

AUTOMORPHIC LIFTS OF PRESCRIBED TYPES

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ABSTRACT. We prove a variety of results on the existence of automorphic Galois representations lifting a residual automorphic Galois representation. We prove a result on the structure of deformation rings of local Galois representations, and deduce from this and the method of Khare and Wintenberger a result on the existence of modular lifts of specified type for Galois representations corresponding to Hilbert modular forms of parallel weight 2. We discuss some conjectures on the weights of n -dimensional mod p Galois representations. Finally, we use recent work of Taylor to prove level raising and lowering results for n -dimensional automorphic Galois representations.

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1. INTRODUCTION

1.1. If f is a cuspidal eigenform, there is a residual representation

$$\bar{\rho}_f : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

attached to f , and one can ask which other cuspidal eigenforms g give rise to the same representation; if one believes the Fontaine-Mazur conjecture, this is equivalent to asking which geometric representations lift $\bar{\rho}_f$ (here “geometric” means unramified outside of finitely many primes and potentially semi-stable at p). These questions amount to issues of level-lowering and level-raising (at places other than p), and to determining the possible Serre weights of $\bar{\rho}_f$ (at p).

In recent years there has been a new approach to these questions, via the use of lifting theorems due to Ramakrishna and Khare-Wintenberger, together with modularity lifting theorems. In this paper, we present several new applications of this method.

In section 2, we prove a result about the structure of the local deformation rings corresponding to a mod p representation of the absolute Galois group of a finite extension of \mathbb{Q}_l , where $l \neq p$. The method of proof is very close to that of [Kis08], where the corresponding (harder) results are proved for the case $l = p$. This result

is precisely the input needed to allow one to prove the existence of modular lifts of specified Galois type at places other than p , and we apply it repeatedly in the following sections.

In section 3 we consider the case of Hilbert modular forms. We are able to prove a strong result on the possible lifts of parallel weight 2, under the usual hypotheses for the application of the Taylor-Wiles method, and a hypothesis on the existence of ordinary lifts; in particular, one can deduce level-lowering results in full generality at places not dividing p (assumed odd). We then apply this result in section 4 to improve on some of our results from [Gee08] (where we use results from this paper to prove many cases of a conjecture of Buzzard, Diamond and Jarvis on the possible Serre weights of mod p Hilbert modular forms). While we are not able to prove the conjectures of Buzzard, Diamond and Jarvis in full generality, we are able to in the case where p splits completely, and the usual Taylor-Wiles hypothesis holds; this is needed in [Kis09a]. We also take the opportunity to discuss some generalisations of existing conjectures on Serre weights, which are suggested to us by our work on Hilbert modular forms. For example, we state a general conjecture about the possible weights of an n -dimensional representation of $G_{\mathbb{Q}}$. We hope to discuss these conjectures more fully in future work.

Finally, in section 5 we apply these methods to n -dimensional Galois representations. Here we make use of the recent work of Taylor (see [Tay08]). We are able to prove level-raising and level-lowering results at places away from p . Note that such results were originally thought to be needed to prove $R = T$ theorems in this context, but were circumvented in [Tay08]. Our proof relies crucially on this work, so does not give a new proof of these $R = T$ theorems. Our results in this final section are more limited than in the 2-dimensional case, because we do not know any $R = T$ theorems over number fields ramified at p . In particular, we cannot at present generalise our approach from [Gee08] to this setting. However, the framework established here would suffice to prove such results if such $R = T$ theorems were proved.

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1.2. Notation. If K is a field, we let G_K denote its absolute Galois group. We fix an algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} , and regard all algebraic extensions of \mathbb{Q} as subfields of $\overline{\mathbb{Q}}$. For each prime p we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\overline{\mathbb{Q}}$ and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, so that if F is a number field and v is a finite place of F we have a homomorphism $G_{F_v} \hookrightarrow G_F$. If K is a finite extension of \mathbb{Q}_p , we let I_K denote the inertia subgroup of G_K , and if L is a finite Galois extension of K we write $I_{L/K}$ for I_K/I_L .

2. LOCAL STRUCTURE OF DEFORMATION RINGS

2.1. In this section we prove some results on the local structure of deformation rings corresponding to a mod p representation of the absolute Galois group of a finite extension of \mathbb{Q}_l , where $l \neq p$. These results are the analogue of those proved in section 3 of [Kis08] in the case $l = p$, and the proofs are almost identical (but simpler in our case). In [Kis08] one considers deformations of weakly admissible modules,

whereas in our case we consider deformations of Weil-Deligne representations; the main difference is that in our case one has a linear (rather than a semi-linear) Frobenius, and we have no analogue of the Hodge filtration. The absence of such a filtration simplifies the computation of the dimensions of our deformation rings. We follow section 3 of [Kis08] extremely closely, only noting the changes to the arguments of *loc. cit.* that are necessary in our setting.

Let K/\mathbb{Q}_l ($l \neq p$) be a finite extension. Suppose that the residue field of K has cardinality l^f . Fix d a positive integer. As in [Kis08], we use the language of groupoids, which is explained in the appendix to [Kis07b].

We firstly recall some results from [Fon94]. By a theorem of Grothendieck, a continuous representation $\rho : G_K \rightarrow \mathrm{GL}_d(E)$, E a finite extension of \mathbb{Q}_p , is automatically potentially semi-stable, in the sense that there is a finite extension L/K such that $\rho|_{I_L}$ is unipotent. Let W_K denote the Weil group of K , and WD_K the Weil-Deligne group. Denote by $\underline{\mathrm{Rep}}_E(WD_K)$ the category of finite dimensional E -linear representations of WD_K . One can view an object of this category as a triple (Δ, ρ_0, N) , where Δ is a finite dimensional E -vector space, $\rho_0 : W_K \rightarrow \mathrm{Aut}_E(\Delta)$ is a homomorphism whose kernel contains an open subgroup of I_K , and $N : \Delta \rightarrow \Delta$ is an E -linear map satisfying

$$\rho_0(w)N = l^{\alpha(w)}N\rho_0(w) \text{ for all } w \in W_K.$$

Here $\alpha : W_K \rightarrow \mathbb{Z}$ is the map sending $w \in W_K$ to $\alpha(w)$ such that w acts on the residue field \bar{k} of \bar{K} as $\sigma^{\alpha(w)}$, where σ is the absolute Frobenius.

Now, fix $\Phi \in W_K$ with $\alpha(\Phi) = -f$. Fix a finite Galois extension L/K . Then it is easy to see that the full subcategory of $\underline{\mathrm{Rep}}_E(WD_K)$ whose objects are triples (Δ, ρ_0, N) as above with $\rho_0|_{I_L}$ trivial is equivalent to the category $\underline{\mathrm{Rep}}_{E,L}(WD_K)$ of triples (Δ, ϕ, N) , where Δ is a finite dimensional E -vector space with an action of $I_{L/K}$, $\phi \in \mathrm{Aut}_E(\Delta)$, and $N \in \mathrm{End}_E(\Delta)$, such that N commutes with the action of $I_{L/K}$, $N\phi = l^f\phi N$, and for all $\gamma \in I_{L/K}$ we have $\Phi\gamma\Phi^{-1} = \phi\gamma\phi^{-1}$ in $\mathrm{Aut}_E(\Delta)$ (this equivalence follows from the fact that if $\psi \in W_K$ then $\psi\Phi^{\alpha(\psi)/f} \in I_K$).

After making a choice of a compatible system of p -power roots of unity in \bar{K} , we see from Propositions 2.3.4 and 1.3.3 of [Fon94] that there is an equivalence of categories between the category of E -linear representations of G_K which become semi-stable over L , and the full subcategory of $\underline{\mathrm{Rep}}_{E,L}(WD_K)$ whose objects are the triples (Δ, ϕ, N) such that the roots of the characteristic polynomial of ϕ are p -adic units (such an equivalence is given by the functor \widehat{WD}_{pst} of section 2.3.7 of [Fon94]). We will refer to such a triple as an *admissible* triple. This is not standard terminology. Note for future reference that an extension of admissible objects is again admissible. For the rest of this section we will freely identify Galois representations with their corresponding Weil-Deligne representations.

Firstly we define two groupoids on the category of \mathbb{Q}_p -algebras A . Let \mathfrak{Mod}_N be the groupoid whose fibre over a \mathbb{Q}_p -algebra A consists of finite free A -modules D_A of rank d , a linear action of $I_{L/K}$ on D_A , and a nilpotent linear operator N on D_A . We require that the actions of $I_{L/K}$ and N commute.

Let $\mathfrak{Mod}_{\phi,N}$ be the groupoid whose fibre over a \mathbb{Q}_p -algebra A consists of a module D_A in \mathfrak{Mod}_N equipped with a linear automorphism ϕ such that $l^f\phi N = N\phi$ and $\phi\gamma\phi^{-1} = \Phi\gamma\Phi^{-1}$ for all $\gamma \in I_{L/K}$. There is a natural morphism $\mathfrak{Mod}_{\phi,N} \rightarrow \mathfrak{Mod}_N$ given by forgetting ϕ .

Given D_A in $\mathfrak{Mod}_{\phi,N}$, let $\text{ad } D_A = \text{Hom}_A(D_A, D_A)$. Give $\text{ad } D_A$ an operator ϕ by $\phi(f) := \phi \circ f \circ \phi^{-1}$, and an operator N given by $N(f) := N \circ f - f \circ N$. These satisfy $l^f \phi N = N \phi$. Give $\text{ad } D_A$ an action of $I_{L/K}$ with $\gamma \in I_{L/K}$ taking f to $\gamma \circ f \circ \gamma^{-1}$. We have an anti-commutative diagram

$$\begin{array}{ccc} (\text{ad } D_A)^{I_{L/K}} & \xrightarrow{1-\phi} & (\text{ad } D_A)^{I_{L/K}} \\ \downarrow N & & \downarrow N \\ (\text{ad } D_A)^{I_{L/K}} & \xrightarrow{l^f \phi - 1} & (\text{ad } D_A)^{I_{L/K}} \end{array}$$

Let $C^\bullet(D_A)$ denote the total complex of this double complex, and $H^\bullet(D_A)$ denote the cohomology of $C^\bullet(D_A)$.

Lemma 2.1.1. *Let A be a local \mathbb{Q}_p -algebra with maximal ideal \mathfrak{m}_A , and $I \subset A$ an ideal with $\text{Im}_A = 0$. Let $D_{A/I}$ be in $\mathfrak{Mod}_{\phi,N}(A/I)$ and set $D_{A/\mathfrak{m}_A} = D_{A/I} \otimes_{A/I} A/\mathfrak{m}_A$.*

- (1) *If $H^2(D_{A/\mathfrak{m}_A}) = 0$ then there exists a module D_A in $\mathfrak{Mod}_{\phi,N}(A)$ whose reduction modulo I is isomorphic to $D_{A/I}$.*
- (2) *The set of isomorphism classes of liftings of $D_{A/I}$ to D_A in $\mathfrak{Mod}_{\phi,N}(A)$ is either empty or a torsor under $H^1(D_{A/\mathfrak{m}_A}) \otimes_{A/\mathfrak{m}_A} I$. Here two liftings D_A, D'_A are isomorphic if there exists a map $D_A \rightarrow D'_A$ which is compatible with the ϕ -, $I_{L/K}$ -, N -actions and which reduces to the identity modulo I .*

Proof. This is very similar to the proof of Proposition 3.1.2. of [Kis08]. Let D_A be a free A -module equipped with an isomorphism $D_A \otimes_A A/I \xrightarrow{\sim} D_{A/I}$. We wish to lift the action of $I_{L/K}$ to D_A and the operator ϕ to an operator $\tilde{\phi}$ on D_A satisfying the same relations; equivalently, we wish to lift the corresponding representation of W_K/I_L . The obstruction to doing this is in $H^2(W_K/I_L, \text{ad } D_{A/\mathfrak{m}_A}) \otimes I$, which vanishes (this follows from the Hochschild-Serre spectral sequence and the fact that the group $I_{L/K}$ is finite). Similarly, $I_{L/K}$ -invariant maps in $\text{Hom}_{A/I}(D_{A/I}, D_{A/I})$ lift to $I_{L/K}$ -invariant maps in $\text{Hom}_A(D_A, D_A)$, so we can lift N to an endomorphism \tilde{N} which commutes with the $I_{L/K}$ -action.

Then $D_{A/I}$ lifts to an object of $\mathfrak{Mod}_{\phi,N}(A)$ if and only if we can choose $\tilde{N}, \tilde{\phi}$ so that $h := \tilde{N} - l^f \tilde{\phi} \tilde{N} \tilde{\phi}^{-1} = 0$. We see that h induces an I_K -invariant map $h : D_{A/\mathfrak{m}_A} \rightarrow D_{A/\mathfrak{m}_A} \otimes_A I$, so if $H^2(D_{A/\mathfrak{m}_A}) = 0$ we can write $h = N(f) + (l^f \phi - 1)(g)$, with $f, g \in (\text{ad } D_{A/\mathfrak{m}_A})^{I_{L/K}} \otimes_{A/\mathfrak{m}_A} I$. If we then set $\tilde{N}' = \tilde{N} + g$, $\tilde{\phi}' = \tilde{\phi} - f \circ \phi$, we see that $\tilde{N}' \tilde{\phi}' = l^f \tilde{\phi}' \tilde{N}'$, which proves (1). The proof of (2) is then formally identical to the proof of part (2) of Proposition 3.1.2. of [Kis08] (after replacing every occurrence of p with l^f , and $G_{L/K}$ with $I_{L/K}$). \square

Corollary 2.1.2. *Let A be a \mathbb{Q}_p -algebra and D_A an object of $\mathfrak{Mod}_{\phi,N}(A)$. Suppose that the morphism $A \rightarrow \mathfrak{Mod}_{\phi,N}$ given by D_A is formally smooth and that $H^2(D_A) = 0$. Then A is formally smooth over \mathbb{Q}_p .*

Proof. Identical to that of Corollary 3.1.3. of [Kis08]. \square

Let E/\mathbb{Q}_p be a finite extension, and let D_E be a finite free E -module of rank d with an action of $I_{L/K}$. Consider the functor which to an E -algebra A assigns the set of pairs (ϕ, N) where ϕ, N are endomorphisms of $D_A := D_E \otimes_E A$ making D_A into an object of $\mathfrak{Mod}_{\phi,N}(A)$. This functor is represented by an E -algebra $B_{\phi,N}$,

with $X_{\phi,N} = \text{Spec } B_{\phi,N}$ a locally closed subscheme of $\text{Hom}_E(D_E, D_E)^2$. Similarly, let B_N represent the analogous functor considering only objects of $\mathfrak{Mod}_N(A)$, and let $X_N = \text{Spec } B_N$. There is a natural map $X_{\phi,N} \rightarrow X_N$.

Lemma 2.1.3. *Let $D_{B_{\phi,N}} = D_E \otimes_{\mathbb{Q}_p} B_{\phi,N}$, a vector bundle on $X_{\phi,N}$.*

- (1) *The morphism of groupoids on E -algebras $B_{\phi,N} \rightarrow \mathfrak{Mod}_{\phi,N}$ is formally smooth.*
- (2) *There is a dense open subset $U \subset X_{\phi,N}$ such that $H^2(D_{B_{\phi,N}})|_U = 0$.*

Proof. This is very similar to the proof of Lemma 3.1.5. of [Kis08]. (1) is immediate by definition. Let $U \subset X_{\phi,N}$ be the complement of the support of $H^2(D_{B_{\phi,N}})$. We must show that U is dense. It suffices to show that U is dense in every fibre of $X_{\phi,N} \rightarrow X_N$. Let $y \in X_N$, and let D_y be the pullback to y of the tautological vector bundle on X_N . Then $(X_{\phi,N})_y$ can be identified with an open subset (given by the non-vanishing of the determinant) of the $\kappa(y)$ -vector space of linear maps $\phi : D_y \rightarrow D_y$ which satisfy the required commutation relations with N and $I_{L/K}$. Then $(X_{\phi,N})_y$ is smooth and connected, so irreducible, so we need only check that if $(X_{\phi,N})_y$ is non-empty, then there is a point of $(X_{\phi,N})_y$ at which $H^2(D_{B_{\phi,N}}) = 0$.

We have a decomposition

$$D_y \xrightarrow{\sim} \bigoplus_{\tau} \text{Hom}_{I_{L/K}}(\tau, D_y) \otimes_{\mathbb{Q}_p} \tau$$

with τ running over the irreducible representations of $I_{L/K}$ on \mathbb{Q}_p -vector spaces. The N -action on D_y induces an action on $D_{y,\tau} := \text{Hom}_{I_{L/K}}(\tau, D_y)$, and the above decomposition is then compatible with N -actions.

For any such τ and i an integer, we define τ^{ϕ^i} , with the same underlying \mathbb{Q}_p -vector space as τ , by letting $\gamma \in I_{L/K}$ act on τ^{ϕ^i} as $\Phi^i \gamma \Phi^{-i}$ acts on τ . Since $(X_{\phi,N})_y$ is nonempty, there exists a map $\phi : D_y \rightarrow D_y$ such that $l^f \phi N = N \phi$. Then ϕ induces an isomorphism $D_{y,\tau} \rightarrow D_{y,\tau^{\phi}}$. Let n_{τ} be the least positive integer such that $\tau^{\phi^{n_{\tau}}} = \tau$. Then since $l^f \phi N = N \phi$, we see that it is possible to write $D_{y,\tau^{\phi^i}} = \bigoplus_j D_{i,j}$ where $D_{i,j}$ is $\kappa(y)^{s_j}$ for some s_j with canonical basis $(e_{i,1}^j, \dots, e_{i,s_j}^j)$, and $N(e_{i,k}^j) = e_{i,k+1}^j$ ($1 \leq k \leq s_j - 1$), $N(e_{i,s_j}^j) = 0$. We then consider the endomorphism ϕ of $\bigoplus_{i=1}^{n_{\tau}} D_{y,\tau^{\phi^i}}$ given by $\phi(e_{i,k}^j) = l^{f(s_j-k)} e_{i+1,k}^j$, where we consider the i -index to be cyclic. This satisfies $l^f \phi N = N \phi$. Equipping D_y with the resulting choice of ϕ , we find that $H^2(D_y) = 0$, as required. \square

Let A° be a complete Noetherian local \mathcal{O}_E -algebra with finite residue field, and $V_{A^{\circ}}$ be a finite free A° -module of rank d equipped with a continuous action of G_K . Write $A = A^{\circ}[1/p]$. Define a *Galois type* τ to be the restriction to I_K of a d -dimensional p -adic Weil-Deligne representation with open kernel, assumed from now on to contain I_L . We say that a p -adic Galois representation is of type τ if the restriction to I_K of the corresponding Weil-Deligne representation is isomorphic to τ .

There is a quotient A^{τ} of A such that for any finite E -algebra B , a map of E -algebras $x : A \rightarrow B$ factors through A^{τ} if and only if $V_B = V_A \otimes_A B$ has type τ (this is clear, because the isomorphism class of a finite-dimensional representation of a finite group in characteristic zero is determined by its trace). It follows easily from the fact that the cohomology of the finite group $I_{L/K}$ with coefficients in characteristic 0 vanishes in positive degree that A^{τ} is a union of irreducible components of A .

Proposition 2.1.4. *Let A be a Noetherian \mathbb{Q}_p -algebra and D_A an object of $\mathfrak{Mod}_{\phi,N}$. If $A \rightarrow \mathfrak{Mod}_{\phi,N}$ is formally smooth, then there is a dense open subset $U \subset \text{Spec } A$ such that U is formally smooth over \mathbb{Q}_p , and the support of $H^2(D_A)$ does not meet U .*

Proof. This is formally identical to the proof of Proposition 3.1.6. of [Kis08], using Corollary 2.1.2 and Lemma 2.1.3. \square

Now let $V_{\mathbb{F}} = V_{A^\circ} \otimes_{A^\circ} \mathbb{F}$. Let $D_{V_{\mathbb{F}}}$ be the groupoid on the category of complete local \mathcal{O}_E -algebras with residue field \mathbb{F} , whose fibre over such an algebra B consists of the deformations of $V_{\mathbb{F}}$ to a G_K -representation on a finite free B -module V_B .

Let \mathfrak{m} be a maximal ideal of A^τ , and E' its residue field. For each $i \geq 1$ the G_K -representation $V_{A^\circ} \otimes_{A^\circ} A^\tau/\mathfrak{m}^i A^\tau$ gives an object of $\mathfrak{Mod}_{\phi,N}(A^\tau/\mathfrak{m}^i A^\tau)$ via taking the corresponding Weil-Deligne representation. Passing to the limit gives a morphism of groupoids on E -algebras $\hat{A}_{\mathfrak{m}}^\tau \rightarrow \mathfrak{Mod}_{\phi,N}$.

Proposition 2.1.5. *Suppose that the morphism $A^\circ \rightarrow D_{V_{\mathbb{F}}}$ is formally smooth. Then the morphism $\hat{A}_{\mathfrak{m}}^\tau \rightarrow \mathfrak{Mod}_{\phi,N}$ of groupoids on E -algebras is formally smooth.*

Proof. This is very similar to the proof of Proposition 3.3.1. of [Kis08]. Let B be an E -algebra, $I \subset B$ an ideal with $I^2 = 0$ and $h : \hat{A}_{\mathfrak{m}}^\tau \rightarrow B/I$ a map of E -algebras. Let $D_{B/I}$ be the object of $\mathfrak{Mod}_{\phi,N}(B/I)$ induced by h , and let D_B be an object in $\mathfrak{Mod}_{\phi,N}(B)$ together with an isomorphism $D_B \otimes_B B/I \xrightarrow{\sim} D_{B/I}$. We need to show that this is induced by a map $\hat{A}_{\mathfrak{m}}^\tau \rightarrow B$ lifting h .

As in the second paragraph of the proof of Proposition 3.3.1. of [Kis08], we may assume that B is a Noetherian complete local E' -algebra with residue field E' .

Let \mathfrak{m}_B be the maximal ideal of B . If $i \geq 1$ then $D_B \otimes_B B/\mathfrak{m}_B^i$ is an object of $\mathfrak{Mod}_{\phi,N}(B/\mathfrak{m}_B^i)$. It is admissible, as it is a repeated extension of $D_{B/I} \otimes_{B/I} B/\mathfrak{m}_B$. Then one can associate to it a finite free B -module V_{B/\mathfrak{m}_B^i} of rank d , equipped with a continuous action of G_K , via the functor \widehat{WD}_{pst} . Because V_{B/\mathfrak{m}_B} has type τ , we see that V_{B/\mathfrak{m}_B^i} has type τ for all i .

The result then follows as in the final paragraph of the proof of Proposition 3.3.1. of [Kis08]. \square

We are now ready to prove the main result of this section. Fix an \mathbb{F} -basis for $V_{\mathbb{F}}$, and denote by $D_{V_{\mathbb{F}}}^\square$ the groupoid on the category of complete local \mathcal{O}_E -algebras with residue field \mathbb{F} , whose fibre over such a B is an object V_B of $D_{V_{\mathbb{F}}}(B)$ together with a lifting of the given basis for $V_{\mathbb{F}}$ to a B -basis for V_B . A morphism $V_B \rightarrow V_{B'}$ in $D_{V_{\mathbb{F}}}^\square$ covering $\epsilon : B \rightarrow B'$ is a B' -linear G_K -equivariant isomorphism $V_B \otimes_B B' \xrightarrow{\sim} V_{B'}$ sending the given basis of V_B to that of $V_{B'}$.

Denote by $|D_{V_{\mathbb{F}}}^\square|$ the functor which to B assigns the set of isomorphism classes of $D_{V_{\mathbb{F}}}^\square(B)$, and similarly for $|D_{V_{\mathbb{F}}}|$. Then $|D_{V_{\mathbb{F}}}^\square|$ is representable by a complete local \mathcal{O}_E -algebra $R_{V_{\mathbb{F}}}^\square$. If $\text{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$, then $|D_{V_{\mathbb{F}}}|$ is representable by a complete local \mathcal{O}_E -algebra $R_{V_{\mathbb{F}}}$.

Theorem 2.1.6. *$\text{Spec}(R_{V_{\mathbb{F}}}^\square[1/p])^\tau$ is equi-dimensional of dimension d^2 and admits a formally smooth, dense open subscheme. If $\text{End}_{\mathbb{F}[G_K]} V_{\mathbb{F}} = \mathbb{F}$ then the same is true of $\text{Spec}(R_{V_{\mathbb{F}}}[1/p])^\tau$, except that it is 1-dimensional.*

Proof. We give the proof for $(R_{V_{\mathbb{F}}}^{\square}[1/p])^{\tau}$, the argument for $(R_{V_{\mathbb{F}}}[1/p])^{\tau}$ being very similar. Let $A = (R_{V_{\mathbb{F}}}^{\square}[1/p])^{\tau}$, and let D_A denote the object of $\mathfrak{Mod}_{\phi, N}(A)$ corresponding to the universal representation over $(R_{V_{\mathbb{F}}}^{\square}[1/p])^{\tau}$. It follows from Propositions 2.1.4 and 2.1.5 that there is a smooth dense open subscheme U of $\text{Spec } A$ such that the support of $H^2(D_A)$ does not meet U .

To compute the dimension of A , it suffices to compute the dimension of the tangent spaces at closed points of U . Let x be a closed point of U with residue field E' , a finite extension of E , and write \mathfrak{m} for the corresponding maximal ideal of A , V_x for the G_K -representation given by specialising the universal representation over A , and D_x for the corresponding object of $\mathfrak{Mod}_{\phi, N}(E')$. Then the dimension of the tangent space at \mathfrak{m} is, by the formula found in section 2.3.4. of [Kis07b],

$$\dim_{E'} \text{Ext}^1(V_x, V_x) + \dim_{E'} \text{ad}_{E'} V_x - \dim_{E'} (\text{ad}_{E'} V_x)^{G_K}.$$

By Lemma 2.1.1 and the fact that $H^2(D_x) = 0$ (as $x \in U$), we see that

$$\begin{aligned} \dim_{E'} \text{Ext}^1(V_x, V_x) &= \dim_{E'} H^1(D_x) \\ &= \dim_{E'} H^0(D_x) \\ &= \dim_{E'} (\text{ad}_{E'} V_x)^{G_K}. \end{aligned}$$

Thus the dimension of the tangent space is $\dim_{E'} (\text{ad}_{E'} V_x) = d^2$, as required. \square

3. HILBERT MODULAR FORMS

3.1. Let $p > 2$ be a prime. Let F be a totally real field, S a finite set of places of F containing the places of F dividing p and the infinite places, and let $\Sigma \subset S$ be the set of finite places in S . Let $F(S)$ denote the maximal extension of F unramified outside of S , and write $G_{F, S}$ for $\text{Gal}(F(S)/F)$.

Let E/\mathbb{Q}_p be a finite extension with ring of integers \mathcal{O} and residue field \mathbb{F} , and let $\bar{\rho} : G_{F, S} \rightarrow \text{GL}(V)$ be a continuous representation on a 2-dimensional \mathbb{F} -vector space V . Assume that $\bar{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible, and that $\bar{\rho}$ is modular. If $p = 5$ and the projective image of $\bar{\rho}$ is $\text{PGL}_2(\mathbb{F}_5)$, assume further that $[F(\zeta_5) : F] \neq 2$. In this section we use the improvements in [Kis05] of the work of Böckle and Khare-Wintenberger on presentations of global universal deformation rings, and we prove a result guaranteeing the existence of modular lifts of $\bar{\rho}$ with certain local properties.

Fix a continuous character $\psi : G_{F, S} \rightarrow \mathcal{O}^{\times}$, so that the composite $\psi : G_{F, S} \rightarrow \mathcal{O}^{\times} \rightarrow \mathbb{F}^{\times}$ is equal to $\det V$. We fix algebraic closures $\bar{\mathbb{Q}}$ and $\bar{\mathbb{Q}}_l$ for all primes l , and embeddings $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_l$. We regard F as a subfield of $\bar{\mathbb{Q}}$, and in a slight abuse of notation we will also write ψ for the characters $G_{F_v} \rightarrow \mathcal{O}^{\times}$ induced by the inclusions $G_{F_v} \hookrightarrow G_{F, S}$.

Let $R_{F, S}$ denote the universal deformation \mathcal{O} -algebra of V (which exists by our assumption that $\bar{\rho}|_{G_{F(\zeta_p)}}$ (and thus $\bar{\rho}$) is absolutely irreducible), and denote by $R_{F, S}^{\psi}$ the quotient corresponding to deformations of determinant ψ . Let $\text{ad}^0 V \subset \text{End}_{\mathbb{F}}(V)$ be the subspace of endomorphisms of V having trace zero.

For each $v \in \Sigma$ we fix a basis β_v of V . Then the functor which assigns to a local Artin \mathcal{O} -algebra A with residue field \mathbb{F} the set of isomorphism classes of pairs $(V_A, \beta_{v, A})$ with V_A a deformation of V (considered as a G_{F_v} -representation) of determinant ψ , and $\beta_{v, A}$ a basis of V_A lifting β_v , is representable by a complete local \mathcal{O} -algebra $R_v^{\square, \psi}$. Let $R_{F, S}^{\square, \psi}$ be the universal deformation \mathcal{O} -algebra

parameterising tuples $(V_A, \{\beta_{v,A}\}_{v \in \Sigma})$ with V_A a deformation of V considered as a $G_{F,S}$ -representation, and $\beta_{v,A}$ as above. Set $R_\Sigma^{\square,\psi} = \widehat{\otimes}_{\mathcal{O}, v \in \Sigma} R_v^{\square,\psi}$. Then we have

Proposition 3.1.1. *Let $s = \sum_{v|\infty} H^0(G_{F_v}, \text{ad}^0 V)$. Then for some $r \geq 0$ and $f_1, \dots, f_{r+s} \in R_\Sigma^{\square,\psi}[[x_1, \dots, x_{r+|\Sigma|-1}]]$ there exists an isomorphism*

$$R_{F,S}^{\square,\psi} \xrightarrow{\sim} R_\Sigma^{\square,\psi}[[x_1, \dots, x_{r+|\Sigma|-1}]]/(f_1, \dots, f_{r+s}).$$

Proof. This follows at once from Proposition 4.1.5. of [Kis05], because the assumption that $\bar{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible implies that $H^0(G_{F,S}, (\text{ad}^0 V)^*(1)) = 0$. \square

We now consider more refined deformation conditions at the places $v \in \Sigma$; specifically, we consider deformations of specific Galois type. Let ϵ denote the p -adic cyclotomic character (which, with a slight abuse of notation, we will regard as a character of various local or global absolute Galois groups). For each $v \in \Sigma$ fix a Galois type τ_v ; that is, a representation $\tau_v : I_{F_v} \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_p)$ with open kernel, that extends to a representation of G_{F_v} . If $v|p$, we attach to any 2-dimensional potentially semi-stable (so, in particular, any potentially Barsotti-Tate) representation $\rho : G_{F_v} \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_p)$ a 2 dimensional $\overline{\mathbb{Q}}_p$ -representation $WD(\rho)$ of the Weil-Deligne group of F_v (we attach this representation covariantly, as in Appendix B of [CDT99]). We say that ρ is potentially Barsotti-Tate of type τ_v if it is potentially Barsotti-Tate, has determinant a finite order character times the cyclotomic character, and $WD(\rho)|_{I_{F_v}}$ is equivalent to τ_v . Note that by Lemma 2.2.1.1 of [BM02], any representation ρ or τ_v as above is actually defined over a finite extension of $\overline{\mathbb{Q}}_p$.

Suppose now that there is a finite order character $\chi : G_{F,S} \rightarrow \mathcal{O}^\times$ such that $\psi = \chi\epsilon$. Assume that for each $v \in \Sigma$ we have $\det \tau_v = \chi|_{I_{F_v}}$.

Proposition 3.1.2. *For each v dividing p , there exists a unique quotient $R_v^{\square,\psi,\tau_v}$ of $R_v^{\square,\psi}$ such that:*

- (1) $R_v^{\square,\psi,\tau_v}$ is p -torsion free, reduced, and all its components are $4 + [F_v : \mathbb{Q}_p]$ -dimensional.
- (2) If E'/E is a finite extension, then a map $x : R_v^{\square,\psi} \rightarrow E'$ factors through $R_v^{\square,\psi,\tau_v}$ if and only if the corresponding E' -representation V_x is potentially Barsotti-Tate of type τ_v .

Proof. This follows at once from Theorem 3.3.8. of [Kis08]. \square

Using the classification of irreducible components of $R_v^{\square,\psi,\tau_v}$ for the case $\tau_v = 1$ (specifically, Corollary 2.5.16(2) of [Kis07b]) one easily sees that each irreducible component of $R_v^{\square,\psi,\tau_v}$ corresponds either only to potentially ordinary representations or only to representations which are not potentially ordinary. Call these components ordinary, non-ordinary respectively.

Proposition 3.1.3. *For each v not dividing p , there exists a unique quotient $R_v^{\square,\psi,\tau_v}$ of $R_v^{\square,\psi}$ such that:*

- (1) $R_v^{\square,\psi,\tau_v}$ is p -torsion free, reduced, and all its components are 4-dimensional.
- (2) If E'/E is a finite extension, then a map $x : R_v^{\square,\psi} \rightarrow E'$ factors through $R_v^{\square,\psi,\tau_v}$ if and only if the corresponding E' -representation V_x is of type τ_v .

Proof. This follows from Theorem 2.1.6. \square

Note that the rings $R_v^{\square, \psi, \tau_v}$ could be zero; in applications, one must check that they are nonzero (e.g. by exhibiting an appropriate deformation).

Assume now that each $R_v^{\square, \psi, \tau_v}$ is nonzero, and for each $v \in \Sigma$ let $\overline{R}_v^{\square, \psi, \tau_v}$ denote a quotient of $R_v^{\square, \psi, \tau_v}$ corresponding to some irreducible component. Set $R_{\Sigma}^{\square, \psi, \tau} = \widehat{\otimes}_{\mathcal{O}, v \in \Sigma} \overline{R}_v^{\square, \psi, \tau_v}$, and $R_{F,S}^{\square, \psi, \tau} = R_{F,S}^{\square, \psi} \otimes_{R_{\Sigma}^{\square, \psi}} R_{\Sigma}^{\square, \psi, \tau}$. Because $\overline{\rho}$ is irreducible, there is a corresponding quotient of $R_{F,S}^{\psi}$, which we denote $R_{F,S}^{\psi, \tau}$. This is the image of $R_{F,S}^{\psi}$ in $R_{F,S}^{\square, \psi, \tau}$.

Proposition 3.1.4. *For some $r \geq 0$ there is an isomorphism*

$$R_{F,S}^{\square, \psi, \tau} \xrightarrow{\sim} R_{\Sigma}^{\square, \psi, \tau} [[x_1, \dots, x_{r+|\Sigma|-1}]] / (f_1, \dots, f_{r+[F:\mathbb{Q}]}) .$$

In addition, $\dim R_{F,S}^{\square, \psi, \tau} \geq 4|\Sigma|$ and $\dim R_{F,S}^{\psi, \tau} \geq 1$.

Proof. Note that $\overline{\rho}$ is odd (because it is assumed modular), so that the integer s in Proposition 3.1.1 is $\sum_{v|\infty} 1 = [F:\mathbb{Q}]$. From Propositions 3.1.2 and 3.1.3 we have

$$\begin{aligned} \dim R_{\Sigma}^{\square, \psi, \tau} &= \sum_{v \in \Sigma} \dim R_v^{\square, \psi, \tau} - (|\Sigma| - 1) \\ &= 4|\Sigma| + \sum_{v|p} [F_v:\mathbb{Q}_p] - (|\Sigma| - 1) \\ &= 3|\Sigma| + [F:\mathbb{Q}] + 1. \end{aligned}$$

Thus the dimension of $R_{F,S}^{\square, \psi, \tau}$ is at least

$$\dim R_{\Sigma}^{\square, \psi, \tau} + (r + |\Sigma| - 1) - (r + [F:\mathbb{Q}]) = 4|\Sigma|.$$

The morphism $R_{F,S}^{\psi, \tau} \rightarrow R_{F,S}^{\square, \psi, \tau}$ is formally smooth of relative dimension $4|\Sigma| - 1$, so

$$\dim R_{F,S}^{\psi, \tau} = \dim R_{F,S}^{\square, \psi, \tau} - (4|\Sigma| - 1) \geq 1. \quad \square$$

We now use an $R = T$ theorem to show that $R_{F,S}^{\psi, \tau}$ is finite over \mathcal{O} of \mathcal{O} -rank at least 1. This allows us to prove the existence of minimal liftings of specified type.

Proposition 3.1.5. *Assume that $R_{\Sigma}^{\square, \psi, \tau} \neq 0$. Assume also that the following hypothesis is satisfied:*

(ord) *Let Z be the set of places v dividing p such that $\overline{R}_v^{\square, \psi, \tau_v}$ is ordinary. Then $\overline{\rho} \cong \overline{\rho}_f$ for f a Hilbert modular form of parallel weight 2, with f potentially ordinary at all places in Z , and the corresponding automorphic representation π_f not special at any place dividing p .*

Then $R_{F,S}^{\psi, \tau}$ is a finite \mathcal{O} -module of rank at least 1.

Proof. We note that it suffices to show that $R_{F,S}^{\psi, \tau}$ is a finite \mathcal{O} -algebra; if this holds, then if $R_{F,S}^{\psi, \tau}$ had rank 0 as an \mathcal{O} -module then it would have dimension 0, contradicting Proposition 3.1.4 (note that $R_{F,S}^{\psi, \tau}$ is certainly nonzero, as it is a completed tensor product of nonzero \mathcal{O} -algebras).

We may now (e.g. as in the proof of Theorem 3.5.5 of [Kis07b]) choose a solvable extension F'/F so that:

- (1) The base change of π_f to F' , denoted $\pi_{f_{F'}}$, is unramified or special at every finite place of F' .
- (2) If v' is a place of F' lying over a place v in Σ , then $\tau_v|_{I_{F'_{v'}}}$ and $\chi|_{G_{F'_{v'}}}$ are trivial.
- (3) $\bar{\rho}|_{G_{F'_{v'}}}$ and $\chi|_{G_{F'_{v'}}}$ are trivial at all places $v'|p$ of F' .
- (4) $[F'(\zeta_p) : F'] = [F(\zeta_p) : F]$, $[F' : \mathbb{Q}]$ is even, and $\bar{\rho}|_{G_{F'(\zeta_p)}}$ is absolutely irreducible.
- (5) $\bar{\rho}|_{G_{F'}}$ is unramified at all places.

After possibly making a further base change and replacing the Hilbert modular form $f_{F'}$ corresponding to $\pi_{f_{F'}}$ with a congruent form, the argument of Corollary 3.1.6 of [Kis07b] (and the use of a Hecke algebra containing the Hecke operators at places dividing p) shows that we may assume that $f_{F'}$ is ordinary at precisely the places lying over places in Z . Let S' denote the set of places of F' lying over places in S , and Σ' denote the set of places of F' lying over places in Σ . Then as in the proof of Theorem 4.2.8. of [Kis05] (or the proof of Lemma 3.6 of [KW08]), $R_{F',S}$ is a finite $R_{F',S'}$ -algebra.

We now define a framed deformation ring which captures the base changes to F' of the deformations parameterised by $R_{F',S}^{\psi,\tau}$. Restriction gives a basis $\beta_{v'}$ of V for each place $v' \in \Sigma'$. For each place $v' \in \Sigma'$ of F' we let $R_{v'}^{\square}$ denote the universal framed deformation ring for $\bar{\rho}|_{G_{F'_{v'}}}$. If $v'|p$, let $R_{v'}^{\square,0,1}$ be the quotient considered in [Kis07b]; so if E' is a finite extension of E , then a map $x : R_{v'}^{\square} \rightarrow E'$ factors through $R_{v'}^{\square,0,1}$ if and only if the corresponding Galois representation V_x is Barsotti-Tate with Hodge-Tate weights equal to 0, 1, and $\det V_x$ is the cyclotomic character.

It is shown in [Gee06] that $\text{Spec } R_{v'}^{\square,0,1}[1/p]$ has precisely 2 components, one corresponding to ordinary representations, and one to non-ordinary representations. Let $\overline{R}_{v'}^{\square,0,1}$ denote the quotient of $R_{v'}^{\square,0,1}$ corresponding to the closure of the non-ordinary component if v does not lie over a place in Z , and to the closure of the ordinary component if v lies over a place in Z . If $v' \nmid p$ but $v' \in \Sigma'$, then let $\overline{R}_{v'}^{\square}$ be the quotient of $R_{v'}^{\square,\psi}$ corresponding to unramified representations, unless there are representations factoring through $\overline{R}_v^{\square,\psi,\tau}$ which are not potentially unramified, in which case let $\overline{R}_{v'}^{\square}$ be the quotient of $R_{v'}^{\square,\psi}$ defined in corollary 2.6.6 of [Kis07b]; so if E' is a finite extension of E , then a map $x : R_{v'}^{\square} \rightarrow E'$ factors through $\overline{R}_{v'}^{\square}$ if and only if the corresponding Galois representation V_x is an extension of the trivial character by the cyclotomic character. In either case $\overline{R}_{v'}^{\square}$ is a domain.

Let $R_{F',S'}^{\square}$ denote the deformation ring which is universal for tuples $(V_A, \{\beta_{v',A}\}_{v' \in \Sigma'})$, where V_A is a deformation of the $G_{F',S'}$ -representation $\bar{\rho}|_{G_{F',S'}}$ with determinant ψ , which is unramified at all places $v' \notin \Sigma'$, together with a basis $\beta_{v',A}$ of V_A lifting $\beta_{v'}$ for all $v' \in \Sigma'$. Let $R_{\Sigma'}^{\square} := \widehat{\otimes}_{\mathcal{O}, v' \in \Sigma'} R_{v'}^{\square}$, $\overline{R}_{\Sigma'}^{\square} := \widehat{\otimes}_{\mathcal{O}, v' \in \Sigma'} \overline{R}_{v'}^{\square}$. Set $\overline{R}_{F',S'}^{\square} := R_{F',S'}^{\square} \otimes_{R_{\Sigma'}^{\square}} \overline{R}_{\Sigma'}^{\square}$, and let $\overline{R}_{F',S'}$ be the corresponding (unframed) deformation ring, so that $\overline{R}_{F',S'}^{\square}$ is formally smooth over $\overline{R}_{F',S'}$ of relative dimension $j := 4|\Sigma'| - 1$.

Now, the proofs of Proposition 3.3.1 and Theorem 3.4.11 of [Kis07b] show that if we identify $\overline{R}_{F',S'}^{\square}$ with a power series ring $\overline{R}_{F',S'}[[y_1, \dots, y_j]]$, then $\overline{R}_{F',S'}^{\square}$ is a

finite $\mathcal{O}[[y_1, \dots, y_j]]$ -algebra, so that $\overline{R}_{F', S'}$ is a finite \mathcal{O} -algebra. Since $R_{F, S}$ is a finite $R_{F', S'}$ -algebra, we see that $R_{F, S}^{\psi, \tau}$ is a finite \mathcal{O} -algebra, as required. \square

Remark 3.1.6. We note that the hypothesis (ord) is automatically satisfied if $F_v = \mathbb{Q}_p$ for every $v \in Z$, by the arguments of Corollary 3.1.6 of [Kis07b] and Lemma 2.14 of [Kis07a].

Corollary 3.1.7. *With the assumptions of Proposition 3.1.5 (in addition to those imposed at the beginning of this section) there exists a modular lifting ρ of $\bar{\rho}$ which is potentially Barsotti-Tate at all places $v|p$, which is unramified outside S , and which has type τ_v at all places $v \in \Sigma$. More precisely, given for each $v \in \Sigma$ a choice of component of $R_v^{\square, \psi, \tau_v}[1/p]$ satisfying (ord), we can choose ρ so that the corresponding point of $R_v^{\square, \psi, \tau_v}[1/p]$ lies on the given component.*

Proof. The existence of such a Galois representation follows at once from Proposition 3.1.5, and its modularity from the proof of Proposition 3.1.5 (note that F'/F is solvable). \square

4. SERRE WEIGHTS

4.1. In this section we use the results of section 3 to prove some of results used in [Gee08], and to discuss some related conjectures. We assume that $p > 2$.

We begin by recalling some standard facts from the theory of quaternionic modular forms; see either [Tay06], section 3 of [Kis07b] or section 2 of [Kis09a] for more details, and in particular the proofs of the results claimed below. We will follow Kisin's approach closely. Let F be a totally real field (with no assumption on the ramification of p) and let D be a quaternion algebra with center F which is ramified at all infinite places of F and at a set Σ of finite places, which contains no places above p . Fix a maximal order \mathcal{O}_D of D and for each finite place $v \notin \Sigma$ fix an isomorphism $(\mathcal{O}_D)_v \xrightarrow{\sim} M_2(\mathcal{O}_{F_v})$. For any finite place v let π_v denote a uniformiser of F_v .

Let $U = \prod_v U_v \subset (D \otimes_F \mathbb{A}_F^f)^\times$ be a compact open subgroup, with each $U_v \subset (\mathcal{O}_D)_v^\times$. Furthermore, assume that $U_v = (\mathcal{O}_D)_v^\times$ for all $v \in \Sigma$, and that $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ if $v|p$.

Take A a topological \mathbb{Z}_p -algebra. For each $v|p$, fix a continuous representation $\sigma_v : U_v \rightarrow \mathrm{Aut}(W_{\sigma_v})$ with W_{σ_v} a finite free A -module. Write $W_\sigma = \otimes_{v|p, A} W_{\sigma_v}$ and let $\sigma = \otimes_{v|p} \sigma_v$. We regard σ as a representation of U in the obvious way (that is, we let U_v act trivially if $v \nmid p$). Assume that there is a character $\psi : (\mathbb{A}_F^f)^\times / F^\times \rightarrow A^\times$ such that for any place v of F , $\sigma|_{U_v \cap \mathcal{O}_{F_v}^\times}$ is multiplication by ψ^{-1} . Then we can think of W_σ as a $U(\mathbb{A}_F^f)^\times$ -module by letting $(\mathbb{A}_F^f)^\times$ act via ψ^{-1} .

Let $S_{\sigma, \psi}(U, A)$ denote the set of continuous functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^f)^\times \rightarrow W_\sigma$$

such that for all $g \in (D \otimes_F \mathbb{A}_F^f)^\times$ we have

$$f(gu) = \sigma(u)^{-1} f(g) \text{ for all } u \in U,$$

$$f(gz) = \psi(z) f(g) \text{ for all } z \in (\mathbb{A}_F^f)^\times.$$

We can write $(D \otimes_F \mathbb{A}_F^f)^\times = \prod_{i \in I} D^\times t_i U (\mathbb{A}_F^f)^\times$ for some finite index set I and some $t_i \in (D \otimes_F \mathbb{A}_F^f)^\times$. Then we have

$$S_{\sigma, \psi}(U, A) \xrightarrow{\sim} \bigoplus_{i \in I} W_\sigma^{(U(\mathbb{A}_F^f)^\times \cap t_i^{-1} D^\times t_i) / F^\times},$$

the isomorphism being given by the direct sum of the maps $f \mapsto \{f(t_i)\}$. From now on we make the following assumption:

For all $t \in (D \otimes_F \mathbb{A}_F^f)^\times$ the group $(U(\mathbb{A}_F^f)^\times \cap t^{-1} D^\times t) / F^\times = 1$.

One can always replace U by a subgroup (satisfying the above assumptions) for which this holds (c.f. section 3.1.1 of [Kis09b]). Under this assumption $S_{\sigma, \psi}(U, A)$ is a finite free A -module, and the functor $W_\sigma \mapsto S_{\sigma, \psi}(U, A)$ is exact in W_σ .

We now define some Hecke algebras. Let S be a set of finite places containing Σ , the places dividing p , and the places v of F such that U_v is not a maximal compact subgroup of D_v^\times . Let $\mathbb{T}_{S, A}^{\text{univ}} = A[T_v, S_v]_{v \notin S}$ be the commutative polynomial ring in the formal variables T_v, S_v . Consider the left action of $(D \otimes_F \mathbb{A}_F^f)^\times$ on W_σ -valued functions on $(D \otimes_F \mathbb{A}_F^f)^\times$ given by $(gf)(z) = f(zg)$. Then we make $S_{\sigma, \psi}(U, A)$ a $\mathbb{T}_{S, A}^{\text{univ}}$ -module by letting S_v act via the double coset $U \begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix} U$ and T_v via $U \begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix} U$. These are independent of the choices of π_v . We will write $\mathbb{T}_{\sigma, \psi}(U, A)$ or $\mathbb{T}_{\sigma, \psi}(U)$ for the image of $\mathbb{T}_{S, A}^{\text{univ}}$ in $\text{End } S_{\sigma, \psi}(U, A)$.

Let \mathfrak{m} be a maximal ideal of $\mathbb{T}_{S, A}^{\text{univ}}$. We say that \mathfrak{m} is in the support of (σ, ψ) if $S_{\sigma, \psi}(U, A)_\mathfrak{m} \neq 0$. Now let \mathcal{O} be the ring of integers in $\overline{\mathbb{Q}}_p$, with residue field $\mathbb{F} = \overline{\mathbb{F}}_p$, and suppose that $A = \mathcal{O}$ in the above discussion, and that σ has open kernel. Consider a maximal ideal $\mathfrak{m} \subset \mathbb{T}_{S, \mathcal{O}}^{\text{univ}}$ which is induced by a maximal ideal of $\mathbb{T}_{\sigma, \psi}(U, \mathcal{O})$. Then there is a semisimple Galois representation $\overline{\rho}_\mathfrak{m} : G_F \rightarrow \text{GL}_2(\mathbb{F})$ associated to \mathfrak{m} which is characterised up to equivalence by the property that if $v \notin S$ and Frob_v is an arithmetic Frobenius at v , then the trace of $\overline{\rho}_\mathfrak{m}(\text{Frob}_v)$ is the image of T_v in \mathbb{F} .

We are now in a position to define what it means for a representation to be modular of some weight. Let F_v have ring of integers \mathcal{O}_v and residue field k_v , and let σ be an $\overline{\mathbb{F}}_p$ -representation of $G := \prod_{v|p} \text{GL}_2(k_v)$. We also denote by σ the representation of $\prod_{v|p} \text{GL}_2(\mathcal{O}_{F_v})$ induced by the surjections $\mathcal{O}_{F_v} \twoheadrightarrow k_v$.

Definition 4.1.1. We say that an irreducible representation $\overline{\rho} : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ is modular of weight σ if for some D, σ, S, U, ψ , and \mathfrak{m} as above we have $S_{\sigma, \psi}(U, \mathbb{F})_\mathfrak{m} \neq 0$ and $\overline{\rho}_\mathfrak{m} \cong \overline{\rho}$.

4.2. It seems that the methods of [Gee08] do not in themselves suffice to completely prove the conjectures of [BDJ10]. We can, however, prove the conjecture completely in the case where $p > 2$ splits completely in F and $\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible, and we do so in section 4.4. Before doing this, we wish to discuss some conjectures which extend those of [BDJ10]. Firstly, we discuss what we regard as the natural generalisation of their conjectures to the case where p is no longer unramified in F . For the rest of this section, we allow $p = 2$.

Let K/\mathbb{Q}_p be a finite extension with ring of integers \mathcal{O}_K and residue field k_K , and let $\overline{\rho}_K : G_K \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Let the absolute ramification degree of K be e . If $\rho : G_K \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p)$ is de Rham (in particular, if it is crystalline) and $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$ we say that the Hodge-Tate weights of ρ with respect to τ are the i for which $gr^{-i}((\rho \otimes_{\mathbb{Q}_p} B_{dR})^{G_K} \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K, 1 \otimes \tau} \overline{\mathbb{Q}}_p) \neq 0$.

We define a set $W(\bar{\rho}_K)$ of irreducible representations of $\mathrm{GL}_2(k_K)$ in the following fashion. Let $\sigma_{\bar{a}, \bar{b}} = \otimes_{\bar{\tau}: k_K \hookrightarrow \bar{\mathbb{F}}_p} \det^{a_{\bar{\tau}}} \otimes \mathrm{Sym}^{b_{\bar{\tau}}-1} k_K^2 \otimes_{\bar{\tau}} \bar{\mathbb{F}}_p$ with $1 \leq b_{\bar{\tau}} \leq p$. For each embedding $\bar{\tau} : k_K \hookrightarrow \bar{\mathbb{F}}_p$, let $\tau^j : K \hookrightarrow \bar{\mathbb{Q}}_p$, $j = 1, \dots, e$ be the embeddings lifting $\bar{\tau}$. Then $W(\bar{\rho}_K)$ is the set of $\sigma_{\bar{a}, \bar{b}}$ such that there is a continuous crystalline lift $\rho_K : G_K \rightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ of ρ_K such that for all $\bar{\tau}$ the Hodge-Tate weights of ρ with respect to τ^j are $a_{\bar{\tau}}$ and $a_{\bar{\tau}} + b_{\bar{\tau}}$ if $j = 1$ and 0 and 1 otherwise. Note that this definition is independent of the choice of τ^1 , as for any j there is an automorphism of $\bar{\mathbb{Q}}_p$ taking τ^1 to τ^j .

Let $\bar{\rho}$ be as above. Then we conjecture

Conjecture 4.2.1. *$\bar{\rho}$ is modular of weight σ if and only if $\sigma = \otimes_{v|p} \sigma_v$, with $\sigma_v \in W(\bar{\rho}|_{G_{F_v}})$ for all $v|p$.*

We now explain the motivation for this conjecture. Let $\sigma = \otimes_{v|p} \sigma_v$, with each $\sigma_v = \sigma_{\bar{a}, \bar{b}}$. Suppose temporarily that all of the $b_{\bar{\tau}}$ are odd, and that $2a_{\bar{\tau}} + b_{\bar{\tau}} = 1$ for all $\bar{\tau}$. Then we may define a representation $\tilde{\sigma}$ of $\prod_{v|p} \mathrm{GL}_2(\mathcal{O}_{F_v})$ by $\tilde{\sigma} = \otimes \tilde{\sigma}_v$, where $\tilde{\sigma}_v = \otimes_{\bar{\tau}: k_v \hookrightarrow \bar{\mathbb{F}}_p} \det^{a_{\bar{\tau}}} \otimes \mathrm{Sym}^{b_{\bar{\tau}}-1} F_v^2 \otimes_{\tau^1} \bar{\mathbb{Q}}_p$ (where k_v is the residue field of F_v). Note that the reduction mod p of $\tilde{\sigma}$ is just σ . The assumption that $2a_{\bar{\tau}} + b_{\bar{\tau}} = 1$ for all $\bar{\tau}$ allows us to lift ψ as above to a finite-order character $\tilde{\psi}$ satisfying the required compatibility with $\tilde{\sigma}$, and Lemma 4.4.2 below, together with the Jacquet-Langlands correspondence and the standard properties of Galois representations associated to Hilbert modular forms, shows that $\bar{\rho}$ is modular of weight σ if and only if $\bar{\rho}$ lifts to a modular representation $\rho : G_F \rightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ such that for all $v|p$, $\rho|_{G_{F_v}}$ is crystalline, and for all $\bar{\tau}$ the Hodge-Tate weights of ρ with respect to τ^j are $a_{\bar{\tau}}$ and $a_{\bar{\tau}} + b_{\bar{\tau}}$ if $j = 1$ and 0 and 1 otherwise.

Thus (under the assumption that $2a_{\bar{\tau}} + b_{\bar{\tau}} = 1$ for all $\bar{\tau}$) we see that the condition on the existence of local crystalline lifts should be a necessary condition. On the other hand, we conjecture that it is also sufficient, on the grounds of a general principle that “the only obstructions to lifting Galois representations should be obvious”, and the Fontaine-Mazur conjecture. Corollary 3.1.7 is a persuasive example of the validity of this principle. In the general case (where, for example, some of the $b_{\bar{\tau}}$ are allowed to be even) this line of reasoning is no longer valid, but we continue to make the conjecture. This may be justified in two ways: firstly, it is generally believed that there should be a purely local formulation of the weight part of any generalisation of Serre’s conjecture, and the obstruction to reasoning as above in the general case is a global one. Secondly, this obstruction is already present in many cases in the original Buzzard-Diamond-Jarvis conjecture (i.e. the case where p is unramified in F), which has been proved in many cases in [Gee08].

One could hope to make this conjecture more explicit by finding conditions on $\bar{\rho}_K$ for such a lift ρ_K to exist; this is a purely local question. One might also hope to use Corollary 3.1.7 to prove cases of Conjecture 4.2.1 for general totally real fields; we do not know whether this is possible, although we intend to investigate the possibility in future work. Note however that in general as the ramification of F increases we expect that the set of possible weights will also increase. The combinatorial arguments of [Gee08] depend on being able to rule out many weights, because lifts of the corresponding types do not exist, and these arguments will not work if more weights occur.

We now wish to propose a more general conjecture that deals with “higher” weights. Namely, suppose that $\sigma = \sigma_{\bar{a}, \bar{b}}$ as above, except that we now drop the

assumption that $b_\tau \leq p$, so that σ is no longer necessarily irreducible. Exactly as above, let K/\mathbb{Q}_p be a finite extension with ring of integers \mathcal{O}_K and residue field k_K , and let $\bar{\rho}_K : G_K \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Let the absolute ramification degree of K be e . We define a set $W(\bar{\rho}_K)$ of representations of $\mathrm{GL}_2(k_K)$ in the following fashion. Let $\sigma_{\bar{a}, \bar{b}} = \otimes_{\bar{\tau}: k_K \hookrightarrow \overline{\mathbb{F}}_p} \det^{a_\tau} \otimes \mathrm{Sym}^{b_\tau - 1} k_K^2 \otimes_{\bar{\tau}} \overline{\mathbb{F}}_p$. For each embedding $\bar{\tau} : k_K \hookrightarrow \overline{\mathbb{F}}_p$, let $\tau^j : K \hookrightarrow \overline{\mathbb{Q}}_p$, $j = 1, \dots, e$ be the embeddings lifting $\bar{\tau}$. Then $W(\bar{\rho}_K)$ is the set of $\sigma_{\bar{a}, \bar{b}}$ such that there is a continuous crystalline lift $\rho_K : G_K \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ of $\bar{\rho}_K$ such that for all $\bar{\tau}$ the Hodge-Tate weights of ρ with respect to τ^j are a_τ and $a_\tau + b_\tau$ if $j = 1$ and 0 and 1 otherwise.

Let $\bar{\rho}$ be as above. Then we conjecture

Conjecture 4.2.2. *$\bar{\rho}$ is modular of weight σ if and only if $\sigma = \otimes_{v|p} \sigma_v$, with $\sigma_v \in W(\bar{\rho}|_{G_{F_v}})$.*

While this formulation is striking, if one believes Conjecture 4.2.1 then this conjecture is equivalent to a purely local one. Indeed, we conjecture:

Conjecture 4.2.3. *Let K/\mathbb{Q}_p be a finite extension with residue field k_K . Let $\bar{\rho}_K : G_K \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Let $\sigma_{\bar{a}, \bar{b}} = \otimes_{\bar{\tau}: k_K \hookrightarrow \overline{\mathbb{F}}_p} \det^{a_\tau} \otimes \mathrm{Sym}^{b_\tau - 1} k_K^2 \otimes_{\bar{\tau}} \overline{\mathbb{F}}_p$, with the $b_\tau \geq 1$. For each embedding $\bar{\tau} : k_K \hookrightarrow \overline{\mathbb{F}}_p$, let $\tau^j : K \hookrightarrow \overline{\mathbb{Q}}_p$, $j = 1, \dots, e$ be the embeddings lifting $\bar{\tau}$. Then the following two statements are equivalent.*

- (1) $\bar{\rho}_K$ has a lift to a continuous crystalline representation $\rho : G_K \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ such that
 - for all $\bar{\tau}$ the Hodge-Tate weights of ρ with respect to τ^j are a_τ and $a_\tau + b_\tau$ if $j = 1$ and 0 and 1 otherwise
- (2) For some Jordan-Hölder factor $\sigma'_{\bar{a}', \bar{b}'} = \otimes_{\bar{\tau}: k_K \hookrightarrow \overline{\mathbb{F}}_p} \det^{a'_\tau} \otimes \mathrm{Sym}^{b'_\tau - 1} k_K^2 \otimes_{\bar{\tau}} \overline{\mathbb{F}}_p$ of $\sigma_{\bar{a}, \bar{b}}$ (so that $1 \leq b'_\tau \leq p$ for all $\bar{\tau}$), $\bar{\rho}_K$ has a lift to a continuous crystalline representation $\rho' : G_K \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ such that
 - for all $\bar{\tau}$ the Hodge-Tate weights of ρ with respect to τ^j are a'_τ and $a'_\tau + b'_\tau$ if $j = 1$ and 0 and 1 otherwise.

This conjecture can be regarded as a weak generalisation of the Breuil-Mézard conjecture ([BM02]). The connection between the two conjectures is as follows: $\bar{\rho}$ is modular of weight σ if and only if $\bar{\rho}$ is modular of weight σ' for σ' some Jordan-Hölder factor of σ . Then Conjecture 4.2.2 follows from Conjectures 4.2.1 and 4.2.3, together with the definition of $W(\bar{\rho}|_{G_{F_v}})$.

We remark that Conjecture 4.2.2 is true provided that $F = \mathbb{Q}$ and $\bar{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible, $\bar{\rho}|_{G_{\mathbb{Q}_p}} \not\cong \begin{pmatrix} \omega^\chi & * \\ 0 & \chi \end{pmatrix}$ for any character χ , and if $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ has scalar semisimplification then it is scalar. The necessity of the condition in the conjecture is clear; twisting, we may assume that $\sigma = \mathrm{Sym}^{k-2} \mathbb{F}_p^2$, $k \geq 2$. If $\bar{\rho}$ is modular of weight σ then it lifts to a characteristic zero modular form of weight k and level prime to p , whose associated Galois representation is crystalline of Hodge-Tate weights $0, k-1$. The converse follows from the proof of the relevant cases of the Breuil-Mézard conjecture in [Kis09a], as one sees that if $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ has a crystalline lift of Hodge-Tate weights $0, k-1$ then $\bar{\rho}$ must be modular of weight σ' for some irreducible subquotient σ' of σ , so that $\bar{\rho}$ must be modular of weight σ .

4.3. We now state a Serre-type conjecture on the possible weights of an n -dimensional mod p representation of $G_{\mathbb{Q}}$. For this section we allow $p = 2$. We anticipate that an entirely analogous statement should be true for any number field, but for simplicity of statement, and the lack of evidence in other cases, we restrict ourselves to the rationals. Such conjectures have also been made in [ADP02] and [Her09], but the set of weights predicted in [ADP02] is incomplete (although one should note that it is never claimed to be a complete list), and [Her09] only considers tame representations.

Suppose that

$$\bar{\rho} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_p)$$

is continuous, odd, and irreducible. Here “odd” means either that $p = 2$, or that $|n_+ - n_-| \leq 1$, where n_+ (respectively n_-) is the number of eigenvalues equal to 1 (respectively -1) of the image of a complex conjugation in $G_{\mathbb{Q}}$. It is conjectured that any such representation should arise from a Hecke eigenclass in some cohomology group. We now make this precise, following [Her09] (although note that our conventions differ from the ones used there).

For any positive integer N with $(N, p) = 1$, we define $S_1(N)$ to be the matrices in $\mathrm{GL}_n^+(\mathbb{Z}_{(N)})$ whose first row is congruent to $(1, 0, \dots, 0) \pmod{N}$, and let $\Gamma_1(N)$ to be the matrices in $\mathrm{SL}_n(\mathbb{Z})$ whose first row is congruent to $(1, 0, \dots, 0) \pmod{N}$. We obtain a Hecke algebra $\mathcal{H}_1(N)$ in the usual fashion: $\mathcal{H}_1(N)$ consists of right $\Gamma_1(N)$ -invariant elements in the free abelian group of right cosets $\Gamma_1(N)s$, $s \in S_1(N)$. Thus any double coset

$$\Gamma_1(N)s\Gamma_1(N) = \coprod_i \Gamma_1(N)s_i$$

(a finite union) is an element of $\mathcal{H}_1(N)$ in the obvious way, and we denote it $[\Gamma_1(N)s\Gamma_1(N)]$. If M is a right $S_1(N)$ -module then the group cohomology modules $H^\bullet(\Gamma_1(N), M)$ have a natural linear action of $\mathcal{H}_1(N)$, determined by

$$[\Gamma_1(N)s\Gamma_1(N)]m = \sum_i s_i m$$

for all $s \in S_1(N)$, $m \in H^0(\Gamma_1(N), M)$.

For any prime $l \nmid N$ and $0 \leq k \leq n$ we define a Hecke operator

$$T_{l,k} = \left[\Gamma_1(N) \begin{pmatrix} l & & & \\ & \ddots & & \\ & & l & \\ & & & \ddots \\ & & & & 1 \end{pmatrix} \Gamma_1(N) \right] \in \mathcal{H}_0(N)$$

(where there are k l 's on the diagonal). Now let M be a right $\overline{\mathbb{F}}_p[S_1(N)]$ -module, and let $\alpha \in H^0(\Gamma_1(N), M)$ be a common eigenvector of the $T_{l,k}$ for all $l \nmid Np$, $0 \leq k \leq n$. We say that $\bar{\rho}$ is *attached* to α if for all $l \nmid Np$ we have $\bar{\rho}$ unramified at l and

$$\det(1 - \bar{\rho}(\mathrm{Frob}_l)X) = \sum_{i=0}^n (-1)^i l^{i(i-1)/2} a(l, i) X^i$$

where $a(l, i)$ is the eigenvalue of $T_{l,i}$ acting on α , and Frob_l is an arithmetic Frobenius at l .

Let F be a simple $\overline{\mathbb{F}}_p[\mathrm{GL}_n(\mathbb{F}_p)]$ -module, and let $S_1(N)$ act on F via the natural projection $S_1(N) \twoheadrightarrow \mathrm{GL}_n(\mathbb{F}_p)$.

Definition 4.3.1. We say that F is a weight for $\bar{\rho}$, or $\bar{\rho}$ has weight F , if there exists an N coprime to p and a non-zero Hecke eigenvector $\alpha \in H^e(\Gamma_1(N), F)$ (for some e) such that $\bar{\rho}$ is attached to α .

We now give a conjectural description of the set of weights for $\bar{\rho}$. Recall that the simple $\overline{\mathbb{F}}_p[\mathrm{GL}_n(\mathbb{F}_p)]$ -modules are the $F(a_1, \dots, a_n)$, where $0 \leq a_1 - a_2, a_2 - a_3, \dots, a_{n-1} - a_n \leq p - 1$, and $F(a_1, \dots, a_n)$ is the socle of the dual Weyl module for $\mathrm{GL}_n(\mathbb{F}_p)$ of highest weight (a_1, \dots, a_n) (see [Her09]). We have $F(a_1, \dots, a_n) \cong F(a'_1, \dots, a'_n)$ if and only if $(a_1, \dots, a_n) - (a'_1, \dots, a'_n) \in (p - 1, \dots, p - 1)\mathbb{Z}$.

Conjecture 4.3.2. $F(a_1, \dots, a_n)$ is a weight for $\bar{\rho}$ if and only if $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ has a crystalline lift with Hodge-Tate weights $a_1 + (n - 1), a_2 + (n - 2), \dots, a_n$.

Note that this is a well-defined conjecture, as if ρ is a crystalline lift of $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ with Hodge-Tate weights $a_1 + (n - 1), \dots, a_n$, then for any k , $\rho \otimes \epsilon^{k(p-1)}$ is also a crystalline lift, with Hodge-Tate weights $a_1 + (n - 1) + (p - 1)k, \dots, a_n + (p - 1)k$.

This conjecture is the natural generalisation of Conjecture 4.2.1 to higher-dimensional representations. It is rather less explicit than the conjecture of [Her09], but it has the advantage of applying to all $\bar{\rho}$, rather than just the $\bar{\rho}$ which are tame at p , and it also applies to all weights, rather than just the regular weights. While it is less explicit, one may well not need a more explicit formulation in applications, for example to modularity lifting theorems.

It is natural to ask whether our conjecture agrees with that of [Her09] in the cases where $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is tamely ramified. We do not know if this is the case in general; one difficulty in checking this is that typically the conjectural crystalline representations are not in the Fontaine-Laffaille range. However, one can in many cases show that we predict at least as many weights as [Her09]. To keep the notation reasonably clear, we restrict ourselves to a single example.

Suppose that $p \geq 5$ and $\bar{\rho}|_{G_{\mathbb{Q}_p}} = 1 \oplus \omega^2 \oplus \omega^4$. Then [Her09] predicts 9 weights for $\bar{\rho}$, namely

$$(a_1, a_2, a_3) = \begin{array}{lll} (2, 1, 0), & (p - 1, p - 2, 4), & (p - 3, 3, 2), \\ (p - 1, 3, 0) & (p + 1, p - 2, 2), & (2p - 4, p, 4), \\ (p + 2, p - 2, 1) & (2p - 3, p, 3), & (2p - 1, p + 2, p - 2) \end{array}$$

(see Proposition 7.3 and Lemma 7.4 of [Her09], and note that these weights were also predicted by [ADP02]). For each of these weights we will write down a crystalline lift of $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ with the appropriate Hodge-Tate weights.

Note that e.g. from Theorem 3.2.1 of [Ber10] there is a 2-dimensional crystalline representation V of $G_{\mathbb{Q}_p}$ with Hodge-Tate weights $0, p + 3$ and $\overline{V} = \omega \oplus \omega^3$, and a crystalline representation W with Hodge-Tate weights $0, 2p - 4$ and $\overline{W} = \omega^{-3} \oplus \omega$ (in the notation of [Ber10], take $V = V_{p+4, p/3}$ and $W = V_{2p-3, -p/4}$). Then the

$$\begin{array}{ll} (2, 1, 0) & \epsilon^4 \oplus \epsilon^2 \oplus 1 \\ (p - 1, p - 2, 4) & \epsilon^{p+1} \oplus \epsilon^{p-1} \oplus \epsilon^4 \\ (p - 3, 3, 2) & \epsilon^{p-1} \oplus \epsilon^4 \oplus \epsilon^2 \\ (p - 1, 3, 0) & \epsilon^{p+1} \oplus \epsilon^4 \oplus 1 \\ (p + 1, p - 2, 2) & \epsilon^{p+3} \oplus \epsilon^{p-1} \oplus \epsilon^2 \\ (2p - 4, p, 4) & \epsilon^{2p-2} \oplus \epsilon^{p+1} \oplus \epsilon^4 \\ (p + 2, p - 2, 1) & \epsilon V \oplus \epsilon^{p-1} \\ (2p - 3, p, 3) & \epsilon^3 W \oplus \epsilon^{p+1} \\ (2p - 1, p + 2, p - 2) & \epsilon^{p-2} V \oplus \epsilon^{p+3} \end{array}$$

appropriate lifts are as follows:

These lifts are all reducible; it seems likely that one can always produce appropriate reducible lifts whenever the 3-dimensional representation $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is semisimple of niveau 1, but it will not be possible in general, of course.

Finally, we remark that it may be possible to make a conjecture for “higher weights”, as we did for GL_2 (see Conjecture 4.2.2). We do not do this for the simple reason that we have no evidence for such a conjecture. In addition, one can make an analogous conjecture for unitary groups, which one might hope to prove via the techniques of [Gee08] and section 5 of this paper. We hope to return to this in future work.

4.4. From now on we suppose that $p > 2$ splits completely in F .

We recall some group-theoretic results from section 3 of [CDT99]. Firstly, recall the irreducible finite-dimensional representations of $\mathrm{GL}_2(\mathbb{F}_p)$ over $\overline{\mathbb{Q}_p}$. Once one fixes an embedding $\mathbb{F}_{p^2} \hookrightarrow M_2(\mathbb{F}_p)$, any such representation is equivalent to one in the following list:

- For any character $\chi : \mathbb{F}_p^\times \rightarrow \overline{\mathbb{Q}_p}^\times$, the representation $\chi \circ \det$.
- For any $\chi : \mathbb{F}_p^\times \rightarrow \overline{\mathbb{Q}_p}^\times$, the representation $\mathrm{sp}_\chi = \mathrm{sp} \otimes (\chi \circ \det)$, where sp is the representation of $\mathrm{GL}_2(\mathbb{F}_p)$ on the space of functions $\mathbb{P}^1(\mathbb{F}_p) \rightarrow \overline{\mathbb{Q}_p}$ with average value zero.
- For any pair $\chi_1 \neq \chi_2 : \mathbb{F}_p^\times \rightarrow \overline{\mathbb{Q}_p}^\times$, the representation

$$I(\chi_1, \chi_2) = \mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi_1 \otimes \chi_2,$$

where $B(\mathbb{F}_p)$ is the Borel subgroup of upper-triangular matrices in $\mathrm{GL}_2(\mathbb{F}_p)$, and $\chi_1 \otimes \chi_2$ is the character

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d).$$

- For any character $\chi : \mathbb{F}_{p^2}^\times \rightarrow \overline{\mathbb{Q}_p}^\times$ with $\chi \neq \chi^p$, the cuspidal representation $\Theta(\chi)$, characterised by

$$\Theta(\chi) \otimes \mathrm{sp} \cong \mathrm{Ind}_{\mathbb{F}_{p^2}^\times}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi.$$

We now recall the reductions mod p of these representations. Let $\sigma_{m,n}$ be the irreducible $\overline{\mathbb{F}_p}$ -representation $\det^m \otimes \mathrm{Sym}^n \overline{\mathbb{F}_p}^2$, with $0 \leq m < p-1$, $0 \leq n \leq p-1$. Then we have:

Lemma 4.4.1. *Let L be a finite free \mathcal{O} -module with an action of $\mathrm{GL}_2(\mathbb{F}_p)$ such that $V = L \otimes_{\mathcal{O}} \overline{\mathbb{Q}_p}$ is irreducible. Let $\tilde{\cdot}$ denote the Teichmüller lift.*

- (1) *If $V \cong \chi \circ \det$ with $\chi(a) = \tilde{a}^m$, then $L \otimes_{\mathcal{O}} \mathbb{F} \cong \sigma_{m,0}$.*
- (2) *If $V \cong \mathrm{sp}_\chi$ with $\chi(a) = \tilde{a}^m$, then $L \otimes_{\mathcal{O}} \mathbb{F} \cong \sigma_{m,p-1}$.*
- (3) *If $V \cong I(\chi_1, \chi_2)$ with $\chi_i(a) = \tilde{a}^{m_i}$ for distinct $m_i \in \mathbb{Z}/(p-1)\mathbb{Z}$, then $L \otimes_{\mathcal{O}} \mathbb{F}$ has two Jordan-Hölder subquotients: $\sigma_{m_2, \{m_1 - m_2\}}$ and $\sigma_{m_1, \{m_2 - m_1\}}$ where $0 < \{m\} < p-1$ and $\{m\} \equiv m \pmod{p-1}$.*
- (4) *If $V \cong \Theta(\chi)$ with $\chi(c) = \tilde{c}^{i+(p+1)j}$ where $1 \leq i \leq p$ and $j \in \mathbb{Z}/(p-1)\mathbb{Z}$, then $L \otimes_{\mathcal{O}} \mathbb{F}$ has two Jordan-Hölder subquotients: $\sigma_{1+j, i-2}$ and $\sigma_{i+j, p-1-i}$. Both occur unless $i = p$ (when only the first occurs), or $i = 1$ (when only the second one occurs), and in either of these cases $L \otimes_{\mathcal{O}} \mathbb{F} \cong \sigma_{1+j, p-2}$.*

Proof. This is Lemma 3.1.1 of [CDT99]. □

In the below, we will sometimes consider the above representations as representations of $\mathrm{GL}_2(\mathbb{Z}_p)$ via the natural projection map.

We now show how one can gain information about the weights associated to a particular Galois representation by considering lifts to characteristic zero. The key is the following lemma.

Lemma 4.4.2. *Let $\psi : F^\times \backslash (\mathbb{A}_F)^\times \rightarrow \mathcal{O}^\times$ be a continuous character, and write $\bar{\psi}$ for the composite of ψ with the projection $\mathcal{O}^\times \rightarrow \mathbb{F}^\times$. Fix a representation σ on a finite free \mathcal{O} -module W_σ , and an irreducible representation σ' on a finite free \mathbb{F} -module $W_{\sigma'}$. Suppose that for each $v|p$ we have $\sigma|_{U_v \cap \mathcal{O}_{F_v}^\times} = \psi^{-1}|_{U_v \cap \mathcal{O}_{F_v}^\times}$ and $\sigma'|_{U_v \cap \mathcal{O}_{F_v}^\times} = \bar{\psi}^{-1}|_{U_v \cap \mathcal{O}_{F_v}^\times}$.*

Let \mathfrak{m} be a maximal ideal of $\mathbb{T}_{S,\mathcal{O}}^{\mathrm{univ}}$.

Suppose that $W_{\sigma'}$ occurs as a $\prod_{v|p} U_v$ -module subquotient of $W_{\bar{\sigma}} := W_\sigma \otimes \mathbb{F}$. If \mathfrak{m} is in the support of $(\sigma', \bar{\psi})$, then \mathfrak{m} is in the support of (σ, ψ) .

Conversely, if \mathfrak{m} is in the support of (σ, ψ) , then \mathfrak{m} is in the support of $(\sigma', \bar{\psi})$ for some irreducible $\prod_{v|p} U_v$ -module subquotient $W_{\sigma'}$ of $W_{\bar{\sigma}}$.

Proof. The first part is proved just as in Lemma 3.1.4 of [Kis07b], and the second part follows from Proposition 1.2.3 of [AS86a]. \square

We now need a very special case (the tame case) of the inertial local Langlands correspondence of Henniart (see the appendix to [BM02]). If $\chi_1 \neq \chi_2 : \mathbb{F}_p^\times \rightarrow \mathcal{O}^\times$, let τ_{χ_1, χ_2} be the inertial type $\chi_1 \oplus \chi_2$ (considered as a representation of $I_{\mathbb{Q}_p}$ via local class field theory). Then we let $\sigma(\tau_{\chi_1, \chi_2})$ be a representation on a finite \mathcal{O} -module given by taking a lattice in $I(\chi_1, \chi_2)$. If $\chi : \mathbb{F}_p^\times \rightarrow \mathcal{O}^\times$, we let $\tau_{\chi, \chi} = \chi \oplus \chi$, and $\sigma(\tau_{\chi, \chi})$ be $\chi \circ \det$. If $\chi : \mathbb{F}_{p^2}^\times \rightarrow \mathcal{O}^\times$ with $\chi \neq \chi^p$, let τ_χ be the inertial type $\chi \oplus \chi^p$ (considered as a representation of $I_{\mathbb{Q}_p} = I_{\mathbb{Q}_{p^2}}$ via local class field theory). Then we let $\sigma(\tau_\chi)$ be a representation on a finite \mathcal{O} -module given by taking a lattice in $\Theta(\chi)$.

The following result then follows from Lemma 4.4.2, the Jacquet-Langlands correspondence, the compatibility of the local and global Langlands correspondences at places dividing p (see [Kis08]), and standard properties of the local Langlands correspondence (see section 4 of [CDT99]); recall that the definition of ‘‘potentially Barsotti-Tate of type τ ’’ includes the requirement that the determinant be a finite order character times the cyclotomic character.

Lemma 4.4.3. *For each $v|p$ fix a tame type τ_v (i.e. $\tau_v = \tau_{\chi_1, \chi_2}$ or τ_χ as above). Suppose that $\bar{\rho}$ is irreducible and modular of weight σ , and that σ is a $\prod_{v|p} \mathrm{GL}_2(\mathcal{O}_v)$ -module subquotient of $\otimes_{v|p} \sigma(\tau_v) \otimes_{\mathcal{O}} \bar{\mathbb{F}}_p$. Then $\bar{\rho}$ lifts to a modular Galois representation which is potentially Barsotti-Tate of type τ_v for each $v|p$.*

Conversely, if $\bar{\rho}$ lifts to a modular Galois representation which is potentially Barsotti-Tate of type τ_v for each $v|p$, then $\bar{\rho}$ is modular of weight σ for some $\prod_{v|p} \mathrm{GL}_2(\mathcal{O}_v)$ -module subquotient σ of $\otimes_{v|p} \sigma(\tau_v) \otimes_{\mathcal{O}} \bar{\mathbb{F}}_p$.

4.5. We need a slight refinement of this result in some cases, to take account of Hecke eigenvalues at places $v|p$. Suppose firstly that $A = \mathbb{F}$ and σ is irreducible. We can extend the action of $\mathrm{GL}_2(\mathcal{O}_v)$ on σ to an action of $\mathrm{GL}_2(F_v) \cap M_2(\mathcal{O}_{F_v})$; in the case that $\sigma = \mathrm{Sym}^n \bar{\mathbb{F}}_p^2$, we extend in the obvious fashion, by thinking of σ as homogeneous polynomials of degree n in two variables X, Y , with $\mathrm{GL}_2(F_v) \cap$

$M_2(\mathcal{O}_{F_v})$ acting by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} F \right) (X, Y) = F(aX + cY, bX + dY),$$

and in the general case that $\sigma = \det^m \text{Sym}^n \overline{\mathbb{F}}_p^2$ we twist the above action by $\chi^m \circ \det$, where χ is the character $\mathbb{Q}_p^\times \rightarrow \overline{\mathbb{F}}_p^\times$ with $\chi(p) = 1$ and $\chi|_{\mathbb{Z}_p^\times}$ equal to reduction mod p . We denote this action by $\tilde{\sigma}$. For each $v|p$ we let $\text{GL}_2(F_v) \cap M_2(\mathcal{O}_{F_v})$ act on the space of functions $(D \otimes_F \mathbb{A}_F^f)^\times \rightarrow W_\sigma$ via $u(f)(g) := \tilde{\sigma}(u)(f(gu))$, so that for each $g \in \text{GL}_2(F_v) \cap M_2(\mathcal{O}_{F_v})$ there is a corresponding Hecke operator $[UgU]$ acting on $S_{\sigma, \psi}(U, \mathbb{F})$. Let $\mathbb{T}_{S, \mathbb{F}}^{\text{univ}} = \mathbb{T}_{S, \mathbb{F}}^{\text{univ}}[T_v, S_v]_{v|p}$, and extend the action of $\mathbb{T}_{S, \mathbb{F}}^{\text{univ}}$ on $S_{\sigma, \psi}(U, \mathbb{F})$ to one of $\mathbb{T}_{S, \mathbb{F}}^{\text{univ}}$ by letting T_v, S_v act via the Hecke operators corresponding to $\begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix}$ respectively. Any maximal ideal \mathfrak{m} of $\mathbb{T}_{S, \mathbb{F}}$ induces a maximal ideal \mathfrak{m}' of $\mathbb{T}_{S, \mathbb{F}}$, and we write $\bar{\rho}_{\mathfrak{m}}$ for $\bar{\rho}_{\mathfrak{m}'}$.

We also need to consider the case that $A = \mathcal{O}$ and $\sigma = \prod_{w|p} \sigma_w$, where for some $v|p$ we have $\sigma_v = \sigma(\tau_v)$, $\tau_v = \chi_{1,v} \oplus \chi_{2,v}$. We suppose in order to simplify the discussion that $\chi_{1,v} = 1$; as above, the operators in the general case are easily defined via twisting. If $\chi_{2,v} = 1$ then $\sigma(\tau_v)$ is trivial, and we can define T_v in the usual fashion. Suppose now that $\chi_{2,v} \neq 1$. Then $\sigma(\tau_v) = \text{Ind}_{B(\mathbb{F}_p)}^{\text{GL}_2(\mathbb{F}_p)} 1 \otimes \chi_{2,v}$ (the \mathcal{O} -valued induction). The dual of $\sigma(\tau_v)$ is $\sigma(\tau_v)^* = \text{Ind}_{B(\mathbb{F}_p)}^{\text{GL}_2(\mathbb{F}_p)} 1 \otimes \chi_{2,v}^{-1}$, and there is a natural $\text{GL}_2(\mathbb{F}_p)$ -equivariant pairing $\langle \cdot, \cdot \rangle_v : \sigma(\tau_v) \times \sigma(\tau_v)^* \rightarrow \mathcal{O}$. Let $L_{\sigma(\tau_v)^*}$ be the underlying \mathcal{O} -module of $\sigma(\tau_v)^*$. Then we let $x_v \in L_{\sigma(\tau_v)^*}$ be the function in $\text{Ind}_{B(\mathbb{F}_p)}^{\text{GL}_2(\mathbb{F}_p)} 1 \otimes \chi_{2,v}^{-1}$ supported on $B(\mathbb{F}_p)$ and sending the identity to $1 \in \mathcal{O}$. Note that this function is Γ_1 -invariant, where Γ_1 is the inverse image in $\text{GL}_2(\mathbb{Z}_p)$ of the subgroup of $\text{GL}_2(\mathbb{F}_p)$ of elements of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Then if $f \in S_{\sigma, \psi}(U, \mathcal{O})$, we can consider the function $f_x : D^\times \backslash (D \otimes_F \mathbb{A}_F^f)^\times \rightarrow \otimes_{w \neq v} \sigma_w$ given by $\langle f, x_v \rangle_v$. The map $f \mapsto f_x$ is an injective map from $S_{\sigma, \psi}(U, \mathcal{O})$ to the space of functions $D^\times \backslash (D \otimes_F \mathbb{A}_F^f)^\times \rightarrow \otimes_{w \neq v} \sigma_w$ which are $\Gamma_{1,v}$ -invariant, where $\Gamma_{1,v} \subset U_v$ is the subgroup defined above. This latter space has a natural action of a Hecke operator $U_v = [\Gamma_{1,v} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_{1,v}]$. In fact, U_v acts on the image of $S_{\sigma, \psi}(U, \mathcal{O})$; an easy way to see this is to note that an alternative description of the map $f \mapsto f_x$ is provided by using Frobenius reciprocity

$$\text{Hom}_{\text{GL}_2(\mathbb{F}_p)}(\cdot, \text{Ind}_{B(\mathbb{F}_p)}^{\text{GL}_2(\mathbb{F}_p)} 1 \otimes \chi_{2,v}) = \text{Hom}_{B(\mathbb{F}_p)}(\cdot, 1 \otimes \chi_{2,v})$$

to identify $S_{\sigma, \psi}(U, \mathcal{O})$ with a space of functions to $\mathcal{O} \otimes \otimes_{w \neq v} \sigma_w$ which transform via $1 \otimes \chi_{2,v}$ under the standard Iwahori subgroup of U_v . This is obviously a subspace of the functions which are $\Gamma_{1,v}$ -invariant, and is easily seen to be U_v -stable.

Assume now that $\chi_{2,v} \neq 1$. Then for some $1 \leq r \leq p-2$ we have $\sigma(\tau_v) \otimes_{\mathcal{O}} \mathbb{F} = \text{Ind}_{B(\mathbb{F}_p)}^{\text{GL}_2(\mathbb{F}_p)} 1 \otimes \delta^r$, where $1 \otimes \delta^r \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) = d^r$. Then (for example, by Lemma 3.2 of [AS86b]) there is an exact sequence

$$0 \rightarrow \sigma_{0,r} \rightarrow \text{Ind}_{B(\mathbb{F}_p)}^{\text{GL}_2(\mathbb{F}_p)} 1 \otimes \delta^r \rightarrow \sigma_{r,p-1-r} \rightarrow 0.$$

Lemma 4.5.1. *Let $\sigma' = \sigma_{0,r} \otimes (\otimes_{w \neq v} \sigma_w \otimes_{\mathcal{O}} \mathbb{F})$. The natural map $S_{\sigma', \psi}(U, \mathbb{F}) \rightarrow S_{\sigma, \psi}(U, \mathbb{F})$ is equivariant for the actions of T_v on the left hand side and U_v on the right hand side.*

Proof. Dually to the above short exact sequence, we have a short exact sequence

$$0 \rightarrow \sigma_{0,p-1-r} \rightarrow \text{Ind}_{B(\mathbb{F}_p)}^{\text{GL}_2(\mathbb{F}_p)} 1 \otimes \delta^{-r} \rightarrow \sigma_{p-1-r,r} \rightarrow 0.$$

Now, the duality pairing induces a $\text{GL}_2(\mathbb{Z}_p)$ -equivariant pairing $\sigma_{0,r} \times \sigma_{p-1-r,r} \rightarrow \overline{\mathbb{F}}_p$. This pairing can be made completely explicit (see the proof of Lemma 3.1 of [AS86b]), and we find that if

$$F(X, Y) = \sum_{j=0}^r a_j X^{r-j} Y^j, \quad G(X, Y) = \sum_{j=0}^r b_j X^{r-j} Y^j,$$

$$\langle F, G \rangle = \sum_{j=0}^r \binom{r}{j}^{-1} (-1)^j a_j b_{r-j}.$$

Then for any homogenous polynomial $F(X, Y) = \sum_{j=0}^r a_j X^{r-j} Y^j$ of degree r and any $i \in \mathbb{Z}_p$ we have

$$\begin{aligned} \left\langle \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} F(X, Y), X^r \right\rangle &= \left\langle \sum_{j=0}^r a_j (pX)^{r-j} (iX + Y)^j, X^r \right\rangle \\ &= \langle a_r (iX + Y)^r, X^r \rangle \\ &= (-1)^r a_r \\ &= \left\langle \sum_{j=0}^r a_j X^{r-j} Y^j, X^r \right\rangle \\ &= \langle F, X^r \rangle, \end{aligned}$$

$$\begin{aligned} \left\langle \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} F, X^r \right\rangle &= \left\langle \sum_{j=0}^r a_j X^{r-j} (pY)^j, X^r \right\rangle \\ &= \langle a_0 X^r, X^r \rangle \\ &= 0. \end{aligned}$$

Now (see for example the proof of Lemma 3.1 of [AS86b]) we can take the map $\text{Ind}_{B(\mathbb{F}_p)}^{\text{GL}_2(\mathbb{F}_p)} 1 \otimes \delta^{-r} \rightarrow \sigma_{p-1-r,r}$ in the short exact sequence above to be given by

$$\phi \mapsto \sum_{i \in \mathbb{F}_p} \phi \left(\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \right) (X - iY)^r + \phi \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) Y^r,$$

so that the image of x_v in $\sigma_{p-1-r,r}$ is just X^r . Taking an explicit set of coset representatives for U_v , we see that if $f \in S_{\sigma_{0,r} \otimes \sigma'' , \psi}(U, \mathbb{F})$, then

$$\begin{aligned}
\langle U_v f(g), X^r \rangle &= \left\langle \sum_{i=0}^{p-1} f(g \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}), X^r \right\rangle \\
&= \left\langle \sum_{i=0}^{p-1} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} f(g \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}), X^r \right\rangle \\
&= \left\langle \sum_{i=0}^{p-1} \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix} f(g \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}) + \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} f(g \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}), X^r \right\rangle \\
&= \langle T_v f(g), X^r \rangle.
\end{aligned}$$

It is easy to check that the map $f \mapsto \langle f, X^r \rangle$ is injective on $S_{\sigma_0, r \otimes \sigma'', \psi}(U, \mathbb{F})$, so we obtain the claimed compatibility of U_v and T_v . \square

4.6. The final ingredient we need before we can deduce results on the conjectures of [BDJ10] is a characterisation of when mod p representations have Barsotti-Tate lifts of particular types. Such results are proved in [Sav05], and we now recall them:

Theorem 4.6.1. *Suppose that $\bar{\rho} : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$. Then:*

- (1) $\bar{\rho}$ has a potentially Barsotti-Tate lift of type $\tilde{\omega}^i \oplus \tilde{\omega}^j$ only if $\bar{\rho}|_{I_p} \cong \begin{pmatrix} \omega^{1+i} & * \\ 0 & \omega^j \end{pmatrix}$, $\begin{pmatrix} \omega^{1+j} & * \\ 0 & \omega^i \end{pmatrix}$ (with $\bar{\rho}$ peu ramifiée if $i = j$) or $\begin{pmatrix} \omega_2^k & 0 \\ 0 & \omega_2^{pk} \end{pmatrix}$ where $k = 1 + \{j - i\} + (p+1)i$. The converse holds if $\bar{\rho}$ is decomposable or has only scalar endomorphisms.
- (2) $\bar{\rho}$ has a potentially Barsotti-Tate lift of type $\tilde{\omega}_2^m \oplus \tilde{\omega}_2^{pm}$, $(p+1) \nmid m$, only if $\bar{\rho}|_{I_p} \cong \begin{pmatrix} \omega_2^{1+m} & 0 \\ 0 & \omega_2^{p(1+m)} \end{pmatrix}$, $\begin{pmatrix} \omega_2^{p+m} & 0 \\ 0 & \omega_2^{1+pm} \end{pmatrix}$, $\begin{pmatrix} \omega^{i+j} & * \\ 0 & \omega^{1+j} \end{pmatrix}$, (where $\bar{\rho}$ is peu ramifiée if $i = 2$) or $\begin{pmatrix} \omega^{1+j} & * \\ 0 & \omega^{i+j} \end{pmatrix}$ (where $\bar{\rho}$ is peu ramifiée if $i = p-1$). Here $m = i + (p+1)j$, $1 \leq i \leq p$, $j \in \mathbb{Z}/(p-1)\mathbb{Z}$. The converse holds if $\bar{\rho}$ has only scalar endomorphisms.

Proof. If $\bar{\rho}$ has only scalar endomorphisms, then this follows from Theorems 1.3 and 1.4 of [Sav05] (apart from the case where the type is a twist of the trivial type, which is standard). In the cases where $\bar{\rho}$ has non-trivial endomorphisms the proof is an easy consequence of the methods of [Sav05]; in the “only if” direction, the result follows from Theorems 6.11 and 6.12 of [Sav05]. In the “if” direction, one may simply consider decomposable lifts. \square

We now state the precise form of the conjecture for totally real fields in which p splits completely. Let $\bar{\rho} : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$.

Definition 4.6.2. We define a set of weights $W(\bar{\rho})$ as follows. Suppose that $\bar{\rho}$ is irreducible. Then $\sigma_{m,n} \in W(\bar{\rho})$ if and only if $\bar{\rho}|_{I_p} \cong \begin{pmatrix} \omega_2^{n+1} & 0 \\ 0 & \omega_2^{p(n+1)} \end{pmatrix} \otimes \omega^m$. If $\bar{\rho}$ is reducible, $\sigma_{m,n} \in W(\bar{\rho})$ only if $\bar{\rho}|_{I_p} \cong \begin{pmatrix} \omega^{n+1} & * \\ 0 & 1 \end{pmatrix} \otimes \omega^m$. If this is the case then $\sigma_{m,n} \in W(\bar{\rho})$ unless $n = 0$ and $\bar{\rho}$ is très ramifiée.

If now $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular, we define a set of weights $W(\bar{\rho})$ in the obvious fashion; that is, $W(\bar{\rho})$ consists of precisely the representations

$$\sigma_{\vec{m}, \vec{n}} = \otimes_{v|p, \sigma_{m_v, n_v} \in W(\bar{\rho})|_{G_{F_v}}} \sigma_{m_v, n_v}$$

of $\mathrm{GL}_2(\mathcal{O}_F/p) = \prod_{v|p} \mathrm{GL}_2(\mathbb{F}_p)$. Assume from now on that $\bar{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. We wish to prove that $W(\bar{\rho})$ is precisely the set of weights for which $\bar{\rho}$ is modular, by using Theorem 4.6.1, Lemma 4.4.1, Lemma 4.4.3 and corollary 3.1.7. Firstly, we will prove that if $\bar{\rho}$ is modular of some weight, then this weight is contained in $W(\bar{\rho})$. We do this by using Lemma 4.4.1 and Lemma 4.4.3 to show that if $\bar{\rho}$ is modular of some weight σ , then $\bar{\rho}$ must have a potentially Barsotti-Tate lift of a particular type, and then using Theorem 4.6.1 to obtain conditions on $\bar{\rho}|_{G_{F_v}}$.

Once we have proved that any weight for $\bar{\rho}$ is contained in $W(\bar{\rho})$, we will use Corollary 3.1.7 to prove the converse. In combination with theorem 4.6.1 we are able to produce modular lifts of some specified types, and then Lemma 4.4.1 gives a list of irreducible representations, at least one of which must be a weight for $\bar{\rho}$. If only one of these weights is contained in $W(\bar{\rho})$, then $\bar{\rho}$ must be modular of this weight. We cannot always find a unique weight in this fashion, but in the cases where we cannot we are able to make additional arguments to complete the proof.

We say that $\bar{\rho}|_{G_{F_v}}$ is modular of weight σ if $\bar{\rho}$ is modular of some weight $\otimes_{w|p} \sigma_w$, with $\sigma_v = \sigma$.

Lemma 4.6.3. *If $\bar{\rho}|_{G_{F_v}}$ is modular of weight $\sigma = \sigma_{m,n}$, then $\sigma \in W(\bar{\rho}|_{G_{F_v}})$.*

Proof. Suppose firstly that $n \neq p-1$. By parts (1) and (3) of Lemma 4.4.1, $\sigma_{m,n}$ is a Jordan-Hölder factor of $\sigma(\tau_{\tilde{\omega}^{m+n}, \tilde{\omega}^m}) \otimes_{\mathcal{O}} \mathbb{F}$ (using the notation explained before Lemma 4.4.3). By Lemma 4.4.3 (which uses local-global compatibility at v), $\bar{\rho}$ has a lift to characteristic zero, which when restricted to G_{F_v} is potentially Barsotti-Tate of type $\tilde{\omega}^{m+n} \oplus \tilde{\omega}^m$. By part (1) of Theorem 4.6.1, we have

$$\bar{\rho}|_{I_{F_v}} \cong \begin{pmatrix} \omega^{m+n+1} & * \\ 0 & \omega^m \end{pmatrix}, \begin{pmatrix} \omega^{m+1} & * \\ 0 & \omega^{m+n} \end{pmatrix}, \text{ or } \begin{pmatrix} \omega_2^{n+1} & 0 \\ 0 & \omega_2^{p(n+1)} \end{pmatrix} \otimes \omega^m,$$

and furthermore if $n = 0$ and $\bar{\rho}|_{G_{F_v}}$ is reducible then $\bar{\rho}|_{G_{F_v}}$ is peu ramifiée. Comparing this to Definition 4.6.2, we are done unless $\bar{\rho}|_{I_{F_v}} \cong \begin{pmatrix} \omega^{m+1} & * \\ 0 & \omega^{m+n} \end{pmatrix}$ and $n \neq 0, p-1$.

Suppose that we are in this case. By part (4) of Lemma 4.4.1, we see that σ is a Jordan-Hölder factor of $\sigma(\tau_{\tilde{\omega}_2^{(m-1)(p+1)+n+2}}) \otimes_{\mathcal{O}} \mathbb{F}$. Again, by Lemma 4.4.3, $\bar{\rho}|_{G_{F_v}}$ has a potentially Barsotti-Tate lift of type $\tilde{\omega}_2^{(m-1)(p+1)+n+2} \oplus \tilde{\omega}_2^{(m-1)(p+1)+p(n+2)}$. Then by part (2) of Theorem 4.6.1, we see that if $\bar{\rho}|_{I_{F_v}}$ is reducible, then $\bar{\rho}|_{I_{F_v}} \cong \begin{pmatrix} \omega^{m+n+1} & * \\ 0 & \omega^m \end{pmatrix}$ or $\begin{pmatrix} \omega^m & * \\ 0 & \omega^{m+n+1} \end{pmatrix}$. This contradiction gives the required result.

Finally, suppose $n = p-1$. Then by Lemma 3.4 of [GS09], $\bar{\rho}|_{G_{F_v}}$ either has a potentially Barsotti-Tate lift of type $\tilde{\omega}^m \oplus \tilde{\omega}^m$, in which case the result is an easy corollary of Theorem 3.4.3 of [Ray74], or $\bar{\rho}|_{G_{F_v}}$ has a potentially semi-stable lift of type $\tilde{\omega}^m \oplus \tilde{\omega}^m$, which is not potentially crystalline. Any such lift is automatically ordinary, and the result again follows easily. \square

We now take care of (most of) the converse. We will prove a slight refinement of the weight conjecture, where we additionally consider Hecke operators at places dividing p . To this end, we have the following slight refinement of (one direction of) Lemma 4.4.3.

Definition 4.6.4. Let X be a set of places lying over p , and for each $v \in X$ let $\bar{\lambda}_v$ be an element of \mathbb{F} . We say that an irreducible representation $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ is modular of weight σ with Hecke eigenvalues $\{\bar{\lambda}_v\}_{v \in X}$ if for some D, σ, X, U ,

and ψ as above there is a maximal ideal \mathfrak{m} of $\mathbb{T}_{S_p, \mathbb{F}}^{\text{univ}}$ such that $(T_v - \bar{\lambda}_v) \in \mathfrak{m}$ for all $v \in X$, and we have $S_{\sigma, \psi}(U, \mathbb{F})_{\mathfrak{m}} \neq 0$ and $\bar{\rho}_{\mathfrak{m}} \cong \bar{\rho}$.

Lemma 4.6.5. *Let the set of places of F dividing p be partitioned as $S = S_1 \amalg S_2$. For each $v \in S_1$ fix a tame type τ_v (i.e. $\tau_v = \tau_{\chi_1, \chi_2}$ or τ_{χ} as above). For each $v \in S_2$ fix a tame type $\tau_v = \tilde{\omega}^{m_v} \oplus \tilde{\omega}^{m_v-1}$.*

Suppose that $\bar{\rho}$ lifts to a modular Galois representation ρ which is potentially Barsotti-Tate of type τ_v for each $v \in S$, and which for each $v \in S_2$ satisfies

$$\rho|_{G_{F_v}} \cong \begin{pmatrix} * & * \\ 0 & \tilde{\omega}^{m_v} nr(\lambda_v) \end{pmatrix},$$

where $nr(\lambda_v)$ is the unramified character taking an arithmetic Frobenius element to λ_v . Let $\sigma_{S_2} = \otimes_{v \in S_2} \sigma_{m_v, p-2}$. Then for some irreducible $\prod_{v \in S_1} \text{GL}_2(\mathcal{O}_v)$ -module subquotient σ_{S_1} of $\otimes_{v \in S_1} \sigma(\tau_v) \otimes_{\mathcal{O}} \mathbb{F}$, $\bar{\rho}$ is modular of weight $\sigma_{S_1} \otimes \sigma_{S_2}$ and Hecke eigenvalues $\{\bar{\lambda}_v\}_{v \in S_2}$.

Proof. Firstly, note that by Lemma 4.4.3, $\bar{\rho}$ is modular of weight σ for some Jordan-Hölder factor σ of $\otimes_{v|p} \sigma(\tau_v) \otimes_{\mathcal{O}} \mathbb{F}$. Suppose $v \in S_2$. By the assumption on the form of $\rho|_{G_{F_v}}$, we see that

$$\bar{\rho}|_{I_{F_v}} \cong \begin{pmatrix} \omega^{m_v} & * \\ 0 & \omega^{m_v} \end{pmatrix}.$$

The two Jordan-Hölder factors of $\sigma(\tau_v) \otimes_{\mathcal{O}} \mathbb{F}$ are, by part (3) of Lemma 4.4.1, isomorphic to $\sigma_{m_v, p-2}$ and $\sigma_{m_v-1, 1}$. Examining Definition 4.6.2, we see that $\sigma_{m_v-1, 1} \notin W(\bar{\rho}|_{G_{F_v}})$, so by Lemma 4.6.3, we see that in fact $\bar{\rho}$ is modular of weight $\sigma_{S_1} \otimes \sigma_{S_2}$ for some σ_{S_1} as claimed.

It remains to check the claim about the Hecke eigenvalues at places in S_2 . This is a standard consequence of local-global compatibility (and the calculation of the action of the Hecke algebra on a principal series representation), together with the compatibility of the T_v and U_v operators proved in Lemma 4.5.1. \square

Proposition 4.6.6. *Suppose that F is a totally real field in which the prime $p > 2$ splits completely. Suppose that $\bar{\rho} : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ is modular, and that $\bar{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. Suppose that $\sigma_{\bar{m}, \bar{n}} = \prod_{v|p} \sigma_v \in W(\bar{\rho})$, where $\sigma_v = \sigma_{m_v, n_v}$. Then $\bar{\rho}$ is modular of weight $\sigma'_{\bar{m}, \bar{n}} = \prod_{v|p} \sigma'_v$, where $\sigma'_v = \sigma_v$ unless $\sigma_v = \sigma_{m_v, p-1}$ and $\sigma_{m_v, 0} \in W(\bar{\rho}|_{G_{F_v}})$, in which case $\sigma'_v = \sigma_{m_v, 0}$. Furthermore, if X is a set of places $v|p$ such that $\bar{\rho}|_{G_{F_v}} \cong \begin{pmatrix} nr(\alpha_v) \omega^{m_v} & 0 \\ 0 & nr(\beta_v) \omega^{m_v} \end{pmatrix}$, then $\bar{\rho}$ is modular of weight $\sigma'_{\bar{m}, \bar{n}}$ and Hecke eigenvalues $\{\alpha_v\}_{v \in X}$.*

Proof. We make use of Corollary 3.1.7. Note that by Remark 3.1.6 we do not have to concern ourselves with questions about the existence of ordinary lifts. We choose a tame type τ_v for each place $v|p$ such that $\bar{\rho}|_{G_{F_v}}$ has a potentially Barsotti-Tate lift of type τ_v . More precisely, suppose firstly that $\bar{\rho}|_{G_{F_v}}$ is irreducible, and that $n_v \neq p-1$. Then by Definition 4.6.2 we see that since $\sigma_{m_v, n_v} \in W(\bar{\rho}|_{G_{F_v}})$ we have

$$\bar{\rho}|_{I_{F_v}} \cong \begin{pmatrix} \omega_2^{(n_v+1)+(p+1)m_v} & 0 \\ 0 & \omega_2^{p(n_v+1)+(p+1)m_v} \end{pmatrix}.$$

We choose $\tau_v = \tilde{\omega}_2^{n_v+2+(p+1)(m_v-1)} \oplus \tilde{\omega}_2^{n_v+2+(p+1)(m_v-1)}$, and note that $\bar{\rho}|_{G_{F_v}}$ has a potentially Barsotti-Tate lift of this type by part (2) of Theorem 4.6.1. Suppose

secondly that $n_v = p - 1$ and $\bar{\rho}|_{I_{F_v}}$ is reducible and très ramifiée; then we choose an arbitrary type τ_v such that $\bar{\rho}|_{G_{F_v}}$ has a lift of type τ_v . Finally, in all other cases we let $\tau_v = \tilde{\omega}^{m_v+n_v} \oplus \tilde{\omega}^{m_v}$ (note that since it is assumed that $\sigma_{m_v, n_v} \in W(\bar{\rho}|_{G_{F_v}})$, it follows from Definition 4.6.2 and Theorem 4.6.1 that $\bar{\rho}|_{G_{F_v}}$ has a potentially Barsotti-Tate lift of this type). If $v \in X$, we note that for some finite-order character ψ_v as in example (E4) of [Gee07], there is a component of a local framed deformation ring $R_v^{\square, \psi_v \epsilon, \tau_v}[1/p]$ such that the corresponding Galois representations are all of the form $\begin{pmatrix} * & \\ 0 & nr(\tilde{\alpha}_v) \tilde{\omega}^m \end{pmatrix}$ where $\tilde{\alpha}_v$ is a lift of α_v .

For each $v|p$ choose a potentially Barsotti-Tate lift ρ_v of $\bar{\rho}|_{G_{F_v}}$ of type τ_v (lying on the specified component if $v \in X$). Now let $\psi_v = \det \rho_v \epsilon^{-1}$, and choose a finite order character $\psi : G_F \rightarrow \overline{\mathbb{Q}}_p^\times$ with $\psi|_{G_{F_v}} = \psi_v$ for all $v|p$ (this is possible by (the proof of) Lemma 4.1.1 of [CHT08]). Let Σ be the set of places at which ψ is ramified, together with the set of places of F dividing p . From Lemma 4.4.2 one sees that $\bar{\rho}$ has a lift of determinant $\psi \epsilon$, and in particular for each $v \in \Sigma$, $v \nmid p$, $\bar{\rho}|_{G_{F_v}}$ has a lift ρ_v of determinant $\psi|_{G_{F_v}} \epsilon$. For each $v \in \Sigma$, choose a component of $R_v^{\square, \psi_v \epsilon, \tau_v}[1/p]$ containing ρ_v . Applying Corollary 3.1.7 to $\bar{\rho}$ and the above choices of ψ , τ_v , we conclude that $\bar{\rho}$ has a modular lift $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ which is potentially Barsotti-Tate of type τ_v for all $v|p$, and which for all $v \in X$ satisfies

$$\rho|_{G_{F_v}} \cong \begin{pmatrix} * & * \\ 0 & \tilde{\omega}^{m_v} nr(\tilde{\alpha}_v) \end{pmatrix},$$

where $\tilde{\alpha}_v$ is a lift of α_v .

Applying Lemma 4.6.5, we see that $\bar{\rho}$ is modular of some weight $\sigma'_{\vec{m}, \vec{n}} = \prod_{v|p} \sigma'_v$ and Hecke eigenvalues $\{\alpha_v\}_{v \in X}$, where $\sigma'_v = \sigma_v$ if $v \in X$, and otherwise σ_v is a Jordan-Hölder factor of $\sigma(\tau_v) \otimes_{\mathcal{O}} \mathbb{F}$. We now examine the choices of τ_v made above. If $\bar{\rho}|_{G_{F_v}}$ is irreducible and $n_v \neq p - 1$ then by part (4) of Lemma 4.4.1 we see that the Jordan-Hölder factors of $\sigma(\tau_v) \otimes_{\mathcal{O}} \mathbb{F}$ are σ_{m_v, n_v} and $\sigma_{m_v+n_v+1, p-3-n_v}$. However, this second weight is not contained in $W(\bar{\rho}|_{G_{F_v}})$ by definition, so by Lemma 4.6.3 we have $\sigma'_v = \sigma_v$.

If $n_v = p - 1$ and $\bar{\rho}|_{I_{F_v}}$ is reducible and très ramifiée, then by definition $W(\bar{\rho}|_{G_{F_v}}) = \sigma_v$, so again by Lemma 4.6.3 we have $\sigma'_v = \sigma_v$. In the other cases where $n_v = p - 1$, and the cases where $n_v = 0$, we have $\sigma(\tau_v) \otimes_{\mathcal{O}} \mathbb{F} = \sigma_{m_v, 0}$ by part (1) of Lemma 4.4.1.

Finally, in the remaining cases, we have that $\bar{\rho}|_{G_{F_v}}$ is reducible, so since $\sigma_{m_v, n_v} \in W(\bar{\rho}|_{G_{F_v}})$ by assumption, we have

$$\bar{\rho}|_{I_{F_v}} \cong \begin{pmatrix} \omega^{m_v+n_v+1} & * \\ 0 & \omega^{m_v} \end{pmatrix}.$$

In addition, by part (3) of Lemma 4.4.1 the Jordan-Hölder factors of $\sigma(\tau_v) \otimes_{\mathcal{O}} \mathbb{F}$ are σ_{m_v, n_v} and $\sigma_{m_v+n_v, p-1-n_v}$. However, this second weight is not contained in $W(\bar{\rho}|_{G_{F_v}})$ (by definition), so by Lemma 4.6.3 we have $\sigma'_v = \sigma_v$. \square

We now use an argument suggested to us by Mark Kisin. Note that although we are assuming throughout this section that p splits completely in F , Lemma 4.6.8 below only requires that $F_v = \mathbb{Q}_p$.

Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$, and let Z be the centre of G . If σ is any representation of KZ on a finite dimensional \mathbb{F} -vector space V_σ , then we let $\mathrm{c}\text{-Ind}_{KZ}^G \sigma$

denote the compact induction of σ , and $\text{Ind}_{KZ}^G \sigma$ the induction of σ with no restriction on supports (note that this notation is not standard). It is easy to check that if σ^* denotes the dual of σ then $\text{Ind}_{KZ}^G \sigma^*$ is dual to $\text{c-Ind}_{KZ}^G \sigma$.

We recall some of the results on Hecke algebras proved in [BL94]. A KZ -bi-invariant function $\phi : G \rightarrow \text{End}_{\mathbb{F}} V_{\sigma}$ is one which satisfies $\phi(h_1 g h_2) = \sigma(h_1) \phi(g) \sigma(h_2)$ for all $g \in G$, $h_1, h_2 \in KZ$. Any such function acts on $\text{c-Ind}_{KZ}^G \sigma$ as follows: if $f \in \text{c-Ind}_{KZ}^G \sigma$, then

$$(\phi(f))(g) = \sum_{KZy \in KZ \backslash G} \phi(gy^{-1})f(y) = \sum_{yKZ \in G/KZ} \phi(y)f(y^{-1}g)$$

(see Proposition 5 of [BL94]). Now give \mathbb{F}^2 an action of KZ with K acting via the natural map $\text{GL}_2(\mathbb{Z}_p) \rightarrow \text{GL}_2(\mathbb{F}_p)$, and $p \in Z$ acting trivially. Let $r \in [0, p-1]$, and set $\sigma = \text{Sym}^r \mathbb{F}^2$. Let T be the endomorphism of $\text{c-Ind}_{KZ}^G \sigma$ corresponding to the KZ -bi-invariant function which is supported on $KZ \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} KZ$ and which takes $\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}$ to $\text{Sym}^r \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. By Proposition 8 of [BL94], $\mathbb{F}[T]$ is the full endomorphism algebra of $\text{c-Ind}_{KZ}^G \sigma$. One obtains a dual action on $\text{Ind}_{KZ}^G \sigma^*$.

For any $\lambda \in \mathbb{F}$, set $\pi(r, \lambda) = \text{c-Ind}_{KZ}^G \sigma / (T - \lambda) \text{c-Ind}_{KZ}^G \sigma$. From Theorem 30 of [BL94] and Theorem 1.1 of [Bre03] we see that $\pi(0, \lambda)$ is irreducible unless $\lambda = \pm 1$, when $\pi(0, \pm 1)$ is a non-trivial extension of $\mu_{\pm 1} \circ \det$ by $\mu_{\pm 1} \circ \det \otimes \text{Sp}$, where $\mu_{\pm 1} : \mathbb{Q}_p^{\times} \rightarrow \mathbb{F}^{\times}$ is the irreducible character sending $p \mapsto \pm 1$, and Sp is a certain irreducible representation. Similarly, $\pi(p-1, \lambda)$ is irreducible unless $\lambda = \pm 1$, when $\pi(p-1, \pm 1)$ is a non-trivial extension of $\mu_{\pm 1} \circ \det \otimes \text{Sp}$ by $\mu_{\pm 1} \circ \det$.

Lemma 4.6.7. *There is a morphism of $\mathbb{F}[T][\text{GL}_2(\mathbb{Q}_p)]$ -modules*

$$\theta : \text{Ind}_{KZ}^G \mathbf{1} \rightarrow \text{Ind}_{KZ}^G \text{Sym}^{p-1} \mathbb{F}^2$$

such that for all $\lambda \in \mathbb{F}$, the kernel of the induced map

$$\theta : (\text{Ind}_{KZ}^G \mathbf{1})[T - \lambda] \rightarrow (\text{Ind}_{KZ}^G \text{Sym}^{p-1} \mathbb{F}^2)[T - \lambda]$$

contains only functions which factor through the determinant.

Proof. In Lemma 1.5.5 of [Kis09a] there is a construction of a morphism of $\mathbb{F}[T][\text{GL}_2(\mathbb{Q}_p)]$ -modules

$$\text{c-Ind}_{KZ}^G \text{Sym}^{p-1} \mathbb{F}^2 \rightarrow \text{c-Ind}_{KZ}^G \mathbf{1}$$

which is nonzero modulo $T - \lambda$ for all $\lambda \in \mathbb{F}$, and which induces an isomorphism modulo $T - \lambda$ for all $\lambda \neq \pm 1$. From the above discussion, we see that if $\lambda = \pm 1$, then we have an induced morphism $\pi(p-1, \pm 1) \rightarrow \pi(0, \pm 1)$ whose cokernel is $\mu_{\pm 1} \circ \det$. Taking duals gives the required morphism. \square

We can now construct a form of weight $\sigma_{m, p-1}$ from one of weight $\sigma_{m, 0}$.

Lemma 4.6.8. *If $\bar{\rho}$ is modular of weight $\sigma = \otimes_{w|p} \sigma_w$ and Hecke eigenvalues $\{\alpha_w\}_{w \in S}$ for some S not containing v , and $\sigma_v = \sigma_{m_v, 0}$, then $\bar{\rho}$ is also modular of weight $\sigma' = \sigma_{m_v, p-1} \otimes_{w \neq v} \sigma_w$ and Hecke eigenvalues $\{\alpha_w\}_{w \in S}$.*

Proof. By definition there is a maximal ideal \mathfrak{m} of $\mathbb{T}_{S^p, \mathbb{F}}^{\text{univ}}$ with $(T_w - \alpha_w) \in \mathfrak{m}$ for all $w \in S$ and some U such that $S_{\sigma, \psi}(U, \mathbb{F})_{\mathfrak{m}} \neq 0$ and $\bar{\rho}_{\mathfrak{m}} \otimes \bar{\mathbb{F}}_p \cong \bar{\rho}$.

Suppose $f \in S_{\sigma, \psi}(U, \mathbb{F})$. Then

$$f : D^{\times} \backslash (D \otimes_F \mathbb{A}_F^f)^{\times} \rightarrow \sigma$$

is such that for all $g \in (D \otimes_F \mathbb{A}_F^f)^\times$ we have

$$\begin{aligned} f(gu) &= \sigma(u)^{-1} f(g) \text{ for all } u \in U \\ f(gz) &= \psi(z) f(g) \text{ for all } z \in (\mathbb{A}_F^f)^\times. \end{aligned}$$

Consider f as a D^\times -invariant function $\tilde{f} : (D \otimes_F \mathbb{A}_F^f)^\times \rightarrow \sigma$. Let $\tilde{U} = GL_2(F_v) \times \prod_{x \neq v} U_x$. Then by Frobenius reciprocity there is a canonical isomorphism

$$\text{Map}_U((D \otimes_F \mathbb{A}_F^f)^\times, \sigma) \xrightarrow{\sim} \text{Map}_{\tilde{U}}((D \otimes_F \mathbb{A}_F^f)^\times, \otimes_{w \neq v} \sigma_w \otimes \text{Ind}_{KZ}^G \sigma),$$

so we see that the map θ of Lemma 4.6.7 induces a map

$$\text{Map}_U((D \otimes_F \mathbb{A}_F^f)^\times, \sigma) \rightarrow \text{Map}_U((D \otimes_F \mathbb{A}_F^f)^\times, \sigma').$$

Furthermore, since Frobenius reciprocity is functorial with respect to the first argument, we easily see that this map is compatible with the action of the Hecke operators at all places other than v for which they are defined and the action of $(\mathbb{A}_F^f)^\times$, and that it takes \tilde{f} to a D^\times -invariant function. Thus we have an induced map of $\mathbb{F}[T]$ -modules

$$S_{\sigma, \psi}(U, \mathbb{F})_{\mathfrak{m}} \rightarrow S_{\sigma', \psi}(U, \mathbb{F})_{\mathfrak{m}}.$$

It remains to be checked that this map is injective. Since $S_{\sigma, \psi}(U, \mathbb{F})_{\mathfrak{m}}$ is a finite $\mathbb{F}[T]$ -module, it suffices to check that for each $\lambda \in \mathbb{F}$ the map

$$S_{\sigma, \psi}(U, \mathbb{F})_{\mathfrak{m}}[T - \lambda] \rightarrow S_{\sigma', \psi}(U, \mathbb{F})_{\mathfrak{m}}[T - \lambda]$$

is injective. If f is in the kernel of the map, then one sees that \tilde{f} is left $SL_2(F_v)$ -invariant by Lemma 4.6.7. Because it is also left D^\times -invariant, strong approximation shows that \tilde{f} is in fact invariant under the subgroup of $(D \otimes_F \mathbb{A}_F^f)^\times$ of elements of reduced norm 1. Thus f factors through the quotient of $(D \otimes_F \mathbb{A}_F^f)^\times$ by the elements of reduced norm 1. If $f \neq 0$ this implies that $\bar{\rho}_{\mathfrak{m}}$ is reducible, a contradiction. \square

Putting all this together, we have proved:

Theorem 4.6.9. *Suppose that F is a totally real field in which the prime $p > 2$ splits completely. Suppose that $\bar{\rho} : G_F \rightarrow GL_2(\overline{\mathbb{F}}_p)$ is modular, and that $\bar{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. Then $\bar{\rho}$ is modular of weight σ if and only if $\sigma \in W(\bar{\rho})$. Furthermore, if $\sigma \in W(\bar{\rho})$ and X is a set of places such that for any $v \in X$ we have $\bar{\rho}|_{G_{F_v}} \cong \begin{pmatrix} nr(\alpha_v)\omega^{m_v} & 0 \\ 0 & nr(\beta_v)\omega^{m_v} \end{pmatrix}$, then $\bar{\rho}$ is modular of weight σ and Hecke eigenvalues $\{\alpha_v\}_{v \in X}$.*

Proof. This follows immediately from Proposition 4.6.6 and Lemma 4.6.8 (applied successively to the places v for which $n_v = p - 1$ and $\sigma_{m_v, 0} \in W(\bar{\rho}|_{G_{F_v}})$). \square

5. AUTOMORPHIC REPRESENTATIONS ON UNITARY GROUPS

5.1. We now extend some of our results on the existence of automorphic liftings of prescribed types to the case of n -dimensional representations. Unfortunately, the results we obtain are rather weaker than those for Hilbert modular forms, for several reasons. Firstly, one can no longer expect to work directly with GL_n , as the Taylor-Wiles method breaks down for GL_n if $n > 2$ (see the introduction to [CHT08]). One needs instead to work with representations satisfying some self-duality conditions; we choose, as in [CHT08] and [Tay08], to work with representations into the disconnected group \mathcal{G}_n , which we define below. Such Galois representations

are associated to automorphic representations on unitary groups, for which the Taylor-Wiles method does work.

We choose to follow the notation of [CHT08] and [Tay08] for the most part, rather than that used in the rest of this paper. We apologise for this, but we hope that this should make it easier for the reader to follow the various references we make to [CHT08], [Tay08] and [Gue09]. In particular, we work with l -adic, as opposed to p -adic, representations. With this in mind, we can state the second major restriction on our results. In section 3 we were able (under a hypothesis on the existence of ordinary lifts) to choose the type of our lifting at any finite place, including those dividing the characteristic of the residual representation. In the n -dimensional case, we are limited to considering places not dividing l which split in the CM field used to define our unitary group. The reason we cannot change types at places dividing l is the absence of an appropriate $R = T$ theorem. In [CHT08], [Tay08], and [Gue09] results are only obtained for representations which are crystalline over an unramified extension of \mathbb{Q}_l , whereas to consider non-trivial types at l one needs to be able to prove an $R = T$ theorem for representations which only become crystalline over a ramified extension. To our knowledge, the only such theorems in the literature are those for weight two Hilbert modular forms in [Kis07b] and [Gee06], and the work of [Kis09a] for modular forms over totally real fields in which p splits completely. However, the work of [Tay08] reduces the task of proving such theorems to a purely local analysis of the irreducible components of certain deformation spaces, and we hope to return to this question in the future. Note that if we had such $R = T$ theorems then the framework presented here would immediately allow us to prove a theorem allowing one to choose the type of an automorphic lift at places dividing l .

When the first draft of this paper was written, the only $R = T$ theorems available for unitary groups were those of [CHT08] and [Tay08], which required a square-integrability hypothesis at a finite place. Thanks to the proof of the fundamental lemma and subsequent work of Chenevier, Harris, Labesse and Shin, this hypothesis has been removed in [Gue09], which has allowed us to strengthen our results while simplifying their proofs. $R = T$ theorems have also been proved in the ordinary case by Geraghty ([Ger09]), and the analogues of the results of this section are proved in [GG09].

We begin by recalling from section 1.1 of [CHT08] the definition of the group \mathcal{G}_n , and the relationship between representations valued in \mathcal{G}_n and essentially self-dual representations valued in GL_n . Let \mathcal{G}_n be the group scheme over \mathbb{Z} which is the semi-direct product of $\mathrm{GL}_n \times \mathrm{GL}_1$ by the group $\{1, j\}$, which acts on $\mathrm{GL}_n \times \mathrm{GL}_1$ via

$$j(g, \mu)j^{-1} = (\mu^t g^{-1}, \mu).$$

Let $\nu : \mathcal{G}_n \rightarrow \mathrm{GL}_1$ be the homomorphism sending $(g, \mu) \mapsto \mu$ and $j \mapsto -1$, and let \mathcal{G}_n^0 be the connected component of \mathcal{G}_n . We let $\mathfrak{g}_n := \mathrm{Lie} \mathrm{GL}_n \subset \mathrm{Lie} \mathcal{G}_n$, and let ad denote the adjoint action of \mathcal{G}_n on \mathfrak{g}_n . Let \mathfrak{g}_n^0 be the trace zero subspace of \mathfrak{g}_n . Let F/F^+ be an extension of number fields of degree 2, with F^+ totally real and F CM. Let $c \in G_{F^+}$ be a complex conjugation. Then we have

Lemma 5.1.1. *Suppose that k is a field, that $\chi : G_{F^+} \rightarrow k^\times$ is a continuous homomorphism, and that*

$$\rho : G_F \rightarrow \mathrm{GL}_n(k)$$

is absolutely irreducible, continuous, and satisfies $\chi\rho^\vee \cong \rho^c$. Then there exists a continuous homomorphism

$$r : G_{F^+} \rightarrow \mathcal{G}_n(k)$$

such that $r|_{G_F} = \rho$, $(\nu \circ r)|_{G_F} = \chi|_{G_F}$, and $r(c) \notin \mathcal{G}_n^0(k)$. There is a bijection between $\mathrm{GL}_n(k)$ -conjugacy classes of such extensions of ρ and $k^\times / (k^\times)^2$, so that in particular if k is algebraically closed then r is unique up to $\mathrm{GL}_n(k)$ -conjugacy.

Proof. This is a special case of Lemma 1.1.4 of [CHT08]. \square

We say that a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ is RACSDC (regular, algebraic, conjugate self dual, cuspidal) if

- $\pi^\vee \cong \pi^c$, and
- π_∞ has the same infinitesimal character as some irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n$.

We wish to define the weight of such a representation. Let $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F, \mathbb{C})}$ satisfy

- $a_{\tau,1} \geq \cdots \geq a_{\tau,n}$
- $a_{\tau c, i} = -a_{\tau, n+1-i}$

for all $\tau \in \mathrm{Hom}(F, \mathbb{C})$, where c again denotes complex conjugation. Let Ξ_a denote the irreducible algebraic representation of $\mathrm{GL}_n^{\mathrm{Hom}(F, \mathbb{C})}$ which is the tensor product over $\tau \in \mathrm{Hom}(F, \mathbb{C})$ of the irreducible representations of GL_n with highest weights a_τ . We say that an RACSDC representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ has weight a if π_∞ has the same infinitesimal character as Ξ_a^\vee .

Let π be an irreducible smooth \mathbb{Q}_l -representation of $\mathrm{GL}_n(K)$, K a finite extension of \mathbb{Q}_p , $p \neq l$. Then as in section 3.1 of [CHT08] we let $r_l(\pi)$ denote the l -adic Galois representation associated to the Weil-Deligne representation $\mathrm{rec}_l(\pi^\vee \otimes |\cdot|^{(1-n)/2})$, provided this exists. Here rec_l is the local Langlands correspondence of [HT01], so that for example if $\pi = \mathrm{n}\text{-Ind}(\chi_1 \times \cdots \times \chi_n)$ is an irreducible principal series representation, then

$$\mathrm{rec}_l(\pi) = \chi_1 \circ \mathrm{Art}_K^{-1} \oplus \cdots \oplus \chi_n \circ \mathrm{Art}_K^{-1}.$$

Proposition 5.1.2. *Let $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Suppose that π is an RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Then there is a continuous representation $r_{l,\iota}(\pi) : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ satisfying:*

- (1) *If $v \nmid l$ then*

$$r_{l,\iota}(\pi)|_{G_{F_v}}^{\mathrm{ss}} = r_l(\iota^{-1}\pi_v)^\vee(1-n)^{\mathrm{ss}}.$$

- (2) *$r_{l,\iota}(\pi)^c = r_{l,\iota}(\pi)^\vee \epsilon^{1-n}$, where ϵ is the l -adic cyclotomic character.*

- (3) *If $v|l$ then $r_{l,\iota}(\pi)|_{G_{F_v}}$ is potentially semi-stable, and if furthermore π_v is unramified then $r_{l,\iota}(\pi)|_{G_{F_v}}$ is crystalline.*

Proof. See Theorem 1.2 of [BLGHT09]. \square

We can take $r_{l,\iota}(\pi)$ to be valued in $\mathrm{GL}_n(\mathcal{O})$ with \mathcal{O} the ring of integers of a finite extension of \mathbb{Q}_l ; reducing modulo the maximal ideal of \mathcal{O} and semisimplifying gives a well defined semisimple representation

$$\bar{r}_{l,\iota}(\pi) : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_l).$$

We say that a continuous semisimple representation $r : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is automorphic of weight a if it is isomorphic to $r_{l,\iota}(\pi)$ for some $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, and some RACSDC automorphic form π of weight a . We say that it is automorphic of weight

a and level prime to l if furthermore π_l is unramified. We say that a continuous, semisimple representation $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is automorphic of weight a if it lifts to a representation $r : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ which is automorphic of weight a and level prime to l .

From Lemma 5.1.1 and Proposition 5.1.2(2), we see that there is a unique extension (up to $\mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ -conjugation) of $r_{l,l}(\pi)$ to a homomorphism

$$r'_{l,l}(\pi) : G_{F^+} \rightarrow \mathcal{G}_n(\overline{\mathbb{Q}}_l)$$

with $\nu \circ r'_{l,l}(\pi) = \epsilon^{1-n} \delta_{F/F^+}^{\mu_\pi}$, and $r'_{l,l}(\pi)(c_v) \notin \mathcal{G}_n^0(\overline{\mathbb{Q}}_l)$ for any infinite place v of F^+ , where c_v denotes complex conjugation at v . Here δ_{F/F^+} is the unique nontrivial character of $\mathrm{Gal}(F/F^+)$, and $\mu_\pi \in \mathbb{Z}/2\mathbb{Z}$. Accordingly, from now on we will work with representations to \mathcal{G}_n .

We now define the deformation rings we work with, following [CHT08], §2.2. Suppose from now on that every place of F^+ dividing l splits in F , and suppose that $l > n$ and l is unramified in F^+ . Take K/\mathbb{Q}_l finite with $\#\mathrm{Hom}(F^+, K) = [F^+ : \mathbb{Q}]$. Let $\mathcal{O} = \mathcal{O}_K$, let λ be the maximal ideal of \mathcal{O} , $k = \mathcal{O}/\lambda$. Let $\mathcal{C}_{\mathcal{O}}^f$ be the category of Artinian local \mathcal{O} -algebras R for which the map $\mathcal{O} \rightarrow R$ induces an isomorphism on residue fields. Let $\mathcal{C}_{\mathcal{O}}$ be the category of complete local \mathcal{O} -algebras with residue field k . We will assume from now on that K is large enough that all representations that we wish to consider which are valued in finite extensions of K are in fact valued in K ; this may be achieved by replacing K by a finite extension.

Fix a continuous homomorphism

$$\bar{r} : G_{F^+} \rightarrow \mathcal{G}_n(k)$$

with $\bar{r}^{-1}(\mathrm{GL}_n(k) \times \mathrm{GL}_1(k)) = G_F$. Let $\bar{\rho} := \bar{r}|_{G_F}$. Assume that $\bar{r}|_{G_F}$ is absolutely irreducible and automorphic of weight a . To be precise, suppose that $\bar{r} = \bar{r}'_{l,l}(\pi)$, where π is a RACSDC automorphic representation of level prime to l and weight a .

Suppose further that for all $\tau \in (\mathbb{Z}^n)^{\mathrm{Hom}(F, \mathbb{C})}$ we have either

$$l - 1 - n \geq a_{\tau,1} \geq \cdots \geq a_{\tau,n} \geq 0$$

or

$$l - 1 - n \geq a_{\tau c,1} \geq \cdots \geq a_{\tau c,n} \geq 0.$$

If R is an object of $\mathcal{C}_{\mathcal{O}}$ then a *lifting* of \bar{r} to R is a continuous homomorphism

$$r : G_{F^+} \rightarrow \mathcal{G}_n(R)$$

with $r \bmod \mathfrak{m}_R = \bar{r}$ and $\nu \circ r = \epsilon^{1-n} \delta_{F/F^+}^{\mu_\pi}$. A *deformation* of \bar{r} to R is a $\ker(\mathcal{G}_n(R) \rightarrow \mathcal{G}_n(k))$ -conjugacy class of liftings.

We wish, as usual, to impose some local conditions on the deformations we consider. Let S_l be the set of places of F^+ above l . Fix T a finite set of finite places of F^+ containing S_l and all places at which \bar{r} is ramified, and assume that all places in T split in F . We choose a set \tilde{T} of places of F above the places in T so that we may write the places of F above places in T as $\tilde{T} \amalg c\tilde{T}$. If $X \subset T$, write \tilde{X} for the set of places in \tilde{T} above places of X . We will sometimes identify F_v^+ and $F_{\tilde{v}}$.

Write $G_{F^+, T} = \mathrm{Gal}(F(T)/F^+)$ where $F(T)$ is the maximal extension of F unramified outside of the places lying over T . From now on, we consider \bar{r}, χ as $G_{F^+, T}$ -representations, and all our global deformations are to $G_{F^+, T}$ -representations. By Proposition 1.2.9 of [CHT08] that there is a universal deformation r_T^{univ} of \bar{r} over an

object R_T^{global} of $\mathcal{C}_{\mathcal{O}}$. Note that since $\bar{r}|_{G_F}$ is absolutely irreducible, $H^0(G_{F^+,T}, \text{ad } \bar{r}) = 0$.

Write

$$T = S_l \coprod Y.$$

For each place $v \in T$ there is a universal lifting of $\bar{r}|_{G_{F_v}}$ over an object R_v^{loc} of $\mathcal{C}_{\mathcal{O}}$. We consider the following $\ker(\text{GL}_n(R_v^{loc}) \rightarrow \text{GL}_n(k))$ -invariant quotients of R_v^{loc} .

- (1) If $v \in S_l$ then R_v is the maximal quotient of R_v^{loc} such that for every Artinian quotient A of R_v the pushforward of the universal lifting of $\bar{r}|_{G_{F_v}}$ to A is in the essential image of the Fontaine-Laffaille functor \mathbb{G}_v of §1.4.1 of [CHT08].
- (2) If $v \in Y$ then choose an inertial type τ_v of I_{F_v} (as in §2), and let R_v be the maximal quotient of R_v^{loc} over which the universal lift of $\bar{r}|_{G_{F_v}}$ is of type τ_v .

Say that a lifting is of type τ if for each $v \in T$ it factors through the ring R_v constructed above. Then there is a universal deformation r_{τ}^{univ} of \bar{r} of type τ over an object R_{τ}^{univ} of $\mathcal{C}_{\mathcal{O}}$. In the terminology of section 2.3 of [CHT08], this is the universal deformation ring $R_{\mathcal{S}_{\tau}}$ for the deformation problem

$$\mathcal{S}_{\tau} = (F/F^+, T, \tilde{T}, \mathcal{O}, \bar{r}, \epsilon^{1-n} \delta_{F/F^+}^{\mu_{\pi}}, \{R_v\}_{v \in T}).$$

As in §1.5 of [CHT08], we say that a subgroup $H \subset \text{GL}_n(k)$ is *big* if:

- H has no l -power order quotients.
- $H^0(H, \mathfrak{g}_n^0(k)) = (0)$,
- $H^1(H, \mathfrak{g}_n^0(k)) = (0)$,
- For all irreducible $k[H]$ -submodules W of $\mathfrak{g}_n(k)$ we can find $h \in H$ and $\alpha \in k$ satisfying the following properties. The α -generalised eigenspace $V_{h,\alpha}$ of h on k^n is one-dimensional. Let $\pi_{h,\alpha} : k^n \rightarrow V_{h,\alpha}$ be the h -equivariant projection of k^n to $V_{h,\alpha}$, and let $i_{h,\alpha} : V_{h,\alpha} \hookrightarrow k^n$ be the h -equivariant injection of $V_{h,\alpha}$ into k^n . Then $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq (0)$.

We say that a subgroup $H \subset \mathcal{G}_n(k)$ is *big* if H surjects onto $\mathcal{G}_n(k)/\mathcal{G}_n^0(k)$ and $H \cap \mathcal{G}_n^0(k)$ is big.

We assume from now on that $\bar{F}^{\ker \text{ad } \bar{r}|_{G_F}}$ does not contain $F(\zeta_l)$, and that $\bar{r}(G_{F^+(\zeta_l)})$ is big.

We wish to obtain a lower bound on the dimension of R_{τ}^{univ} . This is straightforward, and is in fact done in [CHT08].

Lemma 5.1.3. $\dim R_{\tau}^{univ} \geq 1 - \frac{n}{2}[F^+ : \mathbb{Q}](1 + (-1)^{1+n+\mu_{\pi}})$. In particular, if $\mu_{\pi} \equiv n \pmod{2}$, then $\dim R_{\tau}^{univ} \geq 1$.

Proof. By Corollary 2.3.5 of [CHT08],

$$\dim R_{\tau}^{univ} \geq 1 + \sum_{v \in T} (\dim R_v - n^2 - 1) - \dim_k H^0(G_{F^+,T}, \text{ad } \bar{r}(1)) - n \sum_{v|_{\infty}} (n + \chi(c_v))/2,$$

where $\chi = \epsilon^{1-n} \delta_{F/F^+}^{\mu_{\pi}}$.

Now, by Theorem 2.1.6 of this paper and Corollary 2.4.3 of [CHT08], we see that if $v \nmid l$ then $\dim R_v = 1 + n^2$, and if $v|l$ then $\dim R_v = 1 + n^2 + \frac{n(n-1)}{2}[F_v : \mathbb{Q}_l]$.

Thus

$$\begin{aligned} \sum_{v \in T} (\dim R_{\bar{v}} - n^2 - 1) &= \frac{n(n-1)}{2} [F : \mathbb{Q}] \\ &= n \sum_{v | \infty} (n-1)/2. \end{aligned}$$

The assumption that $\bar{r}(G_{F^+(\zeta_l)})$ is big implies that $H^0(G_{F^+, T}, \text{ad } \bar{r}(1)) = 0$, and the result follows. \square

In fact, $\mu_\pi \equiv n \pmod{2}$; this is obtained in [Gue09] as a consequence of the Taylor-Wiles method.

We wish to use the $R = T$ theorems proved in [Gue09] to show that $(R_\tau^{\text{univ}})^{\text{red}}$ is finite over \mathcal{O} . We now make a base change, as in the proof of Theorem 5.2 of [Tay08]. Enlarge Y if necessary, so that $\tilde{T} \cup \mathcal{C}\tilde{T}$ contains all the places at which π is ramified. Because $\bar{F}^{\ker \text{ad } \bar{r}|_{G_F}}$ does not contain $F(\zeta_l)$, we can choose a finite place v_1 of F^+ such that:

- $v_1 \notin T$, and v_1 splits in F .
- v_1 is unramified over a rational prime p for which $[F(\zeta_p) : F] > n$.
- v_1 does not split completely in $F(\zeta_l)$.
- $\text{ad } \bar{r}(\text{Frob}_{v_1}) = 1$.

Now choose a totally real field L^+/F^+ satisfying the following conditions:

- $4|[L^+ : \mathbb{Q}]$.
- L^+/F^+ is Galois and solvable.
- $L = L^+F$ is linearly disjoint from $\bar{F}^{\ker \bar{r}|_{G_F}}(\zeta_l)$ over F .
- L/L^+ is everywhere unramified.
- l is unramified in L^+ .
- v_1 splits completely in L/F^+ .
- Let π_L denote the base change of π to L . If w is a place of L lying over $v \in Y$ then $(\pi_{L,w})^{Iw(w)} \neq (0)$ and $\tau_v|_{I_{L,w}}$ is trivial (here $Iw(w)$ is the Iwahori subgroup of $\text{GL}_n(\mathcal{O}_{L,w})$). In addition, $Nw \equiv 1 \pmod{l}$ and $\bar{r}|_{G_{L,w}}$ is trivial.

Let $S_1(L^+)$ be the places of L^+ above v_1 . Let $S_l(L^+)$ denote the places of L^+ above l . Let $Y(L^+)$ be the places of L^+ above elements of Y . Then let $T(L^+) = S_1(L^+) \cup S_l(L^+) \cup Y(L^+)$, and choose a set $\tilde{T}(L)$ consisting of a choice of a place of L above every place of $T(L^+)$, in such a way that $\tilde{T}(L)$ contains all places of L lying over places in \tilde{T} .

Let $a_L \in (\mathbb{Z}^n)^{\text{Hom}(L, \bar{\mathbb{Q}}_l)}$ be defined by $a_{L,\tau} = a_{\tau|_F}$, so that $\bar{r}|_{G_L}$ is automorphic of weight a_L .

We now consider certain deformations of $\bar{r}|_{G_{L^+}}$. Note that $(\nu \circ \bar{r})|_{G_{L^+}} = \epsilon^{1-n} \delta_{L/L^+}^{\mu_\pi}$, with μ_π as above, and δ_{L/L^+} the unique non-trivial character of $\text{Gal}(L/L^+)$. A lifting of $\bar{r}|_{G_{L^+}}$ to an object R of $\mathcal{C}_{\mathcal{O}}$ will be a continuous homomorphism $r : G_{L^+} \rightarrow \mathcal{G}_n(R)$ with $r \pmod{\mathfrak{m}_R} = \bar{r}$ and $\nu \circ r = \epsilon^{n-1} \delta_{L/L^+}^{\mu_\pi}$. For each $v \in T(L^+)$ there is a universal lifting of $\bar{r}|_{G_{L_{\bar{v}}}}$ over an object $R_{\bar{v}}^{\text{loc}}$. We consider the following $\ker(\text{GL}_n(R_{\bar{v}}^{\text{loc}}) \rightarrow \text{GL}_n(k))$ -invariant quotients $R_{\bar{v}}$ of $R_{\bar{v}}^{\text{loc}}$.

- (1) If $v \in S_l(L^+)$ then $R_{\bar{v}}$ is the maximal quotient of $R_{\bar{v}}^{\text{loc}}$ such that for every Artinian quotient of $R_{\bar{v}}$ the pushforward of the universal lifting of $\bar{r}|_{G_{F_{\bar{v}}}}$ to

A is in the essential image of the Fontaine-Laffaille functor $\mathbb{G}_{\bar{v}}$ of §1.4.1 of [CHT08].

- (2) If $v \in Y(L^+)$ then let $R_{\bar{v}}$ be the maximal quotient of $R_{\bar{v}}^{loc}$ over which for all $\sigma \in I_{L_{\bar{v}}}$ the universal lift of $\bar{r}|_{G_{L_{\bar{v}}}}$ evaluated at σ has characteristic polynomial $(X - 1)^n$.
- (3) If $v \in S_1(L^+)$ then $R_{\bar{v}} = R_{\bar{v}}^{loc}$.

Say that a lifting of $\bar{r}|_{G_{L^+}}$ is of type \mathcal{S} if it is unramified outside $T(L^+)$ and if at each $v \in T(L^+)$ it factors through the ring $R_{\bar{v}}$ defined above. There is a universal deformation $r_{\mathcal{S}}^{univ}$ of type \mathcal{S} over an object $R_{\mathcal{S}}^{univ}$ of $\mathcal{C}_{\mathcal{O}}$. As above, in the terminology of [CHT08] this is the universal deformation ring for the deformation problem

$$\mathcal{S} = (L/L^+, T(L^+), \tilde{T}(L), \mathcal{O}, \bar{r}|_{G_{L^+}}, \epsilon^{1-n} \delta_{L/L^+}^{\mu_{\pi}}, \{R_{\bar{v}}\}_{v \in T(L^+)}).$$

Now, if we choose a representative for the universal deformation $r_{\mathcal{S}_{\tau}}^{univ} : G_{F^+} \rightarrow \mathcal{G}_n(R_{\mathcal{S}_{\tau}}^{univ})$, and restrict it to G_{L^+} , then universal property gives $R_{\mathcal{S}_{\tau}}^{univ}$ the structure of an $R_{\mathcal{S}_{\tau}}^{univ}$ -module.

Theorem 5.1.4. *With the above assumptions, $(R_{\mathcal{S}_{\tau}}^{univ})^{red}$ is a finite \mathcal{O} -module of rank at least 1.*

Proof. By Theorem 3.4 of [Gue09], $\mu_{\pi} \equiv n \pmod{2}$, so that Corollary 5.1.3 shows that $\dim(R_{\mathcal{S}_{\tau}}^{univ})^{red} = \dim R_{\mathcal{S}_{\tau}}^{univ} \geq 1$. As in the proof of Proposition 3.1.5, it suffices to show that $(R_{\mathcal{S}_{\tau}}^{univ})^{red}$ is a finite \mathcal{O} -algebra. We claim that $(R_{\mathcal{S}_{\tau}}^{univ})^{red}$ is a finite $(R_{\mathcal{S}_{\tau}}^{univ})^{red}$ -module; this follows just as in Theorem 4.2.8 of [Kis05] and Lemma 3.6 of [KW08], using Lemma 2.1.12 of [CHT08]. Thus we need only check that $(R_{\mathcal{S}_{\tau}}^{univ})^{red}$ is finite over \mathcal{O} . By Theorem 3.4 of [Gue09] we have $(R_{\mathcal{S}_{\tau}}^{univ})^{red} \xrightarrow{\sim} \mathbb{T}$, where \mathbb{T} is a certain Hecke algebra, which is finite over \mathcal{O} by construction. The result follows. \square

From this, together with Theorem 3.4 of [Gue09], one immediately obtains the following result, where for the convenience of the reader we have incorporated all the assumptions made into the statement of the theorem.

Theorem 5.1.5. *Let $F = F^+E$ be a CM field, where F^+ is totally real and E is imaginary quadratic. Let $n \geq 1$ and let $l > n$ be a prime which is unramified in F^+ and split in E . Suppose that*

$$\bar{\rho} : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_l)$$

is an irreducible representation which is unramified at all places of F lying above primes which do not split in E , and which satisfies the following properties.

- (1) $\bar{\rho}$ is automorphic of weight a , where we assume that for all $\tau \in (\mathbb{Z}^n)^{\mathrm{Hom}(F, \mathbb{C})}$ we have either

$$l - 1 - n \geq a_{\tau,1} \geq \cdots \geq a_{\tau,n} \geq 0$$

or

$$l - 1 - n \geq a_{c\tau,1} \geq \cdots \geq a_{c\tau,n} \geq 0.$$

Note that in particular these conditions imply that $\bar{\rho}^c \cong \bar{\rho}^{\vee} \epsilon^{1-n}$.

- (2) $\overline{F}^{\ker \mathrm{ad} \bar{\rho}}$ does not contain $F(\zeta_l)$.
- (3) $\bar{\rho}(G_{F^+(\zeta_l)})$ is big.

Let Y be a finite set of finite places of F^+ which split in F , which does not contain any places dividing l . For each $v \in Y$, choose an inertial type τ_v and a place \tilde{v} of F above v . Assume that $\bar{\rho}|_{G_{F_{\tilde{v}}}}$ has a lift to characteristic zero of type τ_v .

Then there is an automorphic representation π on $\mathrm{GL}_n(\mathbb{A}_F)$ of weight a and level prime to l such that

- (1) $\bar{r}_{l,\iota}(\pi) \cong \bar{\rho}$.
- (2) $r_{l,\iota}(\pi)|_{G_{F_{\tilde{v}}}}$ has type τ_v for all $v \in Y$.
- (3) π is unramified at all places of F at which $\bar{\rho}$ is unramified, except possibly for the places lying over elements of Y .

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