$, so we will concern ourselves exclusively with the latter case.

(1) This is a straightforward induction on $m' = \max_{i,j}\{m'_{ij}\}$, with the case $m' \leq 0$ serving as the base case. Let us suppose that the statement is true for $m'-1$ and consider the situation for m'. If $x \in \mathrm{Fil}^{\{m_{ij}\}}\mathcal{D}$, we certainly have $f_{ij}(x) \in \mathrm{Fil}^{m_{ij}} D_{K,ij} \subseteq \mathrm{Fil}^{m'_{ij}} D_{K,ij}$, so it suffices to show that $N(x) \in \mathrm{Fil}^{\{m'_{ij}-1\}}\mathcal{D}$. But $N(x) \in \mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$, and $\mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D} \subseteq \mathrm{Fil}^{\{m'_{ij}-1\}}\mathcal{D}$ by induction, which completes the proof.

(2) This is immediate from the definition.

For each of items (3)–(6), we proceed by induction on $m = \max_{i,j}\{m_{ij}\}$. These statements are all trivial if $m \leq 0$, except for (6) where we take $m < 0$ as the base case. For each of these items in turn, let us suppose that the statement is true for $m-1$ and consider the situation for m.

(3) Pick $s \in S_E$ and $x \in \mathrm{Fil}^{\{m_{ij}\}}\mathcal{D}$. It is clear from the definitions that $f_{ij}(sx) = f_{ij}(s)f_{ij}(x) \in \mathrm{Fil}^{m_{ij}} D_{K,ij}$ for all i,j, so it remains to show that $N(sx) = N(s)x + sN(x)$ is inside $\mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$. Since $N(x) \in \mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$, and $\mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$ is an S_E-module by induction, we see that $sN(x) \in \mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$. For the other term, we know from (1) that $\mathrm{Fil}^{\{m_{ij}\}}\mathcal{D} \subseteq \mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$. Therefore x, and hence $N(s)x$, is in $\mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$ (again using the induction hypothesis).

We now prove (6), (5), and (4) in that order.

(6) Let $x = sy$ with $s \in \mathrm{Fil}^1_{i'j'} S_E$ and $y \in \mathrm{Fil}^{\{m_{ij}\}}\mathcal{D}$. We note that $f_{ij}(sy) \in \mathrm{Fil}^{m_{ij}} D_{K,ij}$ if $(i,j) \neq (i',j')$, and $f_{i'j'}(sy) = 0 \in \mathrm{Fil}^{m_{i'j'}+1} D_{K,i'j'}$. It remains to show that $N(x) = N(s)y + sN(y)$ lies in $\mathrm{Fil}^{\{m'_{ij}-1\}}\mathcal{D}$. We have

$N(s)y \in \mathrm{Fil}^{\{m_{ij}\}}\mathcal{D} \subseteq \mathrm{Fil}^{\{m'_{ij}-1\}}\mathcal{D}$ by (3) and (1). On the other hand $N(y)$ is in $\mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$ by definition, and so by the induction hypothesis we have $sN(y) \in \mathrm{Fil}^{\{m'_{ij}-1\}}\mathcal{D}$. This completes the induction.

(5) This follows by an argument essentially identical to the proof of (6), using $f_{ij}(s) = 0$ for all i, j if $s \in \mathrm{Fil}^r S$, and that $N(\mathrm{Fil}^r S) \subseteq \mathrm{Fil}^{r-1} S$ for $r \geq 1$.

(4) Pick $x = (x_{ij}) \in \oplus_{i,j}\mathrm{Fil}^{m_{ij}} D_{K,ij}$. Since $x_{ij} \in \mathrm{Fil}^{m_{ij}} D_{K,ij} \subseteq \mathrm{Fil}^{m_{ij}-1} D_{K,ij}$, by induction there exists $\hat{x} \in \mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$ such that $f_{ij}(\hat{x}) = x_{ij}$ for all i, j. Note that $N(E(u))$ is relatively prime to $E(u)$, so there exists $R(u) \in K_0[u]$ such that $E(u)$ divides $1 + R(u)N(E(u))$. Set $H(u) := R(u)E(u)$ and write $N(H(u)) + 1 = E(u)Q(u)$. Define

$$\hat{y} := \sum_{\ell=0}^{m-1} \frac{H(u)^\ell N^\ell(\hat{x})}{\ell!}.$$

It is easy to check that $f_{ij}(\hat{y}) = f_{ij}(\hat{x}) = x_{ij}$ for all i, j. Now we have

$$\begin{aligned} N(\hat{y}) &= N(\hat{x}) + \sum_{\ell=1}^{m-1}\frac{1}{\ell!}\Big(\ell H(u)^{\ell-1}N(H(u))N^\ell(\hat{x}) + H(u)^\ell N^{\ell+1}(\hat{x})\Big) \\ &= \frac{H(u)^{m-1}}{(m-1)!}N^m(\hat{x}) + \sum_{\ell=1}^{m-1}\frac{(1+N(H(u)))H(u)^{\ell-1}N^\ell(\hat{x})}{(\ell-1)!} \\ &= \frac{H(u)^{m-1}}{(m-1)!}N^m(\hat{x}) + \sum_{\ell=1}^{m-1}\frac{Q(u)R(u)^{\ell-1}E(u)^\ell N^\ell(\hat{x})}{(\ell-1)!} \end{aligned}$$

Since $\hat{x} \in \mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$, repeated applications of (2) and (5) show that $E(u)^\ell N^\ell(\hat{x})$ is in $\mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$ for all ℓ. Since $E(u)$ divides $H(u)$ we also have $H(u)^{m-1}\mathcal{D} \subseteq \mathrm{Fil}^{\{m-1\}}\mathcal{D} \subseteq \mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$. By (5) we conclude that $\hat{y} \in \mathrm{Fil}^{\{m_{ij}\}}\mathcal{D}$ as required, and the induction is complete.

(7) First we show that $f_{ij}(N^\ell(x)) \in \mathrm{Fil}^{\{m_{ij}-1-\ell\}} D_{K,ij}$ for all i, j, proceeding by induction on ℓ. Note that $N(E(u)x) = N(E(u))x + E(u)N(x)$ lies in $\mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$ and so $f_{ij}(N(E(u))x) \in \mathrm{Fil}^{m_{ij}-1} D_{K,ij}$ for all i, j. Since $f_{ij}(N(E(u))) \neq 0$ we see that $f_{ij}(x) \in \mathrm{Fil}^{m_{ij}-1} D_{K,ij}$ for all i, j. This proves the case $\ell = 0$. In general, we have

$$N^{\ell+1}(E(u)x) = \sum_{n=0}^{\ell+1}\binom{\ell+1}{n}N^{\ell+1-n}(E(u))N^n(x) \in \mathrm{Fil}^{\{m_{ij}-1-\ell\}}\mathcal{D},$$

using (2) to deduce the containment. After applying f_{ij} to both sides of the above equation, each term on the right-hand side with $n \neq \ell$ will be known to belong to $\mathrm{Fil}^{\{m_{ij}-1-\ell\}} D_{K,ij}$. For $n < \ell$ this follows from the induction hypothesis, while the term with $n = \ell + 1$ vanishes because $f_{ij}(E(u)) = 0$. We conclude that the same is true for the term with $n = \ell$, and so (again using that $f_{ij}(N(E(u))) \neq 0$) we deduce that $f_{ij}(N^{\ell}(x)) \in \mathrm{Fil}^{\{m_{ij}-1-\ell\}} D_{K,ij}$.

Now, the claim $N^{\ell}(x) \in \mathrm{Fil}^{\{m_{ij}-1-\ell\}} \mathcal{D}$ is automatic for $\ell \geq \max_{i,j}\{m_{ij}\}$. Suppose that we have $N^{\ell}(x) \in \mathrm{Fil}^{\{m_{ij}-1-\ell\}} \mathcal{D}$, or in other words $N(N^{\ell-1}(x)) \in \mathrm{Fil}^{\{m_{ij}-1-\ell\}} \mathcal{D}$. Since we also have $f_{ij}(N^{\ell-1}(x)) \in \mathrm{Fil}^{\{m_{ij}-1-(\ell-1)\}} D_{K,ij}$ from the previous paragraph, we deduce that $N^{\ell-1}(x) \in \mathrm{Fil}^{\{m_{ij}-1-(\ell-1)\}} \mathcal{D}$. Now (7) follows by (reverse) induction on ℓ.

(8) Again we proceed by induction on $m = \max_{ij}\{m_{ij}\}$. Since $E_{i'j'}(u)x \in \mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}$, we have $f_{ij}(E_{i'j'}(u)x) \in \mathrm{Fil}^{m_{ij}} D_{K,ij}$. Then $f_{ij}(x) \in \mathrm{Fil}^{m_{ij}} D_{K,ij}$ for any $(i, j) \neq (i', j')$ because $f_{ij}(E_{i'j'}(u)) \neq 0$ in that case. Note that

$$N(E_{i'j'}(u)x) = N(E_{i'j'}(u))x + E_{i'j'}(u)N(x) \in \mathrm{Fil}^{\{m_{ij}-1\}} \mathcal{D}. \tag{2.1.10}$$

Applying $f_{i'j'}$ to (2.1.10) and noting that $f_{i'j'}(N(E_{i'j'}(u))) \neq 0$, we see that $f_{i'j'}(x) \in \mathrm{Fil}^{m_{i'j'}-1} D_{K,i'j'}$. Thus $f_{ij}(x) \in \mathrm{Fil}^{m'_{ij}} D_{K,ij}$ for all i, j. It remains to show that $N(x) \in \mathrm{Fil}^{\{m'_{ij}-1\}} \mathcal{D}$. Note that $E_{i'j'}(u)x \in \mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}$ implies that $E(u)x \in \mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}$, and (7) with $\ell = 0$ shows that that $x \in \mathrm{Fil}^{\{m_{ij}-1\}} \mathcal{D}$. Hence $N(E_{i'j'}(u))x \in \mathrm{Fil}^{\{m_{ij}-1\}} \mathcal{D}$ and then (2.1.10) gives $E_{i'j'}(u)N(x) \in \mathrm{Fil}^{\{m_{ij}-1\}} \mathcal{D}$. By induction, we see that $N(x) \in \mathrm{Fil}^{\{m'_{ij}-1\}} \mathcal{D}$, and we are done. □

2.1.11 Remark. An argument analogous to the proof of Proposition 2.1.9(1) shows that $\mathrm{Fil}^{\{m_{ij}\}} \mathcal{D} = \mathrm{Fil}^{\{\max(m_{ij},0)\}} \mathcal{D}$.

The next proposition shows that the filtration $\mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}$ is essentially characterized by certain of the properties listed in Proposition 2.1.9.

2.1.12 Proposition. *Fix elements $n_{ij} \in \mathbb{Z} \cup \{\infty\}$ indexed by $i = 0, \ldots, f-1$ and $j = 0, \ldots, e-1$, and let $\mathcal{S}$ be the set of collections of integers $\{m_{ij}\}$ with $m_{ij} \leq n_{ij}$ for all i, j. Suppose that we have another filtration of $\mathcal{D}$ by additive subsets $\widetilde{\mathrm{Fil}}^{\{m_{ij}\}}\mathcal{D}$, defined for all $\{m_{ij}\} \in \mathcal{S}$ and satisfying the following properties.*

1. *$\widetilde{\mathrm{Fil}}^{\{m_{ij}\}}\mathcal{D} = \mathcal{D}$ if $m_{ij} \leq 0$ for all i, j.*

2. *For each $\{m_{ij}\} \in \mathcal{S}$ there exists $r > 0$ such that $(\mathrm{Fil}^r S)\mathcal{D} \subseteq \widetilde{\mathrm{Fil}}^{\{m_{ij}\}}\mathcal{D}$.*

3. *The filtration $\widetilde{\mathrm{Fil}}^{\{m_{ij}\}}\mathcal{D}$ satisfies properties (2) and (4) of Proposition 2.1.9, as well as $E(u) \cdot \widetilde{\mathrm{Fil}}^{\{m_{ij}-1\}}\mathcal{D} \subseteq \widetilde{\mathrm{Fil}}^{\{m_{ij}\}}\mathcal{D}$, for all $\{m_{ij}\} \in \mathcal{S}$.*

Then for all $\{m_{ij}\} \in \mathcal{S}$ we have $\widetilde{\mathrm{Fil}}^{\{m_{ij}\}}\mathcal{D} = \mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}$.

The analogous statement holds for the filtration $\mathrm{Fil}^{\{m_{i,0},\ldots,m_{i,e-1}\}} \mathcal{D}_i$ of 2.1.8, replacing $\mathcal{D}$ by $\mathcal{D}_i$.

Proof. The proof for $\mathcal{D}_i$ is the same as the proof for $\mathcal{D}$, so again we concentrate on the latter case. As usual we proceed by induction on $m = \max_{ij}\{m_{ij}\}$, with the base case $m = 0$ coming from (1). Suppose that the claim holds for $m-1$, and consider the situation for m.

Pick $x \in \widetilde{\mathrm{Fil}}^{\{m_{ij}\}}\mathcal{D}$. By hypotheses 2.1.9(4) and 2.1.9(2) respectively, we have $f_{ij}(x) \in \mathrm{Fil}^{m_{ij}} D_{K,ij}$ and $N(x) \in \widetilde{\mathrm{Fil}}^{\{m_{ij}-1\}}\mathcal{D}$. By induction $\widetilde{\mathrm{Fil}}^{\{m_{ij}-1\}}\mathcal{D} = \mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$, so $x \in \mathrm{Fil}^{\{m_{ij}\}}\mathcal{D}$ by definition, and we have shown that $\widetilde{\mathrm{Fil}}^{\{m_{ij}\}}\mathcal{D} \subseteq \mathrm{Fil}^{\{m_{ij}\}}\mathcal{D}$.

Conversely pick $x \in \mathrm{Fil}^{\{m_{ij}\}}\mathcal{D}$. By hypothesis 2.1.9(4) there exists $y \in \widetilde{\mathrm{Fil}}^{\{m_{ij}\}}\mathcal{D}$ such that $f_{ij}(y) = f_{ij}(x)$ for all i, j. We can (and do) modify y so that $x = y + E(u)z$ with $z \in \mathcal{D}$ and y still in $\widetilde{\mathrm{Fil}}^{\{m_{ij}\}}\mathcal{D}$. This is possible because any $s \in \mathrm{Fil}^1 S$ can be written as $E(u)s' + s''$ with $s' \in S[1/p]$ and $s'' \in \mathrm{Fil}^r S$ for $r \gg 0$; now use hypothesis (2).

Now $E(u)z = x - y \in \mathrm{Fil}^{\{m_{ij}\}}\mathcal{D}$ (from the second paragraph of the proof), so $z \in \mathrm{Fil}^{\{m_{ij}-1\}}\mathcal{D}$ by Proposition 2.1.9(7). Then $z \in \widetilde{\mathrm{Fil}}^{\{m_{ij}-1\}}\mathcal{D}$ by induction, and hypothesis (3) with $r = 1$ shows that $E(u)z \in \widetilde{\mathrm{Fil}}^{\{m_{ij}\}}\mathcal{D}$. We conclude that $x = y + E(u)z \in \widetilde{\mathrm{Fil}}^{\{m_{ij}\}}\mathcal{D}$, as desired. □

2.2. Some general facts about integral *p*-adic Hodge theory Suppose in this section that the Hodge–Tate weights of V lie in the interval $[0, r]$. Let $T \subset V$ be a G_K-stable $\mathcal{O}_E$-lattice, and let $\mathfrak{M}$ be the Kisin module attached to T by the theory of [Kis06, Liu08]; see the statements of [GLS14, Def. 3.1, Thm. 3.2, Def. 3.3, Prop. 3.4] for a concise summary of the definitions and properties that we will need. The object $\mathfrak{M}$ is a finite free $\mathfrak{S}_{\mathcal{O}_E}$-module with rank $d = \dim_E V$, together with an $\mathcal{O}_E$-linear φ-semilinear map $\varphi : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of $1 \otimes \varphi : \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$. As in [GLS14, §3] we use the contravariant functor $T_{\mathfrak{S}}$ to associate Galois representations to Kisin modules.

Set $\mathfrak{M}^* = \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$, which by [Liu08, Cor. 3.2.3] (or [GLS14, Thm. 3.2(4)]) we can view as a subset of $\mathcal{D}$. Define

$$\mathrm{Fil}^{\{m_{ij}\}}\mathfrak{M}^* := \mathfrak{M}^* \cap \mathrm{Fil}^{\{m_{ij}\}}\mathcal{D},$$

and $M_{K,ij} = f_{ij}(\mathfrak{M}^*) \subset D_{K,ij}$. Similarly we define $\mathfrak{M}^*_i = \mathfrak{M}^* \otimes_{\mathfrak{S}_E} \mathfrak{S}_{E,i}$ and $\mathrm{Fil}^{\{m_{i,0},\dots,m_{i,e-1}\}}\mathfrak{M}^*_i := \mathfrak{M}^*_i \cap \mathrm{Fil}^{\{m_{i,0},\dots,m_{i,e-1}\}}\mathcal{D}_i$.. Note that $1 \otimes \varphi : \mathfrak{M}^* \to \mathfrak{M}$ is an $\mathfrak{S}$-linear map. By [Liu08, Cor. 3.2.3] one also has that

$$\mathrm{Fil}^r \mathfrak{M}^* = \{x \in \mathfrak{M}^* | (1 \otimes \varphi)(x) \in E(u)^r \mathfrak{M}\}. \tag{2.2.1}$$

2.2.2 Lemma. *Let $\{m_{ij}\}$ be non-negative integers indexed by $i = 0, \dots, e-1$ and $j = 0, \dots, f-1$. Fix a pair (i', j') and define $m'_{ij} = m_{ij} + 1$ if $(i, j) = (i', j')$ and $m'_{ij} = m_{ij}$ otherwise. Then we have the following.*

1. *$M_{K,ij}$ is an $\mathcal{O}_E$-lattice inside $D_{K,ij}$.*

2. $\mathrm{Fil}^{\{m_{ij}\}}\,\mathfrak{M}^*/\,\mathrm{Fil}^{\{m'_{ij}\}}\,\mathfrak{M}^*$ *is a finite free $\mathcal{O}_E$-module.*

3. *If* $\mathrm{Fil}^{m_{i'j'}+1}\,D_{K,i'j'} = 0$ *then* $\mathrm{Fil}^{\{m'_{ij}\}}\,\mathfrak{M}^* = E_{i'j'}(u)\,\mathrm{Fil}^{\{m_{ij}\}}\,\mathfrak{M}^*$.

Moreover the natural analogues of (2) and (3) hold for the filtration on $\mathfrak{M}^*_i$.

Proof. (1) Note that $M_K := f_\pi(\mathfrak{M}^*) \subset D_K$ is a full $\mathcal{O}_K$-lattice. Indeed, $M_K \simeq \mathfrak{M}^*/E(u)\mathfrak{M}^*$ is a finite free $\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}_E$-module of rank d because $\mathfrak{M}$ is finite $\mathfrak{S}_{\mathcal{O}_E}$-free of rank d. Then (1) follows quickly from this fact.

As usual the proofs of (2) and (3) will be the same for $\mathfrak{M}^*_i$ as for $\mathfrak{M}^*$, and so we concentrate on the latter case.

(2) It is clear from the definition of $\mathrm{Fil}^{\{m_{ij}\}}\,\mathfrak{M}^*$ that $\mathrm{Fil}^{\{m_{ij}\}}\,\mathfrak{M}^*/\,\mathrm{Fil}^{\{m'_{ij}\}}\,\mathfrak{M}^*$ injects into $\mathrm{Fil}^{\{m_{ij}\}}\,\mathcal{D}/\,\mathrm{Fil}^{\{m'_{ij}\}}\,\mathcal{D}$. By Proposition 2.1.9(6), $\mathrm{Fil}^{\{m_{ij}\}}\,\mathcal{D}/\,\mathrm{Fil}^{\{m'_{ij}\}}\,\mathcal{D}$ is an $S_E/\,\mathrm{Fil}^1_{i'j'}\,S_E = E_{i'j'}$-module. In particular it is p-torsion free, so the same is true of $\mathrm{Fil}^{\{m_{ij}\}}\,\mathfrak{M}^*/\,\mathrm{Fil}^{\{m'_{ij}\}}\,\mathfrak{M}^*$. On the other hand $\mathrm{Fil}^{\{m_{ij}\}}\,\mathfrak{M}^*/\,\mathrm{Fil}^{\{m'_{ij}\}}\,\mathfrak{M}^*$ is an $\mathfrak{S}_{\mathcal{O}_E}/\,\mathrm{Fil}^1_{i'j'}\,\mathfrak{S}_{\mathcal{O}_E} = \mathcal{O}_{E_{i'j'}}$-module. Since it is p-torsion free and finitely generated (note that $\mathfrak{M}^*$ is finite $\mathfrak{S}_{\mathcal{O}_E}$-free and $\mathfrak{S}_{\mathcal{O}_E}$ is noetherian), we see that $\mathrm{Fil}^{\{m_{ij}\}}\,\mathfrak{M}^*/\,\mathrm{Fil}^{\{m'_{ij}\}}\,\mathfrak{M}^*$ is a finite free $\mathcal{O}_E$-module.

(3) Let $\mathcal{S}$ be the set of tuples $\{n_{ij}\}$ with $n_{ij} \le m'_{ij}$ for all i, j. Fix any $r > \max_{i,j}\{m'_{ij}\}$, define

$$\widetilde{\mathrm{Fil}}^{\{m'_{ij}\}}\mathcal{D} = (\mathrm{Fil}^1_{i'j'}\,S_E)\,\mathrm{Fil}^{\{m_{ij}\}}\,\mathcal{D} + (\mathrm{Fil}^r\,S)\mathcal{D},$$

and define $\widetilde{\mathrm{Fil}}^{\{n_{ij}\}}\mathcal{D} = \mathrm{Fil}^{\{n_{ij}\}}\,\mathcal{D}$ for all other tuples $\{n_{ij}\} \in \mathcal{S}$. (It is easy to check using (1) and (5) of Proposition 2.1.9 that the above definition does not depend on the choice of r.) Our first goal is to show that $\mathrm{Fil}^{\{m'_{ij}\}}\,\mathcal{D} = \widetilde{\mathrm{Fil}}^{\{m'_{ij}\}}\mathcal{D}$. This will follow from Proposition 2.1.12 once we can show that

(i) $N(\widetilde{\mathrm{Fil}}^{\{m'_{ij}\}}\mathcal{D}) \subseteq \mathrm{Fil}^{\{m'_{ij}-1\}}\,\mathcal{D}$,

(ii) $f_\pi(\widetilde{\mathrm{Fil}}^{\{m'_{ij}\}}\mathcal{D}) = \bigoplus_{i,j} \mathrm{Fil}^{m'_{ij}}\,D_{K,ij}$,

(iii) $E(u)\cdot\mathrm{Fil}^{\{m'_{ij}-1\}}\,\mathcal{D} \subseteq \widetilde{\mathrm{Fil}}^{\{m'_{ij}\}}\mathcal{D}$.

To check (i), let $s \in \mathrm{Fil}^1_{i'j'}\,S_E$ and $x \in \mathrm{Fil}^{\{m_{ij}\}}\,\mathcal{D}$. Then $N(sx) = N(s)x + sN(x)$. We have $N(s)x \in \mathrm{Fil}^{\{m_{ij}\}}\,\mathcal{D} \subseteq \mathrm{Fil}^{\{m'_{ij}-1\}}\,\mathcal{D}$. Similarly $N(x) \in \mathrm{Fil}^{\{m_{ij}-1\}}\,\mathcal{D}$; by Proposition 2.1.9(6) we have $sN(x)$ and so also $N(sx)$ in $\mathrm{Fil}^{\{m'_{ij}-1\}}\,\mathcal{D}$. Since $N((\mathrm{Fil}^r\,S)\mathcal{D}) \subseteq N(\mathrm{Fil}^{\{r\}}\,\mathcal{D}) \subseteq \mathrm{Fil}^{\{m'_{ij}-1\}}\,\mathcal{D}$ as well, this checks (i).

Now let us check property (ii). Certainly f_π does map $(\mathrm{Fil}^1_{i'j'}\,S_E)\,\mathrm{Fil}^{\{m_{ij}\}}\,\mathcal{D}$ into $\bigoplus_{i,j}\mathrm{Fil}^{m'_{ij}}\,D_{K,ij}$, and we have to check that this map is surjective. (Since

$f_\pi(\mathrm{Fil}^r S) = 0$, the $(\mathrm{Fil}^r S)\mathcal{D}$ term does not affect the statement.) Choose an element $(x_{ij}) \in \oplus_{i,j} \mathrm{Fil}^{\{m'_{ij}\}} D_{K,ij}$. Note that $x_{i'j'} = 0$ by our hypothesis that $\mathrm{Fil}^{m_{i'j'}+1} D_{K,i'j'} = 0$. By Proposition 2.1.9(4), there exists $y \in \mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}$ such that $f_{ij}(y) = x_{ij}$ if $i \neq i'$ and $f_{i'j}(y) = \frac{1}{\pi_{i'j}-\pi_{i'j'}} x_{i'j}$ if $j \neq j'$. We see that $f_{ij}(E_{i'j'}(u)y) = x_{ij}$ for all i, j, and this checks property (ii).

Finally, property (iii) is an immediate consequence of of the factorization $E(u) = \prod_{i,j} E_{ij}(u)$ together with Proposition 2.1.9(6) applied to $E(u) \cdot \mathrm{Fil}^{\{m'_{ij}-1\}} \mathcal{D}$ repeatedly at all pairs *other* than our fixed (i', j').

Now turn to the statement we wish to prove. It is clear from the definitions and Proposition 2.1.9(6) that $E_{i'j'}(u) \mathrm{Fil}^{\{m_{ij}\}} \mathfrak{M}^* \subseteq \mathrm{Fil}^{\{m'_{ij}\}} \mathfrak{M}^*$. Conversely, take $x \in \mathrm{Fil}^{\{m'_{ij}\}} \mathfrak{M}^*$. By the first part of the proof we can write $x = sy + s'y'$ with $s \in \mathrm{Fil}^1_{i'j'} S_E$, $s' \in \mathrm{Fil}^r S$, $y \in \mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}$, and $y' \in \mathcal{D}$. Since $\mathfrak{M}^*$ is finite $\mathfrak{S}_{O_E}$-free, we can choose $e_1, \dots, e_d \in \mathfrak{M}^*$ that are an $\mathfrak{S}_{O_E}$-basis of $\mathfrak{M}^*$ (hence also an S_E-basis of $\mathcal{D}$). We write $x = \sum_n a_n e_n$ in terms of this basis, with $a_n \in \mathfrak{S}_{O_E}$ for all n. The expression $x = sy + s'y'$ shows that $a_n \in \mathrm{Fil}^1_{i'j'} S_E + \mathrm{Fil}^r S = \mathrm{Fil}^1_{i'j'} S_E$ as well. Hence $a_n \in \mathrm{Fil}^1_{i'j'} S_E \cap \mathfrak{S}_{O_E} = E_{i'j'}(u)\mathfrak{S}_{O_E}$ by Lemma 2.1.4(3). Now $x = E_{i'j'}(u)y''$ with $y'' \in \mathfrak{M}^*$. To conclude, we need to show that $y'' \in \mathrm{Fil}^{\{m_{ij}\}} \mathcal{D}$, and this follows from Proposition 2.1.9(8). □

As an application, we describe the Kisin modules of crystalline characters.

2.2.3 Lemma. *Suppose that V is a crystalline character such that $\mathrm{HT}_{\kappa_{ij}}(V) = \{r_{ij}\}$ with $r_{ij} \geq 0$ for all i, j. Choose basis elements $e_i \in \mathfrak{M}^*_i$ for all i. Then we have $\varphi(e_{i-1}) = \alpha_i \prod_{j=0}^{e-1} E_{ij}(u)^{r_{ij}} \cdot e_i$ with $\alpha_i \in O_E[\![u]\!]^\times$ for all i.*

Proof. Choose $r \geq \max_{i,j} r_{ij}$. We claim that $\mathrm{Fil}^r \mathfrak{M}^*_i = \prod_{j=0}^{e-1} E_{ij}(u)^{r-r_{ij}} \cdot \mathfrak{M}^*_i$ for all i. It is immediate from the definition that $\mathrm{Fil}^{\{r_{ij}\}} \mathcal{D} = \mathcal{D}$, so that $\mathrm{Fil}^{\{r_{ij}\}} \mathfrak{M}^* = \mathfrak{M}^*$ as well. Now the claim follows by repeatedly applying Lemma 2.2.2(3).

Recall that we have $\mathfrak{M}^* = \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$, so that e_{i-1} (and not e_i) is a generator of $\mathfrak{M}^*_i$. Also recall from [GLS14, Lem. 4.3(1)] that we have $\mathrm{Fil}^r \mathfrak{M}^*_i = \{x \in \mathfrak{M}^*_i : (1 \otimes \varphi)(x) \in E^{\kappa_i}(u)^r \mathfrak{M}_i\}$. Writing $\varphi(e_{i-1}) = \alpha'_i e_i$, we see that $\mathrm{Fil}^r \mathfrak{M}^*_i = \{\beta e_{i-1} : E^{\kappa_i}(u)^r \mid \beta\alpha'\}$. From the shape of $\mathrm{Fil}^r \mathfrak{M}^*$ we deduce that $\alpha'_i = \alpha_i \prod_{j=0}^{e-1} E_{ij}(u)^{r_{ij}}$ for some unit α_i, as desired. □

2.3. Pseudo-Barsotti–Tate representations

2.3.1 Definition. Fix integers $r_i \in [1, p]$ for all i. We say that a two-dimensional crystalline E-representation V of G_K is *pseudo-Barsotti–Tate* (or pseudo-BT) of weight $\{r_i\}$ if for all $0 \leq i \leq f-1$ we have $\mathrm{HT}_{\kappa_{i,0}}(V) = \{0, r_i\}$, and $\mathrm{HT}_{\kappa_{ij}}(V) = \{0, 1\}$ if $j \neq 0$. (Strictly speaking, we should say that V is pseudo-BT with respect to the labelling κ_{ij} of the embeddings $\mathrm{Hom}_{\mathbb{Q}_p}(K, E)$.)

Equivalently, a two-dimensional crystalline representation V is pseudo-BT if and only if

- $\dim_E \mathrm{gr}^0 D_{K,ij} = 1$ for all i, j,
- $\dim_E \mathrm{gr}^{r_i} D_{K,i0} = 1$ with $r_i \in [1, p]$ for all i, and
- $\dim_E \mathrm{gr}^1 D_{K,ij} = 1$ for all $j \neq 0$.

Note that a pseudo-BT representation is actually Barsotti–Tate if and only if $r_i = 1$ for all i. For the remainder of this section, we assume that V is pseudo-BT.

2.3.2. As in the previous subsection, we let $T \subset V$ be an G_K-stable $\mathcal{O}_E$-lattice, and $\mathfrak{M}$ the Kisin module attached to T. Write $M_{K,ij} = f_{ij}(\mathfrak{M}^*) \subset D_{K,ij}$. Note that $\mathfrak{M}^*$ is a rank-two finite free $\mathfrak{S}_{\mathcal{O}_E}$-module. Set $\mathrm{Fil}^m M_{K,ij} := M_{K,ij} \cap \mathrm{Fil}^m D_{K,ij}$. By definition,

- $\mathrm{Fil}^1 M_{K,ij}$ is a saturated rank-one free $\mathcal{O}_E$-submodule of $M_{K,ij}$,
- $\mathrm{Fil}^{r_i} M_{K,i0} = \mathrm{Fil}^1 M_{K,i0}$ and $\mathrm{Fil}^{r_i+1} M_{K,i0} = \{0\}$, and
- $\mathrm{Fil}^2 M_{K,ij} = \{0\}$ if $j \neq 0$.

In the next few pages, we will establish several results about the structure of the submodules $\mathrm{Fil}^{\{n,0,\ldots,0\}} \mathfrak{M}_i^*$ of $\mathfrak{M}_i^*$, following roughly the same strategy as in [GLS14, §4]. We remark that until Corollary 2.3.10, none of these results will actually use the pseudo-BT hypothesis, only that the crystalline representation V has non-negative Hodge–Tate weights and $\mathrm{HT}_{\kappa_{i,0}}(V) = \{0, r_i\}$.

We begin with following proposition, which should be compared with [GLS14, Prop. 4.5].

2.3.3 Proposition. *Suppose that there exists an $\mathfrak{S}_{\mathcal{O}_E,i}$-basis $\{\mathfrak{e}_i, \mathfrak{f}_i\}$ of $\mathfrak{M}_i^*$ such that $\mathfrak{f}_i \in \mathrm{Fil}^{\{r_i,0\ldots,0\}} \mathfrak{M}_i^*$ and $f_{i,0}(\mathfrak{f}_i)$ generates $\mathrm{Fil}^{r_i} M_{K,i0}$. Then for any $n \geq r_i$ we have*

$$\mathrm{Fil}^{\{n,0,\ldots,0\}} \mathfrak{M}_i^* = \mathfrak{S}_{\mathcal{O}_E,i}(u - \pi_{i,0})^n \mathfrak{e}_i \oplus \mathfrak{S}_{\mathcal{O}_E,i}(u - \pi_{i,0})^{n-r_i} \mathfrak{f}_i.$$

Proof. We first prove by induction that for $0 \leq n \leq r_i$ we have

$$\mathrm{Fil}^{\{n,0,\ldots,0\}} \mathcal{D}_i = S_{E,i}(u - \pi_{i,0})^n \mathfrak{e}_i \oplus S_{E,i} \mathfrak{f}_i + (\mathrm{Fil}^n S_{E,i}) \mathcal{D}_i. \tag{2.3.4}$$

The case that $n = 0$ is trivial as $\{\mathfrak{e}_i, \mathfrak{f}_i\}$ is a basis of $\mathfrak{M}_i^*$. Suppose that the statement is valid for $n-1$, and consider the statement for n. Define $\widetilde{\mathrm{Fil}}^{\{n,0,\ldots,0\}} \mathcal{D}_i$ to be equal to the right-hand side of (2.3.4), and set $\widetilde{\mathrm{Fil}}^{\{m,0,\ldots,0\}} \mathcal{D}_i = \mathrm{Fil}^{\{m,0,\ldots,0\}} \mathcal{D}_i$ for $m < n$. The conditions of Proposition 2.1.12 for this filtration are straightforward to check: use $\mathfrak{f}_i \in \mathrm{Fil}^{\{r_i,0,\ldots,0\}} \mathcal{D}_i$ to see that $N(\mathfrak{f}_i) \in \widetilde{\mathrm{Fil}}^{\{n-1,0,\ldots,0\}} \mathcal{D}_i = \mathrm{Fil}^{\{n-1,0,\ldots,0\}} \mathcal{D}_i$, so property (2) of 2.1.9 holds; then to verify property (4) of 2.1.9, note that for $1 \le n \le r_i$, $\mathrm{Fil}^n D_{K,i0} = \mathrm{Fil}^{r_i} D_{K,i0}$ is generated by $f_{i,0}(\mathfrak{f}_i)$ by hypothesis. By Proposition 2.1.12 we deduce that $\widetilde{\mathrm{Fil}}^{\{n,0,\dots,0\}}\mathcal{D}_i = \mathrm{Fil}^{\{n,0,\dots,0\}}\mathcal{D}_i$, as desired.

Now by Lemma 2.2.2 for $\mathcal{D}_i$, it suffices to show that

$$\mathrm{Fil}^{\{r_i,0,\dots,0\}}\mathfrak{M}_i^* = \mathfrak{S}_{O_E,i}(u-\pi_{i,0})^{r_i}\mathfrak{e}_i \oplus \mathfrak{S}_{O_E,i}\mathfrak{f}_i.$$

Evidently $\mathfrak{S}_{O_E,i}(u-\pi_{i,0})^{r_i}\mathfrak{e}_i \oplus \mathfrak{S}_{O_E,i}\mathfrak{f}_i \subseteq \mathrm{Fil}^{\{r_i,0,\dots,0\}}\mathfrak{M}_i^*$. For the converse, let $x \in \mathrm{Fil}^{\{r_i,0,\dots,0\}}\mathfrak{M}_i^*$. Then by (2.3.4) we have $x = (s(u-\pi_{i,0})^{r_i}+t)\mathfrak{e}_i + (s'+t')\mathfrak{f}_i$ with $s, s' \in S_{E,i}$ and $t, t' \in \mathrm{Fil}^{r_i} S_{E,i}$. Since $x \in \mathfrak{M}_i^*$, we see that $s(u-\pi_{i,0})^{r_i}+t \in \mathfrak{S}_{O_E}$ and $s'+t' \in \mathfrak{S}_{O_E}$. Note that $s(u-\pi_{i,0})^{r_i}+t$ is in $\mathrm{Fil}^{r_i} S_{E,i} \cap \mathfrak{S}_{O_E}$, and since this is just $(u-\pi_{i,0})^{r_i}\mathfrak{S}_{O_E}$, we are done. □

Via the identity $\mathfrak{M}^* = \mathfrak{S}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$, we can regard $\mathfrak{M}_{i-1}$ as a $\varphi(\mathfrak{S})$-submodule of $\mathfrak{M}_i^*$ such that $\mathfrak{S}\otimes_{\varphi(\mathfrak{S})}\mathfrak{M}_{i-1} = \mathfrak{M}_i^*$. (Note that the analogous statement immediately preceding the published version of [GLS14, Thm. 4.22] contains a mistake. See Appendix Appendix A for a correction.) The following proposition is our key result on the structure of $\mathrm{Fil}^{\{r_i,0,\dots,0\}}\mathfrak{M}_i^*$, and may be compared with [GLS14, Prop. 4.16].

2.3.5 Proposition.

1. *There exists a basis* $\{\mathfrak{e}_{i-1}, \mathfrak{f}_{i-1}\}$ *of* $\mathfrak{M}_{i-1}$ *such that* $f_{i,0}(\mathfrak{f}_{i-1})$ *generates* $\mathrm{Fil}^{r_i} M_{K,i0}$.
2. *Suppose that* $p \ge 3$. *There exists a basis* $\{\mathfrak{e}_i', \mathfrak{f}_i'\}$ *of* $\mathfrak{M}_i^*$ *such that* $\mathfrak{e}_i' - \mathfrak{e}_{i-1}$ *and* $\mathfrak{f}_i' - \mathfrak{f}_{i-1}$ *are in* $\mathfrak{m}_E\mathfrak{M}_i^*$ *and* $\mathrm{Fil}^{\{p,0,\dots,0\}}\mathfrak{M}_i^* = \mathfrak{S}_{O_E,i}(u-\pi_{i,0})^p\mathfrak{e}_i' \oplus \mathfrak{S}_{O_E,i}(u-\pi_{i,0})^{p-r_i}\mathfrak{f}_i'$.

Proof. (1) We know by Lemma 2.2.2(1) that $f_{i,0}(\mathfrak{M}_i^*)$ is an O_{E_i}-lattice inside $D_{K,i0}$, so there exists an O_{E_i}-basis $\{\bar{\mathfrak{e}}, \bar{\mathfrak{f}}\}$ of $M_{K,i0}$ such that $\bar{\mathfrak{f}}$ is a basis of $\mathrm{Fil}^{r_i} D_{K,i0}$. Pick any $\varphi(\mathfrak{S}_{O_E,i})$-basis $\{\widetilde{\mathfrak{e}}_{i-1}, \widetilde{\mathfrak{f}}_{i-1}\}$ of $\mathfrak{M}_{i-1}$. Then $\{f_{i,0}(\widetilde{\mathfrak{e}}_{i-1}), f_{i,0}(\widetilde{\mathfrak{f}}_{i-1})\}$ forms a basis of $f_{i,0}(\mathfrak{M}_i^*)$. Let $A \in \mathrm{GL}_2(O_E)$ be the matrix such that $(f_{i,0}(\widetilde{\mathfrak{e}}_{i-1}), f_{i,0}(\widetilde{\mathfrak{f}}_{i-1}))A = (\bar{\mathfrak{e}}, \bar{\mathfrak{f}})$. Then $(\mathfrak{e}_{i-1}, \mathfrak{f}_{i-1}) = (\widetilde{\mathfrak{e}}_{i-1}, \widetilde{\mathfrak{f}}_{i-1})A$ is the desired basis.

(2) The proof is a minor variation of the proof of [GLS14, Prop. 4.16]. Let $\mathfrak{e} := \mathfrak{e}_{i-1}$ and $\mathfrak{f} := \mathfrak{f}_{i-1}$ be a basis of $\mathfrak{M}_{i-1}$ as in part (1). In this proof only, we will write $\pi := \pi_{i,0}$ and $r := r_i$, to avoid too many subscripts and to make the discussion easier to compare with the proof of [GLS14, Prop. 4.16]. We consider the following assertion:

($\star$) For each $n = 1, \dots, r$ there exists $\mathfrak{f}^{(n)} \in \mathrm{Fil}^{\{n,0,\dots,0\}}\mathfrak{M}_i^*$ such that

$$\mathfrak{f}^{(n)} = \mathfrak{f} + \sum_{s=1}^{n-1}\pi^{p-s}(u-\pi)^s(a_s^{(n)}\mathfrak{e} + \widetilde{a}_s^{(n)}\mathfrak{f})$$

with $a_s^{(n)}, \widetilde{a}_s^{(n)} \in \mathcal{O}_{E_i}$. Once the assertion $(\star)$ is established, the proposition follows from Proposition 2.3.3, taking $\mathfrak{e}'_i = \mathfrak{e}$ and $\mathfrak{f}'_i = \mathfrak{f}^{(r)}$.

We prove $(\star)$ by induction on n. From the definition of $\mathrm{Fil}^{\{1,0\ldots,0\}}\,\mathcal{D}_i$ and the defining property of $\mathfrak{f}$ from (1), we see that $\mathfrak{f} \in \mathrm{Fil}^{\{1,0,\ldots,0\}}\,\mathfrak{M}_i^*$, so that for the base case $n = 1$ we can take $\mathfrak{f}^{(1)} = \mathfrak{f}$.

Now assume that $(\star)$ is valid for some $1 \le n < r$, and let us consider the case $n+1$. Set $H(u) = \frac{u-\pi}{\pi}$ and

$$\widetilde{\mathfrak{f}}^{(n+1)} := \sum_{\ell=0}^{n} \frac{H(u)^\ell N^\ell(\mathfrak{f}^{(n)})}{\ell!}.$$

As in the proof of [GLS14, Prop. 4.16] (and the proof of Proposition 2.1.9(4) of this paper) one computes that

$$N(\mathfrak{f}^{(n+1)}) = \frac{H(u)^n}{n!} N^{n+1}(\mathfrak{f}^{(n)}) + \sum_{\ell=1}^{n} \frac{(1+N(H(u)))H(u)^{\ell-1}N^\ell(\mathfrak{f}^{(n)})}{(\ell-1)!}.$$

Then, using $1 + N(H(u)) \in \mathrm{Fil}^1_{i,0}\, S_{E,i}$, one deduces that $N(\widetilde{\mathfrak{f}}^{(n+1)}) \in \mathrm{Fil}^{\{n,0,\ldots,0\}}\,\mathcal{D}$ and so $\widetilde{\mathfrak{f}}^{(n+1)} \in \mathrm{Fil}^{\{n+1,0,\ldots,0\}}\,\mathcal{D}$. Now by induction, we have

$$\widetilde{\mathfrak{f}}^{(n+1)} - \mathfrak{f}^{(n)} = \sum_{\ell=1}^{n} \frac{(u-\pi)^\ell}{\pi^\ell \ell!} N^\ell \left(\mathfrak{f} + \sum_{s=1}^{n-1} \pi^{p-s}(u-\pi)^s (a_s^{(n)}\mathfrak{e} + \widetilde{a}_s^{(n)}\mathfrak{f}) \right)$$

$$= \sum_{\ell=1}^{n} \frac{(u-\pi)^\ell}{\pi^\ell \ell!} N^\ell(\mathfrak{f}) +$$

$$\sum_{\ell=1}^{n}\sum_{s=1}^{n-1}\sum_{t=0}^{\ell} \frac{\pi^{p-s}(u-\pi)^\ell}{\pi^\ell \ell!}\binom{\ell}{t} N^{\ell-t}((u-\pi)^s)\left(a_s^{(n)}N^t(\mathfrak{e}) + \widetilde{a}_s^{(n)}N^t(\mathfrak{f})\right). \quad (2.3.6)$$

Write

$$N^t(\mathfrak{e}) = \sum_{m=0}^{\infty}(u-\pi)^m(c_m^t\mathfrak{e} + d_m^t\mathfrak{f}) \quad \text{and} \quad N^t(\mathfrak{f}) = \sum_{m=0}^{\infty}(u-\pi)^m(\widetilde{c}_m^t\mathfrak{e} + \widetilde{d}_m^t\mathfrak{f}) \quad (2.3.7)$$

with $c_m^t, d_m^t, \widetilde{c}_m^t, \widetilde{d}_m^t \in E$. After substituting (2.3.7) into (2.3.6) and expanding the terms $N^{\ell-t}((u-\pi)^s)$ using [GLS14, Lem. 4.13], we can collect terms and write

$$\widetilde{\mathfrak{f}}^{(n+1)} = \mathfrak{f}^{(n)} + \sum_{m=1}^{\infty}(u-\pi)^m(b_m\mathfrak{e} + \widetilde{b}_m\mathfrak{f}) \quad (2.3.8)$$

with each $b_m, \widetilde{b}_m \in E$. Now we delete all terms of $(u-\pi)$-degree at least $n+1$ from this expression, and define

$$\mathfrak{f}^{(n+1)} := \mathfrak{f}^{(n)} + \sum_{m=1}^{n}(u-\pi)^m(b_m\mathfrak{e} + \widetilde{b}_m\mathfrak{f}).$$

It remains to show that $\pi^{p-m} \mid b_m, \widetilde{b}_m$, or in other words that if $m \le n$ then every occurrence of $(u-\pi)^m$ in the terms collected to form (2.3.8) has coefficient divisible by π^{p-m}. A direct examination of these terms (just as in the last part of the proof of [GLS14, Prop. 4.16]) shows that this comes down to the claim that $\pi^{p-m} \mid c^\ell_m, d^\ell_m, \widetilde{c}^\ell_m, \widetilde{d}^\ell_m$ for all $1 \le \ell < p$ and $0 \le m < p$. By [GLS14, Cor. 4.11], this follows from Lemma 2.3.9 below (applied at $a_i = c^\ell_i$, d^ℓ_i, etc.), which generalizes [GLS14, Lem. 4.12]. (We remind the reader that except for [GLS14, Lem. 4.12], in the results of [GLS14, §4.2] our ground field was an arbitrary finite extension of $\mathbb{Q}_p$.) The application of [GLS14, Cor. 4.11] is where we use the hypothesis that $p \ge 3$. □

Define $S' = W(k)[\![u^p, \frac{u^{ep}}{p}]\!]\left[\frac{1}{p}\right] \cap S$ and set $\mathcal{I}_\ell = \sum_{m=1}^{\ell} p^{\ell-m} u^{pm} S' \subset S'$.

2.3.9 Lemma. *Suppose that $y \in \mathcal{I}_\ell$ for some $1 \le \ell \le p$. Write $y = \sum_{i=0}^{\infty} a_i(u-\pi)^i$ with $a_i \in K_0$. Then we have $\pi^{p+(\ell-1)\min(p,e)} \mid a_0$ and $\pi^{p+e-i+(\ell-1)\min(p,e)} \mid a_i$ in $\mathcal{O}_K$ for $1 \le i \le p-1$.*

Proof. For any non-negative integer n, let $e(n) = \lfloor \frac{n}{e} \rfloor$. Note that any $x \in S$ can be written uniquely as $x = \sum_{i=0}^{\infty} a_i \frac{u^i}{e(i)!}$ with $a_i \in W(k)$.

By hypothesis we have $y = \sum_{m=1}^{\ell} p^{\ell-m} u^{pm} z_m$ with $z_m \in S'$. We can write $z_m = \sum_{j=0}^{\infty} \frac{b_{j,m} u^{pj}}{e(pj)!}$ with $b_{j,m} \in W(k)$. Then

$$\begin{aligned} y &= \sum_{m=1}^{\ell} p^{\ell-m} \left(\sum_{j=0}^{\infty} b_{j,m} \frac{u^{p(j+m)}}{e(pj)!} \right) \\ &= \sum_{m=1}^{\ell} \sum_{j=0}^{\infty} p^{\ell-m} b_{j,m} \frac{(u-\pi+\pi)^{p(j+m)}}{e(pj)!} \\ &= \sum_{m=1}^{\ell} \sum_{j=0}^{\infty} p^{\ell-m} \frac{b_{j,m}}{e(pj)!} \left(\sum_{i=0}^{p(j+m)} \binom{p(j+m)}{i} (u-\pi)^i \pi^{p(j+m)-i} \right) \\ &= \sum_{i=0}^{\infty} \left(\sum_{m=1}^{\ell} \sum_{j \ge s_{i,p,m}} \frac{b_{j,m} \pi^{p(j+m)-i} p^{\ell-m}}{e(pj)!} \binom{p(j+m)}{i} \right) (u-\pi)^i, \end{aligned}$$

where $s_{i,p,m} = \max\{0, i/p - m\}$. Since we only consider a_i for $0 \le i \le p$, we have $s_{i,p,m} = 0$ in all cases. Note that $\pi^{pj}/e(pj)! \in \mathcal{O}_K$ for all $j \ge 0$. Let v_π denote the valuation on $\mathcal{O}_K$ such that $v_\pi(\pi) = 1$. We first observe that

$v_\pi(a_0) \geq \min_{1\leq m\leq \ell}(pm + e(\ell - m)) = p + (\ell-1)\min(p,e)$. If $1 \leq i \leq p-1$ then p divides $\binom{p(j+m)}{i}$, so we get

$$v_\pi(a_i) \geq \min_{1\leq m\leq \ell}(pm - i + e(\ell - m) + e) = p + e - i + (\ell-1)\min(p,e)$$

instead. □

In what follows, when we write a product of matrices as $\prod_{j=1}^n A_i$, we mean $A_1A_2\cdots A_n$.

2.3.10 Corollary. *Suppose that $p \geq 3$ and let $\mathfrak{M}$ be the Kisin module corresponding to a lattice in a pseudo-BT representation V of weight $\{r_i\}$. There exist matrices $Z'_{ij} \in \mathrm{GL}_2(\mathcal{O}_E)$ for $j = 1,\dots,e-1$ such that $\mathrm{Fil}^{p,p\dots,p}\mathfrak{M}_i^* = \mathfrak{S}_{\mathcal{O}_E,i}\alpha_{i,e-1} \oplus \mathfrak{S}_{\mathcal{O}_E,i}\beta_{i,e-1}$ with*

$$(\alpha_{i,e-1},\beta_{i,e-1}) = (\mathfrak{e}'_i,\mathfrak{f}'_i)\Lambda'_{i,0}\left(\prod_{j=1}^{e-1} Z'_{ij}\Lambda'_{ij}\right)$$

where $\mathfrak{e}'_i,\mathfrak{f}'_i$ are as in Proposition 2.3.5(2), $\Lambda'_{i,0} = \begin{pmatrix}(u-\pi_{i,0})^p & 0\\ 0 & (u-\pi_{i,0})^{p-r_i}\end{pmatrix}$ and $\Lambda'_{ij} = \begin{pmatrix}(u-\pi_{ij})^p & 0\\ 0 & (u-\pi_{ij})^{p-1}\end{pmatrix}$ for $j = 1,\dots,e-1$.

Proof. Define $\mathfrak{p}_m := \{p,\dots,p,0,\dots,0\}$ where the tuple contains exactly $m+1$ copies of p. We prove by induction on m that there exist matrices $Z'_{ij} \in \mathrm{GL}_2(\mathcal{O}_E)$ for $j = 1,\dots,m$ such that

$$\mathrm{Fil}^{\mathfrak{p}_m}\mathfrak{M}_i^* = \mathfrak{S}_{\mathcal{O}_E,i}\alpha_{i,m} \oplus \mathfrak{S}_{\mathcal{O}_E,i}\beta_{i,m}$$

with $(\alpha_{i,m},\beta_{i,m}) := (\mathfrak{e}'_i,\mathfrak{f}'_i)\Lambda'_{i,0}\left(\prod_{j=1}^m Z'_{ij}\Lambda'_{ij}\right)$. If $m = 0$ then this is Proposition 2.3.5(2).

Suppose the statement holds for $m-1$, and let us consider the statement for m. We first show that $M'_{K,im} := f_{im}(\mathrm{Fil}^{\mathfrak{p}_{m-1}}\mathfrak{M}_i^*)$ is an $\mathcal{O}_{E_{im}}$-lattice inside $D_{K,im}$, or equivalently that $\{f_{im}(\alpha_{i,m-1}), f_{im}(\beta_{i,m-1})\}$ is an E_{im}-basis of $D_{K,im}$. Since $\{\mathfrak{e}'_i,\mathfrak{f}'_i\}$ is a basis of $\mathfrak{M}_i^*$, it suffices to check that $f_{im}\left(\Lambda'_{i,0}\left(\prod_{j=1}^{m-1} Z'_{ij}\Lambda'_{ij}\right)\right)$ is an invertible matrix in $\mathrm{GL}_2(E)$. This holds because $Z'_{ij} \in \mathrm{GL}_2(\mathcal{O}_E)$ and $f_{im}(\Lambda'_{ij}) \in \mathrm{GL}_2(E)$ for $j < m$.

The fact that $M'_{K,im}$ is an $\mathcal{O}_{E_{im}}$-lattice inside $D_{K,im}$ implies that there exists an $\mathcal{O}_{E_i}$-basis $\bar\gamma',\bar\delta'$ of $M'_{K,im}$ such that $\bar\delta'$ generates $\mathrm{Fil}^1 D_{K,im}$. Write $\bar\alpha_{m-1} := f_{im}(\alpha_{i,m-1})$ and $\bar\beta_{m-1} := f_{im}(\beta_{i,m-1})$. Let $Z'_{im} \in \mathrm{GL}_2(\mathcal{O}_E)$ be the matrix such that $(\bar\gamma',\bar\delta') = (\bar\alpha_{i,m-1},\bar\beta_{i,m-1})Z'_{im}$ and define $(\gamma_m,\delta_m) := (\alpha_{i,m-1},\beta_{i,m-1})Z'_{im}$. Let

$\mathfrak{q}_m := \{p, \ldots, p, 1, 0 \ldots, 0\}$ where the tuple contains exactly m copies of p. We claim

$$\mathrm{Fil}^{\mathfrak{q}_m} \mathfrak{M}_i^* = \mathfrak{S}_{O_E,i}(u - \pi_{im})\gamma_m \oplus \mathfrak{S}_{O_E,i}\delta_m. \tag{2.3.11}$$

We first show that $(u - \pi_{im})\gamma_m, \delta_m$ are in $\mathrm{Fil}^{\mathfrak{q}_m} \mathfrak{M}_i^*$. Note that γ_m, δ_m generate $\mathrm{Fil}^{\mathfrak{p}_{m-1}} \mathfrak{M}^*$ by construction, so by Proposition 2.1.9(6) for $\mathcal{D}_i$ it suffices to show that $\delta_m \in \mathrm{Fil}^{\mathfrak{q}_m} \mathcal{D}_i$. Note that $f_{im}(\delta_m) = \bar{\delta}' \in \mathrm{Fil}^1 D_{K,im}$, so we just need to check that $N(\delta_m) \in \mathrm{Fil}^{\{p-1,\ldots,p-1,0,0,\ldots,0\}} \mathcal{D}_i$, where there are m copies of $p-1$ in the superscript (*cf.* Remark 2.1.11). But this follows from the fact that $\alpha_{i,m-1}$ and $\beta_{i,m-1}$ are in $\mathrm{Fil}^{\mathfrak{p}_{m-1}} \mathcal{D}_i$. Therefore, $\mathfrak{S}_{O_{E,i}}(u - \pi_{im})\gamma_m \oplus \mathfrak{S}_{O_E,i}\delta_m \subseteq \mathrm{Fil}^{\mathfrak{q}_m} \mathfrak{M}_i^*$.

Now pick $x \in \mathrm{Fil}^{\mathfrak{q}_m} \mathfrak{M}_i^* \subseteq \mathrm{Fil}^{\mathfrak{p}_{m-1}} \mathfrak{M}_i^*$. We have $x = a\gamma_m + b\delta_m$ with $a, b \in \mathfrak{S}_{O_E,i}$. It suffices to show that $(u - \pi_{im}) \mid a$. Note that $f_{im}(x) = f_{im}(a) f_{im}(\gamma_m) + f_{im}(b) f_{im}(\delta_m) \in \mathrm{Fil}^1 D_{K,im}$. But $(f_{im}(\gamma_m), f_{im}(\delta_m))$ is just $(\bar{\gamma}', \bar{\delta}')$, which is a basis of $D_{K,im}$, and $\bar{\delta}'$ generates $\mathrm{Fil}^1 D_{K,im}$. This forces $f_{im}(a) = 0$ and then $(u - \pi_{im}) \mid a$ by Lemma 2.1.4. This completes the proof of (2.3.11).

Finally, recall that $\mathrm{Fil}^2 D_{K,im} = \{0\}$ since V is pseudo-BT, so the equality (2.3.11) together with Lemma 2.2.2(2) implies that

$$\mathrm{Fil}^{\mathfrak{p}_m} \mathfrak{M}_i^* = \mathfrak{S}_{O_E,i}(u - \pi_{im})^p \gamma_m \oplus (u - \pi_{im})^{p-1} \mathfrak{S}_{O_E,i}\delta_m.$$

That is, $\mathrm{Fil}^{\mathfrak{p}_m} \mathfrak{M}_i^*$ is generated by $(\mathfrak{e}_i', \mathfrak{f}_i')\Lambda_{i,0}' \left(\prod_{j=1}^m Z_{ij}' \Lambda_{ij}'\right)$. This completes the induction on m and proves the proposition. □

2.4. The structure theorem for pseudo-Barsotti–Tate Kisin modules In this subsection we prove our main result about Kisin modules associated to pseudo-BT representations. We retain the notation from the previous subsection.

2.4.1 Theorem. *Suppose that $p \geq 3$ and let $\mathfrak{M}$ be the Kisin module corresponding to a lattice in a pseudo-BT representation V of weight $\{r_i\}$. Then there exists an $O_E[\![u]\!]$-basis $\{\mathfrak{e}_i, \mathfrak{f}_i\}$ of $\mathfrak{M}_i$ for all $0 \leq i \leq f-1$ such that*

$$\varphi(\mathfrak{e}_{i-1}, \mathfrak{f}_{i-1}) = (\mathfrak{e}_i, \mathfrak{f}_i) X_i \left(\prod_{j=1}^{e-1} \Lambda_{i,e-j} Z_{i,e-j} \right) \Lambda_{i,0} Y_i,$$

for $X_i, Y_i \in \mathrm{GL}_2(O_E[\![u]\!])$ with $Y_i \equiv I_2 \pmod{\mathfrak{m}_E}$, matrices $Z_{ij} \in \mathrm{GL}_2(O_E)$ for all j, and $\Lambda_{i,0} = \begin{pmatrix} 1 & 0 \\ 0 & (u - \pi_{i,0})^{r_i} \end{pmatrix}$ and $\Lambda_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & u - \pi_{ij} \end{pmatrix}$ for $j = 1, \ldots, e-1$.

Proof. For all i we let $\{\mathfrak{e}_i, \mathfrak{f}_i\}$ be the $O_E[\![u]\!]$-basis of $\mathfrak{M}_i$ in Proposition 2.3.5(1) and write $\varphi(\mathfrak{e}_{i-1}, \mathfrak{f}_{i-1}) = (\mathfrak{e}_i, \mathfrak{f}_i) A_i$, where A_i is a matrix with coefficients in $O_E[\![u]\!]$. From (2.2.1) we see that $\mathrm{Fil}^p \mathfrak{M}_i^*$ is generated by $(\mathfrak{e}_{i-1}, \mathfrak{f}_{i-1}) B_i$, where B_i is the matrix satisfying $A_i B_i = (E^{\kappa_i}(u))^p I_2$. On the other hand, Corollary 2.3.10 shows that $\mathrm{Fil}^p \mathfrak{M}_i^*$ (which by definition is equal to $\mathrm{Fil}^{\{p,\dots,p\}} \mathfrak{M}_i^*$) is generated by

$$(\alpha_{i,e-1}, \beta_{i,e-1}) = (\mathfrak{e}'_i, \mathfrak{f}'_i)\Lambda'_{i,0}\left(\prod_{j=1}^{e-1} Z'_{ij}\Lambda'_{ij}\right) = (\mathfrak{e}_{i-1}, \mathfrak{f}_{i-1})Y_i^{-1}\Lambda'_{i,0}\left(\prod_{j=1}^{e-1} Z'_{ij}\Lambda'_{ij}\right).$$

Here Y_i is the matrix such that $(\mathfrak{e}'_i, \mathfrak{f}'_i)Y_i = (\mathfrak{e}_{i-1}, \mathfrak{f}_{i-1})$, which by Proposition 2.3.5(2) is congruent to the identity modulo $\mathfrak{m}_E$. Therefore there exists an invertible matrix $X_i \in \mathrm{GL}_2(\mathcal{O}_E[\![u]\!])$ such that $(\mathfrak{e}_{i-1}, \mathfrak{f}_{i-1})B_i = (\alpha_{i,e-1}, \beta_{i,e-1})X_i^{-1}$. Hence we have

$$Y_i^{-1}\Lambda'_{i,0}\left(\prod_{j=1}^{e-1} Z'_{ij}\Lambda'_{ij}\right)X_i^{-1} = B_i.$$

Then the relation $A_i B_i = (E^{\kappa_i}(u))^p I_2$ proves that

$$A_i = X_i\left(\prod_{j=1}^{e-1} \Lambda_{i,e-j} Z_{i,e-j}\right)\Lambda_{i,0} Y_i$$

with $Z_{ij} = (Z'_{ij})^{-1}$ and $\Lambda_{ij} = E_{ij}(u)^p(\Lambda'_{ij})^{-1}$. □

3. Semisimple reductions mod p of pseudo-BT representations

3.1. In this section we use Theorem 2.4.1 to study the semisimple representations that can be obtained as the reduction modulo p of pseudo-BT representations. We begin with the following notation.

3.1.1 Definition. Suppose $s_0, \dots, s_{f-1}$ are non-negative integers and $a \in k_E^\times$. Let $\overline{\mathfrak{M}}(s_0, \dots, s_{f-1}; a)$ be the Kisin module with natural k_E-action (in the sense of [GLS14, §3]) that has rank one over $\mathfrak{S}_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} k_E$ and satisfies

- $\overline{\mathfrak{M}}(s_0, \dots, s_{f-1}; a)_i$ is generated by e_i, and
- $\varphi(e_{i-1}) = (a)_i u^{s_i} e_i$.

Here $(a)_i = a$ if $i \equiv 0 \pmod f$ and $(a)_i = 1$ otherwise. All Kisin modules of rank one have this form (see e.g. [GLS14, Lem. 6.2]).

Write $\overline{\kappa}_i$ for the embedding $k \to \overline{\mathbb{F}}_p$ induced by κ_{ij} (this is independent of j). For brevity we will sometimes write ω_i for the fundamental character $\omega_{\overline{\kappa}_i}$.

We refer the reader to [GLS14, §3] for the definition of the contravariant functors $T_{\mathfrak{S}}$ that associate a representation of G_{K_∞} to each torsion or finite free Kisin module.

3.1.2 Lemma. *We have $T_{\mathfrak{S}}(\overline{\mathfrak{M}}(s_0, \ldots, s_{f-1}; a)) \simeq \overline{\chi}|_{G_{K_\infty}}$ for a unique character $\overline{\chi} : G_K \to k_E^\times$, and $\overline{\chi}$ satisfies $\overline{\chi}|_{I_K} \simeq \prod_{i=0}^{f-1} \omega_i^{s_i}$.*

Proof. Choose any integers $r_{ij} \geq 0$ such that $\sum_j r_{ij} = s_i$. By Lemma 2.2.3 (together with [GLS14, Lem. 6.4] and an analysis of the Kisin modules associated to unramified characters as in the proof of [GLS14, Lem. 6.3]) we see that $\overline{\mathfrak{M}}(s_0, \ldots, s_{f-1}; a)$ is isomorphic to $\mathfrak{M} \otimes_{\mathcal{O}_E} k_E$ for a Kisin module $\mathfrak{M}$ corresponding to a lattice T in a crystalline character V with Hodge–Tate weights $\mathrm{HT}_{\kappa_{ij}}(V) = \{r_{ij}\}$. Then $\overline{\chi} = T \otimes_{\mathcal{O}_E} k_E$. The character $\overline{\chi}$ is unique since K_∞/K is totally wildly ramified, so that restriction to G_{K_∞} is faithful on characters of G_K.

For the last part of the statement it suffices to check that $\overline{\psi}_{ij}|_{I_K} = \omega_i$, where ψ_{ij} is a crystalline character whose $\kappa_{i'j'}$-labeled Hodge–Tate is 1 if $(i', j') = (i, j)$ and is 0 otherwise. For this see [Con11, Prop. B.3] and the proof of [GLS14, Prop. 6.7](1). □

We write $\Delta(\lambda_1, \ldots, \lambda_d)$ for the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_d$.

3.1.3 Proposition. *Assume that $p \geq 3$, let $\mathfrak{M}$ be the Kisin module corresponding to a lattice in a pseudo-BT representation V of weight $\{r_i\}$, and write $\overline{\mathfrak{M}} = \mathfrak{M} \otimes_{\mathcal{O}_E} k_E$.*

Suppose that $\overline{\mathfrak{N}} \subset \overline{\mathfrak{M}}$ is a sub-φ-module such that $\overline{\mathfrak{M}}/\overline{\mathfrak{N}}$ is free of rank one as an $\mathfrak{S}_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} k_E$-module. Then $\overline{\mathfrak{N}} \simeq \overline{\mathfrak{M}}(s_0, \ldots, s_{f-1}; a)$ for some $a \in k_E^\times$, with $s_i = r_i + x_i$ or $s_i = e - 1 - x_i$ for some $x_i \in [0, e-1]$ for all i.

Proof. Choose a basis $\{\mathfrak{e}_i, \mathfrak{f}_i\}$ for $\mathfrak{M}_i$ as in Theorem 2.4.1. Since we will work in $\overline{\mathfrak{M}}$ for the remainder of the proof, no confusion will arise if we write $\{\mathfrak{e}_i, \mathfrak{f}_i\}$ also for the image of that basis in $\overline{\mathfrak{M}}$.

A generator e_{i-1} of $\overline{\mathfrak{N}}_{i-1}$ has the form $(\mathfrak{e}_{i-1}, \mathfrak{f}_{i-1}) \cdot (v, w)^T$ for some $v, w \in k_E[\![u]\!]$, by hypothesis at least one of which is a unit. We know from Theorem 2.4.1 that

$$\varphi(e_{i-1}) = (\mathfrak{e}_i, \mathfrak{f}_i)\overline{X}_i \left(\prod_{j=1}^{e-1} \overline{\Lambda}_{i,e-j} \overline{Z}_{i,e-j} \right) \overline{\Lambda}_{i,0} \cdot (\varphi(v), \varphi(w))^T$$

where $\overline{X}_i$ and $\overline{Z}_{ij}$ are the reductions mod $\mathfrak{m}_E$ of X_i and Z_{ij}, where $\overline{\Lambda}_{i,0} = \Delta(1, u^{r_i})$, and where $\overline{\Lambda}_{ij} = \Delta(1, u)$ for $1 \leq j \leq e - 1$.

Observe that each entry of $(\varphi(v), \varphi(w))^T$ is either a unit or divisible by u^p, and at least one is a unit. Since we have $r_i \leq p$ for all i, it follows that the largest power of u dividing the column vector $\overline{\Lambda}_{i,0} \cdot (\varphi(v), \varphi(w))^T$ is either u^{r_i} or u^0.

For any $s \geq 0$, if y is a column vector of length 2 that is exactly divisible by u^s, it is easy to see that that $\Delta(1, u) \cdot y$ is exactly divisible by either u^s or u^{s+1}. On the other hand if Z is invertible, then $Z \cdot y$ is still exactly divisible by u^s.

Applying these observations iteratively to the invertible matrices $\overline{X}_i$ and $\overline{Z}_{ij}$, and to the matrices $\overline{\Lambda}_{ij} = \Delta(1, u)$ for $1 \le j \le e-1$, we see that $\varphi(e_{i-1})$ is divisible exactly by u^{s_i} where $s_i = r_i + x_i'$ or $s_i = x_i'$ and $0 \le x_i' \le e-1$ is the number of times that we took u^{s+1} rather than u^s when considering the effect of the matrix $\overline{\Lambda}_{ij}$. Setting $x_i = x_i'$ in the first case and $x_i = e - 1 - x_i'$ in the latter case, the proposition follows. □

If V is a pseudo-BT representation of weight $\{r_i\}$ and $\lambda = \overline{\kappa}_i \in \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$, we write $r_\lambda := r_i$.

3.1.4 Theorem. *Assume that $p \ge 3$. Let T be a lattice in a pseudo-BT representation V of weight $\{r_i\}$, and assume that $\overline{T} = T \otimes_{\mathcal{O}_E} k_E$ is reducible. Then there is a subset $J \subseteq \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$ and integers $x_\lambda \in [0, e-1]$ such that*

$$\overline{T}|_{I_K} \simeq \begin{pmatrix} \prod_{\lambda \in J} \omega_\lambda^{r_\lambda + x_\lambda} \prod_{\lambda \notin J} \omega_\lambda^{e-1-x_\lambda} & * \\ 0 & \prod_{\lambda \notin J} \omega_\lambda^{r_\lambda + x_\lambda} \prod_{\lambda \in J} \omega_\lambda^{e-1-x_\lambda} \end{pmatrix}.$$

Proof. Let $\mathfrak{M}$ be the Kisin module associated to the lattice T. We have $\overline{T}|_{G_{K_\infty}} \simeq T_{\mathfrak{S}}(\overline{\mathfrak{M}})$. From (the proof of) [GLS14, Lem. 5.5], we see that $\overline{\mathfrak{M}}$ is reducible and has a submodule $\overline{\mathfrak{N}}$ as in Proposition 3.1.3 such that $\overline{T}$ has a quotient character $\overline{\chi}$ with $\overline{\chi}|_{G_{K_\infty}} \simeq T_{\mathfrak{S}}(\overline{\mathfrak{N}})$. Take $J = \{\overline{\kappa}_i : s_i = e - 1 - x_i\}$. (In particular, if it happens that $e - 1 - x_i = r + x_i$ then we have $i \in J$.) The result follows from Lemma 3.1.2 (and a determinant argument to compute the sub-character of $\overline{T}$). □

We now give the analogue of Theorem 3.1.4 when $\overline{T}$ is absolutely irreducible. Let k_2 denote the unique quadratic extension of k inside the residue field of $\overline{K}$. We say that a subset $J \subseteq \mathrm{Hom}(k_2, \overline{\mathbb{F}}_p)$ is *balanced* if for each $\lambda \in \mathrm{Hom}(k_2, \overline{\mathbb{F}}_p)$ exactly one of λ and λ^q lies in J, with $q = p^f$. If $\lambda \in \mathrm{Hom}(k_2, \overline{\mathbb{F}}_p)$, write r_λ for $r_{\lambda|_k}$. The result is as follows.

3.1.5 Theorem. *Assume that $p \ge 3$. Let T be a lattice in a pseudo-BT representation V of weight $\{r_i\}$, and assume that $\overline{T} = T \otimes_{\mathcal{O}_E} k_E$ is absolutely irreducible. Then there is a balanced subset $J \subseteq \mathrm{Hom}(k_2, \overline{\mathbb{F}}_p)$ and integers $x_\lambda \in [0, e-1]$ so that x_λ depends only on $\lambda|_k$ and*

$$\overline{T}|_{I_K} \simeq \prod_{\lambda \in J} \omega_\lambda^{r_\lambda + x_\lambda} \prod_{\lambda \notin J} \omega_\lambda^{e-1-x_\lambda} \bigoplus \prod_{\lambda \notin J} \omega_\lambda^{r_\lambda + x_\lambda} \prod_{\lambda \in J} \omega_\lambda^{e-1-x_\lambda}. \tag{3.1.6}$$

Proof. Note that V restricted to the unramified quadratic extension K_2 of K remains pseudo-BT, and for each embedding $\kappa' : K_2 \to \overline{\mathbb{Q}}_p$ extending $\kappa : K \to \overline{\mathbb{Q}}_p$ we have $\mathrm{HT}_{\kappa'}(V|_{G_{K_2}}) = \mathrm{HT}_\kappa(V)$. Applying Theorem 3.1.4 to the lattice $T|_{G_{K_2}}$ shows that $\overline{T}|_{I_K}$ has the form (3.1.6), except that J need not be balanced, nor must

$x_\lambda = x_{\lambda^q}$. It remains to be seen that these additional conditions may be taken to hold. Assume that $\overline{\chi} : I_K \to \overline{\mathbb{F}}_p^\times$ is a character of niveau $2f$ such that $\overline{\chi} \oplus \overline{\chi}^q$ is equal to a representation as in the right-hand side of (3.1.6); to complete the proof, we wish to show that $\overline{\chi} \oplus \overline{\chi}^q$ is also equal to a representation as in (3.1.6) with J balanced and $x_\lambda = x_{\lambda^q}$. We apologise to the reader for the argument that follows, which is entirely elementary but long and unenlightening.

It follows from [BLGG13, Cor. 4.1.20] that if $e \geq p$, then as J varies over all balanced sets and the $x_\lambda \in [0, e-1]$ vary over all possibilities with $x_\lambda = x_{\lambda^q}$, the right-hand side of (3.1.6) exhausts all representations $\overline{\chi} \oplus \overline{\chi}^q$ of niveau $2f$ with determinant $\prod_{\lambda \in \mathrm{Hom}(k, \overline{\mathbb{F}}_p)} \omega_\lambda^{r_\lambda + e - 1}$. The theorem is then automatic in this case, since $\overline{T}|_{I_K}$ must have this determinant as well. For the remainder of the argument, then, we assume that $e \leq p-1$.

Fix a character $\overline{\kappa}'_0$ extending $\overline{\kappa}_0$, and define $\overline{\kappa}'_i$ for $i \in \mathbb{Z}$ by $(\overline{\kappa}'_{i+1})^p = \overline{\kappa}'_i$. Write x_i for $x_{\overline{\kappa}'_i}$ for $i \in \mathbb{Z}$. Similarly define r_i for all $i \in \mathbb{Z}$ so that $r_{i+f} = r_i$. Define

$$\begin{cases} J_1 = J \cap (f + J) \\ J_2 = J^c \cap (f + J)^c \\ J_3 = J \cap (f + J)^c \\ J_4 = J^c \cap (f + J) \end{cases}$$

so that J is balanced if and only if $J_1 = J_2 = \varnothing$. To simplify notation we will for instance write $i \in J_1$ in lieu of $\overline{\kappa}'_i \in J_1$, and we will write ω_i for $\omega_{\overline{\kappa}'_i}$. (We stress that the symbol ω_i here denotes a fundamental character of level $2f$, and that the symbols ω_i and x_i are indexed modulo $2f$.) The condition that the two summands on the right-hand side of (3.1.6) are qth powers of one another is equivalent to

$$\prod_{i \in J_1} \omega_i^{r_i + x_i + x_{i+f} - (e-1)} \prod_{i \in J_2} \omega_i^{(e-1) - r_i - x_i - x_{i+f}} \prod_{i \in J_3} \omega_i^{x_i - x_{i+f}} \prod_{i \in J_4} \omega_i^{x_{i+f} - x_i} = 1.$$

Write y_i for the exponent of ω_i in the above expression. Since $r_i \in [1, p]$ for all i and $e \leq p-1$, we have

$$\begin{cases} y_i \in [-p+3, 2p-2] & \text{if } i \in J_1 \\ y_i \in [-2p+2, p-3] & \text{if } i \in J_2 \\ y_i \in [-p+2, p-2] & \text{if } i \in J_3 \\ y_i \in [-p+2, p-2] & \text{if } i \in J_4. \end{cases}$$

Note that $y_i = y_{i+f}$ for all i, so we can consider the y_i's as being labeled cyclically with index taken modulo f. As in the proof of [GLS14, Lem. 7.1], one checks that since $\prod_i \omega_i^{y_i} = 1$ with $y_i \in [-2p+2, 2p-2]$ for all i, the tuple $(y_0, \ldots, y_{f-1})$ must have the shape

$$a_0(p, 0, \ldots, 0, -1) + a_1(-1, p, 0, \ldots, 0) + \cdots + a_{f-1}(0, \ldots, 0, -1, p)$$

with $|a_i| \in \{0, 1, 2\}$ for all i, and in fact either $a_i = 2$ for all i, or $a_i = -2$ for all i, or else $a_i \in \{0, \pm 1\}$ for all i. But if $a_i = 2$ for all i, so that $y_i = 2p - 2$ for all i, we would have to have $J = \{0, \ldots, 2f-1\}$ with $r_i = p$ and $e = x_i = p - 1$ for all i. But then $\overline{\chi}^q = \overline{\chi}$, i.e. $\overline{\chi}$ has niveau f rather than niveau $2f$, a contradiction. The case where $a_i = -2$ for all i is similarly impossible. So in fact we must have $a_i \in \{0, \pm 1\}$ for all i. We now consider separately the case where some a_i is equal to 0, and the case where $a_i = \pm 1$ for all i.

First let us suppose that at least one a_i is equal to 0. The cyclic set of those i with $y_i \neq 0$ (with index i taken modulo f) must break up as a disjoint union of sets of the form $(i, i+1, \ldots, i+j)$ with $y_i = \pm 1$, $y_{i+j} = \pm p$, and $y_{i+1}, \ldots, y_{i+j-1} \in \{\pm p \pm 1\}$. For every such interval $[i, i+j]$, choose a representative of i modulo $2f$ (that we also denote i), and perform the following operation (noting that since $y_i \in [-p+2, p-2]$ for $i \in J_3 \cup J_4$, we have $i+1, \ldots, i+j \in J_1 \cup J_2$):

- Replace J with $J \triangle \{i, \ldots, i+j\}$ if $i \in J_1 \cup J_2$, or else with $J \triangle \{i+1, \ldots, i+j\}$ if $i \in J_3 \cup J_4$;
- Replace x_ℓ with $x_{\ell+f}$ for each $\ell \in [i, i+j]$.

Here $\triangle$ denotes symmetric difference. It is easy to check that this operation does not change $\overline{\chi}$, and that for the new choice of J and x_i's we have $y_i = 0$ for all i. We now have $r_i + x_i = e - 1 - x_{i+f}$ for each $i \in J_1 \cup J_2$, so for each pair $\{i, i+f\} \subseteq J_1 \cup J_2$ we can again replace J with $J \triangle \{i\}$ and x_i with x_{i+f} without changing $\overline{\chi}$. When this operation is complete, our new set J is balanced. Furthermore $y_i = 0$ for all i, and so $x_i = x_{i+f}$ for all i, and this case is complete.

Finally suppose that $a_i = \pm 1$ for all i. Then $J_1 \cup J_2 = \{0, \ldots, 2f-1\}$, and in fact $i \in J_1$ if $a_i = 1$ while $i \in J_2$ if $a_i = -1$. Note that if $i \in J_1$ and $i+1 \in J_2$ or vice-versa, then $r_i + x_i + x_{i+f} - (e-1) = p+1$ (so in particular both x_i, x_{i+f} are nonzero), while if $i,\ i+1 \in J_1$ or $i,\ i+1 \in J_2$, then $r_i + x_i + x_{i+f} - (e-1) = p-1$.

By symmetry we can suppose without loss of generality that $J_1 \neq \varnothing$. If $J_2 = \varnothing$, then some x_i is not equal to $e-1$ (otherwise we have $x_i = e-1$ for all i, so that $r_i = p - e$ for all i, and $\overline{\chi}$ has niveau f); changing our choice of $\overline{\kappa}_0$ (if necessary) we suppose that $x_{f-1} \neq e-1$. On the other hand if $J_2 \neq \varnothing$, then by changing our choice of $\overline{\kappa}_0$ (if necessary) we suppose that $0 \in J_1$ but $f-1 \in J_2$.

Take $J' = \{0, \ldots, f-1\}$, and for each $0 \le i \le f-2$ we set

$$x'_i = x'_{i+f} = \begin{cases} x_i & \text{if } i \in J_1,\ i+1 \in J_1 \\ x_i - 1 & \text{if } i \in J_1,\ i+1 \in J_2 \\ x_{i+f} - 1 & \text{if } i \in J_2,\ i+1 \in J_1 \\ x_{i+f} & \text{if } i \in J_2,\ i+1 \in J_2. \end{cases}$$

We take $x'_{f-1} = x'_{2f-1} = x_{f-1} + 1$ if $f - 1 \in J_1$, or $x'_{f-1} = x'_{2f-1} = x_{2f-1}$ if $f - 1 \in J_2$. (In other words if $i = f - 1$ we take the common value of x'_{f-1}, x'_{2f-1} to be 1 more than the value that would otherwise have been given by the above table.) Note that we have $x'_i \in [0, e - 1]$ for all i by the observations and choices in the two preceding paragraphs.

We claim that $\overline{\chi}' := \prod_{i=0}^{f-1} \omega_i^{r_i + x'_i} \prod_{i=f}^{2f-1} \omega_i^{e-1-x'_i}$ is equal to $\overline{\chi}$. Since J' is balanced and $x'_i = x'_{i+f}$ for all i, this will complete the proof. Checking the claim is somewhat laborious, and we only give an indication of the argument. We wish to show that $\overline{\chi}'\overline{\chi}^{-1}$ is trivial. Write $\overline{\chi}'\overline{\chi}^{-1} = \prod_i \omega_i^{z_i}$ using the defining formulas for $\overline{\chi}$ and $\overline{\chi}'$. The values of the z_i are calculated by considering eight cases, depending on whether or not $i \in \{0, \dots, f - 1\}$, whether or not $i \in J_1$, and whether or not $i + 1 \in J_1$ (as well as making an adjustment by 1 when $i = f - 1,\ 2f - 1$). For instance if $i \in \{f, \dots, 2f - 1\}$, $i \in J_1$, and $i + 1 \in J_2$ then we have

$$z_i = (e - 1 - (x_{i+f} - 1)) - (r_i + x_i) = -p,$$

while if $i \in \{f, \dots, 2f - 1\}$, $i \in J_1$, and $i + 1 \in J_1$ then $z_i = 1 - p$ if $i \neq 2f - 1$ and $z_i = -p$ if $i = 2f - 1$. (In all cases one finds that $|z_i| \in \{0, 1, p-1, p\}$ and z_i depends only on i, J_1, and J_2, not on the r_i's or x_i's.) It is then straightforward to check that $\prod_i \omega_i^{z_i} = 1$ (one just has to "carry" by replacing every $\omega_i^{\pm p}$ with $\omega_{i-1}^{\pm 1}$). □

4. The weight part of Serre's conjecture I: the semisimple case

For a detailed discussion of the weight part of Serre's conjecture for GL(2), we refer the reader to [BLGG13, §4]. In this section, we will content ourselves with a brief explanation of the consequences of the results of the previous sections for the weight part of Serre's conjecture for (definite or indefinite) quaternion algebras over totally real fields; we note that the analogous results for compact at infinity unitary groups U(2) over CM fields follow immediately from [BLGG13, Thm. 5.1.3], together with the discussion here.

4.1. Local Serre weights In order to make our various definitions associated to Serre weights, it will be convenient to work in the local setting of the previous sections.

4.1.1 Definition. A *Serre weight* of $\mathrm{GL}_2(k)$ is by definition an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_2(k)$, which is necessarily of the form

$$\sigma_{a,b} := \otimes_{\lambda \in \mathrm{Hom}(k, \overline{\mathbb{F}}_p)} \det{}^{b_\lambda} \otimes \mathrm{Sym}^{a_\lambda - b_\lambda} k^2 \otimes_{k,\lambda} \overline{\mathbb{F}}_p,$$

for some (uniquely determined) integers a_λ, b_λ with $b_\lambda,\ a_\lambda - b_\lambda \in [0, p - 1]$ for all λ, and not all b_λ equal to $p - 1$.

Note that $\sigma_{a,b}$ has a natural model $\otimes_{\lambda \in \mathrm{Hom}(k,k_E)} \det{}^{b_\lambda} \otimes \mathrm{Sym}^{a_\lambda - b_\lambda} k^2 \otimes_{k,\lambda} k_E$, and it will sometimes be convenient for us to think of $\sigma_{a,b}$ as being defined over k_E (or rather, it will be convenient for us to identify $\mathrm{Hom}(k, k_E)$ with $\mathrm{Hom}(k, \overline{\mathbb{F}}_p)$).

4.1.2. Suppose that $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is continuous. In [BLGG13] and [GK] there are definitions of several sets of Serre weights $W^{\mathrm{explicit}}(\bar{r})$, $W^{\mathrm{BT}}(\bar{r})$, and $W^{\mathrm{cris}}(\bar{r})$. We now recall the definitions of $W^{\mathrm{explicit}}(\bar{r})$ and $W^{\mathrm{cris}}(\bar{r})$; see [GK, Def. 4.5.6] for $W^{\mathrm{BT}}(\bar{r})$.

Write a_i, b_i in place of $a_{\overline{\kappa}_i}$, $b_{\overline{\kappa}_i}$. We say that a de Rham lift r of $\bar{r}$ has *Hodge type* $\sigma_{a,b}$ if for all $0 \le i \le f-1$ we have $\mathrm{HT}_{\kappa_{i,0}}(r) = \{b_i, a_i+1\}$, and if $\mathrm{HT}_{\kappa_{ij}}(r) = \{0,1\}$ when $j \neq 0$. Note that a crystalline representation of Hodge type $\sigma_{a,0}$ is pseudo-BT of weight $\{a_i+1\}$.

4.1.3 Definition. ([BLGG13, Def. 4.1.4]) $W^{\mathrm{cris}}(\bar{r})$ is the set of Serre weights $\sigma_{a,b}$ for which $\bar{r}$ has a crystalline lift of Hodge type $\sigma_{a,b}$.

As in the previous section let k_2 denote the unique quadratic extension of k inside the residue field of $\overline{K}$.

4.1.4 Definition. ([BLGG13, Def. 4.1.23]) If $\bar{r}$ is irreducible, then $W^{\mathrm{explicit}}(\bar{r})$ is the set of Serre weights $\sigma_{a,b}$ such that there is a balanced subset $J \subset \mathrm{Hom}(k_2, \overline{\mathbb{F}}_p)$, and for each $\lambda \in \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$ an integer $0 \le x_\lambda \le e-1$ such that if we write x_λ for $x_{\lambda|_k}$ when $\lambda \in \mathrm{Hom}(k_2, \overline{\mathbb{F}}_p)$, then

$$\bar{r}|_{I_K} \cong \begin{pmatrix} \prod_{\lambda\in J}\omega_\lambda^{a_\lambda+1+x_\lambda}\prod_{\lambda\notin J}\omega_\lambda^{b_\lambda+e-1-x_\lambda} & 0 \\ 0 & \prod_{\lambda\notin J}\omega_\lambda^{a_\lambda+1+x_\lambda}\prod_{\lambda\in J}\omega_\lambda^{b_\lambda+e-1-x_\lambda} \end{pmatrix}.$$

If $\bar{r}$ is reducible, then $W^{\mathrm{explicit}}(\bar{r})$ is the set of weights $\sigma_{a,b}$ for which $\bar{r}$ has a crystalline lift of type $\sigma_{a,b}$ of the form

$$\begin{pmatrix} \chi' & * \\ 0 & \chi \end{pmatrix}.$$

In particular (see the remark after [BLGG13, Def. 4.1.14]), if $\sigma_{a,b} \in W^{\mathrm{explicit}}(\bar{r})$ then it is necessarily the case that there is a subset $J \subseteq \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$ and for each $\lambda \in \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$ there is an integer $0 \le x_\lambda \le e-1$ such that

$$\bar{r}|_{I_K} \cong \begin{pmatrix} \prod_{\lambda\in J}\omega_\lambda^{a_\lambda+1+x_\lambda}\prod_{\lambda\notin J}\omega_\lambda^{b_\lambda+x_\lambda} & * \\ 0 & \prod_{\lambda\notin J}\omega_\lambda^{a_\lambda+e-x_\lambda}\prod_{\lambda\in J}\omega_\lambda^{b_\lambda+e-1-x_\lambda} \end{pmatrix},$$

and when $\bar{r}$ is a sum of two characters this is necessary and sufficient.

4.1.5. The inclusion $W^{\mathrm{explicit}}(\bar{r}) \subseteq W^{\mathrm{cris}}(\bar{r})$ was proved in [BLGG13, Prop. 4.1.25] and was conjectured there to be an equality ([BLGG13, Conj. 4.1.26]); this equality is the main local result of this paper.

The definition of $W^{\mathrm{BT}}(\bar{r})$ is unimportant for us in this paper; the only fact that we will need is that by [GK, Cor. 4.5.7], the inclusion $W^{\mathrm{explicit}}(\bar{r}) \subseteq W^{\mathrm{cris}}(\bar{r})$ can be refined to inclusions

$$W^{\mathrm{explicit}}(\bar{r}) \subseteq W^{\mathrm{BT}}(\bar{r}) \subseteq W^{\mathrm{cris}}(\bar{r}),$$

so that our main result will show that both these inclusions are equalities.

Observe that the definitions of the sets $W^{\mathrm{cris}}(\bar{r})$ and $W^{\mathrm{explicit}}(\bar{r})$ involve our fixed choice of embeddings $\kappa_{i,0}$. We will prove that these sets in fact do not depend on this choice. Indeed the definition of $W^{\mathrm{BT}}(\bar{r})$ does not involve any choice of embeddings $\kappa_{i,0}$, so that once we have proved the equality $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$ it follows automatically that $W^{\mathrm{explicit}}(\bar{r})$ and $W^{\mathrm{cris}}(\bar{r})$ do not depend on any choice of embeddings either. On the other hand this will also follow easily and directly from the arguments in this paper, and so we will give a direct proof as well.

In the above language, the main local result of this section is the following.

4.1.6 Theorem. *Suppose that $p \geq 3$ and that $\bar{r}$ is semisimple. Then we have $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$. Moreover, these sets do not depend on our choice of embeddings $\kappa_{i,0}$.*

Proof. Suppose that $\sigma_{a,b} \in W^{\mathrm{cris}}(\bar{r})$. Twisting, we may assume that $b_\lambda = 0$ for all λ. If $r : G_K \to \mathrm{GL}_2(\mathcal{O}_E)$ is a pseudo-BT lift of $\bar{r}$ of Hodge type $\sigma_{a,0}$, we can freely enlarge the coefficient field E so that it satisfies our usual hypotheses, and so that $r \otimes_{\mathcal{O}_E} k_E$ is either reducible or absolutely irreducible.

Now for the first part of the result it remains to show that $W^{\mathrm{cris}}(\bar{r}) \subseteq W^{\mathrm{explicit}}(\bar{r})$; this is immediate from Theorems 3.1.4 and 3.1.5 and Definitions 4.1.3 and 4.1.4, taking $r_i = a_i + 1$ for all i. The second part is automatic since the definition of $W^{\mathrm{explicit}}(\bar{r})$ when $\bar{r}$ is semisimple does not depend on the choice of embeddings $\kappa_{i,0}$. □

4.2. Global Serre weights Let F be a totally real field, and continue to assume that $p > 2$. Let $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be continuous, absolutely irreducible, and modular (in the sense that it is isomorphic to the reduction modulo p of a p-adic Galois representation associated to a Hilbert modular eigenform of parallel weight two). Let k_v be the residue field of F_v for each $v|p$.

A *global Serre weight* is by definition an irreducible representation of the group $\prod_{v|p} \mathrm{GL}_2(k_v)$, which is necessarily of the form $\sigma = \otimes_{v|p}\sigma_v$ with σ_v a Serre weight of $\mathrm{GL}_2(k_v)$ as above. Let D be a quaternion algebra with centre F, which is split at all places dividing p and at zero or one infinite places. Then [GK, Def. 5.5.2] explains what it means for $\overline{\rho}$ to be modular for D of weight σ. (There is a possible local obstruction at the finite places of F at which D is ramified to $\overline{\rho}$ being modular for D for any weight at all; following [GK, Def. 5.5.3], we say that $\overline{\rho}$ is *compatible* with D if this obstruction vanishes. Any $\overline{\rho}$ will be compatible with some D; indeed, we could take D to be split at all finite places of F.) The following is a theorem of Gee–Kisin [GK, Cor. 5.5.4].

4.2.1 Theorem. *Assume that $p > 2$, that $\overline{\rho}$ is modular and compatible with D, that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible, and if $p = 5$ assume further that the projective image of $\overline{\rho}|_{G_{F(\zeta_p)}}$ is not isomorphic to A_5.*

Then $\overline{\rho}$ is modular for D of weight σ if and only if $\sigma_v \in W^{\mathrm{BT}}(\overline{\rho}|_{G_{F_v}})$ for all $v|p$.

The main global result of this section is the following (which is an immediate consequence of Theorem 4.1.6 in combination with the above result of Gee–Kisin).

4.2.2 Theorem. *Assume that $p > 2$, that $\overline{\rho}$ is modular and compatible with D, that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible, and if $p = 5$ assume further that the projective image of $\overline{\rho}|_{G_{F(\zeta_p)}}$ is not isomorphic to A_5.*

Assume that $\overline{\rho}|_{G_{F_v}}$ is semisimple for each place $v|p$. Then $\overline{\rho}$ is modular for D of weight σ if and only if $\sigma_v \in W^{\mathrm{explicit}}(\overline{\rho}|_{G_{F_v}})$ for all $v|p$.

Modulo the hypotheses on the image of $\overline{\rho}$ (the usual hypotheses needed in order to apply the Taylor–Wiles–Kisin method), Theorem 4.2.2 is the main conjecture of Schein [Sch08].

Note that we have thus far said nothing about the case where $\overline{\rho}|_{G_{F_v}}$ is reducible but non-split, where $W^{\mathrm{explicit}}(\overline{\rho}|_{G_{F_v}})$ will depend on the extension class of $\overline{\rho}|_{G_{F_v}}$. Our treatment of this more delicate case will occupy the remainder of the paper.

4.2.3 Remark. If $\overline{\rho}|_{G_{F_v}}$ is generic in a suitable sense, then [DS, Thm. 4.5] implies that $W^{\mathrm{explicit}}(\overline{\rho}|_{G_{F_v}}) = W^{\mathrm{BT}}(\overline{\rho}|_{G_{F_v}}) \cap W^{\mathrm{explicit}}(\overline{\rho}|^{\mathrm{ss}}_{G_{F_v}})$. On the other hand we have inclusions

$$W^{\mathrm{BT}}(\overline{\rho}|_{G_{F_v}}) \subseteq W^{\mathrm{cris}}(\overline{\rho}|_{G_{F_v}}) \subseteq W^{\mathrm{cris}}(\overline{\rho}|^{\mathrm{ss}}_{G_{F_v}}) = W^{\mathrm{explicit}}(\overline{\rho}|^{\mathrm{ss}}_{G_{F_v}})$$

where the equality is an application of Theorem 4.1.6, and so we see in this case that $W^{\mathrm{explicit}}(\overline{\rho}|_{G_{F_v}}) = W^{\mathrm{BT}}(\overline{\rho}|_{G_{F_v}})$. In particular we can already extend Theorem 4.2.2 to the case where $\overline{\rho}|_{G_{F_v}}$ is either semisimple or generic for all $v|p$.

(We refer the reader [DS, Def. 3.5] for the definition of genericity that we use here, but note that genericity implies that $e \le (p-1)/2$.)

5. The weight part of Serre's conjecture II: the non-cyclotomic case

In this section we make a detailed study of the extensions of rank-one Kisin modules and use these results to prove that if $p \ge 3$ then $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$ for reducible representations $\bar{r} \simeq \begin{pmatrix} \overline{\chi}' & * \\ 0 & \overline{\chi} \end{pmatrix}$ with $\overline{\chi}^{-1}\overline{\chi}' \neq \overline{\varepsilon}$ (see Theorem 5.4.1 below).

5.1. Extensions of rank-one Kisin modules We begin with some basic results on extensions of rank-one Kisin modules, mildly generalising the results in [GLS14, §7]. We begin with the following notation.

5.1.1 Definition. If $\overline{\mathfrak{N}} = \overline{\mathfrak{M}}(s_0, \ldots, s_{f-1}; a)$, define $\alpha_i(\overline{\mathfrak{N}}) := \frac{1}{p^f-1} \sum_{j=1}^{f} p^{f-j} s_{j+i}$.

It is immediate from the definition that these constants satisfy the relations $\alpha_i(\overline{\mathfrak{N}}) + s_i = p\alpha_{i-1}(\overline{\mathfrak{N}})$ for $i = 0, \ldots, f-1$ and indeed are uniquely defined by them. We have the following easy lemma.

5.1.2 Lemma. *Write* $\overline{\mathfrak{N}} = \overline{\mathfrak{M}}(s_0, \ldots, s_{f-1}; a)$ *and* $\overline{\mathfrak{N}}' = \overline{\mathfrak{M}}(s'_0, \ldots, s'_{f-1}; a')$. *There exists a nonzero map* $\overline{\mathfrak{N}} \to \overline{\mathfrak{N}}'$ *if and only if* $\alpha_i(\overline{\mathfrak{N}}) - \alpha_i(\overline{\mathfrak{N}}') \in \mathbb{Z}_{\geq 0}$ *for all i, and* $a = a'$.

Proof. Such a map, if it exists, must take the form $e_i \mapsto c u^{\alpha_i(\overline{\mathfrak{N}}) - \alpha_i(\overline{\mathfrak{N}}')} e'_i$ for all i, where e'_i is the basis element for $\overline{\mathfrak{N}}'_i$ as in Definition 3.1.1, and $c \in k_E^\times$. □

We can now check the following analogue of [GLS14, Prop. 7.4].

5.1.3 Proposition. *Let* $\overline{\mathfrak{N}} = \overline{\mathfrak{M}}(s_0, \ldots, s_{f-1}; a)$ *and* $\overline{\mathfrak{P}} = \overline{\mathfrak{M}}(t_0, \ldots, t_{f-1}; b)$ *be rank-one Kisin modules, and let* $\overline{\mathfrak{M}}$ *be an extension of* $\overline{\mathfrak{N}}$ *by* $\overline{\mathfrak{P}}$. *Then we can choose bases* e_i, f_i *of the* $\overline{\mathfrak{M}}_i$ *so that* φ *has the form*

$$\begin{aligned} \varphi(e_{i-1}) &= (b)_i u^{t_i} e_i \\ \varphi(f_{i-1}) &= (a)_i u^{s_i} f_i + y_i e_i \end{aligned}$$

with $y_i \in k_E[\![u]\!]$ *a polynomial with* $\deg(y_i) < s_i$, *except that when there is a nonzero map* $\overline{\mathfrak{N}} \to \overline{\mathfrak{P}}$ *we must also allow* y_j *to have a term of degree* $s_j + \alpha_j(\overline{\mathfrak{N}}) - \alpha_j(\overline{\mathfrak{P}})$ *for any one choice of j.*

Proof. The proof is essentially identical to the proof of [GLS14, Prop. 7.4] except for the analysis of the exceptional terms. Namely, it is possible to use a simultaneous change of basis of the form $f'_i = f_i + z_i e_i$ for $i = 0, \ldots, f-1$ to eliminate all terms of degree at least s_i in the y_i's, except that if there is a sequence of integers $d_i \geq s_i$ satisfying

$$d_i = p(d_{i-1} - s_{i-1}) + t_i \tag{5.1.4}$$

for all i (with subscripts taken modulo f), and if $a = b$, then for some j the term of degree d_j in y_j may survive. (Note that taking z_{i-1} to be a suitable monomial of degree $d_{i-1} - s_{i-1}$ will eliminate the term of degree d_{i-1} in y_{i-1} but yield a term of degree d_i in y_i instead. This is more-or-less what we called a *loop* in the proof of [GLS14, Prop. 7.4].)

Comparing with the defining relations for $\alpha_i(\overline{\mathfrak{N}})$, $\alpha_i(\overline{\mathfrak{P}})$, we must have

$$d_i = p\alpha_{i-1}(\overline{\mathfrak{N}}) - \alpha_i(\overline{\mathfrak{P}}) = \alpha_i(\overline{\mathfrak{N}}) - \alpha_i(\overline{\mathfrak{P}}) + s_i,$$

and these are integers greater than or equal to s_i provided that $\alpha_i(\overline{\mathfrak{N}}) - \alpha_i(\overline{\mathfrak{P}}) \in \mathbb{Z}_{\geq 0}$ for all i. The result now follows from Lemma 5.1.2. □

Next we combine Proposition 5.1.3 with Theorem 2.4.1 to study the extensions of Kisin modules that can arise from the reduction mod p of pseudo-BT representations; this is our analogue of [GLS14, Thm. 7.9].

5.1.5 Theorem. *Suppose that $p \geq 3$. Let T be a G_K-stable $\mathcal{O}_E$-lattice in a pseudo-BT representation of weight $\{r_i\}$. Let $\mathfrak{M}$ be the Kisin module associated to T, and let $\overline{\mathfrak{M}} := \mathfrak{M} \otimes_{\mathcal{O}_E} k_E$.*

Assume that the $k_E[G_K]$-module $\overline{T} := T \otimes_{\mathcal{O}_E} k_E$ is reducible, so that there exist rank-one Kisin modules $\overline{\mathfrak{N}} = \overline{\mathfrak{M}}(s_0, \dots, s_{f-1}; a)$ and $\overline{\mathfrak{P}} = \overline{\mathfrak{M}}(t_0, \dots, t_{f-1}; b)$ such that $\overline{\mathfrak{M}}$ is an extension of $\overline{\mathfrak{N}}$ by $\overline{\mathfrak{P}}$. Then for all i there is an integer $x_i \in [0, e-1]$ such that $\{s_i, t_i\} = \{r_i + x_i, e - 1 - x_i\}$.

We can choose bases e_i, f_i of the $\overline{\mathfrak{M}}_i$ so that φ has the form

$$\begin{aligned} \varphi(e_{i-1}) &= (b)_i u^{t_i} e_i \\ \varphi(f_{i-1}) &= (a)_i u^{s_i} f_i + y_i e_i \end{aligned}$$

where

- *$y_i \in k_E[\![u]\!]$ is a polynomial with* $\deg(y_i) < s_i$,
- *if $t_i < r_i$ then the nonzero terms of y_i have degrees in the set $\{t_i\} \cup [r_i, s_i - 1]$,*
- *except that when there is a nonzero map $\overline{\mathfrak{N}} \to \overline{\mathfrak{P}}$ we must also allow y_j to have a term of degree $s_j + \alpha_j(\overline{\mathfrak{N}}) - \alpha_j(\overline{\mathfrak{P}})$ for any one choice of j.*

Proof. The fact that $\overline{\mathfrak{M}}$ is an extension of two rank-one Kisin modules follows e.g. from [GLS14, Lem. 5.5], and the fact that $\{s_i, t_i\} = \{r_i + x_i, e - 1 - x_i\}$ is an immediate consequence of Proposition 3.1.3 and determinant considerations. After applying Proposition 5.1.3, it remains to check that if $t_i < r_i$ then the nonzero terms of y_i have degrees in the set $\{t_i\} \cup [r_i, s_i - 1]$ (except possibly for an exceptional term arising from the existence of a nonzero map $\overline{\mathfrak{N}} \to \overline{\mathfrak{P}}$).

It is an immediate consequence of Theorem 2.4.1 that $\varphi(\overline{\mathfrak{M}}_{i-1})$ contains a saturated element (i.e. an element not divisible by u in $\varphi(\overline{\mathfrak{M}}_{i-1})$) that is divisible by u^{r_i} in $\overline{\mathfrak{M}}_i$. Such an element must be a saturated $\varphi(\mathfrak{S})$-linear combination of $\varphi(e_{i-1})$ and $\varphi(f_{i-1})$, i.e. we must have γ, δ, at least one of which is a unit, such that u^{r_i} divides

$$\varphi(\gamma) \cdot ((a)_i u^{s_i} f_i + y_i e_i) + \varphi(\delta) \cdot (b)_i u^{t_i} e_i.$$

The assumptions on $\{s_i, t_i\}$ imply that one of them is at least r_i. Suppose, then, that $t_i < r_i$. It follows that $s_i \geq r_i$, so we have to have

$$u^{r_i} \mid \varphi(\gamma)y_i + \varphi(\delta)(b)_i u^{t_i}.$$

If γ were a non-unit, then we would have $u^p \mid \varphi(\gamma)$, so that u^{r_i} divides both terms in the above sum separately. But $t_i < r_i$, and so δ must also be a non-unit, a contradiction. It follows that γ is a unit. Replacing δ with $\delta\gamma^{-1}$ we can suppose $\gamma = 1$, and we must be able to choose δ so that u^{r_i} divides $y_i + \varphi(\delta)(b_i)u^{t_i}$. That is, all terms of y_i must have degree at least r_i, except for terms that could be canceled by the addition of $\varphi(\delta)(b_i)u^{t_i}$. These are the terms of degree $t_i + pn$ with $n \in \mathbb{Z}_{\geq 0}$. But $t_i + p \geq r_i$, so the only extra term we may get this way is a term of degree t_i. □

5.1.6 Definition. Fix integers $r_i \in [1, p]$. Suppose that $\overline{\mathfrak{N}} = \overline{\mathfrak{M}}(s_0, \ldots, s_{f-1}; a)$ and $\overline{\mathfrak{P}} = \overline{\mathfrak{M}}(t_0, \ldots, t_{f-1}; b)$ are rank-one Kisin modules with $\{s_i, t_i\} = \{r_i + x_i, e - 1 - x_i\}$. We let $\mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}, \overline{\mathfrak{P}})$ denote the subset of $\mathrm{Ext}^1(\overline{\mathfrak{N}}, \overline{\mathfrak{P}})$ defined by the conditions of Theorem 5.1.5. (The integers $\{r_i\}$ will be implicit in the notation.)

From the proof of Theorem 5.1.5 we see that $\mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}, \overline{\mathfrak{P}})$ can be characterised as the set of classes $\overline{\mathfrak{M}} \in \mathrm{Ext}^1(\overline{\mathfrak{N}}, \overline{\mathfrak{P}})$ such that each $\varphi(\overline{\mathfrak{M}}_{i-1})$ contains a saturated element that is divisible by u^{r_i} in $\overline{\mathfrak{M}}_i$.

In the remainder of Section 5, we prove that if $p \geq 3$ then $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$ for reducible representations $\bar{r} \simeq \begin{pmatrix} \overline{\chi}' & * \\ 0 & \overline{\chi} \end{pmatrix}$ with $\overline{\chi}^{-1}\overline{\chi}' \neq \overline{\varepsilon}$. We follow the same basic strategy as in [GLS14, §§8-9], which relies on (but, as we will explain, is necessarily somewhat more complicated than) a comparision between the the dimension of the space $\mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}, \overline{\mathfrak{P}})$ and the dimension of an appropriate local Bloch–Kato group $H^1_f(G_K, -)$. Perhaps the main difference between the arguments here and the arguments of [GLS14, §§8-9] is that certain combinatorial issues which we were able to address in ad hoc ways in the unramified case (particularly those of [GLS14, §8.2]) are rather more involved in the ramified case, and so have had to be addressed systematically (*cf.* Section 5.3).

5.2. Comparison of extension classes We give a simple example that illustrates why the proof that $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$ in the indecomposable case is more complicated than one might initially expect, and in particular cannot follow immediately from comparing the dimension of the space $\mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}, \overline{\mathfrak{P}})$ with the dimension of an appropriate local Bloch–Kato group $H^1_f(G_K, -)$.

5.2.1 Example. Take $K = \mathbb{Q}_p$. The group $H^1_f(G_{\mathbb{Q}_p}, \mathcal{O}_E(\varepsilon^{1-p}))$ is torsion (since $H^1_f(G_{\mathbb{Q}_p}, E(\varepsilon^{1-p}))$ is trivial) and its ϖ-torsion has rank one, corresponding to the congruence $\varepsilon^{1-p} \equiv 1 \pmod{\varpi}$. It follows that the subspace of $\mathrm{Ext}^1_{k_E[G_{\mathbb{Q}_p}]}(\overline{1}, \overline{1})$ arising from crystalline extensions of ε^{p-1} by 1 is one-dimensional. (This extension comes from the reduction mod ϖ of a nontrivial extension T of $\mathcal{O}_E(\varepsilon^{p-1})$ by $\mathcal{O}_E$ inside the split representation $\varepsilon^{p-1} \oplus 1$.) By Lemma 5.4.2 below, the subspace of $\mathrm{Ext}^1_{k_E[G_{K_\infty}]}(\overline{1}, \overline{1})$ arising from crystalline extensions of $\mathcal{O}_E(\varepsilon^{p-1})$ by $\mathcal{O}_E$ is also one-dimensional.

On the other hand, by Proposition 5.1.3 there are no nontrivial extensions of $\overline{\mathfrak{M}}(0; 1)$ by $\overline{\mathfrak{M}}(p-1; 1)$ at all. Why is this not a contradiction, given that the Kisin modules corresponding to 1 and ε^{p-1} reduce to $\overline{\mathfrak{M}}(0; 1)$ and $\overline{\mathfrak{M}}(p-1; 1)$ respectively? The point is that although the functor $T_{\mathfrak{S}}$ is an equivalence of categories, the inverse functor from lattices to Kisin modules need not be exact; and indeed the reduction mod ϖ of the Kisin module corresponding to the lattice T of the previous paragraph turns out to be an extension of $\overline{\mathfrak{M}}(p-1; 1)$ by $\overline{\mathfrak{M}}(0; 1)$ rather than the reverse.

5.2.2. Suppose that $\bar{r}$ is the reduction mod ϖ of an $\mathcal{O}_E$-lattice in a pseudo-BT representation V of weight $\{r_i\}$, so that by Theorem 5.1.5 there exist $\overline{\mathfrak{N}}$ and $\overline{\mathfrak{P}}$ such that $\bar{r}|_{G_{K_\infty}} \simeq T_{\mathfrak{S}}(\overline{\mathfrak{M}})$ for some $\overline{\mathfrak{M}} \in \mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}, \overline{\mathfrak{P}})$. To prove that $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$, we wish to show that there exist crystalline characters $\chi', \chi : G_K \to \mathcal{O}_E^\times$ and an extension T of $\mathcal{O}_E(\chi)$ by $\mathcal{O}_E(\chi')$ such that $T[1/p]$ is pseudo-BT of weight $\{r_i\}$ and $\bar{r} \simeq T \otimes_{\mathcal{O}_E} k_E$.

It is natural to try to argue by choosing χ' and χ so that their corresponding rank-one Kisin modules $\mathfrak{N}, \mathfrak{P}$ are lifts of $\overline{\mathfrak{N}}, \overline{\mathfrak{P}}$ respectively, and then comparing the spaces $H^1_f(G_K, \mathcal{O}_E(\chi^{-1}\chi'))$ and $\mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}, \overline{\mathfrak{P}})$ by a counting argument. Unfortunately, Example 5.2.1 shows that the Kisin module corresponding to an element of the first group may not reduce to an element of the latter set, and so an additional argument is needed.

We consider instead all pairs of crystalline characters $\widetilde{\chi}', \widetilde{\chi} : G_K \to \mathcal{O}_E^\times$ with reductions $\overline{\chi}', \overline{\chi}$, and such that $\widetilde{\chi}' \oplus \widetilde{\chi}$ is pseudo-BT of weight $\{r_i\}$. We will show that there is a preferred choice $\chi'_{\min}, \chi_{\max}$ (with Kisin modules $\mathfrak{N}_{\min}, \mathfrak{P}_{\max}$ respectively) with the property that the reduction mod ϖ of any element of any $H^1_f(G_K, \mathcal{O}_E(\widetilde{\chi}^{-1}\widetilde{\chi}'))$ can be shown to occur as the image under $T_{\mathfrak{S}}$ of an element of $\mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}_{\min}, \overline{\mathfrak{P}}_{\max})$. Then we *can* proceed by comparing $H^1_f(G_K, \mathcal{O}_E(\chi^{-1}_{\max}\chi'_{\min}))$ with $\mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}_{\min}, \overline{\mathfrak{P}}_{\max})$.

The construction of $\chi'_{\min}, \chi_{\max}$ can be found in the proof of Theorem 5.4.1. In the rest of this subsection we begin to carry out the above strategy by proving the following proposition, which will allow us to compare the spaces $\mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}, \overline{\mathfrak{P}})$ (or at least the generic fibres of the Kisin modules in those spaces) for certain varying choices of $\overline{\mathfrak{N}}$ and $\overline{\mathfrak{P}}$.

5.2.3 Proposition. *Suppose that we are given Kisin modules*

$$\overline{\mathfrak{N}} := \overline{\mathfrak{M}}(s_0, \ldots, s_{f-1}; a) \quad \textit{and} \quad \overline{\mathfrak{P}} := \overline{\mathfrak{M}}(t_0, \ldots, t_{f-1}; b)$$

as well as

$$\overline{\mathfrak{N}}' := \overline{\mathfrak{M}}(s'_0, \ldots, s'_{f-1}; a) \quad \textit{and} \quad \overline{\mathfrak{P}}' := \overline{\mathfrak{M}}(t'_0, \ldots, t'_{f-1}; b)$$

such that

- *there exist nonzero maps* $\overline{\mathfrak{P}} \to \overline{\mathfrak{P}}'$ *and* $\overline{\mathfrak{N}}' \to \overline{\mathfrak{N}}$, *and*
- $s_i + t_i = s'_i + t'_i = r_i + e - 1$ *and* $\max\{s_i, t_i\}, \max\{s'_i, t'_i\} \geq r_i$ *for all* i.

For each $\overline{\mathfrak{M}} \in \mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}, \overline{\mathfrak{P}})$ *there exists* $\overline{\mathfrak{M}}' \in \mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}', \overline{\mathfrak{P}}')$ *such that* $T_{\mathfrak{S}}(\overline{\mathfrak{M}}) \cong T_{\mathfrak{S}}(\overline{\mathfrak{M}}')$.

Proof. The proof has two steps. First we construct $\overline{\mathfrak{M}}' \in \mathrm{Ext}^1(\overline{\mathfrak{N}}', \overline{\mathfrak{P}}')$ such that $T_{\mathfrak{S}}(\overline{\mathfrak{M}}) \cong T_{\mathfrak{S}}(\overline{\mathfrak{M}}')$, and then we check that in fact $\overline{\mathfrak{M}}' \in \mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}', \overline{\mathfrak{P}}')$.

We can choose bases e_i, f_i of the $\overline{\mathfrak{M}}_i$ so that φ has the form

$$\begin{aligned} \varphi(e_{i-1}) &= (b)_i u^{t_i} e_i \\ \varphi(f_{i-1}) &= (a)_i u^{s_i} f_i + y_i e_i \end{aligned}$$

with the elements y_i as in Theorem 5.1.5. First we define the Kisin module $\overline{\mathfrak{M}}'' \in \mathrm{Ext}^1(\overline{\mathfrak{N}}', \overline{\mathfrak{P}})$ by the formulas

$$\begin{aligned} \varphi(e''_{i-1}) &= (b)_i u^{t_i} e''_i \\ \varphi(f''_{i-1}) &= (a)_i u^{s'_i} f''_i + y''_i e''_i \end{aligned}$$

with $y''_i := u^{p(\alpha_{i-1}(\overline{\mathfrak{N}}') - \alpha_{i-1}(\overline{\mathfrak{N}}))} y_i$. It is easy to check that there is a morphism $g : \overline{\mathfrak{M}}'' \to \overline{\mathfrak{M}}$ sending $e''_i \mapsto e_i$ and $f''_i \mapsto u^{\alpha_i(\overline{\mathfrak{N}}') - \alpha_i(\overline{\mathfrak{N}})} f_i$ for all i, and since g induces an isomorphism $\overline{\mathfrak{M}}''[1/u] \xrightarrow{\sim} \overline{\mathfrak{M}}[1/u]$ of étale φ-modules, it induces an isomorphism $T_{\mathfrak{S}}(\overline{\mathfrak{M}}) \xrightarrow{\sim} T_{\mathfrak{S}}(\overline{\mathfrak{M}}'')$.

Next define the Kisin module $\overline{\mathfrak{M}}' \in \mathrm{Ext}^1(\overline{\mathfrak{N}}', \overline{\mathfrak{P}}')$ by the formulas

$$\begin{aligned} \varphi(e'_{i-1}) &= (b)_i u^{t'_i} e'_i \\ \varphi(f'_{i-1}) &= (a)_i u^{s'_i} f'_i + y'_i e'_i \end{aligned}$$

with $y'_i := u^{\alpha_i(\overline{\mathfrak{P}}) - \alpha_i(\overline{\mathfrak{P}}')} y''_i$. Again, it is easy to check that there is a morphism $g' : \overline{\mathfrak{M}}'' \to \overline{\mathfrak{M}}'$ sending $e''_i \mapsto u^{\alpha_i(\overline{\mathfrak{P}}) - \alpha_i(\overline{\mathfrak{P}}')} e'_i$ and $f''_i \mapsto f'_i$, and that g' induces an isomorphism $T_{\mathfrak{S}}(\overline{\mathfrak{M}}') \xrightarrow{\sim} T_{\mathfrak{S}}(\overline{\mathfrak{M}}'')$. Combining these calculations shows that indeed $T_{\mathfrak{S}}(\overline{\mathfrak{M}}) \cong T_{\mathfrak{S}}(\overline{\mathfrak{M}}'')$.

It remains to check that $\overline{\mathfrak{M}}' \in \mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}', \overline{\mathfrak{P}}')$. Using the characterisation following Definition 5.1.6, we wish to show for each i that there is a saturated element of $\varphi(\overline{\mathfrak{M}}'_{i-1})$ that is divisible by u^{r_i} in $\overline{\mathfrak{M}}'_i$. When $t'_i \geq r_i$ this is obvious (since $\varphi(e'_{i-1})$ will do), so we can suppose that $t'_i < r_i$ and so $s'_i \geq r_i$.

Recall that $y'_i = u^{\alpha_i(\overline{\mathfrak{P}})-\alpha_i(\overline{\mathfrak{P}}')+p(\alpha_{i-1}(\overline{\mathfrak{N}}')-\alpha_{i-1}(\overline{\mathfrak{N}}))}y_i$. If $\alpha_{i-1}(\overline{\mathfrak{N}}') > \alpha_{i-1}(\overline{\mathfrak{N}})$ then we are done, since u^p (hence also u^{r_i}) divides y'_i. We may therefore assume that $\alpha_{i-1}(\overline{\mathfrak{N}}') = \alpha_{i-1}(\overline{\mathfrak{N}})$. Since $s_i + t_i = s'_i + t'_i$ for all i, it follows that $\alpha_{i-1}(\overline{\mathfrak{P}}') = \alpha_{i-1}(\overline{\mathfrak{P}})$ as well. Note that whenever there is a map $\overline{\mathfrak{P}} \to \overline{\mathfrak{P}}'$ with $\alpha_{i-1}(\overline{\mathfrak{P}}') = \alpha_{i-1}(\overline{\mathfrak{P}})$, we must have $t_i \leq t'_i$. Indeed we have $t_i = p\alpha_{i-1}(\overline{\mathfrak{P}}) - \alpha_i(\overline{\mathfrak{P}})$ and $t'_i = p\alpha_{i-1}(\overline{\mathfrak{P}}') - \alpha_i(\overline{\mathfrak{P}}')$, so that $t'_i - t_i = \alpha_i(\overline{\mathfrak{P}}) - \alpha_i(\overline{\mathfrak{P}}') \geq 0$ by Lemma 5.1.2.

In particular we also have $t_i < r_i$, and the assumption that $\overline{\mathfrak{M}} \in \mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}, \overline{\mathfrak{P}})$ implies that every term of y_i has degree at least r_i except possibly for a term of degree t_i. (Note that in the case that there is an extra term of degree d_i, we have $d_i \geq s_i$, which is at least r_i since $t_i < r_i$.) Since y_i divides y'_i, we see that every term of y'_i has degree at least r_i except possibly for a term of degree

$$t_i + \alpha_i(\overline{\mathfrak{P}}) - \alpha_i(\overline{\mathfrak{P}}') + p(\alpha_{i-1}(\overline{\mathfrak{N}}') - \alpha_{i-1}(\overline{\mathfrak{N}})) = t_i + \alpha_i(\overline{\mathfrak{P}}) - \alpha_i(\overline{\mathfrak{P}}').$$

This quantity is easily seen to be congruent to $t'_i \pmod p$, so is either equal to t'_i or is at least r_i, and we are done, because we can subtract a constant multiple of $\varphi(e'_{i-1})$ from $\varphi(f'_{i-1})$ to obtain an element divisible by u^{r_i}. □

5.3. Maximal and minimal models We now construct the maximal and minimal Kisin modules to which we alluded in 5.2.2.

5.3.1 Lemma. *Fix integers $r_i \in [1, p]$. Suppose that $\overline{\chi} : G_K \to k_E^\times$ is a character and let $\mathcal{S}$ be the space of rank-one Kisin modules $\overline{\mathfrak{P}} = \overline{\mathfrak{M}}(t_0, \ldots, t_{f-1}; b)$ such that*

- $T_{\mathfrak{S}}(\overline{\mathfrak{P}}) = \overline{\chi}|_{G_{K_\infty}}$, *and*
- $t_i \in [0, e-1] \cup [r_i, r_i + e - 1]$ *for all i.*

If $\mathcal{S} \neq \varnothing$, then $\mathcal{S}$ contains a maximal model. That is, there exists $\overline{\mathfrak{P}}_{\max} \in \mathcal{S}$ such that there is a nontrivial map $\overline{\mathfrak{P}} \to \overline{\mathfrak{P}}_{\max}$ for all $\overline{\mathfrak{P}} \in \mathcal{S}$.

Proof. Assume that $\mathcal{S}$ is non-empty. Then there exists some $\overline{\mathfrak{M}}(t'_0, \ldots, t'_{f-1}; b) \in \mathcal{S}$, and every other element of $\mathcal{S}$ has the form $\overline{\mathfrak{M}}(t_0, \ldots, t_{f-1}; b)$ (i.e. the b is the same).

Write $\overline{\chi}|_{I_K} = \prod_i \omega_i^{m_i}$ with $m_i \in [0, p-1]$ and not all equal to $p-1$. For $0 \leq i \leq f-2$, let v_i be the f-tuple $(0, \ldots, -1, p, \ldots, 0)$ with the -1 in the ith position (where the leftmost position is the zeroth), and similarly let v_{f-1} be the f-tuple $(p, 0, \ldots, 0, -1)$. It is straightforward to see from Lemma 3.1.2 that if $\overline{\mathfrak{M}}(t_0, \ldots, t_{f-1}; b) \in \mathcal{S}$, then

$$(t_0, \ldots, t_{f-1}) = (m_0, \ldots, m_{f-1}) + \sum_{i=0}^{f-1} c_i v_i \tag{5.3.2}$$

with $c_i \in \mathbb{Z}_{\geq 0}$ for all i.

If we have $m_i \in \mathcal{I}_i := [0, e-1] \cup [r_i, r_i + e - 1]$ for all i, then it is clear that $\overline{\mathfrak{P}}_{\max} := \overline{\mathfrak{M}}(m_0, \ldots, m_{f-1}; b)$ is a maximal model, e.g. because $\alpha_i(\overline{\mathfrak{P}}_{\max}) \leq \alpha_i(\overline{\mathfrak{P}})$ for all $\overline{\mathfrak{P}} \in \mathcal{S}$. This always holds for instance if $e \geq p$, or if $\overline{\chi}$ is unramified, so let us assume for the rest of the proof that $e \leq p-1$ and that $\overline{\chi}$ is ramified, so that not every m_i is equal to 0.

With these additional hypothesis, we have $r_i + e - 1 \leq 2p - 2$ for all i, and also the integers c_i in (5.3.2) must all be 0 or 1. If $\overline{\mathfrak{P}} = \overline{\mathfrak{M}}(t_0, \ldots, t_{f-1}; b)$ with integers c_i as in (5.3.2), write $J(\overline{\mathfrak{P}}) := \{i : c_i \neq 0\}$. To complete the proof, it suffices to show that there exists a subset $J \subseteq \{0, \ldots, f-1\}$ such that

- if $\overline{\mathfrak{P}} \in \mathcal{S}$ then $J \subseteq J(\overline{\mathfrak{P}})$, and
- there exists $\overline{\mathfrak{P}}' \in \mathcal{S}$ such that $J = J(\overline{\mathfrak{P}}')$,

for then $\overline{\mathfrak{P}}'$ is $\overline{\mathfrak{P}}_{\max}$. We construct the set J as follows. Note that if x is a non-negative integer with $x \notin \mathcal{I}_i$, then $x \geq e$, so that also $x + p \notin \mathcal{I}_i$.

Let $K \subseteq \{0, \ldots, f-1\}$ be any set with the property that if $\overline{\mathfrak{P}} \in \mathcal{S}$ then $K \subseteq J(\overline{\mathfrak{P}})$. Write $(m'_0, \ldots, m'_{f-1}) = (m_0, \ldots, m_{f-1}) + \sum_{i \in K} v_i$. Let $\overline{\mathfrak{P}} = \overline{\mathfrak{M}}(t_0, \ldots, t_{f-1}; b)$ be any element of $\mathcal{S}$ (here we use our assumption that $\mathcal{S}$ is non-empty). Suppose first that $i \in K$. Observe that $t_i \in \{m'_i, m'_i + p\}$, and since $t_i \in \mathcal{I}_i$ it follows by the last sentence in the previous paragraph that $m'_i \in \mathcal{I}_i$. Now suppose instead that $i \notin K$. Observe that $t_i \in \{m'_i, m'_i + p, m'_i - 1, m'_i + p - 1\} \cap \mathcal{I}_i$. If in fact $m'_i \notin \mathcal{I}_i$, it follows by the last sentence in the previous paragraph that $t_i \neq m'_i, m'_i + p$, and therefore $i \in J(\overline{\mathfrak{P}})$. Write $\Delta(K) = \{i \notin K : m'_i \notin \mathcal{I}_i\}$; it follows that $K \cup \Delta(K)$ is another set that is contained in $J(\overline{\mathfrak{P}})$ for all $\overline{\mathfrak{P}} \in \mathcal{S}$.

Set $J_0 = \varnothing$, and iteratively define $J_i = J_{i-1} \cup \Delta(J_{i-1})$ for $i \geq 1$. Eventually this process stabilizes at some J_n. By construction $J_n \subseteq J(\overline{\mathfrak{P}})$ for all $\overline{\mathfrak{P}} \in S$. Again write $(m'_0, \ldots, m'_{f-1}) = (m_0, \ldots, m_{f-1}) + \sum_{i \in J_n} v_i$. We claim that $m'_i \in \mathcal{I}_i$ for all i, so that J_n is the desired set J: this is automatic if $i \in J_n$ (by the first observation in the previous paragraph), and if $i \notin J_n$ follows from the fact that $\Delta(J_n) = \varnothing$. □

Evidently we must also have the following lemma, which essentially is dual to the previous one.

5.3.3 Lemma. *Fix integers $r_i \in [1, p]$. Suppose that $\overline{\chi}' : G_K \to k_E^\times$ is a character and let $\mathcal{S}$ be the space of rank-one Kisin modules $\overline{\mathfrak{N}} = \overline{\mathfrak{M}}(s_0, \ldots, s_{f-1}; a)$ such that*

- $T_{\mathfrak{S}}(\overline{\mathfrak{N}}) = \overline{\chi}'|_{G_{K_\infty}}$, *and*
- $s_i \in [0, e-1] \cup [r_i, r_i + e - 1]$ *for all i.*

If $\mathcal{S} \neq \varnothing$, then $\mathcal{S}$ contains a minimal model. That is, there exists $\overline{\mathfrak{N}}_{\min} \in \mathcal{S}$ such that there is a nontrivial map $\overline{\mathfrak{N}}_{\min} \to \overline{\mathfrak{N}}$ for all $\overline{\mathfrak{N}} \in \mathcal{S}$.

Proof. Indeed, if $\overline{\chi} : G_K \to k_E^\times$ is a character such that $\overline{\chi\chi'}|_{I_K} = \prod_i \omega_i^{r_i+e-1}$ and $\overline{\mathfrak{M}}(t_0, \ldots, t_{f-1}; b)$ is the maximal model for $\overline{\chi}$ given by the previous lemma, then the desired minimal model is given by $\overline{\mathfrak{M}}(s_0, \ldots, s_{f-1}; a)$ with $s_i = (r_i + e - 1) - t_i$. □

Combining Proposition 5.2.3 with the above lemmas, we obtain the following result, which was promised in 5.2.2.

5.3.4 Proposition. *Fix characters $\overline{\chi}', \overline{\chi} : G_K \to k_E^\times$ and integers $r_i \in [1, p]$. There exist rank-one Kisin modules $\overline{\mathfrak{N}}_{\min}$ and $\overline{\mathfrak{P}}_{\max}$ with the following property: if $\bar{r} \in \mathrm{Ext}^1_{k_E[G_K]}(\overline{\chi}, \overline{\chi}')$ is the reduction modulo ϖ of an O_E-lattice in a pseudo-BT representation of weight $\{r_i\}$, then $\bar{r}|_{G_{K_\infty}} \simeq T_{\mathfrak{S}}(\overline{\mathfrak{M}})$ for some $\overline{\mathfrak{M}} \in \mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}_{\min}, \overline{\mathfrak{P}}_{\max})$.*

Moreover, if we write $\overline{\mathfrak{N}}_{\min} = \overline{\mathfrak{M}}(s_0, \ldots, s_{f-1}; a)$ and $\overline{\mathfrak{P}}_{\max} = \overline{\mathfrak{M}}(t_0, \ldots, t_{f-1}; b)$, then $\overline{\mathfrak{N}}_{\min}$ and $\overline{\mathfrak{P}}_{\max}$ can be chosen so that for all i we have $s_i + t_i = r_i + e - 1$ and $s_i, t_i \in [0, e-1] \cup [r_i, r_i + e - 1]$.

Proof. If no such extensions $\bar{r}$ exist, then the Proposition is vacuously true, so we may suppose that some $\bar{r}$ exists as in the statement of the Proposition. As in the proof of Theorem 3.1.4, there exist rank-one Kisin modules $\overline{\mathfrak{N}}, \overline{\mathfrak{P}}$ such that $\bar{r}|_{G_{K_\infty}} \simeq T_{\mathfrak{S}}(\overline{\mathfrak{M}})$ for some $\overline{\mathfrak{M}} \in \mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}, \overline{\mathfrak{P}})$. In particular the set $\mathcal{S}$ of Lemma 5.3.1 is nonempty (it contains $\overline{\mathfrak{P}}$, or possibly $\overline{\mathfrak{N}}$ if $\bar{r}|_{G_{K_\infty}}$ is split), and similarly the set $\mathcal{S}$ of Lemma 5.3.3 is nonempty. Let $\overline{\mathfrak{P}}_{\max}$ and $\overline{\mathfrak{N}}_{\min}$ be the maximal and minimal models given, respectively, by those lemmas. (Note that these depend only on $\overline{\chi}, \overline{\chi}'$, and the integers $\{r_i\}$, and not on $\bar{r}$.)

The fact that $s_i, t_i \in [0, e-1] \cup [r_i, r_i + e - 1]$ is given to us by Lemmas 5.3.1 and 5.3.3, and the equality $s_i + t_i = r_i + e - 1$ comes from the proof of Lemma 5.3.3. Now an application of Proposition 5.2.3 gives the claim in the first paragraph of the Proposition when $\bar{r}|_{G_{K_\infty}}$ is non-split, while if $\bar{r}|_{G_{K_\infty}}$ is split we can take $\overline{\mathfrak{M}} = \overline{\mathfrak{P}}_{\max} \oplus \overline{\mathfrak{N}}_{\min}$. □

5.4. The non-cyclotomic case of the weight part of Serre's conjecture We are now ready to prove the following result.

5.4.1 Theorem. *Suppose that $p \geq 3$ and that $\bar{r} : G_K \to \mathrm{GL}_2(k_E)$ is reducible, and write $\bar{r} \simeq \begin{pmatrix} \overline{\chi}' & * \\ 0 & \overline{\chi} \end{pmatrix}$. If $\overline{\chi}^{-1}\overline{\chi}' \neq \overline{\varepsilon}$, then $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$. Moreover, these sets do not depend on our choice of embeddings $\kappa_{i,0}$.*

Proof. Suppose that $\sigma \in W^{\mathrm{cris}}(\bar{r})$. We may freely enlarge E so that $\bar{r}$ has a crystalline lift of Hodge type σ defined over E. After twisting, we may assume that σ has the shape $\otimes_i \mathrm{Sym}^{r_i-1} k^2 \otimes_{k,\bar{\kappa}_i} \overline{\mathbb{F}}_p$ with integers $r_i \in [1, p]$, and the hypothesis that $\sigma \in W^{\mathrm{cris}}(\bar{r})$ means that $\bar{r}$ is the reduction modulo p of a lattice in a pseudo-BT representation of weight $\{r_i\}$. Let $\overline{\mathfrak{N}}_{\min} = \overline{\mathfrak{M}}(s_0, \ldots, s_{f-1}; a)$ and $\overline{\mathfrak{P}}_{\max} = \overline{\mathfrak{M}}(t_0, \ldots, t_{f-1}; b)$ be the rank-one Kisin modules given to us by Proposition 5.3.4 applied to $\overline{\chi}', \overline{\chi}$, and $\{r_i\}$.

We construct a pair of crystalline characters $\chi'_{\min}, \chi_{\max} : G_K \to \mathcal{O}_E^\times$ as follows. If $t_i \geq r_i$, we take the ordered pair $(\mathrm{HT}_{\kappa_{i,0}}(\chi_{\max}), \mathrm{HT}_{\kappa_{i,0}}(\chi'_{\min})) = (r_i, 0)$; and for $j > 0$ we take the pair $(\mathrm{HT}_{\kappa_{ij}}(\chi_{\max}), \mathrm{HT}_{\kappa_{ij}}(\chi'_{\min}))$ to be $(1, 0)$ for exactly $t_i - r_i$ values of j and to be $(0, 1)$ for exactly s_i values of j. On the other hand if $t_i < r_i$, we take the pair $(\mathrm{HT}_{\kappa_{i,0}}(\chi_{\max}), \mathrm{HT}_{\kappa_{i,0}}(\chi'_{\min})) = (0, r_i)$; and for $j > 0$ we take the pair $(\mathrm{HT}_{\kappa_{ij}}(\chi_{\max}), \mathrm{HT}_{\kappa_{ij}}(\chi'_{\min}))$ to be $(1, 0)$ for exactly t_i values of j and to be $(0, 1)$ for exactly $s_i - r_i$ values of j. It is then possible to choose the unramified parts of $\chi'_{\min}, \chi_{\max}$ so that they reduce to $\overline{\chi}'$ and $\overline{\chi}$ respectively.

Let us consider the extensions T of $\mathcal{O}_E(\chi_{\max})$ by $\mathcal{O}_E(\chi'_{\min})$ such that $T[1/p]$ is crystalline. Each of these is pseudo-BT of weight $\{r_i\}$, and so (by the defining property of $\overline{\mathfrak{N}}_{\min}, \overline{\mathfrak{P}}_{\max}$ from Proposition 5.3.4) we must have $\overline{T}|_{G_{K_\infty}} \simeq T_{\mathfrak{S}}(\overline{\mathfrak{M}})$ for some $\overline{\mathfrak{M}} \in \mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}_{\min}, \overline{\mathfrak{P}}_{\max})$.

The space of crystalline extensions of $\chi_{\max}$ by $\chi'_{\min}$, which we identify with $H^1_f(G_K, E(\chi_{\max}^{-1}\chi'_{\min}))$, has dimension equal to the number of labeled Hodge–Tate weights of $\chi'_{\min}$ that exceed the corresponding weight of $\chi_{\max}$. This is precisely

$$d = \sum_{i\,:\,t_i \geq r_i} s_i + \sum_{i\,:\,t_i < r_i} (1 + s_i - r_i).$$

It follows as in [GLS14, Lem. 9.3] that the image of $H^1_f(G_K, \mathcal{O}_E(\chi_{\max}^{-1}\chi'_{\min}))$ in $H^1(G_K, \overline{\chi}^{-1}\overline{\chi}')$ has dimension d if $\overline{\chi} \neq \overline{\chi}'$ and dimension $d + 1$ if $\overline{\chi} = \overline{\chi}'$.

By Lemma 5.4.2 below, the restriction map $H^1(G_K, \overline{\chi}^{-1}\overline{\chi}') \to H^1(G_{K_\infty}, \overline{\chi}^{-1}\overline{\chi}')$ is injective. (The application of Lemma 5.4.2 is the only place in the argument that we need our assumption that $\overline{\chi}^{-1}\overline{\chi}' \neq \overline{\varepsilon}$.) It follows that the number of elements of $H^1(G_{K_\infty}, \overline{\chi}^{-1}\overline{\chi}')$ that come from a crystalline extension of $\chi_{\max}$ by $\chi'_{\min}$ is exactly $|k_E|^{d+\delta}$, where $\delta = 0$ if $\overline{\chi} \neq \overline{\chi}'$ and $\delta = 1$ otherwise.

On the other hand, we know from Proposition 5.3.4 that the number of elements of $H^1(G_{K_\infty}, \overline{\chi}^{-1}\overline{\chi}')$ that come from the reduction mod p of some pseudo-BT representation of weight $\{r_i\}$ is *at most* $\#\mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}_{\min}, \overline{\mathfrak{P}}_{\max})$. One easily checks by counting, in the explicit description of the extensions in Theorem 5.1.5, the number of terms in each y_i that are permitted to be non-zero (and noting that if $\overline{\chi} = \overline{\chi}'$ there must exist a map $\overline{\mathfrak{N}}_{\min} \to \overline{\mathfrak{P}}_{\max}$, by the maximality of $\overline{\mathfrak{P}}_{\max}$) that $\#\mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}_{\min}, \overline{\mathfrak{P}}_{\max}) = |k_E|^{d+\delta}$ as well.

In particular every element of $H^1(G_{K_\infty}, \overline{\chi}^{-1}\overline{\chi}')$ that comes from the reduction mod p of some pseudo-BT representation of weight $\{r_i\}$ must in fact come from a lattice in a crystalline extension of $\chi_{\max}$ by $\chi'_{\min}$. Applying Lemma 5.4.2 again, the same must be true of every element of $H^1(G_K, \overline{\chi}^{-1}\overline{\chi}')$ that comes from the reduction mod p of some pseudo-BT representation of weight $\{r_i\}$. Since in particular $\bar{r}$ is such an element, we deduce that $\sigma \in W^{\mathrm{explicit}}(\bar{r})$, as desired.

Finally, note that the Kisin modules $\overline{\mathfrak{N}}_{\min}$ and $\overline{\mathfrak{P}}_{\max}$ do not depend on the choice of embeddings $\kappa_{i,0}$. Then the independence of $W^{\mathrm{explicit}}(\bar{r})$ from the choice of embeddings $\kappa_{i,0}$ follows from the characterisation that $\sigma \in W^{\mathrm{explicit}}(\bar{r})$ if and only if $\bar{r}|_{G_{K_\infty}} = T_{\mathfrak{S}}(\overline{\mathfrak{M}})$ for some $\overline{\mathfrak{M}} \in \mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}_{\min}, \overline{\mathfrak{P}}_{\max})$. □

5.4.2 Lemma. *Let $\overline{\chi} : G_K \to k_E^\times$ be a continuous character. If $\overline{\chi} \neq \overline{\varepsilon}$, then the restriction map*

$$H^1(G_K, \overline{\chi}) \to H^1(G_{K_\infty}, \overline{\chi})$$

is injective. If $\overline{\chi} = \overline{\varepsilon}$, then the kernel of the restriction map is the très ramifiée line determined by the fixed uniformiser π.

Proof. If $\overline{\chi} \neq 1, \overline{\varepsilon}$, this is a special case of [EGS, Lem. 7.4.3]. If $\overline{\chi} = 1$, then $H^1(G_K, \overline{\chi}) = \mathrm{Hom}(G_K, k_E)$ and $H^1(G_{K_\infty}, \overline{\chi}) = \mathrm{Hom}(G_{K_\infty}, k_E)$, so if the kernel of the restriction map is nonzero, there must be a Galois extension of K of degree p contained in K_∞. This can only happen if K contains a primitive pth root of unity, in which case $\overline{\varepsilon} = 1$, so $\overline{\chi} = \overline{\varepsilon}$.

Finally, suppose that $\overline{\chi} = \overline{\varepsilon}$. Kummer theory identifies the restriction map with the natural map

$$K^\times/(K^\times)^p \to K_\infty^\times/(K_\infty^\times)^p,$$

and the kernel of this map is evidently generated by π. □

6. The weight part of Serre's conjecture III: the general case

6.1. $(\varphi, \hat{G})$-modules In order to complete our arguments in the remaining case, we will need to make use of the second author's theory of $(\varphi, \hat{G})$-modules. We refer the reader to [GLS14, §5.1] (specifically, from the start of that section up to the statement of Theorem 5.2) for the definitions and notation that we will use, as well as to [GLS14, (4.8)] for the definition of the operator τ.

6.1.1. Consider a $(\varphi, \hat{G})$-module with natural k_E-action $\hat{\overline{\mathfrak{M}}}$, sitting in an extension of $(\varphi, \hat{G})$-modules with natural k_E-action

$$0 \to \hat{\overline{\mathfrak{P}}} \to \hat{\overline{\mathfrak{M}}} \to \hat{\overline{\mathfrak{N}}} \to 0,$$

where the underlying Kisin modules $\overline{\mathfrak{N}}, \overline{\mathfrak{P}}$ are given by $\overline{\mathfrak{N}} = \overline{\mathfrak{M}}(s_0, \ldots, s_{f-1}; a)$ and $\overline{\mathfrak{P}} = \overline{\mathfrak{M}}(t_0, \ldots, t_{f-1}; b)$ for some a, b, with $\{s_i, t_i\} = \{r_i + x_i, e - 1 - x_i\}$, and the underlying extension of Kisin modules is in $\mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}, \overline{\mathfrak{P}})$. We say that such a $(\varphi, \hat{G})$-module is of *reducible pseudo-BT type and weight* $\{r_i\}$ if for all $x \in \overline{\mathfrak{M}}$ there exist $\alpha \in R$ and $y \in R \otimes_{\varphi, S} \overline{\mathfrak{M}}$ such that $\tau(x) - x = \alpha y$ and $v_R(\alpha) \geq \frac{p}{p-1} + \frac{p}{e}$.

6.1.2 Lemma. *Suppose that $p \geq 3$, and that $\bar{r} : G_K \to \mathrm{GL}_2(k_E)$ is reducible and arises as the reduction mod p of a pseudo-BT representation r of weight $\{r_i\}$. Then there is a $(\varphi, \hat{G})$-module with natural k_E-action $\hat{\overline{\mathfrak{M}}}$ such that $\hat{\overline{\mathfrak{M}}}$ is of reducible pseudo-BT type and weight $\{r_i\}$, and $\hat{T}(\hat{\overline{\mathfrak{M}}}) \cong \bar{r}$.*

Proof. Let $\hat{\overline{\mathfrak{M}}}$ be the $(\varphi, \hat{G})$-module arising as the reduction mod p of the $(\varphi, \hat{G})$-module corresponding to r by [GLS14, Thm. 5.2(2)]. Then the underlying Kisin module $\overline{\mathfrak{M}}$ is of the required kind by Theorem 5.1.5, and this extension can be extended to an extension of $(\varphi, \hat{G})$-modules by [GLS14, Lem. 5.5]. Finally, the claim about the action of τ is immediate from [GLS14, Cor. 5.10]. □

We have the following generalisation of [GLS14, Lem. 8.1].

6.1.3 Lemma. *Suppose that $p \geq 3$ and take $\overline{\mathfrak{M}} \in \mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}, \overline{\mathfrak{P}})$. Except possibly for the case that $r_i = p$ and $t_i = 0$ for all $i = 0, \ldots, f-1$, there is at most one $(\varphi, \hat{G})$-module $\hat{\overline{\mathfrak{M}}}$ of reducible pseudo-BT type and weight $\{r_i\}$ with underlying Kisin module $\overline{\mathfrak{M}}$.*

Proof. We follow the proof of [GLS14, Lem. 8.1]. Since by definition $\overline{\mathfrak{M}}$ is contained in the H_K-invariants of $\hat{\overline{\mathfrak{M}}}$, it suffices to show that the τ-action on $\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \overline{\mathfrak{M}}$ is uniquely determined. As usual we write e_i, f_i for a basis of $\overline{\mathfrak{M}}_i$ as given by Theorem 5.1.5. We can write

$$\tau(e_{i-1}, f_{i-1}) = (e_{i-1}, f_{i-1}) \begin{pmatrix} \delta_i & \beta_i \\ 0 & \gamma_i \end{pmatrix}$$

with $\delta_i, \beta_i, \gamma_i \in (\widehat{\mathcal{R}}/p\widehat{\mathcal{R}}) \otimes_{\mathbb{F}_p} k_E \subset R \otimes_{\mathbb{F}_p} k_E$. If $\zeta \in R \otimes_{\mathbb{F}_p} k_E$ is written $\zeta = \sum_{i=1}^n y_i \otimes z_i$ with $z_1, \ldots, z_n \in k_E$ linearly independent over $\mathbb{F}_p$, write $v_R(\zeta) = \min_i \{v_R(y_i)\}$. By assumption, we have $v_R(\delta_i - 1), v_R(\gamma_i - 1), v_R(\beta_i) \geq \frac{p}{p-1} + \frac{p}{e}$ for all i.

Recalling that $\overline{\mathfrak{M}}$ is regarded as a $\varphi(\mathfrak{S})$-submodule of $\widehat{\mathcal{R}}\otimes_{\varphi,\mathfrak{S}}\overline{\mathfrak{M}}$, we may write $\varphi(e_{i-1}, f_{i-1}) = (e_i, f_i)\varphi(A_i)$ with $A_i = \begin{pmatrix} (b)_i u^{t_i} & x_i \\ 0 & (a)_i u^{s_i} \end{pmatrix}$. Since φ and τ commute, we have

$$\varphi(A_i)\begin{pmatrix} \varphi(\delta_i) & \varphi(\beta_i) \\ 0 & \varphi(\gamma_i) \end{pmatrix} = \begin{pmatrix} \delta_{i+1} & \beta_{i+1} \\ 0 & \gamma_{i+1} \end{pmatrix}\tau(\varphi(A_i)).$$

We obtain the following formulas:

$$u^{pt_i}\varphi(\delta_i) = \delta_{i+1}(\underline{\epsilon}u)^{pt_i}, \quad u^{ps_i}\varphi(\gamma_i) = (\underline{\epsilon}u)^{ps_i}\gamma_{i+1} \tag{6.1.4}$$

and

$$(b)_i u^{pt_i}\varphi(\beta_i) + \varphi(x_i)\varphi(\gamma_i) = \delta_{i+1}\tau(\varphi(x_i)) + (a)_i(\underline{\epsilon}u)^{ps_i}\beta_{i+1} \tag{6.1.5}$$

where for succinctness we have written $(a)_i$, $(b)_i$ in lieu of $1\otimes(a)_i$, $1\otimes(b)_i$ in the preceding equation.

Let $\eta \in R$ be the element defined in [GLS14, Lem. 6.6(2)], so that $\varphi^f(\eta) = \underline{\epsilon}\eta$. ($K/\mathbb{Q}_p$ is assumed to be unramified throughout [GLS14, §6], but it is easily checked that [GLS14, Lem. 6.6(2)] remains valid with the same proof in our setting.) From (6.1.4) we see that $\varphi^f(\delta_i) = \delta_i\underline{\epsilon}^{\sum_{j=0}^{f-1} p^{f-j}t_{i+j}}$, and now [GLS14, Lem. 6.6(2)] together with the requirement that $v_R(\delta_i - 1) > 0$ implies that

$$\delta_i = \eta^{\sum_{j=0}^{f-1} p^{f-j}t_{i+j}} \otimes 1$$

for all i. Similarly we must have $\gamma_i = \eta^{\sum_{j=0}^{f-1} p^{f-j}s_{i+j}} \otimes 1$ for all i. So at least the δ_i, γ_i are uniquely determined.

Now suppose that there exists some other extension of $\overline{\mathfrak{M}}$ to a $(\varphi, \hat{G})$-module $\hat{\overline{\mathfrak{M}}}'$. Then the τ-action on $\hat{\overline{\mathfrak{M}}}'$ is given by some δ'_i, β'_i and γ'_i that also satisfy (6.1.4) and (6.1.5), and indeed we have already seen that $\delta'_i = \delta_i$ and $\gamma'_i = \gamma_i$.

Let $\tilde{\beta}_i = \beta_i - \beta'_i$. Taking the difference between (6.1.5) for $\hat{\overline{\mathfrak{M}}}$ and $\hat{\overline{\mathfrak{M}}}'$ gives

$$(b)_i u^{pt_i}\varphi(\tilde{\beta}_i) = (a)_i(\underline{\epsilon}u)^{ps_i}\tilde{\beta}_{i+1},$$

which implies that

$$bu^{\sum_{j=0}^{f-1} u^{p^{f-j}t_{i+j}}}\varphi^f(\tilde{\beta}_i) = a(\underline{\epsilon}u)^{\sum_{j=0}^{f-1} u^{p^{f-j}s_{i+j}}}\tilde{\beta}_i.$$

Considering the valuations of both sides, and recalling that $v_R(\underline{\pi}) = 1/e$, we see that if $\widetilde{\beta_i} \neq 0$ then

$$v_R(\tilde{\beta}_i) = \frac{1}{e(p^f - 1)}\sum_{j=0}^{f-1} p^{f-j}(s_{i+j} - t_{i+j}). \tag{6.1.6}$$

But since $s_i - t_i \leq r_i + e - 1$ is at most $p + e - 1$ with equality if and only if $t_i = 0$ and $r_i = p$, the right-hand side of (6.1.6) is at most $\frac{p}{e(p-1)}(p + e - 1) = \frac{p}{p-1} + \frac{p}{e}$ with equality if and only if $t_i = 0$ and $r_i = p$ for all i. In particular, since $v_R(\beta_i), v_R(\beta'_i) \geq \frac{p}{p-1} + \frac{p}{e}$, either $\beta_i = \beta'_i$ for all i, or else $t_i = 0$ and $r_i = p$ for all i, as required. □

6.1.7 Proposition. *Suppose that $p \geq 3$ and let $\hat{\overline{\mathfrak{M}}}$ be a $(\varphi, \hat{G})$-module of reducible pseudo-BT type and weight $\{r_i\}$ with underlying Kisin module $\overline{\mathfrak{M}}$. Suppose we have $\overline{\mathfrak{N}}', \overline{\mathfrak{P}}'$ as in the statement of Proposition 5.2.3 and let $\overline{\mathfrak{M}}'$ be the Kisin module provided by that proposition. Then there is a $(\varphi, \hat{G})$-module $\hat{\overline{\mathfrak{M}}}'$ of reducible pseudo-BT type and weight $\{r_i\}$ with underlying Kisin module $\overline{\mathfrak{M}}'$, such that $\hat{T}(\hat{\overline{\mathfrak{M}}}) \cong \hat{T}(\hat{\overline{\mathfrak{M}}}')$.*

Proof. From the proof of Proposition 5.2.3, we see that there is a Kisin module $\overline{\mathfrak{M}}''$ and morphisms $g : \overline{\mathfrak{M}}'' \to \overline{\mathfrak{M}}$, $g' : \overline{\mathfrak{M}}'' \to \overline{\mathfrak{M}}'$, both of which induce isomorphisms after inverting u. Using these isomorphisms, the $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \overline{\mathfrak{M}}$ induces a $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \overline{\mathfrak{M}}'[1/u]$, and it is enough to show that this preserves $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \overline{\mathfrak{M}}'$ and makes it a $(\varphi, \hat{G})$-module of reducible pseudo-BT type and weight $\{r_i\}$.

Since H_K acts trivially on u and $\overline{\mathfrak{M}}$, and since $\tau(u) = \underline{\epsilon}u$ and $(\epsilon - 1) \in I_+$, we see that we only need to check that $\tau(\overline{\mathfrak{M}}') \subset \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \overline{\mathfrak{M}}'$, and that for all $x \in \overline{\mathfrak{M}}'$ there exists $\alpha \in R$ and $y \in R \otimes_{\varphi,\mathfrak{S}} \overline{\mathfrak{M}}'$ such that $\tau(x) - x = \alpha y$ and $v_R(\alpha) \geq \frac{p}{p-1} + \frac{p}{e}$.

Take bases e_i, f_i of $\overline{\mathfrak{M}}_i$ and e'_i, f'_i of $\overline{\mathfrak{M}}'_i$ as in the proof of Proposition 5.2.3. Writing

$$\tau(e_{i-1}, f_{i-1}) = (e_{i-1}, f_{i-1}) \begin{pmatrix} \delta_i & \beta_i \\ 0 & \gamma_i \end{pmatrix},$$

an easy calculation shows that

$$\tau(e'_{i-1}, f'_{i-1}) = (e'_{i-1}, f'_{i-1}) \begin{pmatrix} \delta'_i & \beta'_i \\ 0 & \gamma'_i \end{pmatrix},$$

where

$$\delta'_i = \delta_i \underline{\epsilon}^{p(\alpha_{i-1}(\overline{\mathfrak{P}}') - \alpha_{i-1}(\overline{\mathfrak{P}}))},$$

$$\beta'_i = \beta_i (u^{p(\alpha_{i-1}(\overline{\mathfrak{N}}') - \alpha_{i-1}(\overline{\mathfrak{N}}) + \alpha_{i-1}(\overline{\mathfrak{P}}) - \alpha_{i-1}(\overline{\mathfrak{P}}'))} \otimes 1) \underline{\epsilon}^{p(\alpha_{i-1}(\overline{\mathfrak{N}}') - \alpha_{i-1}(\overline{\mathfrak{N}}))},$$

$$\gamma'_i = \gamma_i \underline{\epsilon}^{p(\alpha_{i-1}(\overline{\mathfrak{N}}') - \alpha_{i-1}(\overline{\mathfrak{N}}))}.$$

(The factors of p in the above exponents come from the fact that τ acts on the left-hand side of the twisted tensor product $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \overline{\mathfrak{M}}'$.) Now, $\underline{\epsilon}$ is a unit, so it is enough to check that the exponent of u in the expression for β'_i is non-negative; but this is immediate from Lemma 5.1.2. □

We are now in a position to prove that $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$ in the reducible cyclotomic case, and so to deduce this equality in full generality (for $p \neq 2$).

6.1.8 Theorem. *Suppose that $p \geq 3$ and that $\bar{r} : G_K \to \mathrm{GL}_2(k_E)$ is a continuous representation. Then $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{BT}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$. Moreover, these sets do not depend on our choice of embeddings $\kappa_{i,0}$.*

Proof. Recall that we have inclusions $W^{\mathrm{explicit}}(\bar{r}) \subseteq W^{\mathrm{BT}}(\bar{r}) \subseteq W^{\mathrm{cris}}(\bar{r})$ by [GK, Cor. 4.5.7], so it is enough to check that $W^{\mathrm{cris}}(\bar{r}) \subseteq W^{\mathrm{explicit}}(\bar{r})$. Suppose that $\sigma \in W^{\mathrm{cris}}(\bar{r})$. As in the proof of Theorem 5.4.1, we may assume that σ has the shape $\otimes_i \mathrm{Sym}^{r_i-1} k^2 \otimes_{k,\bar{\kappa}_i} \overline{\mathbb{F}}_p$ with integers $r_i \in [1, p]$, and that $\bar{r}$ is the reduction modulo p of a lattice in a pseudo-BT representation of weight $\{r_i\}$. By Theorem 4.1.6 we may assume that $\bar{r}$ is an extension of $\overline{\chi}$ by $\overline{\chi}'$; by Theorem 5.4.1, we may assume that $\overline{\chi}^{-1}\overline{\chi}' = \bar{\varepsilon}$.

Suppose first that not all of the r_i are equal to p. We may replace the appeals to Lemma 5.4.2 in the proof of Theorem 5.4.1 with appeals to Lemma 6.1.2, Lemma 6.1.3 and Proposition 6.1.7; then the count of Kisin modules remains valid, and the argument goes through as before. (Recall that the only place that the assumption that $\overline{\chi}^{-1}\overline{\chi}' \neq \bar{\varepsilon}$ was used in the proof of Theorem 5.4.1 was in the appeals to Lemma 5.4.2. In particular the construction of $\chi_{\max}$ and $\chi'_{\min}$ carries over to the case that $\overline{\chi}^{-1}\overline{\chi}' = \varepsilon$.)

Finally, in the case that all of the r_i are equal to p, the same argument applies unless $\bar{r}$ has a model where all of the $t_i = 0$. In this case, we see that the character $\chi_{\max}$ in the proof of Theorem 5.4.1 is unramified, while $\mathrm{HT}_{\kappa_{i,0}}(\chi'_{\min}) = p$ and $\mathrm{HT}_{\kappa_{i,j}}(\chi'_{\min}) = 1$ if $j > 0$.

Then every extension of $\chi_{\max}$ by an unramified twist of $\chi'_{\min}$ is automatically crystalline. So, it suffices to show that *any* extension of $\overline{\chi}$ by $\overline{\chi}'$ lifts to an extension of $\chi_{\max}$ by a twist of $\chi'_{\min}$ by an unramified character with trivial reduction. This may be proved by exactly the same argument as [GLS14, Lem. 9.4] (*cf.* [GLS12, Prop. 5.2.9], which proves the claim in the case that $K/\mathbb{Q}_p$ is totally ramified). □

6.1.9 Remark. Suppose that $\bar{r}$ is an extension of $\overline{\chi}$ by $\overline{\chi}'$, and let $\chi_{\max}, \chi'_{\min}$ be the crystalline lifts of $\overline{\chi}, \overline{\chi}'$ constructed in the proof of Theorem 5.4.1. Recall that there is a choice in this construction, namely $\chi_{\max}$ and $\chi'_{\min}$ are only specified up to twist by unramified characters with trivial reduction. It follows immediately from our arguments that this choice is immaterial, i.e. that the image of the map $H^1_f(G_K, \mathcal{O}_E(\chi_{\max}^{-1}\chi'_{\min})) \to H^1(G_K, \overline{\chi}^{-1}\overline{\chi}')$ does not depend on the particular choice of $\chi_{\max}, \chi'_{\min}$, *except* for the case where $r_i = p$ for all i and $\chi_{\max}$ is unramified (*cf.* [BDJ10, Rem. 3.10]).

6.2. Conclusion of the proof of the weight part of Serre's conjecture. We now extend the results of Section 4, using our analysis of the extension classes of reducible lifts to complete the proof of the weight part of Serre's conjecture.

Continue to assume that $p > 2$, let F be a totally real field, and let $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be continuous, irreducible, and modular. Again, let D be a quaternion algebra with centre F, which is split at all places dividing p and at at most one infinite place. The main global result of this paper is the following.

6.2.1 Theorem. *Assume that $p > 2$, that $\overline{\rho}$ is modular and compatible with D, that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible, and if $p = 5$ assume further that the projective image of $\overline{\rho}|_{G_{F(\zeta_p)}}$ is not isomorphic to A_5.*

Then $\overline{\rho}$ is modular for D of weight $\sigma = \otimes_{v|p}\sigma_v$ if and only if $\sigma_v \in W^{\mathrm{explicit}}(\overline{\rho}|_{G_{F_v}})$ for all $v|p$.

Proof. The result is immediate from Theorems 4.2.1 and 6.1.8. □

6.3. Dependence on the restriction to inertia We conclude the paper by checking (when $p \geq 3$) that the local weight set $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{cris}}(\bar{r})$ depends only on the restriction to inertia $\bar{r}|_{I_K}$. This statement is part of the formulation of some versions of the weight part of Serre's conjecture. For instance when $K/\mathbb{Q}_p$ is unramified, the definition of the weight set $W^{\mathrm{BDJ}}(\bar{r})$ in [BDJ10] is modified in certain cases to ensure that $W^{\mathrm{BDJ}}(\bar{r})$ depends only on $\bar{r}|_{I_K}$ (*cf.* the definition immediately preceding [BDJ10, Rem. 3.10]). An immediate consequence of the following result (and its proof) is that the local weight set defined in [BDJ10] is equal to the local weight set considered in this paper (again, when $p \geq 3$).

6.3.1 Proposition. *Suppose $p \geq 3$. Let $\bar{r}, \bar{r}' : G_K \to \mathrm{GL}_2(k_E)$ be two continuous representations with $\bar{r}|_{I_K} \simeq \bar{r}'|_{I_K}$. Then $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{explicit}}(\bar{r}')$.*

Proof. When $K/\mathbb{Q}_p$ is unramified, the corresponding statement for the weight set $W^{\mathrm{BDJ}}(\bar{r})$ is proved in [BDJ10, Prop. 3.13]. The proof carries over to this context nearly word for word, except that in the special case where $\bar{r}^{\mathrm{ss}} \simeq \overline{\chi} \oplus \overline{\chi}$ is scalar and the class in $H^1(G_K, k_E)$ defining $\bar{r}$ as an extension of $\overline{\chi}$ by $\overline{\chi}$ is unramified, it is built into the definitions that $W^{\mathrm{BDJ}}(\bar{r}) = W^{\mathrm{BDJ}}(\overline{\chi} \oplus \overline{\chi})$. (In the notation of [BDJ10], this is the containment $L_{\mathrm{ur}} \subseteq L_\alpha$ in this case.) In our context, it remains to show that $W^{\mathrm{explicit}}(\bar{r}) = W^{\mathrm{explicit}}(\overline{\chi} \oplus \overline{\chi})$.

We must prove that if $\overline{\chi} \oplus \overline{\chi}$ has a pseudo-BT lift of weight $\{r_i\}$, then so does $\bar{r}$. Let $\overline{\mathfrak{N}}_{\min}$ and $\overline{\mathfrak{P}}_{\max}$ be the rank-one Kisin modules given to us by (the proof of) Proposition 5.3.4, with corresponding parameters $s_0, \ldots, s_{f-1}$ and $t_0, \ldots, t_{f-1}$ as in that Proposition. In the special case where $r_i = p$ and $t_i = 0$ for all i (which is only possible when $\overline{\varepsilon} = 1$ on G_K), the proof of Theorem 6.1.8 already shows that *every* extension of $\overline{\chi}$ by $\overline{\chi}$ has a pseudo-BT lift of weight $\{r_i\}$, so for the rest of the proof we assume that we are not in this case.

Write $\mathcal{E}$ for the set $\mathcal{E}_{\psi\mathrm{BT}}(\overline{\mathfrak{N}}_{\min}, \overline{\mathfrak{P}}_{\max})$. Note that by construction (since $\overline{\mathfrak{P}}_{\max}$ and $\overline{\mathfrak{N}}_{\min}$ have the same generic fibre $\overline{\chi}$) there exists a nonzero map $\overline{\mathfrak{N}}_{\min} \to \overline{\mathfrak{P}}_{\max}$. We can therefore define a nontrivial element $\overline{\mathfrak{M}} \in \mathcal{E}$ by taking $y_i = 0$ for all i (in the notation of Theorem 5.1.5), except that y_0 is taken to have a nonzero term of degree $d := s_0 + \alpha_0(\overline{\mathfrak{N}}_{\min}) - \alpha_0(\overline{\mathfrak{P}}_{\max})$. Indeed we obtain a line's worth of such elements.

Let K_p be the unramified extension of K of degree p, with residue field k_p, and suppose without loss of generality that k_p embeds into k_E. Define $\overline{\mathfrak{M}}' = k_p \otimes_k \overline{\mathfrak{M}}$, with φ extended k_p-semilinearly to $\overline{\mathfrak{M}}'$. It is straightforward to check that the Kisin module $\overline{\mathfrak{M}}'$ is split. We briefly indicate how to see this. Write the Kisin module $\overline{\mathfrak{M}}'$ as an extension as in Proposition 5.1.3. The field K_p has inertial degree pf. The extension parameters y'_i defining $\overline{\mathfrak{M}}'$ are all zero, except for terms of degree d in $y'_0, \dots, y'_{(p-1)f}$, all with the same coefficient. Since terms of this degree are part of a loop (in the terminology of the proof of [GLS14, Prop. 7.4]) there is a change of variables which replaces y'_{if} with 0 for each $0 < i < p$ and replaces y'_0 with $y'_0 + \dots y'_{(p-1)f} = py'_0 = 0$.

Let T be a lattice in a pseudo-BT representation of G_K of weight $\{r_i\}$ such that $\overline{T}|_{G_{K_\infty}} \simeq T_{\mathfrak{S}}(\overline{\mathfrak{M}})$. Since $T_{\mathfrak{S}}$ is faithful on $\mathcal{E}$, the representation $\overline{T}$ is non-split. By Lemmas 6.1.2 and 6.1.3 there is a unique $(\varphi, \hat{G})$-module $\hat{\overline{\mathfrak{M}}}$ of reducible pseudo-BT type and weight $\{r_i\}$ with $\hat{T}(\hat{\overline{\mathfrak{M}}}) \simeq \overline{T}$. Evidently we have $\overline{T}|_{G_{(K_p)_\infty}} \simeq T_{\mathfrak{S}}(\overline{\mathfrak{M}}')$. Since $T|_{G_{K_p}}$ is still pseudo-BT but $\overline{\mathfrak{M}}'$ is split, another application of Lemma 6.1.3 shows that $\overline{T}|_{G_{K_p}}$ itself must be split. (We remark that if $\overline{\varepsilon} \neq 1$ on G_K, the fact that $\overline{T}|_{G_{K_p}}$ is split can be deduced more easily from Lemma 5.4.2.) It follows that the extension class in $H^1(G_K, k_E)$ defining $\overline{T}$ as an extension of $\overline{\chi}$ by $\overline{\chi}$ lies in the kernel of the restriction map $H^1(G_K, k_E) \to H^1(G_{K_p}, k_E)$, i.e. it is unramified. Since the unramified subspace of $H^1(G_K, k_E)$ is a line, we have $\overline{T} \simeq \bar{r}$, and $\bar{r}$ has the desired pseudo-BT lift (namely T). □

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Appendix A. Corrigendum to [GLS14]

We take this opportunity to correct a minor but unfortunate mistake in [GLS14]. In the sentence preceding the published version of [GLS14, Thm. 4.22], we write that when we regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule of $\mathfrak{M}^*$, we are regarding $\mathfrak{M}_{s+1}$ as a submodule of $\mathfrak{M}_s^*$. As noted in Section 2.3 of the present paper, this should be $\mathfrak{M}_{s-1}$ rather than $\mathfrak{M}_{s+1}$. Although the main results of [GLS14] about the weight part of Serre's conjecture are unaffected, the statement of [GLS14, Thm. 4.22] and many of the ensuing Kisin module and $(\varphi, \hat{G})$-module formulas in [GLS14, §§6–8], have some indices that are off by one. (This issue does not affect the structure or content of any of the arguments in the paper, only the statements.)

For instance, where we write in [GLS14, Thm. 4.22] that

$$\varphi(e_{1,s}, \ldots, e_{d,s}) = (e_{1,s+1}, \ldots, e_{d,s+1}) X_s \Lambda_s Y_s$$

we should instead have

$$\varphi(e_{1,s-1}, \ldots, e_{d,s-1}) = (e_{1,s}, \ldots, e_{d,s}) X_s \Lambda_s Y_s,$$

and the other corrections are all of a similar nature. Corrected versions of the paper are available on our websites and on the arXiv. (Alternately, many of the calculations in [GLS14, §§6–8] are generalised by results in Sections 3, 5, and 6 of the present paper.)

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