CONGRUENCES BETWEEN HILBERT MODULAR FORMS: CONSTRUCTING ORDINARY LIFTS

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Abstract. Under mild hypotheses, we prove that if F is a totally real field, and $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is irreducible and modular, then there is a finite solvable totally real extension F'/F such that $\overline{\rho}|_{G_{F'}}$ has a modular lift which is ordinary at each place dividing l. We deduce a similar result for $\overline{\rho}$ itself, under the assumption that at places $v|l$ the representation $\overline{\rho}|_{G_{F_v}}$ is reducible. This allows us to deduce improvements to results in the literature on modularity lifting theorems for potentially Barsotti-Tate representations and the Buzzard-Diamond-Jarvis conjecture. The proof makes use of a novel lifting technique, going via rank 4 unitary groups.

Contents

1. Introduction.

1.1. If l is a prime and f is a cuspidal newform of weight $2 \leq k \leq l+1$ and level prime to l, with associated mod l Galois representation $\overline{\rho}_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_l)$, then there are two possibilities for the local representation $\overline{\rho}_f|_{G_{\mathbb{Q}_l}}$; either it is reducible or irreducible. Furthermore, there is a simple criterion distinguishing these two cases: $\overline{\rho}_f|_{G_{\mathbb{Q}_l}}$ is reducible if and only if f is ordinary at l, in the sense that a_l, the eigenvalue of the Hecke operator T_l, is an l-adic unit. It is relatively straightforward to deduce from this result that if $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_l)$ is continuous, irreducible and modular (or equivalently odd, by Serre's conjecture) and $\overline{\rho}|_{G_{\mathbb{Q}_l}}$ is reducible, then $\overline{\rho}$ has an ordinary modular lift $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$. This can be generalised to prove the analogous statement for modular representations of G_{F^+}, where F^+ is a totally real field in which l splits completely (see Lemma 2.14 of [Kis07a] together with Lemma 6.1.6 of this paper).

If l does not split completely in F^+ the situation is rather more complicated. It is, for example, quite possible for the reduction mod l of a non-ordinary Galois

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representation to be ordinary; one easy way to see this is that by solvable base change one may even find non-ordinary representations $\rho : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ such that $\overline{\rho}|_{G_{F^+,v}}$ is trivial for each $v|l$, simply by taking any non-ordinary modular representation and making a sufficiently large base change. Moreover, as far as we know nothing is known about the existence of global ordinary lifts other than Lemma 2.14 of [Kis07a], even in the case that l is unramified in F^+ (despite the claims made in section 3 of [Gee06b]). There are, however, several reasons to want to prove such results in greater generality: in particular, the very general modularity lifting theorems for two-dimensional potentially Barsotti-Tate representations proved in [Kis07c] and [Gee06b] are subject to a hypothesis on the existence of ordinary lifts which limits their applicability.

In the present paper we resolve this situation by proving that ordinary lifts exist in considerable generality; all that we require is that $l \geq 5$, and a mild hypothesis on the image $\overline{\rho}(G_{F^+})$. The techniques that we employ are somewhat novel, and are inspired by our work on the Sato-Tate conjecture ([BLGG09]); in particular, they make use of automorphy lifting theorems for rank 4 unitary groups.

As in our earlier work on related questions (e.g. [Gee06a] and [GG09]) we construct the lifts that we seek by considering a universal deformation ring, and proving that it has points in characteristic zero. This gives a candidate Galois representation lifting $\overline{\rho}$, and one can then hope to prove that this representation is modular by applying modularity lifting theorems. In fact, the modularity of the representation usually follows from the proof that the universal deformation ring has points in characteristic zero (although not always; there is also a method due to Ramakrishna ([Ram02]) of a purely Galois-cohomological nature). This method is originally due to Khare–Wintenberger (cf. Theorem 3.7 of [KW08]). In these arguments one uses base change to simplify or modify the local deformation conditions.

However, in the present case no advantage is gained by making a base change, as any base change of an ordinary representation is still ordinary, and components of local crystalline deformation rings contain either only ordinary points or only non-ordinary points. Instead, we proceed by a method related to our work on the Sato-Tate conjecture ([BLGG09]). Rather than employ base change, we use a different functoriality, that of automorphic induction from GL_2 to GL_4.

We now sketch the main argument. Using the arguments of Khare–Wintenberger mentioned above, it is enough to prove that there is a finite solvable extension F_1^+/F^+ of totally real fields such that $\overline{\rho}|_{G_{F_1^+}}$ has an ordinary modular Barsotti-Tate lift. We begin by choosing F_1^+/F^+ a finite solvable extension of totally real fields such that $\overline{\rho}|_{G_{F_{1,v}^+}}$ is trivial for each place $v|l$ of F_1^+, and such that $\overline{\rho}$ has a modular Barsotti-Tate lift $\rho^{\mathrm{BT}} : G_{F_1^+} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ which is unramified at all places not dividing l, and which is non-ordinary at all places dividing l.

We now choose a CM quadratic extension M/F_1^+ in which every place dividing l splits, and an algebraic character $\theta : G_M \to \overline{\mathbb{Q}}_l^\times$ such that $\mathrm{Ind}_{G_M}^{G_{F_1^+}} \theta$ is Barsotti-Tate and non-ordinary at each place dividing l. We also construct an algebraic character $\theta' : G_M \to \overline{\mathbb{Q}}_l^\times$ such that $\mathrm{Ind}_{G_M}^{G_{F_1^+}} \theta'$ is crystalline and ordinary with Hodge–Tate weights 0 and 2 at each place dividing l, with θ and θ' congruent mod l. We now consider two deformation problems: firstly, we let $R_2^{\mathrm{univ,ord}}$ denote the universal deformation ring for deformations of $\overline{\rho}$ which are unramified at all places not dividing

l and which are crystalline and ordinary at all places dividing l with Hodge–Tate weights 0 and 2. Secondly, we let R_4^{univ} denote the universal deformation ring for certain self-dual lifts of the 4-dimensional representation $\overline{\rho}\otimes\mathrm{Ind}_{G_M}^{G_{F_1^+}}\bar{\theta}$, which are unramified at all places not dividing l, and which are crystalline at all places dividing l with Hodge–Tate weights 0, 1, 2 and 3, and which furthermore satisfy an additional condition: for each place $v|l$, the global deformations give rise to local liftings on the same components of the appropriate local crystalline lifting ring as the representation

$$(\rho^{\mathrm{BT}}\otimes(\mathrm{Ind}_{G_M}^{G_{F_1^+}}\theta'))|_{G_{F_{1,v}^+}}.$$

Now, note that (because the local crystalline ordinary lifting rings are connected) a putative $\overline{\mathbb{Q}}_l$-point of $R_2^{\mathrm{univ,ord}}$ would correspond to a representation $\rho: G_{F_1^+}\to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ such that the representation $\rho\otimes(\mathrm{Ind}_{G_M}^{G_{F_1^+}}\theta)$ gives a $\overline{\mathbb{Q}}_p$-point of R_4^{univ}. Thus we obtain (from the universal properties defining R_4^{univ} and $R_2^{\mathrm{univ,ord}}$) a natural map $R_4^{\mathrm{univ}}\to R_2^{\mathrm{univ,ord}}$. We show that this map is finite.

Furthermore, we show that R_4^{univ} is a finite $\mathbb{Z}_l$-algebra, by identifying its reduced quotient with a Hecke algebra, using the automorphy lifting theorem proved in [BLGG09], and the automorphy of $\rho^{\mathrm{BT}}\otimes\mathrm{Ind}_{G_M}^{G_{F_1^+}}\theta'$ (which follows from standard properties of automorphic induction). Thus $R_2^{\mathrm{univ,ord}}$ is a finite $\mathbb{Z}_l$-algebra. Standard Galois cohomology calculations show that it has dimension at least 1, so it has $\mathbb{Z}_l$-rank at least one, and thus has $\overline{\mathbb{Q}}_l$-points. Let $\rho^{\mathrm{ord}}: G_{F_1^+}\to\mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ be the lift of $\overline{\rho}|_{G_{F_1^+}}$ corresponding to such a point. By construction, this is ordinary at each place $v|l$, and since $(R_4^{\mathrm{univ}})^{red}$ has been identified with a Hecke algebra, we see that $\rho^{\mathrm{ord}}\otimes\mathrm{Ind}_{G_M}^{G_{F_1^+}}\theta$ is automorphic. The modularity of ρ^{ord} then follows from an idea of Harris ([Har07], although the version we use is based on that employed in [BLGHT09]). Finally, a basic argument with Hida families allows us to replace ρ^{ord} with a modular ordinary Barsotti-Tate representation.

In fact, we have glossed over several technical difficulties in the above argument. It is more convenient to work over a quadratic CM extension of F_1^+ for much of the argument, only descending to F_1^+ at the end. It is also convenient to make several further solvable base changes during the argument; for example, we prefer to work in situations where all representations considered are unramified outside of l, in order to avoid complications due to non-smooth points on local lifting rings. We also need to ensure that the various hypotheses of the automorphy lifting theorems used are satisfied; this requires us to assume that $l>4$ for most of the paper (because we use automorphy lifting theorems for GL_4), and to assume various "big image" hypotheses.

We now explain the structure of the paper. In section 2 we construct the characters θ and θ'. The arguments of this section are very similar to the corresponding arguments in [BLGHT09] and [BLGG09]. Section 3 contains our main results on global deformation rings, including the finiteness results discussed above, and a discussion of oddness. We take care to work in considerable generality, with a view to further applications; in particular, in future work we will apply the machinery of this section to generalisations of the weight part of Serre's conjecture. In section 4 we examine the condition of a subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_l)$ being 2-big in some detail;

this allows us to obtain concrete conditions under which our main theorems hold. Section 5 contains a slight variant on the trick of Harris mentioned above, and is very similar to the corresponding section of [BLGG09]. The main theorems are obtained in section 6. In addition to the argument explained above, we give a more elementary argument in the case of residually dihedral representations, which allows us to handle some cases with $l = 3$.

We would like to thank Brian Conrad for his assistance with the proof of Lemma 6.1.6. This paper was written simultaneously with [BLGGT10], and we would like to thank Richard Taylor for a number of helpful discussions.

Note added in proof. Since the preparation of the manuscript for this paper, Jack Thorne has relaxed the 'bigness' condition attached to the various modularity lifting theorems we rely on (see [Tho10]). This in turn would allow certain simplifications to be made to parts of this paper. We have not implemented these changes here, as to do so would be involve significant modifications to the paper, and in particular would make changes to results to which our more recent papers (such as [BLGGT10] and [BLGG11]) refer. However, we note that it should ultimately be possible to improve the results of this paper by replacing every occurrence of the adjectives 'big' or '2-big' with 'adequate', at least if $l \geq 7$ (and even for $l = 3, 5$ if one uses the results of [BLGG11]).

In addition, the paper [BLGGT10] has been completed, which proves analogous results to those of this paper in arbitrary dimension. In particular, Proposition 4.1.1 of [BLGGT10] is similar to the main results of this paper. However, the results of this paper cannot be immediately be deduced from those of [BLGGT10]. Firstly, this paper proves results for $l = 3, 5$, whereas Proposition 4.1.1 of [BLGGT10] requires that $l \geq 7$. It is also necessary to prove that two-dimensional modular representations necessarily have potentially diagonalizable lifts, which we do in the course of the proof of Proposition 6.1.3 of this paper, to relate deformation rings over totally real fields to those over imaginary CM fields, and to prove that reducible mod l representations of the absolute Galois group of a finite extension of $\mathbb{Q}_l$ admit reducible potentially Barsotti–Tate lifts, all of which we do in Section 6. Finally, certain results in [BLGGT10] actually depend on arguments from this paper, for example those in Section 5 of this paper.

1.2. Notation. If M is a field, we let G_M denote its absolute Galois group. We write all matrix transposes on the left; so ${}^tA$ is the transpose of A. Let ϵ denote the l-adic cyclotomic character, and $\bar{\epsilon}$ or ω the mod l cyclotomic character. If M is a finite extension of $\mathbb{Q}_p$ for some p, we write I_M for the inertia subgroup of G_M. If R is a local ring we write $\mathfrak{m}_R$ for the maximal ideal of R.

We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and regard all algebraic extensions of $\mathbb{Q}$ as subfields of $\overline{\mathbb{Q}}$. For each prime p we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. In this way, if v is a finite place of a number field F, we have a homomorphism $G_{F_v} \hookrightarrow G_F$.

We normalise the definition of Hodge–Tate weights so that all the Hodge–Tate weights of the l-adic cyclotomic character are -1. We refer to a crystalline representation with all Hodge–Tate weights equal to 0 or 1 as a Barsotti-Tate representation (this is somewhat non-standard terminology, but it will be convenient).

We will use some of the notation and definitions of [CHT08] without comment. In particular, we will use the notions of RACSDC and RAESDC automorphic representations, for which see sections 4.2 and 4.3 of [CHT08]. We will also use the notion of a RAECSDC automorphic representation, for which see section 1 of [BLGHT09]. If π is a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, F a totally real field, and $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, then we let $r_{l,\iota}(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ denote the corresponding Galois representation. Similarly, if π is a RAECSDC or RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, F a CM field (in this paper, all CM fields are totally imaginary), and $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, then we let $r_{l,\iota}(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ denote the corresponding Galois representation. For the properties of $r_{l,\iota}(\pi)$, see Theorems 1.1 and 1.2 of [BLGHT09].

2. Another character-building exercise.

2.1. In this section, we will complete a trivial but rather technical exercise in class field theory, which will allow us to construct characters with prescribed properties which will be useful to us throughout our arguments. Similar calculations appeared in [BLGHT09] and [BLGG09]; we apologize that our earlier efforts were not carried out in sufficient generality for us to be able to to simply cite them.

We first recall the following definition from [BLGHT09, Definition 7.2].

Definition 2.1.1. Let $k/\mathbb{F}_l$ be algebraic and let m and n be positive integers. We say that a subgroup H of $\mathrm{GL}_n(k)$ is *m-big* if the following conditions are satisfied.

- H has no l-power order quotient.
- $H^0(H, \mathfrak{sl}_n(k)) = (0)$.
- $H^1(H, \mathfrak{sl}_n(k)) = (0)$.
- For all irreducible $k[H]$-submodules W of $\mathfrak{gl}_n(k)$ we can find $h \in H$ and $\alpha \in k$ such that:
  - α is a simple root of the characteristic polynomial of h, and if β is any other root then $\alpha^m \neq \beta^m$.
  - Let $\pi_{h,\alpha}$ (respectively $i_{h,\alpha}$) denote the h-equivariant projection from k^n to the α-eigenspace of h (respectively the h-equivariant injection from the α-eigenspace of h to k^n). Then $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq 0$.

Also, we say that H is *big* if it is 1-big.

We now turn to the main result of this section.

Lemma 2.1.2. *Suppose that:*

- *F^+ is a totally real field,*
- *n is a positive integer,*
- *l is a rational prime, with $l > 2$ and $l > 2n - 2$,*
- *m is a positive even integer, with $l \nmid m$,*
- *$F^{(\mathrm{avoid})}$ is a finite extension of F^+,*
- *T is a finite set of primes of F^+, not containing places above l,*
- *$\eta : G_{F^+} \to \overline{\mathbb{Z}}_l^\times$ is a finite order character, such that η takes some fixed value $\gamma \in \{\pm 1\}$ on every complex conjugation, and is unramified at each prime of T, and*
- *$\eta' : G_{F^+} \to \overline{\mathbb{Z}}_l^\times$ is another finite order character, congruent to η (mod l), and is unramified at each prime of T. (Note that η' therefore also takes the value γ on every complex conjugation.)*

Suppose further that for each embedding τ of F^+ into $\overline{\mathbb{Q}}_l$, and each integer i, $1 \leq i \leq m$, we are given an integer $h_{i,\tau}$ and another integer $h'_{i,\tau}$, Finally, suppose that *there are integers w, w' such that for each i and τ, the integers $h_{i,\tau}$, $h'_{i,\tau}$ satisfy:*

$$h_{i,\tau} + h_{m+1-i,\tau} = w \quad \text{and} \quad h'_{i,\tau} + h'_{m+1-i,\tau} = w'.$$

Then we can find a cyclic, degree m CM extension M of F^+ which is linearly disjoint from $F^{(\mathrm{avoid})}$ over F^+, and continuous characters

$$\theta, \theta' : G_M \to \overline{\mathbb{Z}}_l^\times$$

such that θ and θ' are de Rham at all primes above l, and furthermore:

1. *θ, θ' are congruent (mod l).*
2. *For any $\bar{r} : G_{F^+} \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$, a continuous Galois representation ramified only at primes in T and above l, which satisfies $\overline{F^+}^{\ker \bar{r}}(\zeta_l) \subset F^{(\mathrm{avoid})}$:*
   - *If $\bar{r}(G_{F^+(\zeta_l)})$ is m-big, then $(\bar{r} \otimes \mathrm{Ind}_{G_M}^{G_{F^+}} \bar{\theta})(G_{F^+(\zeta_l)}) = (\bar{r} \otimes \mathrm{Ind}_{G_M}^{G_{F^+}} \bar{\theta}')(G_{F^+(\zeta_l)})$ is big.*
   - *If $[\overline{F^+}^{\ker \mathrm{ad}\, \bar{r}}(\zeta_l) : \overline{F^+}^{\ker \mathrm{ad}\, \bar{r}}] > m$ then the fixed field of the kernel of $\mathrm{ad}(\bar{r} \otimes \mathrm{Ind}_{G_M}^{G_{F^+}} \bar{\theta}) = \mathrm{ad}(\bar{r} \otimes \mathrm{Ind}_{G_M}^{G_{F^+}} \bar{\theta}')$ will not contain ζ_l.*
3. *We can put a perfect pairing on $\mathrm{Ind}_{G_M}^{G_{F^+}} \theta$ satisfying*
   (a) *$\langle v_1, v_2 \rangle = (-1)^w \gamma \langle v_2, v_1 \rangle$.*
   (b) *For $\sigma \in G_{F^+}$, we have*
   $$\langle \sigma v_1, \sigma v_2 \rangle = \epsilon(\sigma)^{-w} \eta(\sigma) \langle v_1, v_2 \rangle.$$

   Thus, in particular,
   $$(\mathrm{Ind}_{G_M}^{G_{F^+}} \theta) \cong (\mathrm{Ind}_{G_M}^{G_{F^+}} \theta)^\vee \otimes \epsilon^{-w} \eta.$$
   (Note that the character on the right hand side takes the value $(-1)^w \gamma$ on complex conjugations.)
4. *We can put a perfect pairing on $\mathrm{Ind}_{G_M}^{G_{F^+}} \theta'$ satisfying*
   (a) *$\langle v_1, v_2 \rangle = (-1)^w \gamma \langle v_2, v_1 \rangle$.*
   (b) *For $\sigma \in G_{F^+}$, we have*
   $$\langle \sigma v_1, \sigma v_2 \rangle = \epsilon(\sigma)^{-w'} \tilde{\omega}(\sigma)^{w'-w} \eta'(\sigma) \langle v_1, v_2 \rangle$$
   where $\tilde{\omega}$ is the Teichmüller lift of the mod l cyclotomic character.

   Thus, in particular,
   $$(\mathrm{Ind}_{G_M}^{G_{F^+}} \theta') \cong (\mathrm{Ind}_{G_M}^{G_{F^+}} \theta')^\vee \otimes \epsilon^{-w'} \tilde{\omega}^{w'-w} \eta'.$$
   (Note that the character on the right hand side takes the value $(-1)^w \gamma$ on complex conjugations.)
5. *For each v above l, the representation $(\mathrm{Ind}_{G_M}^{G_{F^+}} \theta)|_{G_{F_v^+}}$ is conjugate to a representation which breaks up as a direct sum of characters:*
   $$(\mathrm{Ind}_{G_M}^{G_{F^+}} \theta)|_{G_{F_v^+}} \cong \chi_1^{(v)} \oplus \chi_2^{(v)} \oplus \cdots \oplus \chi_m^{(v)}$$
   where, for each i, $1 \le i \le m$, and each embedding $\tau : F_v^+ \to \overline{\mathbb{Q}}_l$ we have that:
   $$\mathrm{HT}_\tau(\chi_i^{(v)}) = h_{i,\tau}.$$
   Similarly, the representation $(\mathrm{Ind}_{G_M}^{G_{F^+}} \theta')|_{G_{F_v^+}}$ is conjugate to a representation which breaks up as a direct sum of characters:
   $$(\mathrm{Ind}_{G_M}^{G_{F^+}} \theta')|_{G_{F_v^+}} \cong \chi_1'^{(v)} \oplus \chi_2'^{(v)} \oplus \cdots \oplus \chi_m'^{(v)}$$

where, for each i, $1 \le i \le m$, and each embedding $\tau : F_v^+ \to \overline{\mathbb{Q}}_l$ we have that:

$$\mathrm{HT}_\tau(\chi_i'^{(v)}) = h'_{i,\tau}.$$

(6) *M/F^+ is unramified at each prime of T; and θ, θ' are unramified above each prime of T. Thus $\mathrm{Ind}_{G_M}^{G_{F^+}} \theta$ and $\mathrm{Ind}_{G_M}^{G_{F^+}} \theta'$ are unramified at each prime of T. Each prime of F^+ above l splits completely in M.*

(7) *If η is unramified at each place dividing l, then θ is crystalline.*

Proof. Throughout this proof, we will use $\bar{F}$ as a shorthand for $\overline{F^+}$.

Step 1: Finding a suitable field M. We claim that there exists a surjective character $\chi : \mathrm{Gal}(\bar{F}/F^+) \to \mu_m$ (where μ_m is the group of m-th roots of unity in $\overline{\mathbb{Q}}^\times$) such that

- χ is unramified with $\chi(\mathrm{Frob}_v) = 1$ at all places v of F^+ above l.
- $\chi(c_v) = -1$ for each infinite place v (where c_v denotes a complex conjugation at v).
- $\bar{F}^{\ker \chi}$ is linearly disjoint from $F^{(\mathrm{avoid})}$ over F^+.
- χ is unramified at all primes of T.

We construct the character χ as follows. First, we find using weak approximation a totally negative element $\alpha \in (F^+)^\times$ which is a v-adic unit and a quadratic residue mod v for each prime v of F^+ above l, and which is a w-adic unit for each prime w in T. Let χ_0 be the quadratic character associated to the extension we get by adjoining the square root of this element. Then:

- χ_0 is unramified with $\chi_0(\mathrm{Frob}_v) = 1$ at all places v of F^+ above l.
- $\chi_0(c_v) = -1$ for each infinite place v (where c_v denotes a complex conjugation at v).
- χ_0 is unramified at all places in T.

Now choose a cyclic totally real extension $M_1/\mathbb{Q}$ of degree m such that:

- $M_1/\mathbb{Q}$ is unramified at all the rational primes where $\bar{F}^{\ker \chi_0} F^{(\mathrm{avoid})}/\mathbb{Q}$ is ramified, and at all rational primes which lie below a prime of T.
- l splits completely in M_1.

Since $\bar{F}^{\ker \chi_0} F^{(\mathrm{avoid})}/\mathbb{Q}$ and $M_1/\mathbb{Q}$ ramify at disjoint sets of primes, they are linearly disjoint, and we can find a rational prime p which splits completely in $F^{(\mathrm{avoid})}\bar{F}^{\ker \chi_0}$ but such that Frob_p generates $\mathrm{Gal}(M_1/\mathbb{Q})$. Since $M_1/\mathbb{Q}$ is cyclic, we may pick an isomorphism between $\mathrm{Gal}(M_1/\mathbb{Q})$ and μ_m, and we can think of M_1 as determining a character $\chi_1 : G_\mathbb{Q} \to \mu_m$ such that:

- χ_1 is trivial on $G_{\mathbb{Q}_l}$.
- χ_1 is trivial on complex conjugation.
- $\chi_1(\mathrm{Frob}_p) = \zeta_m$, a primitive mth root of unity.
- χ_1 is unramified at all rational primes lying below primes of T.

Then, set $\chi = (\chi_1|_{G_{F^+}})\chi_0$. Note that this maps onto μ_m, even when we restrict to $G_{F^{(\mathrm{avoid})}}$ (since p splits completely in $F^{(\mathrm{avoid})}$ and if $\wp$ is a place of $F^{(\mathrm{avoid})}$ over p, we have $\chi_0(\mathrm{Frob}_\wp) = 1$ while $\chi_1(\mathrm{Frob}_\wp) = \zeta_m$). The remaining properties are clear.

Having shown χ exists, we set $M = \bar{F}^{\ker \chi}$; note that this is a CM field, and a cyclic extension of F^+ of degree m. Write σ_{M/F^+} for a generator of $\mathrm{Gal}(M/F^+)$. Write M^+ for the maximal totally real subfield of M. Note also our choices have ensured that almost all the points in conclusion 6 of the Lemma currently being proved will hold; all that remains from there to be checked is that θ and θ' are unramified at all primes above prime of T.

Step 2: An auxiliary prime q. Choose a rational prime q such that

- no prime of T lies above q,
- $q \neq l$,
- q splits completely in M,
- q is unramified in $F^{(\text{avoid})}$,
- $q - 1 > 2n$, and
- η and η' are both unramified at all primes above q.

Also choose a prime $\mathfrak{q}$ of F^+ above q, and a prime $\mathfrak{Q}$ of M above $\mathfrak{q}$.

Step 3: Defining certain algebraic characters ϕ, ϕ'. For each prime v of F^+, choose a prime $\tilde{v}$ of M lying above it. Then for each embedding τ of F^+ into $\overline{\mathbb{Q}}_l$, select an embedding $\tilde{\tau}$ of M into $\overline{\mathbb{Q}}_l$ extending it, in such a way that $\tilde{\tau}$ corresponds to the prime $\tilde{v}$ if τ corresponds to v.

Note we now have a convenient notation for all the embeddings extending τ; in particular, they can be written as $\tilde{\tau} \circ \sigma^j_{M/F^+}$ for $j = 0, \ldots, m-1$. (Recall that we are writing σ_{M/F^+} for the generator of $\mathrm{Gal}(M/F^+)$.)

We are now forced into a slight notational ugliness. Write $\tilde{M}_0$ for the Galois closure of M over $\mathbb{Q}$, and $\tilde{M}$ for $\tilde{M}_0$ with the $\#\eta(G_{F^+})$th roots of unity adjoined, so $\tilde{M} = \tilde{M}_0(\mu_{\#\eta(G_{F^+})})$. (Thus $\mathrm{Gal}(\tilde{M}/\mathbb{Q})$ is in bijection with embeddings $\tilde{M} \to \overline{\mathbb{Q}}$.) Let us fix ι^*, an embedding of $\tilde{M}$ into $\overline{\mathbb{Q}}_l$, and write v^* for the prime of M below this.[1] Using ι^*, we can and will abuse notation by thinking of η as being valued in $\tilde{M}$. Given any embedding ι' of M into $\overline{\mathbb{Q}}_l$, we can choose an element $\sigma_{\iota^* \rightsquigarrow \iota'}$ in $\mathrm{Gal}(\tilde{M}/\mathbb{Q})$ such that $\iota' = (\iota^* \circ \sigma_{\iota^* \rightsquigarrow \iota'})|_M$.

We claim that there exists an extension M' of $\tilde{M}$, and a character $\phi : \mathbb{A}_M^\times \to (M')^\times$ with open kernel such that:

- For $\alpha \in M^\times$,

$$\phi(\alpha) = \prod_{\tau \in \mathrm{Hom}(F^+, \overline{\mathbb{Q}}_l)} \prod_{j=0}^{(m/2)-1} (\sigma_{\iota^* \rightsquigarrow \tilde{\tau} \circ \sigma^{-j}_{M/F^+}}(\alpha))^{h_{j+1,\tau}} (\sigma_{\iota^* \rightsquigarrow \tilde{\tau} \circ \sigma^{-j-(m/2)}_{M/F^+}}(\alpha))^{h_{m-j,\tau}}.$$

- For $\alpha \in (\mathbb{A}_{M^+})^\times$, we have

$$\phi(\alpha) = (\prod_{v \nmid \infty} |\alpha_v| \prod_{v | \infty} \mathrm{sgn}_v(\alpha_v))^{-w} \delta_{M/M^+}(\mathrm{Art}_{M^+}(\alpha))^{-w+(\gamma-1)/2} \eta|_{G_{M^+}}(\mathrm{Art}_{M^+}(\alpha)),$$

  where δ_{M/M^+} is the quadratic character of G_{M^+} associated to M. (Note that, in the right hand side, we really think of α as an element of $\mathbb{A}_{M^+}$, not just as an element of $\mathbb{A}_M$ which happens to lie in $\mathbb{A}_{M^+}$; so for instance v runs over places of M^+, and the local norms are appropriately normalized to reflect us thinking of them as places of M^+.)
- If η is unramified at l, then ϕ is unramified at l.
- ϕ is unramified at all primes above primes of T.
- $q | \#\phi(\mathcal{O}^\times_{M,\mathfrak{Q}})$, but ϕ is unramified at primes above $\mathfrak{q}$ other than $\mathfrak{Q}$ and $\mathfrak{Q}^c$

[1] The choice of this ι^* will affect the choice of the algebraic characters ϕ, ϕ' below, but will be cancelled out—at least concerning the properties we care about—when we pass to the l-adic characters θ, θ' below.

This is an immediate consequence of Lemma 2.2 of [HSBT06], as follows. We must verify that the formula for $\phi(\alpha)$ for $\alpha \in \mathbb{A}_{M^+}^\times$ in the second bullet point is trivial on $-1 \in M_v^+$ for each infinite place v and that the conditions in the various bullet points are compatible. The former is immediate, as may be verified by a direct calculation. The only difficult part in checking that the bullet points are compatible is comparing the first and second, which we may verify as follows. Suppose that $\alpha \in (M^+)^\times$; then we must check that the expression for $\phi(\alpha)$ from the first bullet point ($\phi_0(\alpha)$, say), matches that from the second ($\phi_1(\alpha)$, say). It will suffice to show that $\iota^*(\phi_0(\alpha)) = \iota^*(\phi_1(\alpha))$.

We have:

$$\iota^*(\phi_0(\alpha)) = \prod_{\tau\in\mathrm{Hom}(F^+,\overline{\mathbb{Q}}_l)} \prod_{j=0}^{(m/2)-1} (\tilde{\tau}(\sigma_{M/F^+}^{-j}(\alpha)))^{h_{j+1,\tau}}(\tilde{\tau}(\sigma_{M/F^+}^{-j-(m/2)}(\alpha)))^{h_{m-j,\tau}}$$

and then using the facts that $\sigma_{F^+/M}^{m/2}$ fixes M^+ and $h_{i,\tau}+h_{m+1-i,\tau} = w$, this

$$\begin{aligned}
&= \prod_{\tau\in\mathrm{Hom}(F^+,\overline{\mathbb{Q}}_l)} \prod_{j=0}^{(m/2)-1} (\tilde{\tau}(\sigma_{M/F^+}^{-j}(\alpha)))^{h_{j+1,\tau}}(\tilde{\tau}(\sigma_{M/F^+}^{-j}(\alpha)))^{h_{m-j,\tau}}\\
&= \prod_{\tau\in\mathrm{Hom}(F^+,\overline{\mathbb{Q}}_l)} \prod_{j=0}^{(m/2)-1} (\tilde{\tau}(\sigma_{M/F^+}^{-j}(\alpha)))^{w}\\
&= \prod_{\tau_M\in\mathrm{Hom}(M^+,\overline{\mathbb{Q}}_l)} \tau_M(\alpha)^w\\
&= i_{\mathbb{Q}\to\overline{\mathbb{Q}}_l}(N_{M^+/\mathbb{Q}}(\alpha)^w)\\
&= i_{\mathbb{Q}\to\overline{\mathbb{Q}}_l}\left(\prod_{\substack{v \text{ a finite}\\ \text{place of } \mathbb{Q}}} |N_{M^+/\mathbb{Q}}(\alpha)|_v^w \,\mathrm{sgn}(N_{M^+/\mathbb{Q}}(\alpha))^w\right)\\
&= i_{\mathbb{Q}\to\overline{\mathbb{Q}}_l}((\prod_{v\nmid\infty}|\alpha_v|\prod_{v|\infty}\mathrm{sgn}_v(\alpha_v))^{-w}) \qquad (v \text{ ranges over places of } M^+)\\
&= i_{\mathbb{Q}\to\overline{\mathbb{Q}}_l}(\prod_{v\nmid\infty}|\alpha_v|\prod_{v|\infty}\mathrm{sgn}_v(\alpha_v))^{-w}\delta_{M/M^+}(\mathrm{Art}_{M^+}(\alpha))^{-w+(\gamma-1)/2}\eta(\mathrm{Art}_{M^+}(\alpha)),
\end{aligned}$$

(where $i_{\mathbb{Q}\to\overline{\mathbb{Q}}_l}$ is the inclusion $\mathbb{Q}\hookrightarrow\overline{\mathbb{Q}}_l$) since $\mathrm{Art}(\alpha)$ is trivial; this is as required.

Similarly, we construct a character $\phi' : (\mathbb{A}_M)^\times \to (M')^\times$ (enlarging M' if necessary) with open kernel such that:

- For $\alpha \in M^\times$,

$$\phi'(\alpha) = \prod_{\tau\in\mathrm{Hom}(F^+,\overline{\mathbb{Q}}_l)} \prod_{j=0}^{(m/2)-1} (\sigma_{\iota^*\rightsquigarrow\tilde{\tau}\circ\sigma_{M/F^+}^{-j}}(\alpha))^{h'_{j+1,\tau}}(\sigma_{\iota^*\rightsquigarrow\tilde{\tau}\circ\sigma^{-j-(m/2)}}(\alpha))^{h'_{m-j,\tau}}.$$

- For $\alpha \in (\mathbb{A}_{M^+})^\times$, we have

$$\phi'(\alpha) = (\prod_{v\nmid\infty}|\alpha_v|\prod_{v|\infty}\mathrm{sgn}_v(\alpha_v))^{-w'}\delta_{M/M^+}(\mathrm{Art}_{M^+}(\alpha))^{-w'+(\gamma-1)/2}\eta'|_{G_{M^+}}(\mathrm{Art}_{M^+}(\alpha)).$$

(Again, we think of α in the right hand side as a bona fide member of $\mathbb{A}_{M^+}$.)

- ϕ' is unramified at all primes above primes of T.

Again, this follows from Lemma 2.2 of [HSBT06].

Step 4: Defining the characters θ, θ'. Write $\tilde{M}'$ for the Galois closure of M' over $\mathbb{Q}$, and extend $\iota^* : \tilde{M} \to \overline{\mathbb{Q}}_l$ to an embedding $\iota^{**} : \tilde{M}' \to \overline{\mathbb{Q}}_l$. Define l-adic characters $\theta, \theta'_0 : \mathrm{Gal}(\overline{M}/M) \to \overline{\mathbb{Z}}_l^\times$ by the following expressions (here $\alpha \in \mathbb{A}_M$, and if $\tilde{\tau}$ is a map $M \hookrightarrow \overline{\mathbb{Q}}_l$, corresponding to a place $v(\tilde{\tau})$ of M, then $\tilde{\tau}(\alpha)$ is a shorthand for $\alpha_{v(\tilde{\tau})}$, mapped into $\overline{\mathbb{Q}}_l$ via the unique continuous extension of $\tilde{\tau}$ to a map $M_{v(\tilde{\tau})} \hookrightarrow \overline{\mathbb{Q}}_l$):

$$\theta(\mathrm{Art}\,\alpha) = \iota^{**}(\phi(\alpha)) \prod_{\tau \in \mathrm{Hom}(F^+, \overline{\mathbb{Q}}_l)} \prod_{j=0}^{(m/2)-1} (\tilde{\tau}\sigma_{M/F^+}^{-j})(\alpha)^{-h_{j+1,\tau}} (\tilde{\tau}\sigma_{M/F^+}^{-j-m/2})(\alpha)^{-h_{m-j,\tau}}$$

$$\theta'_0(\mathrm{Art}\,\alpha) = \iota^{**}(\phi'(\alpha)) \prod_{\tau \in \mathrm{Hom}(F^+, \overline{\mathbb{Q}}_l)} \prod_{j=0}^{(m/2)-1} (\tilde{\tau}\sigma_{M/F^+}^{-j})(\alpha)^{-h'_{j+1,\tau}} (\tilde{\tau}\sigma_{M/F^+}^{-j-m/2})(\alpha)^{-h'_{m-j,\tau}}$$

where v runs over places of F dividing l. (It is easy to check that the expressions on the right hand sides are unaffected when α is multiplied by an element of $M^\times$.) Observe then that they enjoy the following properties:

- $\theta \circ V_{M/M^+} = \epsilon^{-w} \delta_{M/M^+}^{-w+(\gamma-1)/2} \eta|_{G_M^+}$ where V_{M/M^+} is the transfer map $G_{M^+}^{\mathrm{ab}} \to G_M^{\mathrm{ab}}$. In particular, $\theta\theta^c = \epsilon^{-w}\eta|_{G_M}$.
- $\theta'_0 \circ V_{M/M^+} = \epsilon^{-w'} \delta_{M/M^+}^{-w'+(\gamma-1)/2} \eta'|_{G_M^+}$ and hence, $\theta'_0 {\theta'_0}^c = \epsilon^{-w'}\eta'|_{G_M}$.
- For τ an embedding of F^+ into $\overline{\mathbb{Q}}_l$ and $0 \le j \le (m/2)-1$, the Hodge–Tate weight of θ at the embedding $\tilde{\tau}\sigma_{M/F^+}^{-j}$ is $h_{j+1,\tau}$ and the Hodge–Tate weight of θ at the embedding $\tilde{\tau}\sigma_{M/F^+}^{-j-m/2}$ is $h_{m-j,\tau}$.
- For τ an embedding of F^+ into $\overline{\mathbb{Q}}_l$ and $0 \le j \le (m/2)-1$, the Hodge–Tate weight of θ'_0 at the embedding $\tilde{\tau}\sigma_{M/F^+}^{-j}$ is $h'_{j+1,\tau}$ and the Hodge–Tate weight of θ'_0 at the embedding $\tilde{\tau}\sigma_{M/F^+}^{-j-m/2}$ is $h'_{m-j,\tau}$.
- $q | \#\theta(I_{\mathfrak{Q}})$, but θ is unramified at all primes above $\mathfrak{q}$ except $\mathfrak{Q}, \mathfrak{Q}^c$.

We now define $\theta' = \theta'_0(\tilde{\theta}/\tilde{\theta}'_0)$—where $\tilde{\theta}$ (resp $\tilde{\theta}'_0$) denotes the Teichmüller lift of the reduction mod l of θ (resp θ'_0)—and observe that:

- $\theta' \pmod{l} = \theta \pmod{l}$.
- $\theta'(\theta')^c = \epsilon^{-w'}\tilde{\omega}^{w'-w}\eta'|_{G_M}$.

Step 5: Properties of $\mathrm{Ind}_{G_M}^{G_{F^+}} \theta$ and $\mathrm{Ind}_{G_M}^{G_{F^+}} \theta'$. We begin by addressing point 3. We define a pairing on $\mathrm{Ind}_{G_M}^{G_{F^+}} \theta$ by the formula

$$\langle \lambda, \lambda' \rangle = \sum_{\sigma \in \mathrm{Gal}(\overline{M}/M) \backslash \mathrm{Gal}(\overline{M}/F)} \epsilon_l(\sigma)^w \eta(\sigma)^{-1} \lambda(\sigma)\lambda'(c\sigma)$$

where c is any complex conjugation. One easily checks that this is well defined and perfect, and that the properties (a) and (b) hold.

We can address point 4 in a similar manner, defining a pairing on $\operatorname{Ind}_{G_M}^{G_{F^+}} \theta'$ via:

$$\langle \lambda, \lambda' \rangle = \sum_{\sigma \in \operatorname{Gal}(\overline{M}/M) \backslash \operatorname{Gal}(\overline{M}/F)} \epsilon_l(\sigma)^{w'} \tilde{\omega}(\sigma)^{w-w'} \eta'(\sigma)^{-1} \lambda(\sigma) \lambda'(c\sigma)$$

and checking the required properties.

Next, we address point 5. We will give the argument for $\operatorname{Ind}_{G_M}^{G_{F^+}} \theta$; the argument for $\operatorname{Ind}_{G_M}^{G_{F^+}} \theta'$ is similar. Since the primes of F^+ above l split in M, we immediately have that for each such prime v, $(\operatorname{Ind}_{G_M}^{G_{F^+}} \theta)|_{G_{F_v^+}}$ will split as required as a direct sum of characters $\chi_i^{(v)}$, with each character $\chi_i^{(v)}$ corresponding to a prime of M above v. We recall that we chose above a prime $\tilde{v}$ of M above v for each such prime v; we without loss of generality assume that the $\chi_i^{(v)}$ are numbered so that for $0 \leq i \leq (m/2) - 1$, we have that $\chi_{i+1}^{(v)}$ corresponds to the prime $\sigma_{M/F^+}^i \tilde{v}$ and $\chi_{m-i}^{(v)}$ corresponds to the prime $\sigma_{M/F^+}^{i+m/2} \tilde{v}$.

Then for τ an embedding of $F_v^+ \to \overline{\mathbb{Q}}_l$, we have that

$$\operatorname{HT}_\tau(\chi_{i+1}^{(v)}) = \operatorname{HT}_{\tilde{\tau}\sigma_{M/F^+}^{-i}}(\theta) = h_{i+1,\tau}$$
$$\operatorname{HT}_\tau(\chi_{m-i}^{(v)}) = \operatorname{HT}_{\tilde{\tau}\sigma_{M/F^+}^{-i-m/2}}(\theta) = h_{m-i,\tau}$$

Step 6: Establishing the big image/avoid ζ_l properties. All that remains is to prove the big image and avoiding ζ_l properties; that is, point (2). We will just show the stated properties concerning $\operatorname{Ind}_{G_M}^{G_{F^+}} \theta$; the statement for $\operatorname{Ind}_{G_M}^{G_{F^+}} \theta'$ then follows since θ and θ' are congruent.

Let $\bar{r} : G_{F^+} \to \operatorname{GL}_n(\overline{\mathbb{F}}_l)$ be a continuous Galois representation such that the following properties hold:

- $\bar{r}$ is ramified only at primes of T and above l, and
- we have $\bar{F}^{\ker \bar{r}}(\zeta_l) \subset F^{(\text{avoid})}$.

We may now apply Lemma 4.1.2 of [BLGG09] (with F in that lemma equal to our current F^+). (To be completely precise, Lemma 4.1.2 as written only applies to a characteristic 0 representation r rather than the characteristic l representation $\bar{r}$ as we have here. But the proof goes through exactly the same if one starts with a characteristic l representation.)

If we assume that $\bar{r}(G_{F^+(\zeta_l)})$ is m-big, then applying part 2 of that lemma will give that $(\bar{r} \otimes \operatorname{Ind}_{G_M}^{G_{F^+}} \bar{\theta})|_{G_{F^+(\zeta_l)}}$ has big image, (the first part of point (2) to be proved). Similarly, applying part 1 will give the fact that we avoid ζ_l (the second part of point (2)). All that remains is to check the hypotheses of Lemma 4.1.2 of [BLGG09].

The fact that M is linearly disjoint from $\overline{F}^{\ker \bar{r}}(\zeta_l)$ (common to both parts) comes from the fact that $\overline{F}^{\ker \bar{r}}(\zeta_l) \subset F^{(\text{avoid})}$ and M was chosen to be linearly disjoint from $F^{(\text{avoid})}$. The fact that every place of F^+ above l is unramified in M follows from the construction of M (in fact, they all split completely).

We turn now to the particular hypotheses of the second part. That $\bar{r}|_{G_{F(\zeta_l)}}$ has m-big image is by assumption. The properties we require of $\mathfrak{q}$ follow directly from the bullet points established in Step 2, the properties of $\bar{r}$ just above, and the first and last bullet points (concerning $\theta\theta^c$ and $\#\theta(I_{\mathfrak{Q}})$ respectively) in the list of properties of θ given immediately after θ is introduced in step 4. The fact that $\theta\theta^c$ can be extended to G_{F^+} comes from the first bullet point in the list of properties of θ in step 4, the fact that η is given as a character of G_{F^+}, and the fact that the cyclotomic character obviously extends in this way. □

3. Lifting a Galois representation.

3.1. Notation.

3.1.1. *The group $\mathcal{G}_n$.* Let n be a positive integer and let $\mathcal{G}_n$ be the group scheme over $\operatorname{Spec}\mathbb{Z}$ defined in section 2.1 of [CHT08], that is, the semi-direct product of $\mathrm{GL}_n \times \mathrm{GL}_1$ by the group $\{1, j\}$ acting on $\mathrm{GL}_n \times \mathrm{GL}_1$ by $j(g,\mu)j^{-1} = (\mu {}^t g^{-1}, \mu)$. Let $\nu : \mathcal{G}_n \to \mathrm{GL}_1$ be the homomorphism which sends (g,μ) to μ and j to -1.

Let F be an imaginary CM field with totally real subfield F^+ and let n be a positive integer. Fix a complex conjugation $c \in G_{F^+}$ and let $\delta_{F/F^+} : G_{F^+} \to \{\pm 1\}$ be the quadratic character associated to the extension F/F^+. If R is a topological ring, then by Lemma 2.1.1 of [CHT08] there is a natural bijection between the set of continuous homomorphisms $\rho : G_{F^+} \to \mathcal{G}_n(R)$ with $\rho^{-1}(\mathrm{GL}_n(R) \times \mathrm{GL}_1(R)) = G_F$ and the set of triples $(r, \chi, \langle\ ,\ \rangle)$ where $r : G_F \to \mathrm{GL}_n(R)$ and $\chi : G_{F^+} \to R^\times$ are continuous homomorphisms and $\langle\ ,\ \rangle : R^n \times R^n \to R$ is a perfect R-bilinear pairing such that

- $\langle x, y\rangle = -\chi(c)\langle y, x\rangle$ for all $x, y \in R^n$, and
- $\langle r(\sigma)x, r(c\sigma c)y\rangle = \chi(\sigma)\langle x, y\rangle$ for all $\sigma \in G_F$ and $x, y \in R^n$.

If ρ corresponds to the triple $(r, \chi, \langle\ ,\ \rangle)$, then $\chi = \nu \circ \rho$, $\rho|_{G_F} = (r, \chi|_{G_F})$ and if we write $\rho(c) = (A, -\chi(c))j$, then $\langle x, y\rangle = {}^t x A^{-1} y$ for all $x, y \in R^n$. We say that ρ extends r.

If R is a ring and $\rho : G_{F^+} \to \mathcal{G}_n(R)$ a homomorphism with $G_F = \rho^{-1}(\mathrm{GL}_n(R) \times \mathrm{GL}_1(R)$ and $(r, \chi, \langle\ ,\ \rangle)$ is the corresponding triple, we will often abuse notation and write $\rho|_{G_F}$ for $r : G_F \to \mathrm{GL}_n(R)$.

3.1.2. *Oddness (CM case).* Let k be a topological field and let $r : G_F \to \mathrm{GL}_n(k)$ be a continuous representation. We say that r is *essentially conjugate-self-dual* (ECSD), if there exists a continuous character $\mu : G_F \to k^\times$ such that $r^c \cong r^\vee\mu$ and such that μ can be extended to a character $\chi : G_{F^+} \to k^\times$ with $\chi(c_v)$ independent of $v|\infty$ (where c_v denotes a complex conjugation at a place $v|\infty$). Note that if such an extension χ exists then there is one other, namely $\chi\delta_{F/F^+}$, and hence there is one totally odd extension and one totally even extension.

Assume that $r : G_F \to \mathrm{GL}_n(k)$ is ECSD and let $\mu : G_F \to k^\times$ be as above. Let $\langle\ ,\ \rangle : k^n \times k^n \to k$ be a perfect bilinear pairing giving rise to the isomorphism $r^c \cong r^\vee\mu$ in the sense that $\langle r(\sigma)x, r(c\sigma c)y\rangle = \mu(\sigma)\langle x, y\rangle$ for all $\sigma \in G_F$ and $x, y \in k^n$. We say that the triple $(r, \mu, \langle\ ,\ \rangle)$ is *odd* if $\langle\ ,\ \rangle$ is symmetric. If r is absolutely irreducible, then $\langle\ ,\ \rangle$ is unique up to scaling by elements of $k^\times$. In this case we say that (r, μ) is odd if $(r, \mu, \langle\ ,\ \rangle)$ is odd for one (hence any) choice of $\langle\ ,\ \rangle$. If μ is clear from the context, we will sometimes just say that r is odd if (r, μ) is odd.

Let r and μ be as in the previous paragraph and assume that r is absolutely irreducible. Then there is a bilinear pairing $\langle\ ,\ \rangle : k^n \times k^n \to k$, unique up to scaling, giving rise to the isomorphism $r^c \cong r^\vee\mu$. This pairing satisfies $\langle x, y\rangle = (-1)^a\langle y, x\rangle$ for some $a \in \mathbb{Z}/2\mathbb{Z}$. Let $\chi : G_{F^+} \to k^\times$ be the unique extension of μ to G_{F^+} with $\chi(c_v) = (-1)^{a+1}$ for all $v|\infty$. Then the triple $(r, \chi, \langle\ ,\ \rangle)$ corresponds to a continuous homomorphism $G_{F^+} \to \mathcal{G}_n(k)$ extending r. Moreover, any other extension of r to a homomorphism $G_{F^+} \to \mathcal{G}_n(k)$ corresponds to a triple $(r, \chi, \alpha\langle\ ,\ \rangle)$ for some $\alpha \in k^\times$. In particular, r is odd if and only if for one (hence any) extension $\rho : G_{F^+} \to \mathcal{G}_n(k)$ of r, the character $\chi = \nu \circ \rho$ is odd.

3.1.3. *Oddness (totally real case).* Let k be a topological field and let $r : G_{F^+} \to \mathrm{GL}_n(k)$ be a continuous representation. We say that r is *essentially-self-dual* (ESD), if there exists a continuous character $\mu : G_{F^+} \to k^\times$ such that $r \cong r^\vee \mu$ and such that $\mu(c_v)$ is independent of $v|\infty$.

Assume that $r : G_{F^+} \to \mathrm{GL}_n(k)$ is ESD and let $\mu : G_{F^+} \to k^\times$ be as above. Assume also that the isomorphism $r \cong r^\vee \mu$ is realised by a perfect bilinear pairing $(\ ,\ ) : k^n \times k^n \to k$ satisfying $(r(\sigma)x, r(\sigma)y) = \mu(\sigma)(x, y)$ for all $\sigma \in G_{F^+}$ and $x, y \in k^n$. We say that the triple $(r, \mu, (\ ,\ ))$ is *odd* if either (i) $(\ ,\ )$ is symmetric and μ is even, or (ii) $(\ ,\ )$ is alternating and μ is odd.

We now explain the relation between oddness in the totally real case and oddness in the CM case. Let $(r, \mu, (\ ,\ ))$ be a triple as in the previous paragraph (not assumed to be odd). Then $r|_{G_F}$ satisfies $(r|_{G_F})^c \cong r|_{G_F} \cong (r|_{G_F})^\vee \mu|_{G_F}$. Moreover, let $J \in M_n(k)$ be the matrix with $(x, y) = {}^t x J y$ and define a new pairing $\langle\ ,\ \rangle : k^n \times k^n \to k$ by $\langle x, y\rangle = {}^t x J r(c) y$. Then $\langle r(\sigma)x, r(c\sigma c)y\rangle = \mu(\sigma)\langle x, y\rangle$ for all $\sigma \in G_{F^+}$ and $x, y \in k^n$ and hence $\langle\ ,\ \rangle$ realises the isomorphism $(r|_{G_F})^c \cong (r|_{G_F})^\vee \mu|_{G_F}$. Note that $\langle\ ,\ \rangle$ is symmetric if and only if ${}^t J = \mu(c) J$ which occurs if and only (i) or (ii) above hold. In particular, we see that $(r, \mu, (\ ,\ ))$ is odd if and only if $(r|_{G_F}, \mu|_{G_F}, \langle\ ,\ \rangle)$ is odd.

Suppose r and μ are as in the previous two paragraphs. If r is absolutely irreducible, then up to scaling there is a unique pairing $(\ ,\ )$ giving rise to the isomorphism $r \cong r^\vee \mu$. This pairing is either symmetric or alternating. We say that (r, μ) is odd if $(r, \mu, (\ ,\ ))$ is odd. If μ is clear from the context we will say that r is odd if this holds.

Note that parts (3) and (4) of Lemma 2.1.2 give rise to odd triples.

3.1.4. *Standard basis for tensor products.*

Definition 3.1.1. Let R be a ring and let V and W be two finite free R-modules of rank n and m respectively. Let $e_1, \ldots, e_n$ be an R-basis of V and $f_1, \ldots, f_m$ an R-basis of W. Let *the basis of* $V \otimes_R W$ *inherited from the bases* $\{e_i\}$ *and* $\{f_j\}$ be the ordered R-basis of $V \otimes_R W$ given by the vectors $e_i \otimes f_j$, ordered lexicographically. Let G be a group let and $r : G \to \mathrm{GL}_n(R)$ and $s : G \to \mathrm{GL}_m(R)$ represent R-linear actions of G on V and W with respect to the bases $\{e_i\}$ and $\{f_j\}$. Then we let $r \otimes s : G \to \mathrm{GL}_{nm}(R)$ denote the action of G on $V \otimes_R W$ with respect to the inherited basis of $V \otimes_R W$.

3.1.5. *Galois deformations.* Let l be a prime number and fix an algebraic closure $\overline{\mathbb{Q}}_l$ of $\mathbb{Q}_l$. Let K be a finite extension of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$ with ring of integers $\mathcal{O}$ and residue field k. Let $\mathfrak{m}_{\mathcal{O}}$ denote the maximal ideal of $\mathcal{O}$. Let $\mathcal{C}_{\mathcal{O}}$ be the category of complete local Noetherian $\mathcal{O}$-algebras with residue field isomorphic to k via the structural homomorphism. As in section 3 of [BLGHT09], we consider an object R of $\mathcal{C}_{\mathcal{O}}$ to be geometrically integral if for all finite extensions K'/K, the algebra $R \otimes_{\mathcal{O}} \mathcal{O}_{K'}$ is an integral domain.

Local lifting rings. Let M be a finite extension of $\mathbb{Q}_p$ for some prime p possibly equal to l and let $\overline{\rho} : G_M \to \mathrm{GL}_n(k)$ be a continuous homomorphism. Then the functor from $\mathcal{C}_{\mathcal{O}}$ to *Sets* which takes $A \in \mathcal{C}_{\mathcal{O}}$ to the set of continuous liftings $\rho : G_M \to \mathrm{GL}_n(A)$ of $\overline{\rho}$ is represented by a complete local Noetherian $\mathcal{O}$-algebra $R^{\square}_{\overline{\rho}}$. We call this ring the universal $\mathcal{O}$-lifting ring of $\overline{\rho}$. We write $\rho^{\square} : G_M \to \mathrm{GL}_n(R^{\square}_{\overline{\rho}})$ for the universal lifting.

Assume now that $p = l$ so that M is a finite extension of $\mathbb{Q}_l$. Assume also that K contains the image of every embedding $M \hookrightarrow \overline{K}$.

Definition 3.1.2. Let $(\mathbb{Z}^n_+)^{\mathrm{Hom}(M,K)}$ denote the subset of $(\mathbb{Z}^n)^{\mathrm{Hom}(M,K)}$ consisting of elements λ which satisfy

$$\lambda_{\tau,1} \geq \lambda_{\tau,2} \geq \ldots \geq \lambda_{\tau,n}$$

for every embedding τ.

Let a be an element of $(\mathbb{Z}^n_+)^{\mathrm{Hom}(M,K)}$. We associate to a an l-adic Hodge type $\mathbf{v}_a$ in the sense of section 2.6 of [Kis08] as follows. Let D_K denote the vector space K^n. Let $D_{K,M} = D_K \otimes_{\mathbb{Q}_l} M$. For each embedding $\tau : M \hookrightarrow K$, we let $D_{K,\tau} = D_{K,M} \otimes_{K\otimes M, 1\otimes\tau} K$ so that $D_{K,M} = \oplus_\tau D_{K,\tau}$. For each τ choose a decreasing filtration $\mathrm{Fil}^i D_{K,\tau}$ of $D_{K,\tau}$ so that $\dim_K \mathrm{gr}^i D_{K,\tau} = 0$ unless $i = (j-1) + a_{\tau,n-j+1}$ for some $j = 1, \ldots, n$ in which case $\dim_K \mathrm{gr}^i D_{K,\tau} = 1$. We define a decreasing filtration of $D_{K,M}$ by $K \otimes_{\mathbb{Q}_l} M$-submodules by setting

$$\mathrm{Fil}^i D_{K,M} = \oplus_\tau \mathrm{Fil}^i D_{K,\tau}.$$

Let $\mathbf{v}_a = \{D_K, \mathrm{Fil}^i D_{K,M}\}$.

We now recall some results of Kisin. Let a be an element of $(\mathbb{Z}^n_+)^{\mathrm{Hom}(M,K)}$ and let $\mathbf{v}_a = \{D_K, \mathrm{Fil}^i D_{K,M}\}$ be the associated l-adic Hodge type.

Definition 3.1.3. If B is a finite K-algebra and V_B is a free B-module of rank n with a continuous action of G_M that makes V_B into a de Rham representation, then we say that V_B *is of l-adic Hodge type* $\mathbf{v}_a$ if for each i there is an isomorphism of $B \otimes_{\mathbb{Q}_l} M$-modules

$$\mathrm{gr}^i (V_B \otimes_{\mathbb{Q}_l} B_{dR})^{G_M} \tilde{\to} (\mathrm{gr}^i D_{K,M}) \otimes_K B.$$

Corollary 2.7.7 of [Kis08] implies that there is a unique l-torsion free quotient $R^{\mathbf{v}_a,cr}_{\overline{\rho}}$ of $R^{\square}_{\overline{\rho}}$ with the property that for any finite K-algebra B, a homomorphism of $\mathcal{O}$-algebras $\zeta : R^{\square}_{\overline{\rho}} \to B$ factors through $R^{\mathbf{v}_a,cr}_{\overline{\rho}}$ if and only if $\zeta \circ \rho^{\square}$ is crystalline of l-adic Hodge type $\mathbf{v}_a$. Moreover, Theorem 3.3.8 of [Kis08] implies that $\mathrm{Spec}\, R^{\mathbf{v}_a,cr}_{\overline{\rho}}[1/l]$ is formally smooth over K and equidimensional of dimension $n^2 + \frac{1}{2}n(n-1)[M : \mathbb{Q}_l]$.

Definition 3.1.4. Suppose that $\rho_1, \rho_2 : G_M \to \mathrm{GL}_n(\mathcal{O})$ are two continuous lifts of $\overline{\rho}$. Then we say that $\rho_1 \sim \rho_2$ if the following hold.

(1) There is an $a \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(M,K)}$ such that ρ_1 and ρ_2 both correspond to points of $R^{\mathbf{v}_a,cr}_{\overline{\rho}}$ (that is, $\rho_1 \otimes_{\mathcal{O}} K$ and $\rho_2 \otimes_{\mathcal{O}} K$ are both crystalline of l-adic Hodge type $\mathbf{v}_a$).
(2) For every minimal prime ideal $\wp$ of $R^{\mathbf{v}_a,cr}_{\overline{\rho}}$, the quotient $R^{\mathbf{v}_a,cr}_{\overline{\rho}}/\wp$ is geometrically integral.
(3) ρ_1 and ρ_2 give rise to closed points on a common irreducible component of $\mathrm{Spec}\, R^{\mathbf{v}_a,cr}_{\overline{\rho}}[1/l]$.

In (3) above, note that the irreducible component is uniquely determined by either of ρ_1, ρ_2 because Spec $R_{\overline{\rho}}^{\mathbf{v}_a,cr}[1/l]$ is formally smooth. Note also that we can always ensure that (2) holds by replacing $\mathcal{O}$ with the ring of integers in a finite extension of K.

Global deformation rings. Let F/F^+ be a totally imaginary quadratic extension of a totally real field F^+. Let c denote the non-trivial element of $\mathrm{Gal}(F/F^+)$. Assume that K contains the image of every embedding $F \hookrightarrow \overline{\mathbb{Q}}_l$ and that the prime l is odd. Assume that every place in F^+ dividing l splits in F. Let S denote a finite set of finite places of F^+ which split in F, and assume that S contains every place dividing l. Let S_l denote the set of places of F^+ lying over l. Let $F(S)$ denote the maximal algebraic extension of F unramified away from S. Let $G_{F^+,S} = \mathrm{Gal}(F(S)/F^+)$ and $G_{F,S} = \mathrm{Gal}(F(S)/F)$. For each $v \in S$ choose a place $\widetilde{v}$ of F lying over v and let $\widetilde{S}$ denote the set of $\widetilde{v}$ for $v \in S$. For each place $v|\infty$ of F^+ we let c_v denote a choice of a complex conjugation at v in $G_{F^+,S}$. For each place w of F we have a $G_{F,S}$-conjugacy class of homomorphisms $G_{F_w} \to G_{F,S}$. For $v \in S$ we fix a choice of homomorphism $G_{F_{\widetilde{v}}} \to G_{F,S}$.

Fix a continuous homomorphism

$$\overline{\rho} : G_{F^+,S} \to \mathcal{G}_n(k)$$

such that $G_{F,S} = \overline{\rho}^{-1}(\mathrm{GL}_n(k) \times \mathrm{GL}_1(k))$ and fix a continuous character $\chi : G_{F^+,S} \to \mathcal{O}^\times$ such that $\nu \circ \overline{\rho} = \overline{\chi}$. Let $(\bar{r}, \overline{\chi}, \langle\ ,\ \rangle)$ be the triple corresponding to $\overline{\rho}$ (see section 3.1.1). Assume that $\bar{r}$ is absolutely irreducible. As in Definition 1.2.1 of [CHT08], we define

- a *lifting* of $\overline{\rho}$ to an object A of $\mathcal{C}_\mathcal{O}$ to be a continuous homomorphism $\rho : G_{F^+,S} \to \mathcal{G}_n(A)$ lifting $\overline{\rho}$ and with $\nu \circ \rho = \chi$;
- two liftings ρ, ρ' of $\overline{\rho}$ to A to be *equivalent* if they are conjugate by an element of $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$;
- a *deformation* of $\overline{\rho}$ to an object A of $\mathcal{C}_\mathcal{O}$ to be an equivalence class of liftings.

For each place $v \in S$, let $R^\square_{\bar{r}|_{G_{F_{\widetilde{v}}}}}$ denote the universal $\mathcal{O}$-lifting ring of $\bar{r}|_{G_{F_{\widetilde{v}}}}$ and let $R_{\widetilde{v}}$ denote a quotient of $R^\square_{\bar{r}|_{G_{F_{\widetilde{v}}}}}$ which satisfies the following property:

(*) let A be an object of $\mathcal{C}_\mathcal{O}$ and let $\zeta, \zeta' : R^\square_{\bar{r}|_{G_{F_{\widetilde{v}}}}} \to A$ be homomorphisms corresponding to two lifts r and r' of $\bar{r}|_{G_{F_{\widetilde{v}}}}$ which are conjugate by an element of $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$. Then ζ factors through $R_{\widetilde{v}}$ if and only if ζ' does.

We consider the *deformation problem*

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{\rho}, \chi, \{R_{\widetilde{v}}\}_{v\in S})$$

(see sections 2.2 and 2.3 of [CHT08] for this terminology). We say that a lifting $\rho : G_{F^+,S} \to \mathcal{G}_n(A)$ is *of type* $\mathcal{S}$ if for each place $v \in S$, the homomorphism $R^\square_{\bar{r}|_{G_{F_{\widetilde{v}}}}} \to A$ corresponding to $\rho|_{G_{F_{\widetilde{v}}}}$ factors through $R_{\widetilde{v}}$. We also define deformations of type $\mathcal{S}$ in the same way. Let $\mathrm{Def}_\mathcal{S}$ be the functor $\mathcal{C}_\mathcal{O} \to$ *Sets* which sends an algebra A to the set of deformations of $\bar{r}$ to A of type $\mathcal{S}$. By Proposition 2.2.9 of [CHT08] this functor is represented by an object $R^{\mathrm{univ}}_\mathcal{S}$ of $\mathcal{C}_\mathcal{O}$. The next lemma follows from Lemma 3.2.3 of [GG09].

Lemma 3.1.5. *Let M be a finite extension of $\mathbb{Q}_p$ for some prime p. Let $\bar{r} : G_M \to \mathrm{GL}_n(k)$ be a continuous homomorphism. If $p \neq l$, let R be a quotient of the maximal l-torsion free quotient of $R^{\square}_{\bar{r}}$ corresponding to a union of irreducible components. If $p = l$, assume that K contains the image of each embedding $M \hookrightarrow \overline{\mathbb{Q}}_l$ and let R be a quotient of $R^{\mathbf{v}_a,cr}_{\bar{r}}$, for some $a \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(M,K)}$, corresponding to a union of irreducible components. Then R satisfies property $(*)$ above.*

3.2. Relative finiteness for deformation rings: restriction and tensor products. Let m and n be positive integers, and let $l > mn$ be a rational prime. Let $F \subset F'$ be imaginary CM fields with maximal totally real subfields F^+ and $(F')^+$ respectively. Let S be a finite set of finite places of F^+ which split in F and let $\tilde{S}$ be a set of places of F consisting of exactly one place lying over each place of S. Assume that each place of F^+ which lies over l is in S (and hence splits in F). Let S' and $\tilde{S}'$ be the sets of places of $(F')^+$ and F' which lie over S and $\tilde{S}$ respectively. If v is a place in S (respectively S') we write $\tilde{v}$ for the unique place of $\tilde{S}$ (respectively $\tilde{S}'$) lying over v. Fix a choice of complex conjugation $c \in G_{(F')^+}$.

Let $K \subset \overline{\mathbb{Q}}_l$ be a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}$ and residue field k. Assume that K contains the image of every embedding $F' \hookrightarrow \overline{\mathbb{Q}}_l$.

Suppose that $\bar{r} : G_{F,S} \to \mathrm{GL}_n(k)$ and $r' : G_{F',S'} \to \mathrm{GL}_m(\mathcal{O})$ are continuous representations. Let $\bar{r}'' = \bar{r}|_{G_{F'}} \otimes_k \bar{r}' : G_{F',S'} \to \mathrm{GL}_{mn}(k)$ (we are using the conventions of Definition 3.1.1 to regard $\bar{r}''$ as a $\mathrm{GL}_{mn}(k)$-valued homomorphism). Suppose that $\chi : G_{F^+,S} \to \mathcal{O}^\times$, $\chi', \chi'' : G_{(F')^+,S'} \to \mathcal{O}^\times$ are continuous characters, and assume that

1. $\bar{r}$, $\bar{r}'$ and $\bar{r}''$ are absolutely irreducible.
2. $\bar{r}^c \cong \bar{r}^\vee \overline{\chi}|_{G_F}$.
3. $(r' \otimes_{\mathcal{O}} K)^c \cong (r' \otimes_{\mathcal{O}} K)^\vee \chi'|_{G_{F'}}$.
4. $\chi''|_{G_{F'}} = \chi|_{G_{F'}} \chi'|_{G_{F'}}$.
5. χ, χ' and χ'' are each totally odd or totally even.

The first and second assumptions imply that we can find a perfect bilinear pairing $\langle\ ,\ \rangle : k^n \times k^n \to k$, unique up to scaling by elements of $k^\times$, such that $\langle \bar{r}(\sigma)x, \bar{r}(c\sigma c)y \rangle = \overline{\chi}(\sigma)\langle x, y\rangle$ for all $\sigma \in G_F$, $x, y \in k^n$. We must have $\langle x, y\rangle = (-1)^a \langle y, x\rangle$ for some $a \in \mathbb{Z}/2\mathbb{Z}$. Note that $r' \otimes_{\mathcal{O}} K$ is absolutely irreducible by the first assumption. This and the third assumption then imply that we can find a perfect symmetric bilinear pairing $\langle\ ,\ \rangle' : K^m \times K^m \to K$, unique up to scaling by elements of $K^\times$, with $\langle r'(\sigma)x, r'(c\sigma c)y\rangle' = \chi'(\sigma)\langle x, y\rangle'$ for all $\sigma \in G_{F'}$, $x, y \in K^m$. The first assumption implies that the dual lattice of $\mathcal{O}^m$ under this pairing is $\lambda^a \mathcal{O}^m$ for some $a \in \mathbb{Z}$. Replacing $\langle\ ,\ \rangle'$ by $\alpha\langle\ ,\ \rangle'$ where $\alpha \in K$ satisfies $\lambda^a = (\alpha)$, we may assume that $\mathcal{O}^m$ is self-dual under $\langle\ ,\ \rangle'$. We necessarily have $\langle x, y\rangle' = (-1)^b \langle y, x\rangle'$ for some $b \in \mathbb{Z}/2\mathbb{Z}$. We assume

6. $\chi(c) = (-1)^{a+1}$, $\chi'(c) = (-1)^{b+1}$ and $\chi''(c) = (-1)^{a+b+1}$.

The discussion of section 3.1.1 shows that the triple $(\bar{r}, \overline{\chi}, \langle\ ,\ \rangle)$ corresponds to a homomorphism $\overline{\rho} : G_{F^+,S} \to \mathcal{G}_n(k)$ with $\overline{\rho}|_{G_F} = (\bar{r}, \overline{\chi}|_{G_F})$ and $\nu \circ \overline{\rho} = \overline{\chi}$. Similarly, the triple $(r, \chi', \langle\ ,\ \rangle')$ corresponds to a homomorphism $\rho' : G_{(F')^+,S'} \to \mathcal{G}_m(\mathcal{O})$ with $\rho'|_{G_{F'}} = (r, \chi'|_{G_{F'}})$ and $\nu \circ \rho' = \chi'$.

Note that $(\bar{r}'')^c \cong (\bar{r}'')^\vee \overline{\chi}|_{G_{F'}} \overline{\chi}'|_{G_{F'}} = (\bar{r}'')^\vee \overline{\chi}''|_{G_{F'}}$. The pairings $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ induce a perfect bilinear pairing $\langle\ ,\ \rangle'' : k^{mn} \times k^{mn} \to k$ with $\langle x \otimes y, w \otimes z\rangle'' = \langle x, w\rangle\langle y, z\rangle'$ for all $x, w \in k^n$ and $y, z \in k^m$ (we are identifying $k^n \otimes_k k^m$ with k^{mn} as above). This pairing satisfies $\langle \bar{r}''(\sigma)u, \bar{r}''(c\sigma c)v\rangle'' = \overline{\chi}''(\sigma)\langle u, v\rangle''$ for all $\sigma \in G_{F'}$,

$u, v \in k^{mn}$. Since $\langle x, y\rangle'' = (-1)^{a+b}\langle x, y\rangle''$ and $\chi''(c) = (-1)^{a+b+1}$, we see that the triple $(\bar{r}'', \overline{\chi}'', \langle\ ,\ \rangle'')$ corresponds to a homomorphism $\overline{\rho}'' : G_{(F')^+,S'} \to \mathcal{G}_{mn}(k)$ with $\overline{\rho}''|_{G_{F'}} = (\bar{r}'', \overline{\chi}|_{G_{F'}})$ and $\nu \circ \overline{\rho}'' = \overline{\chi}''$.

We now turn to deformation theory. For each place $v \in S$ (respectively $w \in S'$), let $R^{\square}_{\bar{r}|_{G_{F_{\tilde{v}}}}}$ (respectively $R^{\square}_{\bar{r}''|_{G_{F'_{\tilde{w}}}}}$) be the universal $\mathcal{O}$-lifting ring of $\bar{r}|_{G_{F_{\tilde{v}}}}$ (respectively $\bar{r}''|_{G_{F'_{\tilde{w}}}}$). Consider the deformation problem

$$\mathcal{S}_0 = (F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{\rho}, \chi, \{R^{\square}_{\bar{r}|_{G_{F_{\tilde{v}}}}}\}_{v\in S}).$$

See section 3.1.5 for this notation. Let $R^{\mathrm{univ}}_{\mathcal{S}_0}$ be the $\mathcal{O}$-algebra representing the corresponding deformation functor, and let $\rho^{\mathrm{univ}}_{\mathcal{S}_0} : G_{F^+,S} \to \mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{S}_0})$ be a representative of the universal deformation. Similarly, consider the deformation problem

$$\mathcal{S}''_0 = (F'/(F')^+, S', \widetilde{S}', \mathcal{O}, \overline{\rho}'', \chi'', \{R^{\square}_{\bar{r}''|_{G_{F'_{\tilde{w}}}}}\}_{w\in S'}).$$

Let $R^{\mathrm{univ}}_{\mathcal{S}''_0}$ be the $\mathcal{O}$-algebra representing the corresponding deformation functor, and let $\rho^{\mathrm{univ}}_{\mathcal{S}''_0} : G_{(F')^+,S'} \to \mathcal{G}_{mn}(R^{\mathrm{univ}}_{\mathcal{S}''_0})$ be a representative for the universal deformation.

Suppose we are given an object A of $\mathcal{C}_{\mathcal{O}}$ and a continuous homomorphism $\rho_A : G_{F^+,S} \to \mathcal{G}_n(A)$ lifting $\overline{\rho}$ and with $\nu \circ \rho_A = \chi$. Let $(r_A, \chi, \langle\ ,\ \rangle_A)$ be the corresponding triple (see section 3.1.1). Identifying $A^n \otimes_A A^m$ with A^{mn} as in Definition 3.1.1, we obtain a homomorphism $r''_A := (r_A|_{G_{F'}}) \otimes_A (r' \otimes_{\mathcal{O}} A) : G_{F'} \to \mathrm{GL}_{mn}(A)$ and a perfect bilinear pairing $\langle\ ,\ \rangle''_A : A^{mn} \times A^{mn} \to A$ with $\langle x \otimes y, w \otimes z\rangle''_A = \langle x, w\rangle_A \langle y, z\rangle'$ for all $x, w \in A^n$ and $y, z \in A^m$. This pairing satisfies $\langle r''_A(\sigma)u, r''_A(c\sigma c)v\rangle''_A = \chi''(\sigma)\langle u, v\rangle''_A$ for all $\sigma \in G_{F'}$, $u, v \in A^{mn}$. The triple $(r''_A, \chi'', \langle\ ,\ \rangle''_A)$ therefore gives rise to a homomorphism $\rho''_A : G_{(F')^+,S'} \to \mathcal{G}_{mn}(A)$ lifting $\overline{\rho}''$. We also denote ρ''_A by $(\rho_A|_{G_{(F')^+}}) \otimes (\rho' \otimes_{\mathcal{O}} A)$. Taking $A = R^{\mathrm{univ}}_{\mathcal{S}_0}$ and $\rho_A = \rho^{\mathrm{univ}}_{\mathcal{S}_0}$, we obtain an $\mathcal{O}$-algebra homomorphism

$$\theta : R^{\mathrm{univ}}_{\mathcal{S}''_0} \to R^{\mathrm{univ}}_{\mathcal{S}_0}$$

with the property that the homomorphisms $\theta \circ \rho^{\mathrm{univ}}_{\mathcal{S}''_0}$ and $(\rho^{\mathrm{univ}}_{\mathcal{S}_0}|_{G_{(F')^+}}) \otimes (\rho' \otimes_{\mathcal{O}} R^{\mathrm{univ}}_{\mathcal{S}_0})$ are $\ker(\mathrm{GL}_{mn}(R^{\mathrm{univ}}_{\mathcal{S}_0}) \to \mathrm{GL}_{mn}(k))$-conjugate.

Results of the following type appear in the work of Khare-Wintenberger and de Jong.

Lemma 3.2.1. *Let*

- $\bar{r} : G_{F,S} \to \mathrm{GL}_n(k)$, $r' : G_{F',S'} \to \mathrm{GL}_m(\mathcal{O})$, $\bar{r}'' = \bar{r}|_{G_{F'}} \otimes \bar{r}' : G_{F',S} \to \mathrm{GL}_{mn}(k)$
- $\chi : G_{F^+,S} \to \mathcal{O}^\times$, $\chi' : G_{(F')^+,S'} \to \mathcal{O}^\times$, *and* $\chi'' : G_{(F')^+,S'} \to \mathcal{O}^\times$

be continuous homomorphisms satisfying assumptions (1)-(6) above. Then the homomorphism $\theta : R^{\mathrm{univ}}_{\mathcal{S}''_0} \to R^{\mathrm{univ}}_{\mathcal{S}_0}$ *constructed above is finite.*

Proof. Let $\wp$ denote a prime ideal of $R^{\mathrm{univ}}_{\mathcal{S}_0}/\theta(\mathfrak{m}_{R^{\mathrm{univ}}_{\mathcal{S}''_0}})$ and let $\bar{R} = (R^{\mathrm{univ}}_{\mathcal{S}_0}/\theta(\mathfrak{m}_{R^{\mathrm{univ}}_{\mathcal{S}''_0}}))/\wp$. It suffices to show that $\bar{R}$ is a finite k-algebra (for any choice of $\wp$), since if this holds then we see that $R^{\mathrm{univ}}_{\mathcal{S}_0}/\theta(\mathfrak{m}_{R^{\mathrm{univ}}_{\mathcal{S}''_0}})$ is a Noetherian ring of dimension 0 and hence is Artinian. The result then follows from the topological version of Nakayama's lemma.

Let $\rho_{\bar{R}} = \rho^{\mathrm{univ}}_{\mathcal{S}_0} \otimes_{R^{\mathrm{univ}}_{\mathcal{S}_0}} \bar{R}$ and $\rho''_{\bar{R}} = (\rho_{\bar{R}}|_{G_{(F')^+}}) \otimes (\overline{\rho}' \otimes_k \bar{R})$. Note that $\rho''_{\bar{R}}$ is $\ker(\mathrm{GL}_{mn}(\bar{R}) \to \mathrm{GL}_{mn}(k))$ conjugate to $\overline{\rho}'' \otimes_k \bar{R}$ and in particular has finite image.

Let $r_{\bar{R}} : G_F \to \mathrm{GL}_n(\bar{R})$ denote $\rho_{\bar{R}}|_{G_F}$ composed with the projection $\mathrm{GL}_n(\bar{R}) \times \mathrm{GL}_1(\bar{R}) \to \mathrm{GL}_n(\bar{R})$. Then $r_{\bar{R}}|_{G_{F'}} \otimes \bar{r}' : G_{F'} \to \mathrm{GL}_{mn}(\bar{R})$ has finite image. From this it follows easily that $r_{\bar{R}}$ has finite image. However, Lemma 2.1.12 of [CHT08] implies that $\bar{R}$ is topologically generated as a k-algebra by $\mathrm{tr}(r_{\bar{R}}(G_F))$. Since $r_{\bar{R}}$ has finite image, we deduce that for each $\sigma \in G_F$, $\mathrm{tr}(r_{\bar{R}}(\sigma))$ is a sum of roots of unity of bounded order. It follows that $\bar{R}$ is a finite k-algebra, as required. $\square$

We also consider refined deformation problems as follows. For each place $v \in S$ let $R_{\tilde{v}}$ be a quotient of $R^{\square}_{\bar{r}|_{G_{F_{\tilde{v}}}}}$ satisfying the condition (*) of section 3.1.5. For each place $w \in S'$, let $R_{\tilde{w}}$ be a quotient of $R^{\square}_{\bar{r}''|_{G_{F'_{\tilde{w}}}}}$ satisfying the same property (*). Furthermore, if v denotes the place of S lying under w, assume that given any object A of $\mathcal{C}_{\mathcal{O}}$ and any $\mathcal{O}$-algebra homomorphism $R_{\tilde{v}} \to A$ corresponding to a lift r_A of $\bar{r}|_{G_{F_{\tilde{v}}}}$, the lift $r_A|_{G_{F'_{\tilde{w}}}} \otimes_A (r'|_{G_{F'_{\tilde{w}}}} \otimes_{\mathcal{O}} A)$ (regarded as a homomorphism to $\mathrm{GL}_{mn}(A)$ using the conventions of Definition 3.1.5) of $\bar{r}''|_{G_{F'_{\tilde{w}}}}$ gives rise to an $\mathcal{O}$-algebra homomorphism $R^{\square}_{\bar{r}''|_{G_{F'_{\tilde{w}}}}} \to A$ which factors through $R_{\tilde{w}}$. We let

$$\begin{aligned}\mathcal{S} &= (F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{\rho}, \chi, \{R_{\tilde{v}}\}_{v\in S})\\ \mathcal{S}'' &= (F'/(F')^+, S', \widetilde{S}', \mathcal{O}, \overline{\rho}'', \chi'', \{R_{\tilde{w}}\}_{w\in S'})\end{aligned}$$

and let $R^{\mathrm{univ}}_{\mathcal{S}}$ and $R^{\mathrm{univ}}_{\mathcal{S}''}$ be objects representing the corresponding deformation problems. The compatibility between the rings $R_{\tilde{v}}$ and $R_{\tilde{w}}$ for $v \in S$, $w \in S'$ implies that the map $\theta : R^{\mathrm{univ}}_{\mathcal{S}''_0} \to R^{\mathrm{univ}}_{\mathcal{S}_0}$ gives rise to a map

$$\overline{\theta} : R^{\mathrm{univ}}_{\mathcal{S}''} \to R^{\mathrm{univ}}_{\mathcal{S}}.$$

The following result follows immediately from 3.2.1.

Lemma 3.2.2. *The map $\overline{\theta} : R^{\mathrm{univ}}_{\mathcal{S}''} \to R^{\mathrm{univ}}_{\mathcal{S}}$ is finite.*

3.3. Relative finiteness for deformation rings: induction. The results of this section are not needed in this paper, but are used in [BLGGT10].

Let n be a positive integer. Let F_1 and F_2 be imaginary CM fields with $F_1^+ \subset F_2^+$. Let $m := [F_2^+ : F_1^+]$. Assume also that every prime of F_i^+ above l splits in F_i for $i = 1, 2$. For $i = 1, 2$, let S_i be a finite set of finite places of F_i^+ which split in F_i with every place above l being contained in S_i. Let $\widetilde{S}_i$ be a set of places of F_i consisting of exactly one place of F_i lying over each place of S_i. If $v \in S_i$, we denote by the $\widetilde{v}$ the unique place of $\widetilde{S}_i$ lying over v. Assume that S_1 contains the restriction to F_1^+ of every place in S_2. Assume also that S_1 contains every prime of F_1^+ that ramifies in F_2. Fix a choice of complex conjugation $c \in G_{F_2^+}$.

Let $K \subset \overline{\mathbb{Q}}_l$ be a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}$ and residue field k. Assume that K contains the image of every embedding $F_2 \hookrightarrow \overline{\mathbb{Q}}_l$.

Let

$$\begin{aligned}\bar{r}_2 &: G_{F_2,S_2} \to \mathrm{GL}_n(k)\\ \chi &: G_{F_1^+,S_1} \to \mathcal{O}^\times\end{aligned}$$

be continuous representations and let

$$\bar{r}_1 = \mathrm{Ind}_{G_{F_2}}^{G_{F_1^+}} \bar{r}_2 : G_{F_1^+,S_1} \to \mathrm{GL}_{2mn}(k).$$

Assume that

(1) $\bar{r}_1|_{G_{F_1}}$ and $\bar{r}_2$ are absolutely irreducible.
(2) $\bar{r}_2^c \cong \bar{r}_2^\vee \overline{\chi}|_{G_{F_2}}$
(3) χ is totally odd or totally even.

Choose a perfect bilinear pairing $\overline{\psi}_2 : k^n \otimes k^n \to k$ such that

$$\overline{\psi}_2(\bar{r}_2(\sigma)x, \bar{r}_2(c\sigma c)y) = \overline{\chi}(\sigma)\overline{\psi}_2(x,y)$$

for all $x, y \in k^n$ and $\sigma \in G_{F_2}$. (Such a pairing exists and is unique up to scaling as $\bar{r}_2$ is absolutely irreducible.) Assume further that

(4) $\overline{\psi}_2(x,y) = -\overline{\chi}(c)\overline{\psi}_2(y,x)$ for all $x, y \in k^n$.

Then the triple $(\bar{r}_2, \chi|_{G_{F_2^+}}, \overline{\psi}_2)$ gives rise to a homomorphism $\overline{\rho}_2 : G_{F_2^+,S_2} \to \mathcal{G}_n(k)$ with $G_{F_2,S_2} = \overline{\rho}_2^{-1}(\mathrm{GL}_n(k) \times \mathrm{GL}_1(k))$.

For each $v \in S_2$, let $R^{\square}_{\bar{r}_2|_{G_{F_{2,\tilde{v}}}}}$ denote the universal $\mathcal{O}$-lifting ring of $\bar{r}_2|_{G_{F_{2,\tilde{v}}}}$. Let $\mathcal{S}_2$ denote the deformation problem

$$\mathcal{S}_2 = (F_2/F_2^+, S_2, \widetilde{S}_2, \mathcal{O}, \overline{\rho}_2, \chi|_{G_{F_2^+}}, \{R^{\square}_{\bar{r}_2|_{G_{F_{2,v}}}}\}).$$

Let $\rho_2^{\mathrm{univ}} : G_{F_2^+} \to \mathcal{G}_n(R^{\mathrm{univ}}_{\mathcal{S}_2})$ represent the universal deformation of type $\mathcal{S}_2$ and let $(r_2^{\mathrm{univ}}, \chi|_{G_{F_2^+}}, \psi_2^{\mathrm{univ}})$ be the corresponding triple. Define a pairing ψ_1 on $r_1 := \mathrm{Ind}_{G_{F_2}}^{G_{F_1^+}} r_2^{\mathrm{univ}}$ by setting

$$\psi_1(f,g) = \sum_{\sigma \in G_{F_2} \backslash G_{F_1^+}} \overline{\chi}(\sigma)^{-1}\overline{\psi}_2(f(\sigma), g(c\sigma))$$

for all $f, g \in r_1$. This pairing is perfect and alternating and satisfies

$$\psi_1(r_1(\tau)f, r_1(\tau)g) = \chi(\tau)\psi_1(f,g)$$

for all $\tau \in G_{F_1^+}$ and $f, g \in r_1$. By Lemma 2.1.2 of [CHT08], there is a continuous homomorphism $\rho_1 : G_{F_1^+,S_1} \to \mathcal{G}_{2mn}(R^{\mathrm{univ}}_{\mathcal{S}_2})$ with

- $G_{F_1,S_1} = \rho_1^{-1}(\mathrm{GL}_n(R^{\mathrm{univ}}_{\mathcal{S}_2}) \times \mathrm{GL}_1(R^{\mathrm{univ}}_{\mathcal{S}_2}))$;
- $\rho_1|_{G_{F_1}} = (r_1|_{G_{F_1}}, \chi|_{G_{F_1}})$;
- $\rho_1(c) = (r_1(c)J^{-1}, -\chi(c))j$, where $J \in M_{2mn}(R^{\mathrm{univ}}_{\mathcal{S}_2})$ is the matrix with $\psi_1(x,y) = {}^t x J y$;
- $\nu \circ \rho_1 = \chi$.

Let $\overline{\rho}_1 = \rho_1 \mod \mathfrak{m}_{R^{\mathrm{univ}}_{\mathcal{S}_2}} : G_{F_1^+} \to \mathcal{G}_{2mn}(k)$. (Note that $\overline{\rho}_1$ corresponds to the triple $(\bar{r}_1|_{G_{F_1}}, \overline{\chi}, \overline{\psi}_1')$ where $\overline{\psi}_1'$ is the pairing associated to the matrix $\overline{J}\bar{r}_1(c)$.)

For each place $v \in S_1$, let $R^{\square}_{\bar{r}_1|_{G_{F_{1,\tilde{v}}}}}$ denote the universal $\mathcal{O}$-lifting ring of $\bar{r}_2|_{G_{F_{2,\tilde{v}}}}$. Let $\mathcal{S}_1$ denote the deformation problem

$$\mathcal{S}_1 = (F_1/F_1^+, S_1, \widetilde{S}_1, \mathcal{O}, \overline{\rho}_1, \{R^{\square}_{\bar{r}_1|_{G_{F_{1,v}}}}\}).$$

The lift ρ_1 of $\overline{\rho}_1$ is of type $\mathcal{S}_1$, and hence gives rise to a map $\iota : R^{\mathrm{univ}}_{\mathcal{S}_1} \to R^{\mathrm{univ}}_{\mathcal{S}_2}$ (note that $R^{\mathrm{univ}}_{\mathcal{S}_1}$ exists as $\bar{r}_1|_{G_{F_1}}$ is absolutely irreducible).

Lemma 3.3.1. *Let*

- $\bar{r}_2 : G_{F_2,S_2} \to \mathrm{GL}_n(k)$,
- $\bar{r}_1 := \mathrm{Ind}_{G_{F_2}}^{G_{F_1^+}} \bar{r}_2 : G_{F_1^+,S_1} \to \mathrm{GL}_{2mn}(k)$, *and*

- $\overline{\chi} : G_{F_1^+} \to \mathcal{O}^\times$

be continuous homomorphisms satisfying assumptions (1)-(4) above. Then the homomorphism $\iota : R^{\mathrm{univ}}_{\mathcal{S}_1} \to R^{\mathrm{univ}}_{\mathcal{S}_2}$ *constructed above is finite.*

Proof. Let $\wp$ denote a prime ideal of $R^{\mathrm{univ}}_{\mathcal{S}_2}/\iota(\mathfrak{m}_{R^{\mathrm{univ}}_{\mathcal{S}_1}})$. As in the proof of Lemma 3.2.1, it suffices to show that $\bar{R} := (R^{\mathrm{univ}}_{\mathcal{S}_2}/\iota(\mathfrak{m}_{R^{\mathrm{univ}}_{\mathcal{S}_1}}))/\wp$ is a finite k-algebra. Let $r_{2,\bar{R}} = r_2^{\mathrm{univ}} \otimes_{R^{\mathrm{univ}}_{\mathcal{S}_2}} \bar{R} : G_{F_2} \to \mathrm{GL}_n(\bar{R})$. As in the proof of Lemma 3.2.1 again, it suffices to show that $r_{2,\bar{R}}$ has finite image. However, $(\mathrm{Ind}_{G_{F_2}}^{G_{F_1^+}} r_{2,\bar{R}})|_{G_{F_1}} : G_{F_1} \to \mathrm{GL}_{2mn}(\bar{R})$ is equivalent to $\bar{r}_1|_{G_{F_1}} \otimes_k \bar{R}$ and hence has finite image. It follows easily that $r_{2,\bar{R}}$ has finite image. $\square$

3.4. Finiteness of a deformation ring. Let F be a CM field with maximal totally real subfield F^+. Let n be a positive integer and $l > n$ an odd prime such that $\zeta_l \notin F$. Let S be a set of places of F^+ which split in F, and assume that S contains all places of F^+ lying over l. Let $\tilde{S}$ be a set of places of F consisting of exactly one place lying over each place of S. If v is a place in S we write $\tilde{v}$ for the unique place of $\tilde{S}$ lying over v. Fix a choice of complex conjugation $c \in G_{F^+}$.

Let $K \subset \overline{\mathbb{Q}}_l$ be a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}$ and residue field k. Assume that K contains the image of every embedding $F \hookrightarrow \overline{\mathbb{Q}}_l$.

Suppose that $\bar{r} : G_F \to \mathrm{GL}_n(k)$ is a continuous absolutely irreducible representation which is unramified at all places not lying over a place in S. Suppose that $\chi : G_{F^+} \to \mathcal{O}^\times$ is a continuous totally odd crystalline character, unramified away from S, such that

$$\bar{r}^c \cong \bar{r}^\vee \overline{\chi}|_{G_F}.$$

Assume also that $\bar{r}$ is odd in the sense of section 3.1.2. Thus, there exists a non-degenerate symmetric bilinear pairing $\langle\ ,\ \rangle : k^n \times k^n \to k$ such that $\langle \bar{r}(\sigma)x, \bar{r}(c\sigma c)y\rangle = \overline{\chi}(\sigma)\langle x, y\rangle$ for all $\sigma \in G_F$, $x, y \in k^n$. The triple $(\bar{r}, \overline{\chi}, \langle\ ,\ \rangle)$ then corresponds to a continuous homomorphism $\overline{\rho} : G_{F^+} \to \mathcal{G}_n(k)$ (see section 3.1.1).

For each place $v \in S$ not dividing l, assume that for every minimal prime ideal $\wp$ of $R^\square_{\bar{r}|_{G_{F_{\tilde{v}}}}}$, the quotient $R^\square_{\bar{r}|_{G_{F_{\tilde{v}}}}}/\wp$ is geometrically integral. Note that this can be achieved by replacing K by a finite extension. Let $R_{\tilde{v}}$ be one such irreducible component of $R^\square_{\bar{r}|_{G_{F_{\tilde{v}}}}}$ which is of characteristic 0. For each place $v \in \tilde{S}$ dividing l, fix an element $a_{\tilde{v}} \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F_{\tilde{v}}, \overline{\mathbb{Q}}_l)}$ with corresponding l-adic Hodge type $\mathbf{v}_{a_{\tilde{v}}}$ and assume that the ring $R^{\mathbf{v}_{a_{\tilde{v}}}, cr}_{\bar{r}|_{G_{F_{\tilde{v}}}}}$ is non-zero and moreover that for every minimal prime $\wp$, the irreducible component $R^{\mathbf{v}_{a_{\tilde{v}}}, cr}_{\bar{r}|_{G_{F_{\tilde{v}}}}}/\wp$ is geometrically integral. Let $R_{\tilde{v}}$ be one of these irreducible components. Consider the deformation problem

$$\mathcal{S} = (F/F^+, S, \tilde{S}, \mathcal{O}, \overline{\rho}, \chi, \{R_{\tilde{v}}\}_{v \in S})$$

(see section 3.1.5 for this terminology) and let $R^{\mathrm{univ}}_{\mathcal{S}}$ represent the corresponding deformation functor. Note that the rings $R_{\tilde{v}}$ satisfy property (*) of section 3.1.5 by Lemma 3.1.5.

Lemma 3.4.1. *Suppose that*

(i) $\bar{r}|_{G_{F(\zeta_l)}}$ *has big image.*
(ii) $\bar{F}^{\ker \mathrm{ad}\, \bar{r}}$ *does not contain* ζ_l

Suppose that there is a continuous representation $r' : G_F \to \mathrm{GL}_n(\mathcal{O})$ lifting $\bar{r}$ such that:

(1) There is an isomorphism $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$, a RAECSDC automorphic representation π' of $\mathrm{GL}_n(\mathbb{A}_F)$ and a continuous character $\mu' : \mathbb{A}_{F^+}^\times/(F^+)^\times$ such that
- $r_{l,\iota}(\pi') \cong r' \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$
- $(\pi')^c \cong (\pi')^\vee \otimes (\mu' \circ \mathbf{N}_{F/F^+} \circ \det)$
- $\chi' := \epsilon^{1-n} r_{l,\iota}(\mu')$ *is a lift of* $\overline{\chi}$. *(Note that* $(r')^c \cong (r')^\vee \chi'|_{G_F}$*.)*
- π' *is unramified above* l *and outside the set of places of* F *lying above* S.

(2) For each place $v \in S$, $r'|_{G_{F_{\tilde{v}}}}$ corresponds to a closed point of $R_{\tilde{v}}[1/l]$. Moreover, if $v \nmid l$ then $r'|_{G_{F_{\tilde{v}}}}$ does not give rise to a closed point on any other irreducible component of $R^{\square}_{\bar{r}|_{G_{F_{\tilde{v}}}}}[1/l]$.

Then the universal deformation ring $R^{\mathrm{univ}}_{\mathcal{S}}$ defined above is finite over $\mathcal{O}$. Moreover $R^{\mathrm{univ}}_{\mathcal{S}}[1/l]$ is non-zero and any $\overline{\mathbb{Q}}_l$-point of this ring gives rise to a representation of G_F which is automorphic of level prime to l.

Proof. When μ' is trivial, the result follows from Proposition 3.6.3 of [BLGG09]. We essentially reduce to this case by a twisting argument:

Step 1: Twisting. Choose a non-degenerate symmetric bilinear pairing $\langle\ ,\ \rangle : \mathcal{O}^n \times \mathcal{O}^n \to \mathcal{O}$ lifting the pairing also denoted $\langle\ ,\ \rangle$ on k^n and such that $\langle r'(\sigma)x, r'(c\sigma c)y\rangle = \chi'(\sigma)\langle x, y\rangle$ for all $\sigma \in G_F$ and $x, y \in \mathcal{O}^n$.

By Lemma 4.1.5 of [CHT08], extending K if necessary, we can and do choose a continuous algebraic character $\psi : G_F \to \mathcal{O}^\times$ such that (i) $\psi\psi^c = (\chi^{-1}\epsilon^{1-n})|_{G_F}$, and (ii) ψ is crystalline above l. By Lemma 4.1.6 of [CHT08], again extending K if necessary, we can and do choose a continuous algebraic character $\psi' : G_F \to \mathcal{O}^\times$ such that (i) $\psi'(\psi')^c = ((\chi')^{-1}\epsilon^{1-n})|_{G_F}$, (ii) ψ' is crystalline above l, (iii) $\psi'|_{I_{F_{\tilde{v}}}} = \psi|_{I_{F_{\tilde{v}}}}$ for each $\tilde{v} \in \tilde{S}$, and (iv) $\overline{\psi}' = \overline{\psi}$.

Let $r'_{\psi'} = r' \otimes \psi'$. Note that $\langle r'_{\psi'}(\sigma)x, r'_{\psi'}(c\sigma c)y\rangle = (\psi'(\psi')^c\chi')(\sigma)\langle x, y\rangle = \epsilon^{1-n}(\sigma)\langle x, y\rangle$ for all $\sigma \in G_F$ and $x, y \in \mathcal{O}^n$. Let $\bar{r}_{\overline{\psi}} = \bar{r} \otimes \overline{\psi} = \bar{r}'_{\overline{\psi}'}$. The triple $(\bar{r}_{\overline{\psi}}, \bar{\epsilon}^{1-n}\delta^n_{F/F^+}, \langle\ ,\ \rangle)$ corresponds to a continuous homomorphism $\overline{\rho}_{\overline{\psi}} : G_{F^+} \to \mathcal{G}_n(k)$. The triple $(r'_{\psi'}, \epsilon^{1-n}\delta^n_{F/F^+}, \langle\ ,\ \rangle)$ gives rise to a homomorphism $\rho'_{\psi'} : G_{F^+} \to \mathcal{G}_n(\mathcal{O})$ lifting $\overline{\rho}_{\overline{\psi}}$.

For each $v \in S$, twisting by $\psi|_{G_{F_{\tilde{v}}}}$ defines an isomorphism between the lifting problems for $\bar{r}|_{G_{F_{\tilde{v}}}}$ and $\bar{r}_{\overline{\psi}}|_{G_{F_{\tilde{v}}}}$. This gives rise to isomorphisms $R^{\square}_{\bar{r}_{\overline{\psi}}|_{G_{F_{\tilde{v}}}}} \xrightarrow{\sim} R^{\square}_{\bar{r}|_{G_{F_{\tilde{v}}}}}$. For each $v \in S$, let $R_{\tilde{v},\psi}$ be the irreducible quotient of $R^{\square}_{\bar{r}_{\overline{\psi}}|_{G_{F_{\tilde{v}}}}}$ corresponding to $R_{\tilde{v}}$ under this isomorphism. Note that $r'_{\psi'}|_{G_{F_{\tilde{v}}}}$ gives rise to a closed point of $\operatorname{Spec} R^{\square}_{\bar{r}_{\overline{\psi}}|_{G_{F_{\tilde{v}}}}}[1/l]$ contained in $\operatorname{Spec} R_{\tilde{v},\psi}[1/l]$ and no other irreducible component. (This follows easily from the fact that $(\psi'/\psi)|_{G_{F_{\tilde{v}}}}$ is unramified and has trivial reduction.)

Choose a finite solvable totally real extension L^+/F^+ such that, if we set $L = FL^+$, then the following hold:

- $\psi|_{G_L}$ and $\psi'|_{G_L}$ are unramified at all places not lying above S.
- Each place $v \in S$ splits completely in L^+.
- L is linearly disjoint from $\overline{F}^{\ker \bar{r}}(\zeta_l)$ over F.

Let S_L and $\tilde{S}_L$ denote the set of places of L^+ and L above S and $\tilde{S}$ respectively. For each $w \in S_L$, let $\tilde{w}$ denote the unique place of $\tilde{S}_L$ lying above w. For $w \in S_L$ lying over $v \in S$, let $R_{\tilde{w},\psi}$ denote $R_{\tilde{v},\psi}$ considered as a quotient of $R^{\square}_{\bar{r}_{\overline{\psi}}|_{G_{L_{\tilde{w}}}}}$. Let

$$\mathcal{S}_{L,\overline{\psi}} = (L/L^+, S_L, \tilde{S}_L, \mathcal{O}, \overline{\rho}_{\overline{\psi}}|_{G_{L^+}}, \epsilon^{1-n}\delta^n_{L/L^+}, \{R_{\tilde{w},\psi}\}_{w\in S_L})$$

and let $R^{\mathrm{univ}}_{\mathcal{S}_{L,\overline{\psi}}}$ represent the corresponding deformation functor. For $w \in S_L$, let $R_{\tilde{w}}$ denote $R_{\tilde{v}}$ regarded as a quotient of $R^{\square}_{\bar{r}|_{G_{L_{\tilde{w}}}}}$. Let $\mathcal{S}_L$ denote the restriction of the deformation problem $\mathcal{S}$ to L/L^+:

$$\mathcal{S}_L = (L/L^+, S_L, \tilde{S}_L, \mathcal{O}, \overline{\rho}|_{G_{L^+}}, \chi|_{G_{L^+}}, \{R_{\tilde{w}}\}_{w\in S_L}).$$

Twisting by $\psi|_{G_L}$ defines an isomorphism between the deformation problems associated to $\mathcal{S}_L$ and $\mathcal{S}_{L,\overline{\psi}}$ and hence we have an isomorphism $R^{\mathrm{univ}}_{\mathcal{S}_{L,\overline{\psi}}} \xrightarrow{\sim} R^{\mathrm{univ}}_{\mathcal{S}_L}$.

Step 2: Finishing off. Proposition 3.6.3 of [BLGG09] implies that $R^{\mathrm{univ}}_{\mathcal{S}_{L,\overline{\psi}}}$ is a finite $\mathcal{O}$-algebra and that any $\overline{\mathbb{Q}}_l$-point of $R^{\mathrm{univ}}_{\mathcal{S}_{L,\overline{\psi}}}$ gives rise to a representation which is automorphic of level prime to l. By Lemma 3.2.1, the natural map $R^{\mathrm{univ}}_{\mathcal{S}_L} \to R^{\mathrm{univ}}_{\mathcal{S}}$ (coming from restriction to G_{L^+}) is finite. It follows that $R^{\mathrm{univ}}_{\mathcal{S}}$ is finite over $\mathcal{O}$ and that any $\overline{\mathbb{Q}}_l$-point of $R^{\mathrm{univ}}_{\mathcal{S}}$ gives rise to a representation $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Z}}_l)$ whose restriction to G_L is automorphic of level prime to l. Since L/F is solvable, Lemma 1.4 of [BLGHT09] implies that any such r is automorphic. Moreover, since each place of F lying above l splits completely in L, we see that such an r must be automorphic of level prime to l. Finally, Lemma 3.2.4 of [GG09] shows that the Krull dimension of $R^{\mathrm{univ}}_{\mathcal{S}}$ is at least 1. Since $R^{\mathrm{univ}}_{\mathcal{S}}$ is finite over $\mathcal{O}$, we deduce that $R^{\mathrm{univ}}_{\mathcal{S}}[1/l] \neq 0$. □

3.5. A lifting result. Let m and n be positive integers and let $l > mn$ be an odd prime. Let F be a CM field with maximal totally real subfield F^+ Let S_l be the set of places of F^+ which lie over l, and assume that they all split in F. Let $\tilde{S}_l$ be a set of places of F consisting of exactly one place lying over each place of S_l. Let $(F')^+/F^+$ be a finite extension of totally real fields, let $F' = (F')^+F$, and let S'_l (respectively $\tilde{S}'_l$) be the set of places of $(F')^+$ (respectively F') lying over places in S_l (respectively $\tilde{S}_l$). If v is a place in S_l (respectively S'_l) we write $\tilde{v}$ for the unique place of $\tilde{S}_l$ (respectively $\tilde{S}'_l$) lying over v. Finally, assume that $\zeta_l \notin F'$.

Let $K \subset \overline{\mathbb{Q}}_l$ be a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}$ and residue field k. Assume that K contains the image of every embedding $F' \hookrightarrow \overline{\mathbb{Q}}_l$.

Fix a continuous absolutely irreducible representation $\bar{r} : G_F \to \mathrm{GL}_n(k)$. For each place $v \in \tilde{S}_l$, fix an element $a_{\tilde{v}} \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F_{\tilde{v}},\overline{\mathbb{Q}}_l)}$ with corresponding l-adic Hodge type $\mathbf{v}_{a_{\tilde{v}}}$. Assume that the ring $R^{\mathbf{v}_{a_{\tilde{v}}},cr}_{\bar{r}|_{G_{F_{\tilde{v}}}}}$ is non-zero and moreover that for every minimal prime $\wp$, the irreducible component $R^{\mathbf{v}_{a_{\tilde{v}}},cr}_{\bar{r}|_{G_{F_{\tilde{v}}}}}/\wp$ is geometrically integral (this can always be arranged by extending K). Fix an irreducible component $R_{\tilde{v}}$ of $R^{\mathbf{v}_{a_{\tilde{v}}},cr}_{\bar{r}|_{G_{F_{\tilde{v}}}}}$.

Theorem 3.5.1. *Use the assumptions and notation established above. Suppose that there exist continuous crystalline totally odd characters $\chi : G_{F^+} \to \mathcal{O}^\times$, $\chi', \chi'' : G_{(F')^+} \to \mathcal{O}^\times$ and continuous representations $r' : G_{F'} \to \mathrm{GL}_m(\mathcal{O})$ and $r'' : G_{F'} \to \mathrm{GL}_{nm}(\mathcal{O})$ such that:*

(1) $\bar{r}^c \cong \bar{r}^\vee \overline{\chi}|_{G_F}$ *and* $\bar{r}$ *is odd.*
(2) $\bar{r}, r', r'', \chi, \chi'$ *and* χ'' *are all unramified away from* l*.*
(3) $(r')^c \cong (r')^\vee \chi'|_{G_{F'}}$*.*
(4) r' *is crystalline at all places of* F' *lying over* l*.*
(5) r'' *is automorphic of level prime to* l *and* $(r'')^c \cong (r'')^\vee \chi''|_{G_{F'}}$*. Note that* r'' *is necessarily crystalline at all places of* F' *dividing* l*.*
(6) $\bar{r}'' = \bar{r}|_{G_{F'}} \otimes_k \bar{r}'$ *and* $\chi''|_{G_{F'}} = \chi|_{G_{F'}} \chi'|_{G_{F'}}$*.*
(7) $\bar{r}''|_{G_{F'(\zeta_l)}}$ *has big image.*
(8) $\bar{F}^{\ker \operatorname{ad} \bar{r}''}$ *does not contain* ζ_l*.*
(9) For all places $w \in S'_l$*, lying over a place* v *of* F^+*, if* $\rho_v : G_{F_{\tilde{v}}} \to \mathrm{GL}_n(\mathcal{O})$ *is a lifting of* $\bar{r}|_{G_{F_{\tilde{v}}}}$ *corresponding to a closed point on* $\operatorname{Spec} R_{\tilde{v}}[1/l]$*, then* $(\rho_v|_{G_{F'_{\tilde{w}}}}) \otimes_{\mathcal{O}} r'|_{G_{F'_{\tilde{w}}}} \sim r''|_{G_{F'_{\tilde{w}}}}$ *(see Definition 3.1.4, noting that we are using Definition 3.1.1 to regard the tensor product as a lift of* $\bar{r}''$*). Note that if this is true for one such* ρ_v *then it is true for all choices of* ρ_v*.*

Then, after possibly extending $\mathcal{O}$*, there is a continuous lifting* $r : G_F \to \mathrm{GL}_n(\mathcal{O})$ *of* $\bar{r}$ *such that:*

(a) $r^c \cong r^\vee \chi|_{G_F}$*.*
(b) r *is unramified at all places of* F *not lying over* l*.*
(c) For each place $v \in \tilde{S}_l$*,* $r|_{G_{F_{\tilde{v}}}}$ *corresponds to a closed point on* $\operatorname{Spec} R_{\tilde{v}}[1/l]$*.*
(d) $r|_{G_{F'}} \otimes_{\mathcal{O}} r'$ *is automorphic of level prime to* l*.*

Proof. Choose a complex conjugation $c \in G_{(F')^+}$. We can choose a symmetric bilinear pairing $\langle\ ,\ \rangle : k^n \times k^n \to k$, unique up to scaling by elements of $k^\times$ such that $\langle \bar{r}(\sigma)x, \bar{r}(c\sigma c)\rangle = \overline{\chi}(\sigma)\langle x, y\rangle$ for all $\sigma \in G_F$, $x, y \in k^n$. The triple $(\bar{r}, \overline{\chi}, \langle\ ,\ \rangle)$ gives rise to a homomorphism $\overline{\rho} : G_{F^+} \to \mathcal{G}_n(k)$. Let $\mathcal{S}$ be the deformation problem

$$\mathcal{S} = (F/F^+, S_l, \tilde{S}_l, \mathcal{O}, \overline{\rho}, \chi, \{R_{\tilde{v}}\}_{v \in S_l})$$

and let $R^{\mathrm{univ}}_{\mathcal{S}}$ be an object of $\mathcal{C}_{\mathcal{O}}$ representing the corresponding deformation functor.

By a similar argument we can extend r' to a continuous homomorphism $\rho' : G_{(F')^+} \to \mathcal{G}_m(\mathcal{O})$ corresponding to a triple $(r', \chi', \langle\ ,\ \rangle')$. The pairings $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ give rise to a non-degenerate symmetric pairing $\langle\ ,\ \rangle''$ on k^{mn} and the triple $(\bar{r}'', \overline{\chi}'', \langle\ ,\ \rangle'')$ corresponds to a homomorphism $\overline{\rho}'' : G_{(F')^+} \to \mathcal{G}_{mn}(k)$.

Let $a \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F', \mathbb{C})}_c$ be the weight of the automorphic representation π. For $w \in S'_l$, let $a_{\tilde{w}}$ be the element of $(\mathbb{Z}^n_+)^{\mathrm{Hom}(F'_{\tilde{w}}, \overline{\mathbb{Q}}_l)}$ with $(a_{\tilde{w}})_\tau = a_{\iota \circ \tau|_{F'}}$ for each $\tau : F'_w \hookrightarrow \overline{\mathbb{Q}}_l$. Then $r''|_{G_{F'_{\tilde{w}}}}$ is crystalline of l-adic Hodge type $\mathbf{v}_{a_{\tilde{w}}}$. Let $R_{\tilde{w}}$ be the irreducible component of $R^{\mathbf{v}_{a_{\tilde{w}}}, cr}_{\bar{r}''|_{G_{F'_{\tilde{w}}}}}$ determined by $r''|_{G_{F'_{\tilde{w}}}}$. Let

$$\mathcal{S}'' = (F'/(F')^+, S'_l, \tilde{S}'_l, \mathcal{O}, \overline{\rho}'', \chi'', \{R_{\tilde{w}}\}_{w \in S_l})$$

and let $R^{\mathrm{univ}}_{\mathcal{S}''}$ be an object representing the deformation functor associated to $\mathcal{S}''$.

As in section 3.2, 'restricting to $G_{(F')^+}$ and tensoring with ρ'' gives rise to a homomorphism $R^{\mathrm{univ}}_{\mathcal{S}''} \to R^{\mathrm{univ}}_{\mathcal{S}}$ which is finite by Lemma 3.2.2. Lemma 3.4.1 (applied to $\bar{r}''$ and $R^{\mathrm{univ}}_{\mathcal{S}''}$) implies that $R^{\mathrm{univ}}_{\mathcal{S}''}$ is finite over $\mathcal{O}$ and any $\overline{\mathbb{Q}}_l$-point of this ring gives rise to a lift $G_{F'} \to \mathrm{GL}_{mn}(\overline{\mathbb{Z}}_l)$ of $\bar{r}''$ which automorphic of level prime to l. It follows that $R^{\mathrm{univ}}_{\mathcal{S}}$ is finite over $\mathcal{O}$ and moreover that any $\overline{\mathbb{Q}}_l$-point of this ring gives rise to a lift $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Z}}_l)$ satisfying (a)-(d) of the statement of the theorem being proved. The existence of such a $\overline{\mathbb{Q}}_l$-point follows from Lemma 3.2.4 of [GG09] (which shows that $R_{\mathcal{S}}^{\mathrm{univ}}$ has Krull dimension at least 1) and the fact that $R_{\mathcal{S}}^{\mathrm{univ}}$ is finite over $\mathcal{O}$. □

4. Big image for GL_2.

4.1. In this section we elucidate the meaning of the condition "2-big" in the special case of subgroups of $\mathrm{GL}_2(\overline{\mathbb{F}}_l)$ (see Definition 2.1.1). We will give concrete criteria under which the 2-big condition may be realised. We will prove results both for subgroups containing $\mathrm{SL}_2(k')$ for some field, and results for subgroups whose projective image is tetrahedral, octahedral or icosahedral.

Before we do so, let us first state a few results which will be useful to us in the sequel. These results are due to Snowden and Wiles for bigness (see [SW09, §2]), and the generalization to m-bigness are due to White (see [Whi10, §2, Propositions 2.1 and 2.2]).

Proposition 4.1.1 (Snowden-Wiles,White)**.** *Let $k/\mathbb{F}_l$ be algebraic, and m be an integer. If $G \subset \mathrm{GL}_2(k)$ has no l-power order quotients, and has a normal subgroup H which is m-big, then G is m-big.*

Proposition 4.1.2 (Snowden-Wiles,White)**.** *Let $k/\mathbb{F}_l$ be algebraic, m be an integer, and $G \subset \mathrm{GL}_2(k)$. Then G is m-big if and only if $k^\times G$ is.*

4.2. Subgroups containing $\mathrm{SL}_2(k')$.

Lemma 4.2.1. *Let $l > 3$ be prime, and let k be an algebraic extension of $\mathbb{F}_l$. Let k' be a finite subfield of k, and suppose that H is a subgroup of $\mathrm{GL}_2(k)$ with*

$$\mathrm{SL}_2(k') \subset H \subset k^\times \mathrm{GL}_2(k').$$

If the cardinality of k' is greater than 5 then H is 2-big.

Proof. All but the last condition in the definition follow at once from Lemma 2.5.6 of [CHT08]. To check the last condition, take $h = \mathrm{diag}(\alpha, \alpha^{-1})$, where $\alpha \in k'$ satisfies $\alpha^4 \neq 1$ (such an α exists because the cardinality of k' is greater than 5). Then the roots of the characteristic polynomial of h are α and α^{-1}, and $\alpha^2 \neq \alpha^{-2}$. Furthermore $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq 0$ for any irreducible $k[H]$-submodule W of $\mathfrak{gl}_2(k)$, just as in the proof of Lemma 2.5.6 of [CHT08]. □

Lemma 4.2.2. *Let $l > 3$ be prime, and let k be an algebraic extension of $\mathbb{F}_l$. Let k' be a finite subfield of k, and suppose that H is a finite subgroup of $\mathrm{GL}_2(k)$ with*

$$\mathrm{SL}_2(k') \subset H.$$

If the cardinality of k' is greater than 5 then H is 2-big.

Proof. Let $\bar{H}$ be the image of H in $\mathrm{PGL}_2(k)$. By Theorem 2.47(b) of [DDT97], $\bar{H}$ is either conjugate to a subgroup of the upper triangular matrices, or is conjugate to $\mathrm{PSL}_2(k'')$ or $\mathrm{PGL}_2(k'')$ for some finite extension k'' of k, or is isomorphic to A_4, S_4, A_5, or a dihedral group. Since $\mathrm{PSL}_2(k')$ is a simple group, and k' has cardinality greater than 5, we see that $\bar{H}$ must be conjugate to $\mathrm{PSL}_2(k'')$ or $\mathrm{PGL}_2(k'')$ for some finite extension k'' of k'. Then

$$\mathrm{SL}_2(k'') \subset H \subset k^\times \mathrm{GL}_2(k''),$$

and the result follows from Lemma 4.2.1. □

We will be able to give a strengthening of this result once we have considered the icosahedral, octahedral and tetrahedral cases in the next subsection.

4.3. Icosahedral, octahedral and tetrahedral cases. In this section, let us fix at the outset a subgroup $H \subset \mathrm{GL}_2(k)$ whose image $Hk^\times/k^\times$ in $\mathrm{PGL}_2(k)$ is isomorphic to A_5, S_4 or A_4. Let us also suppose $l > 5$. Under the assumption that k contains a primitive cube root of unity, we will show that H is 2-big. Since these groups clearly have no l-power order quotients, to show H is 2-big, it will suffice (by proposition 4.1.1) to show that the commutator subgroup $[H, H]$ is 2-big, since this is a normal subgroup of H. If $Hk^\times/k^\times = S_4$, the image of this commutator subgroup in $\mathrm{PGL}_2(k)$ will be isomorphic to $[S_4, S_4] = A_4$. Thus, by replacing H with $[H, H]$ in this case, we may suppress the possibility that $Hk^\times/k^\times \cong S_4$. Thus $H/Z(H) = A_n$, for $n = 4$ or 5.

Now, since both $Hk^\times/k^\times$ and $k^\times \cap H$ have order prime to l (in the latter case, as $k^\times$ does), H itself has order prime to l. Thus, if we think of the inclusion $H \hookrightarrow \mathrm{GL}_2(k)$ as a representation of H, this representation will have a lift to characteristic zero, say $r_l : H \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Z}}_l)$. Given an element $g \in A_n = H/Z(H)$, we can lift to an element $\tilde{g} \in H$, and map this to an element $r_l(\tilde{g})$. Making a series of such choices (and choosing $1 \in H$ as the lift of $1 \in H/Z(H)$), we get a map $P : A_n \to \mathrm{GL}_2(\overline{\mathbb{Z}}_l)$ sending $g \mapsto r_l(\tilde{g})$. This will be a projective representation in the sense of the (unnumbered) definition given at the beginning of chapter 1 of [HH92].

Recall that $n = 4$ or 5. By the last sentence of chapter 2 of [HH92] (on p23, just after the unnumbered remark after Theorem 2.12) we see the construction of a group, called there $\tilde{A}_n$, which is a representation group for A_n. (A *representation group* of G, in the terminology of [HH92], is a stem extension G^* of G such that the newly adjoined central elements are isomorphic to the Schur multiplier $M(G)$ of G.) This is defined as a certain subgroup of a certain group $\tilde{S}_n$, which is given a presentation just before Theorem 2.8 of *loc. cit.*, on p18. Comparing this presentation to the discussion in §2.7.2 of [Wil09], we see that $\tilde{A}_n$ is the same group as the group called $2.A_n$ in [Wil09] (since this seems to be the more standard name for this group, we shall call this group $2.A_n$ from now on). Examining the discussion in §5.6.8 and §5.6.2 of [Wil09], we see that $2.A_5$ and $2.A_4$ are respectively the binary icosahedral and tetrahedral groups. (Thus if we consider A_5 as the group of symmetries of a icosahedron, a subgroup of SO(3), then $2.A_5$ is the inverse image of A_5 under the natural 2 to 1 map $\mathrm{SU}(2) \to \mathrm{SO}(3)$; and similarly for A_4 and the group of symmetries of the tetrahedron.)

Following the discussion at the top of p7 of [HH92], we choose a map $r : A_n \to 2.A_n$ (*not* a homomorphism) sending each element of A_n to a lift in the representation group, and sending 1 to 1.

Then by Theorem 1.3 of [HH92], applied to the projective representation P above, we can find a representation

$$R : 2.A_n \to \mathrm{GL}_2(\overline{\mathbb{Z}}_l)$$

such that for each $a \in A_n$, we have $R(r(a)) = P(a)\beta(a)$, where $\beta(a) \in \overline{\mathbb{Z}}_l$ is a scalar. (The statement of Theorem 1.3 is for representations over $\mathbb{C}$, but the proof generalizes to our case.) We will write $\bar{R}$ for the reduction of R mod l. We then claim that $k^\times H = k^\times \bar{R}(2.A_n)$. To see this, note that if $h \in H$, then h mod $Z(H)$ will be some element x of A_n; then we have $h = \alpha\bar{P}(x) = \alpha\beta\bar{R}(r(x))$, where α and

β are scalars and $\bar{P}(a)$ is the reduction of $P(a)$ mod l. Thus the image of $\bar{R}(2.A_n)$ in $\mathrm{PGL}_2(k^\times)$ is A_n.

Now, note also that by Proposition 4.1.2, we have that

$$H \text{ 2-big} \Leftrightarrow k^\times H \text{ 2-big} \Leftrightarrow k^\times \bar{R}(2.A_n) \text{ 2-big} \Leftrightarrow \bar{R}(2.A_n) \text{ 2-big}$$

and we may therefore replace H by $\bar{R}(2.A_n)$, and therefore assume that H is of the form $\bar{R}(2.A_n)$ where $\bar{R}$ is some characteristic l representation of $2.A_n$, whose kernel is contained in the center of $2.A_n$ (that is, the representation is 'faithful on the A_n quotient'; recall that $2.A_n/Z(2.A_n) = A_n$). Indeed, $\bar{R}$ must either be faithful or factor through A_n, and we can think of $\bar{R}$ as a faithful mod l representation *either* of A_n *or* $2.A_n$. Recall that $n = 4$ or 5.

Since $\bar{R}$ comes from characteristic zero, it will be a direct sum of irreducible representations. Since A_4, A_5, $2.A_4$ and $2.A_5$ are non-abelian, no direct sum of one dimensional representations can ever be faithful, so we can in fact see that $\bar{R}$ is a a faithful irreducible 2 dimensional representation of A_4, A_5, $2.A_4$ or $2.A_5$. It will thus suffice to show that the images of all such representations are 2-big.

Thus, in order to establish the following:

Proposition 4.3.1. *Suppose $l > 5$ is a prime, and k is an algebraic extension of $\mathbb{F}_l$ which contains a primitive cube root of 1. Suppose H is a subgroup of $\mathrm{GL}_2(k)$, and the image of H in $\mathrm{PGL}_2(k)$ is isomorphic to A_5, S_4 or A_4. Then H is 2-big.*

It will suffice to establish the following:

Proposition 4.3.2. *Suppose $l > 5$ is a prime, and k is an algebraic extension of $\mathbb{F}_l$ which contains a primitive cube root of 1. Suppose G is one of A_5, $2.A_5$, A_4 or $2.A_4$. Suppose $r : G \to \mathrm{GL}_2(k)$ is a faithful irreducible representation. Then $r(G)$ is 2-big.*

Before we do so, the following lemma will be useful.

Lemma 4.3.3. *Suppose G is a group, and l is a prime such that that $\mathrm{hcf}(l, |G|) = 1$. Suppose that $r : G \to \mathrm{GL}_2(k)$ is a faithful irreducible representation. Suppose further that the following hold.*

- *$\mathrm{ad}^0 r$ is irreducible.*
- *There exists an odd prime p such that:*
  - *G has some non-central element of order p, and*
  - *k has a primitive p-th root of unity.*

Then $r(G)$ is 2-big.

Proof. The fact that $\mathrm{hcf}(l, |G|) = 1$ implies that $r(G)$ has no quotients of l-power order and also that $H^1(r(G), \mathfrak{sl}_2(k)) = (0)$ (the latter claim follows immediately from Corollary 1 of section VIII.2 of [Ser79]). Since $\mathrm{ad}^0 r$ is irreducible and has dimension 3, it cannot contain any copy of the trivial representation and hence $H^0(G, \mathrm{ad}^0 r) = (0)$; that is, $H^0(r(G), \mathfrak{sl}_2(k)) = (0)$.

Finally, let g be a non-central element of G of order p. Note that $r(g)$ has order p and must be non-central (since r is faithful). If the two roots of the characteristic polynomial were equal, then $r(g)$ would be a scalar matrix and hence central. ($r(g)$ must be diagonalizable, as we assume that $\mathrm{hcf}(l, |G|) = 1$ and non-semisimple elements of $\mathrm{GL}_2(k)$ have order divisible by l.) Thus the roots of the characteristic polynomial of $r(g)$ are distinct p-th roots of unity, α, and α', with at least one of them (α, say) primitive. Since p is odd, we have that $\alpha^2 \neq (\alpha')^2$. Since k contains the p-th roots of unity, $\alpha \in k$.

Now using the fact that $\operatorname{ad}^0 r$ is irreducible again, we see that the only irreducible $k[r(G)]$ submodules of $\mathfrak{gl}_2(k)$ are $\mathfrak{sl}_2(k)$ and $\langle I \rangle$. Checking that $\pi_{r(g),\alpha} \circ W \circ i_{r(g),\alpha} \neq 0$ for each of these is trivial. □

Proof of Proposition 4.3.2. Since $l > 5$, $\operatorname{hcf}(|r(G)|, l) = 1$. Then given Lemma 4.3.3, using our assumption that k has a 3rd root of unity, and noting that A_4 and A_5 have elements of order 3, we see that it suffices to check that $\operatorname{ad}^0 r$ is irreducible.

Since $\operatorname{hcf}(|r(G)|, l) = 1$, r must be the reduction of a characteristic zero representation, and we can replace r with a characteristic zero lift, and prove that $\operatorname{ad}^0 r$ is irreducible for this new r.

Let us first deal with the case $G = 2.A_5$. The character table of $2.A_5$ can be found on page 5 of [CCN+], and we can immediately see that there are precisely two irreducible representations of dimension 2, corresponding to the characters χ_6 and χ_7 there. (The first of these corresponds to $\rho_{\mathrm{nat},2.A_5}$, the natural representation we get by thinking of $2.A_5$ as the binary icosahedral group, and hence a subgroup of SU(2); the second is $\rho^{(12)}_{\mathrm{nat},2.A_5}$) Since the character χ_6 is real, the dual representation of ρ_{nat} has the same character and $\operatorname{ad}^0 \rho_{\mathrm{nat},2.A_5}$ has character $\chi_6^2 - 1$. We recognize this character as χ_2 from the table. Thus $\operatorname{ad}^0 \rho_{\mathrm{nat}}$ is irreducible. Similarly $\operatorname{ad}^0 \rho^{(12)}_{\mathrm{nat},2.A_5}$ has character χ_3, which is irreducible.

We can also deal with the case $G = A_5$; since A_5 is a quotient of $2.A_5$, every representation of A_5 will occur in the character table for $2.A_5$; but in fact the only two dimensional representations we saw do not factor through the center of $2.A_5$.

The character table of $2.A_4$ is standard, but since we have been unable to locate a convenient published reference for them, we reproduce it here. (See Figure 1. In calculating this table, James Montaldi's web page, which provides a real character table for $2.A_4$, was very helpful.) Here are some notes on the construction of the table, which should provide enough detail that the reader can straightforwardly check its accuracy:

- We have labelled conjugacy classes by thinking of $2.A_4$ as the binary tetrahedral group, and hence as a subgroup of SU(2); we have written elements of SU(2) as unit quaternions.
- 1 stands for the trivial representation.
- ω stands for the character $2.A_4 \to A_4 \to A_4/K_4 \xrightarrow{\sim} \mathbb{Z}/3\mathbb{Z} \to \langle e^{2\pi/3} \rangle$.
- $\rho_{2,2.A_4}$ stands for the natural 2D representation we get by thinking of $2.A_4$ as a subgroup of SU(2).
- $\rho_{3,2.A_4}$ stands for the natural 3D representation we get by thinking of $2.A_4$ as a subgroup of SU(2) then mapping $\mathrm{SU}(2)/\{\pm 1\} \to \mathrm{SO}(3)$.

We note that A_4 has no irreducible 2D representations (all 2D irreducible representations of $2.A_4$ fail to factor through A_4) and the only irreducible 2D representations of $2.A_4$ are $\rho_{2,2.A_4}$, $\rho_{2,2.A_4} \otimes \omega$ and $\rho_{2,2.A_4} \otimes \omega^{\otimes 2}$; tensoring any of these with its dual gives $\rho^{\otimes 2}_{2,2.A_4}$, which is $1 \oplus \rho_{3,2.A_4}$. Thus, for each of these representations, the ad^0 is $\rho_{3,2.A_4}$ and hence irreducible. □

4.4. A synoptic result. We now combine the results of the previous two sections to give a somewhat explicit characterization of big subgroups of $\mathrm{GL}_2(k)$. Our first result is an extension of Lemma 4.2.2.

class	e	[-1]	$[i]$	$[\frac{1+i+j+k}{2}]$	$[\frac{1+i+j-k}{2}]$	$[\frac{-1+i+j+k}{2}]$	$[\frac{-1+i+j-k}{2}]$
size	1	1	6	4	4	4	4
1	1	1	1	1	1	1	1
ω	1	1	1	$e^{2\pi/3}$	$e^{-2\pi/3}$	$e^{-2\pi/3}$	$e^{2\pi/3}$
$\omega^{\otimes 2}$	1	1	1	$e^{-2\pi/3}$	$e^{2\pi/3}$	$e^{2\pi/3}$	$e^{-2\pi/3}$
$\rho_{2,2.A_4}$	2	-2	0	1	1	-1	-1
$\rho_{2,2.A_4} \otimes \omega$	2	-2	0	$e^{2\pi/3}$	$e^{-2\pi/3}$	$-e^{-2\pi/3}$	$-e^{2\pi/3}$
$\rho_{2,2.A_4} \otimes \omega^{\otimes 2}$	2	-2	0	$e^{-2\pi/3}$	$e^{2\pi/3}$	$-e^{2\pi/3}$	$-e^{-2\pi/3}$
$\rho_{3,2.A_4}$	3	3	-1	0	0	0	0

FIGURE 1. Character table for $2.A_4$.

Lemma 4.4.1. *Let $l > 5$ be prime, and let k be an algebraic extension of $\mathbb{F}_l$. Suppose that k contains a primitive cube root of 1. If H is a finite subgroup of $\mathrm{GL}_2(k)$, acting irreducibly on k^2, then we have the following alternative. Either:*

(1) the image of H in $\mathrm{PGL}_2(k)$ is a dihedral group, or
(2) H is 2-big.

Proof. Let $\bar{H}$ be the image of H in $\mathrm{PGL}_2(k)$. By Theorem 2.47(b) of [DDT97], $\bar{H}$ is either conjugate to a subgroup of the upper triangular matrices, or is conjugate to $\mathrm{PSL}_2(k'')$ or $\mathrm{PGL}_2(k'')$ for some finite extension k'' of k, or is isomorphic to A_4, S_4, A_5, or a dihedral group. We saw in the proof of Lemma 4.2.2 that we are done in the case $\bar{H}$ is conjugate to $\mathrm{PSL}_2(k'')$ or $\mathrm{PGL}_2(k'')$ for some finite extension k'' of k, since then H is 2-big. The possibility that $\bar{H}$ is conjugate to a subgroup of the upper triangular matrices is excluded by our assumption that H acts irreducibly on k^2. If $\bar{H}$ is dihedral then we are certainly done; thus we may assume $\bar{H}$ is isomorphic to A_4, S_4, A_5, whence we are done by Proposition 4.3.1. □

We also prove a modified version of this result tailored for working with Galois representations.

Lemma 4.4.2. *Let $l > 5$ be prime, and let k be an algebraic extension of $\mathbb{F}_l$. Suppose that k contains a primitive cube root of 1. Suppose K is a number field, and $r : G_K \to \mathrm{GL}_2(k)$ a continuous absolutely irreducible representation. Then we have the following alternative. Either:*

(1) $r(G_{K(\zeta_l)})$ is 2-big, or
(2) there are field extensions $K_2/K_1/K$, with
- *K_1/K either trivial or cubic (and hence cyclic) Galois, and*
- *K_2/K_1 quadratic*

and, after possibly replacing k by its quadratic extension, a continuous character $\bar{\theta} : K_2 \to k^\times$ such that $r|_{G_{K_1}} \cong \mathrm{Ind}_{G_{K_2}}^{G_{K_1}} \bar{\theta}$.

Proof. We begin by claiming that in any case where we can construct an extension K_1/K, either trivial or cubic, such that the image I of $r(G_{K_1})$ in $\mathrm{PGL}_2(k)$ is isomorphic to a dihedral group D, then we are done. For then, writing $R \lhd D$ for the rotations in D, we have:

$$G_{K_1} \xrightarrow{r} r(G_{K_1}) \twoheadrightarrow I \xrightarrow{\sim} D \twoheadrightarrow D/R \xrightarrow{\sim} \langle -1 \rangle$$

is a nontrivial quadratic character of K_1, defining a field extension K_2/K_1. Then $r(G_{K_2})$ has cyclic image in $\mathrm{PGL}_2(k)$ (as R is cyclic), and hence $r(G_{K_2}) \subset \mathrm{GL}_2(k)$

is abelian (as any central extension of a cyclic group is abelian). It follows, after possibly replacing k by its quadratic extension, that $r|_{G_{K_2}}$ is reducible. If $r|_{G_{K_2}}$ is indecomposable, we see easily that r is reducible, contradicting our assumptions. Similarly, if $r|_{G_{K_2}}$ decomposes as a direct sum of characters and this decomposition is preserved under conjugation by a non-trivial element of $\mathrm{Gal}(K_2/K_1)$, then using the fact that the extension K_1/K is either trivial or cubic, we see that r is reducible. Thus $r|_{G_{K_2}}$ is a direct sum of characters and conjugation by a nontrivial element of $\mathrm{Gal}(K_2/K_1)$ swaps these characters. Thus, in this case,

$$r|_{G_{K_1}} \cong \mathrm{Ind}_{G_{K_2}}^{G_{K_1}} \bar{\theta}$$

and we have the second alternative in the statement of the theorem.

Now, let $\bar{H}$ be the image of $r(G_K)$ in $\mathrm{PGL}_2(k)$. Let $\bar{H}'$ be the image of $r(G_{K(\zeta_l)})$ in $\mathrm{PGL}_2(k)$. Since $r(G_{K(\zeta_l)}) \triangleleft G_K$, with cyclic quotient, we have that $\bar{H} \triangleleft \bar{H}'$ with cyclic quotient. Again, by Theorem 2.47(b) of [DDT97], we have the following possibilities for $\bar{H}$:

- *$\bar{H}$ is conjugate to a subgroup of the upper triangular matrices.* This is ruled out by our assumption that r is irreducible.
- *$\bar{H}$ is isomorphic to a dihedral group.* In this case, we may take $K_1 = K$ and we are done by the first paragraph of the proof.
- *$\bar{H}$ is conjugate to $\mathrm{PSL}_2(k'')$ or $\mathrm{PGL}_2(k'')$ for some finite extension k'' of k.* In this case, since $\bar{H}'$ is a normal subgroup of $\bar{H}$, then the simplicity of $\mathrm{PSL}_2(k'')$ tells us that $\bar{H}'$ is conjugate to $\mathrm{PSL}_2(k'')$ or $\mathrm{PGL}_2(k'')$ too. We see, as in the proof of Lemma 4.2.2 (with our $r(G_{K(\zeta_l)})$ and $\bar{H}'$ taking the roles of H and $\bar{H}$ in the proof of Lemma 4.2.2) that this means that $r(G_{K(\zeta_l)})$ is 2-big, and we are done.
- *$\bar{H}$ is conjugate to A_4, S_4, or A_5.* In this case the image of $\bar{H}'$ under the isomorphism must be a normal subgroup with cyclic quotient; that is, one of K_4, A_4, S_4, A_5. In any of the cases except for $\bar{H}' \cong K_4$, we then conclude via Proposition 4.3.1 that $r(G_{K(\zeta_l)})$ is 2-big. In the remaining case, we have $i : \bar{H} \xrightarrow{\sim} A_4$ and $i(\bar{H}') = K_4 \subset A_4$. Then

  $$G_K \xrightarrow{r} r(G_K) \twoheadrightarrow \bar{H} \xrightarrow{\sim} A_4 \twoheadrightarrow A_4/K_4 = \mathbb{Z}/3\mathbb{Z} \xrightarrow{\sim} \langle e^{2\pi i/3} \rangle$$

  is a cubic character of G_K, corresponding to a cubic Galois extension K_1. Moreover the image of $r|_{G_{K_1}}$ in $\mathrm{PGL}_2(k)$ is isomorphic to K_4, which is isomorphic in turn to the dihedral group with 4 elements. Thus we are done by the first paragraph of the proof.

Since we are done in every case, the lemma is proved. □

5. Untwisting: the CM case.

5.1. We now prove a slight variant of Proposition 5.2.1 of [BLGG09], working over a CM base field, rather than a totally real base field. The proof is extremely similar.

Let F be a CM field with maximal totally real subfield F^+. Let M be a cyclic CM extension of F^+ of degree m, linearly disjoint from F over F^+. Suppose that $\theta : M^\times \backslash \mathbb{A}_M^\times \to \mathbb{C}^\times$ is an algebraic character, and that Π is a RAECSDC automorphic representation of $\mathrm{GL}_{mn}(\mathbb{A}_F)$ for some n. Let $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ be an isomorphism.

Proposition 5.1.1. *Assume that there is a continuous representation $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ such that $r|_{G_{FM}}$ is irreducible and*

$$r_{l,\iota}(\Pi) \cong r \otimes (\mathrm{Ind}_{G_M}^{G_{F^+}} r_{l,\iota}(\theta))|_{G_F}.$$

Then r is automorphic. If we assume furthermore that Π has level prime to l, then r is automorphic of level prime to l.

Proof. Note that

$$(\mathrm{Ind}_{G_M}^{G_{F^+}} r_{l,\iota}(\theta))|_{G_F} \cong \mathrm{Ind}_{G_{FM}}^{G_F}(r_{l,\iota}(\theta))|_{G_{FM}}).$$

Let σ denote a generator of $\mathrm{Gal}(FM/F)$, and κ a generator of $\mathrm{Gal}(FM/F)^\vee$. Then we have

$$\begin{aligned} r_{l,\iota}(\Pi \otimes (\kappa \circ \mathrm{Art}_F \circ \det)) &= r_{l,\iota}(\Pi) \otimes r_{l,\iota}(\kappa \circ \mathrm{Art}_F) \\ &\cong r \otimes (r_{l,\iota}(\kappa \circ \mathrm{Art}_F) \otimes \mathrm{Ind}_{G_{FM}}^{G_F}(r_{l,\iota}(\theta)|_{G_{FM}}) \\ &\cong r \otimes \mathrm{Ind}_{G_{FM}}^{G_F}(r_{l,\iota}(\kappa \circ \mathrm{Art}_F)|_{G_{FM}} \otimes r_{l,\iota}(\theta)|_{G_{FM}}) \\ &\cong r \otimes \mathrm{Ind}_{G_{FM}}^{G_F} r_{l,\iota}(\theta)|_{G_{FM}} \\ &\cong r_{l,\iota}(\Pi), \end{aligned}$$

so that $\Pi \otimes (\kappa \circ \mathrm{Art}_F \circ \det) \cong \Pi$.

We claim that for each intermediate field $FM \supset N \supset F$ there is a regular cuspidal automorphic representation Π_N of $\mathrm{GL}_{n[FM:N]}(\mathbb{A}_N)$ such that

$$\Pi_N \otimes (\kappa \circ \mathrm{Art}_N \circ \det) \cong \Pi_N$$

and $BC_{N/F}(\Pi)$ is equivalent to

$$\Pi_N \boxplus \Pi_N^\sigma \boxplus \cdots \boxplus \Pi_N^{\sigma^{[N:F]-1}}$$

in the sense that for all places v of N, the base change from $F_{v|_F}$ to N_v of $\Pi_{v|_F}$ is

$$\Pi_{N,v} \boxplus \Pi_{N,v}^\sigma \boxplus \cdots \boxplus \Pi_{N,v}^{\sigma^{[N:F]-1}}.$$

We prove this claim by induction on $[N : F]$. Suppose that $FM \supset M_2 \supset M_1 \supset F$ with M_2/M_1 cyclic of prime degree, and that we have already proved the result for M_1. Since

$$\Pi_{M_1} \otimes (\kappa \circ \mathrm{Art}_{M_1} \circ \det) \cong \Pi_{M_1}$$

we see from Theorems 3.4.2 and 3.5.1 of [AC89] (together with Lemma VII.2.6 of [HT01] and the main result of [Clo82]) that there is a cuspidal automorphic representation Π_{M_2} of $\mathrm{GL}_{n[FM:M_2]}(\mathbb{A}_{M_2})$ such that $BC_{M_2/F}(\Pi)$ is equivalent to

$$\Pi_{M_2} \boxplus \Pi_{M_2}^\sigma \boxplus \cdots \boxplus \Pi_{M_2}^{\sigma^{[M_2:F]-1}}.$$

Since Π is regular, Π_{M_2} is regular. The representation $\Pi_{M_2} \otimes (\kappa \circ \mathrm{Art}_{M_2} \circ \det)$ satisfies the same properties (because $\Pi \otimes (\kappa \circ \mathrm{Art}_F \circ \det) \cong \Pi$), so we see (by strong multiplicity one for isobaric representations) that we must have

$$\Pi_{M_2} \otimes (\kappa \circ \mathrm{Art}_{M_2} \circ \det) \cong \Pi_{M_2}^{\sigma^i}$$

for some $0 \le i \le [FM : M_2] - 1$. If $i > 0$ then

$$\Pi_{M_2} \boxplus \Pi_{M_2}^\sigma \boxplus \cdots \boxplus \Pi_{M_2}^{\sigma^{[M_2:F]-1}}$$

cannot be regular (note that of course κ is a character of finite order), a contradiction, so in fact we must have $i = 0$. Thus

$$\Pi_{M_2} \otimes (\kappa \circ \mathrm{Art}_{M_2} \circ \det) \cong \Pi_{M_2}$$

and the claim follows.

Let $\pi := \Pi_{FM}$. Note that the representations π^{σ^i} for $0 \le i \le m-1$ are pairwise non-isomorphic (because Π is regular). Note also that $\pi \otimes |\det|^{(n-nm)/2}$ is regular algebraic (again, because Π is regular algebraic).

Since Π is RAECSDC, there is an algebraic character χ of $(F^+)^\times \backslash \mathbb{A}_{F^+}^\times$ such that $\Pi^{c,\vee} \cong \Pi \otimes (\chi \circ N_{F/F^+} \circ \det)$. It follows (by strong multiplicity one for isobaric representations) that for some $0 \le i \le m-1$ we have

$$\pi^{c,\vee} \cong \pi^{\sigma^i} \otimes (\chi \circ N_{FM/F^+} \circ \det).$$

Then we have

$$\begin{aligned}
\pi &\cong (\pi^{c,\vee})^{c,\vee} \\
&\cong (\pi^{\sigma^i} \otimes (\chi \circ N_{FM/F^+} \circ \det))^{c,\vee} \\
&\cong (\pi^{c,\vee})^{\sigma^i} \otimes (\chi^{-1} \circ N_{FM/F^+} \circ \det)) \\
&\cong (\pi^{\sigma^i} \otimes (\chi \circ N_{FM/F^+} \circ \det))^{\sigma^i} \otimes (\chi^{-1} \circ N_{FM/F^+} \circ \det)) \\
&\cong \pi^{\sigma^{2i}}
\end{aligned}$$

so that either $i = 0$ or $i = m/2$. We wish to rule out the latter possibility. Assume for the sake of contradiction that

$$\pi^{c,\vee} \cong \pi^{\sigma^{m/2}} \otimes (\chi \circ N_{FM/F^+} \circ \det).$$

Since F^+ is totally real, there is an integer w such that $\chi|\cdot|^{-w}$ has finite image. Then $\pi \otimes |\det|^{w/2}$ has unitary central character, so is itself unitary. Since $\pi \otimes |\det|^{(n-nm)/2}$ is regular algebraic, we see that for places $v|\infty$ of FM the conditions of Lemma 7.1 of [BLGHT09] are satisfied for $\pi_v|\det|_v^{w/2}$, so that

$$\begin{aligned}
\pi_v \boxplus \pi_v^{\sigma^{m/2}} &\cong \pi_v \boxplus \pi_v^{c,\vee} \otimes (\chi^{-1} \circ N_{FM/F^+} \circ \det) \\
&\cong \pi_v \boxplus ((\pi_v \otimes |\det|^{w/2})^{c,\vee} \otimes (|\cdot|^{w/2} \circ \det)) \otimes (\chi^{-1} \circ N_{FM/F^+} \circ \det) \\
&\cong \pi_v \boxplus (\pi_v \otimes |\det|^{w/2}) \otimes (|\cdot|^{w/2} \circ \det)) \otimes (\chi^{-1} \circ N_{FM/F^+} \circ \det) \\
&\cong \pi_v \boxplus (\pi_v \otimes (\chi^{-1}|\cdot|^{w} \circ N_{FM/F^+} \circ \det))
\end{aligned}$$

which contradicts the regularity of $\Pi_{v|_F}$. Thus we have $i = 0$, so that

$$\pi^{c,\vee} \cong \pi \otimes (\chi \circ N_{FM/F^+} \circ \det).$$

Thus $\pi \otimes |\det|^{(n-nm)/2}$ is a RAECSDC representation, so that we have a Galois representation $r_{l,\iota}(\pi \otimes |\det|^{(n-nm)/2})$. The condition that $BC_{FM/F}(\Pi)$ is equivalent to

$$\pi \boxplus \pi^\sigma \boxplus \cdots \boxplus \pi^{\sigma^{m-1}}$$

translates to the fact that

$$r_{l,\iota}(\Pi)|_{G_{FM}} \cong r_{l,\iota}(\pi \otimes |\det|^{(n-nm)/2}) \oplus \cdots \oplus r_{l,\iota}(\pi \otimes |\det|^{(n-nm)/2})^{\sigma^{m-1}}.$$

By hypothesis, we also have

$$r_{l,\iota}(\Pi)|_{G_{FM}} \cong (r|_{G_{FM}} \otimes r_{l,\iota}(\theta)|_{G_{FM}}) \oplus \cdots \oplus (r|_{G_{FM}} \otimes r_{l,\iota}(\theta)|_{G_{FM}}^{\sigma^{m-1}}).$$

Since $r|_{G_{FM}}$ is irreducible, there must be an i such that

$$r|_{G_{FM}} \cong r_{l,\iota}(\pi \otimes |\det|^{(n-nm)/2}) \otimes r_{l,\iota}(\theta)|^{\sigma^{-i}}_{G_{FM}},$$

so that $r|_{G_{FM}}$ is automorphic. The result now follows from Lemma 1.4 of [BLGHT09]. $\square$

6. Main Results.

6.1. Let $l > 2$ be a fixed prime. Fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. In this section, F^+ always denotes a totally real field. A continuous representation $\bar{r} : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is said to be *modular* if there exists a regular algebraic automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_{F^+})$ such that $\bar{r}_{l,\iota}(\pi) \cong \bar{r}$. It is a standard fact that if such a π exists, then it may be taken to have weight 0.

We begin with some preliminary results.

Lemma 6.1.1. *Let $\bar{r} : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ be an irreducible modular representation. Suppose that $\overline{F^+}^{\ker \mathrm{ad}\, \bar{r}}$ does not contain $F^+(\zeta_l)$. There exists a finite solvable extension of totally real fields L^+/F^+ such that*

- *L^+ is linearly disjoint from $\overline{F^+}^{\ker \bar{r}}(\zeta_l)$ over F^+.*
- *$\bar{r}|_{G_{L^+_v}}$ is trivial for all places $v|l$ of L^+.*
- *For each place $v|l$, $[L^+_v : \mathbb{Q}_l] \geq 2$.*
- *There is a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_{L^+})$ of weight 0 such that*
  - *$\bar{r}_{l,\iota}(\pi) \cong \bar{r}|_{G_{L^+}}$.*
  - *π is unramified at all finite places.*
  - *For all places $v|l$ of L^+, $r_{l,\iota}(\pi)|_{G_{L^+_v}}$ is non-ordinary.*

Proof. Choose a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_{F^+})$ of weight 0 such that $\bar{r}_{l,\iota}(\pi) \cong \bar{r}$. Let $r = r_{l,\iota}(\pi) : G_{F^+} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$. Choose F_1^+/F^+ a finite solvable extension of totally real fields such that:

- F_1^+ is linearly disjoint from $\overline{F^+}^{\ker \bar{r}}(\zeta_l)$ over F^+.
- $\bar{r}|_{G_{F^+_{1,v}}}$ is trivial for all places $v|l$ of F_1^+.
- $\bar{r}|_{G_{F^+_{1,v}}}$ is unramified for all finite places v of F_1^+.
- $[F_1^+ : \mathbb{Q}]$ is even.
- The base change $\pi_{F_1^+}$ of π to F_1^+ is unramified or Steinberg at each finite place.
- If $\pi_{F_1^+}$ is ramified at a prime $v \nmid l$ of F_1^+, then $\mathbf{N}v \equiv 1 \mod l$.

Let B be a quaternion algebra with centre F_1^+ which is ramified at precisely the infinite places. We now introduce l-adic automorphic forms on $B^\times$. Let K be a finite extension of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$ with ring of integers $\mathcal{O}$ and residue field k, and assume that K contains the images of all embeddings $F_1^+ \hookrightarrow \overline{\mathbb{Q}}_l$. Fix a maximal order $\mathcal{O}_B$ in B and for each finite place v of F_1^+ fix an isomorphism $i_v : \mathcal{O}_{B,v} \xrightarrow{\sim} M_2(\mathcal{O}_{F^+_{1,v}})$. For each embedding $\tau : F_1^+ \hookrightarrow \overline{\mathbb{Q}}_l$ we let $\iota\tau$ denote the real place of F_1^+ corresponding to the embedding $\iota \circ \tau$. Similarly, if $\sigma : F_1^+ \hookrightarrow \mathbb{R}$ is an embedding we let $\iota^{-1}\tau$ denote the corresponding embedding $\iota^{-1} \circ \tau$.

For each $v|l$ let τ_v denote a smooth representation of $\mathrm{GL}_2(\mathcal{O}_{F^+_{1,v}})$ on a finite free $\mathcal{O}$-module W_{τ_v}. Let τ denote the representation $\otimes_{v|l}\tau_v$ of $\mathrm{GL}_2(\mathcal{O}_{F_1^+,l})$ on $W_\tau := \otimes_{v|l} W_{\tau_v}$. Suppose that $\psi : (F_1^+)^\times\backslash(\mathbb{A}^\infty_{F_1^+})^\times \to \mathcal{O}^\times$ is a continuous character so that for each prime $v|l$, the action of the centre $\mathcal{O}^\times_{F^+_{1,v}}$ of $\mathcal{O}^\times_{B,v}$ on W_{τ_v} is given by $\psi^{-1}|_{\mathcal{O}^\times_{F^+_{1,v}}}$. Such a character ψ is necessarily of finite order.

Let $U = \prod_v U_v \subset (B \otimes_{\mathbb{Q}} \mathbb{A}^\infty)^\times$ be a compact open subgroup with $U_v \subset \mathcal{O}^\times_{B,v}$ for all v and $U_v = \mathcal{O}^\times_{B,v}$ for $v|l$. We let $S_{0,\tau,\psi}(U,\mathcal{O})$ denote the space of functions

$$f : B^\times\backslash(B \otimes_{\mathbb{Q}} \mathbb{A}^\infty)^\times \to W_\tau$$

with $f(gu) = \tau(u_l)^{-1} f(g)$ and $f(gz) = \psi(z) f(g)$ for all $u \in U$, $z \in (\mathbb{A}^\infty_{F_1^+})^\times$ and $g \in (B \otimes_{\mathbb{Q}} \mathbb{A}^\infty)^\times$. Writing $U = U^l \times U_l$, we let

$$S_{0,\tau,\psi}(U_l,\mathcal{O}) = \varinjlim_{U^l} S_{0,\tau,\psi}(U^l \times U_l, \mathcal{O})$$

and we let $(B \otimes_{\mathbb{Q}} \mathbb{A}^{l,\infty})^\times$ act on this space by right translation.

Let $\psi_{\mathbb{C}} : (F_1^+)^\times\backslash\mathbb{A}^\times_{F_1^+} \to \mathbb{C}^\times$ be the algebraic Hecke character defined by $\psi_{\mathbb{C}}(z) = \iota(\psi(z^\infty))$. Let $W_{\tau,\mathbb{C}} = W_\tau \otimes_{\mathcal{O},\iota} \mathbb{C}$. We have an isomorphism of $(B \otimes_{\mathbb{Q}} \mathbb{A}^{l,\infty})^\times$-modules

$$S_{0,\tau,\psi}(U_l,\mathcal{O}) \otimes_{\mathcal{O},\iota} \mathbb{C} \xrightarrow{\sim} \bigoplus_{\Pi} \mathrm{Hom}_{\mathcal{O}^\times_{B,l}}(W^\vee_{\tau,\mathbb{C}}, \Pi_l) \otimes \Pi^{\infty,l} \tag{6.1.1}$$

where the sum is over all automorphic representations Π of $(B \otimes_{\mathbb{Q}} \mathbb{A})^\times$ of weight 0 and central character $\psi_{\mathbb{C}}$ (see for instance the proof of Lemma 1.3 of [Tay06]).

Let U be as above and let R denote a finite set of places of F_1^+ containing all those places v where $U_v \neq \mathcal{O}^\times_{B,v}$. Let $\mathbb{T}^R$ denote the polynomial algebra $\mathcal{O}[T_v, S_v]$ where v runs over all places of F_1^+ away from l and R. For such v we let T_v and S_v act on $S_{0,\tau,\psi}(U,\mathcal{O})$ via the double coset operators

$$\left[U i_v^{-1}\begin{pmatrix}\varpi_v & 0\\ 0 & 1\end{pmatrix} U\right] \quad\text{and}\quad \left[U i_v^{-1}\begin{pmatrix}\varpi_v & 0\\ 0 & \varpi_v\end{pmatrix} U\right]$$

respectively, where ϖ_v is a uniformizer in $\mathcal{O}_{F^+_{1,v}}$.

Let $\widetilde{\pi}$ denote the automorphic representation of $(B \otimes_{\mathbb{Q}} \mathbb{A})^\times$ of weight 0 corresponding to $\pi_{F_1^+}$ under the Jacquet-Langlands correspondence. Choose a place v_0 of F_1^+ such that

- B is split at v_0;
- v_0 does not split completely in $F_1^+(\zeta_l)$;
- $\widetilde{\pi}$ is unramified at v_0 and $\mathrm{ad}\,\bar{r}(\mathrm{Frob}_{v_0}) = 1$;
- for every non-trivial root of unity ζ in a quadratic extension of F_1^+, $\zeta+\zeta^{-1} \not\equiv 2 \mod v_0$.

The second and third conditions imply that $H^2(G_{F^+_{1,v_0}}, \mathrm{ad}^0\,\bar{r})$ is trivial and that every deformation of $\bar{r}|_{G_{F^+_{1,v_0}}}$ is unramified. Let $U = \prod_v U_v \subset (B \otimes_{\mathbb{Q}} \mathbb{A}^\infty)^\times$ be the compact open subgroup with

- $U_v = \mathcal{O}^\times_{B,v}$ for all $v|l$, and all $v \neq v_0$ where $\widetilde{\pi}_v$ is unramified;
- $U_{v_0} = \iota^{-1}_{v_0}\mathrm{Iw}_1(v_0)$ where $\mathrm{Iw}_1(v_0)$ is the subgroup of $\mathrm{Iw}(v_0)$ consisting of all elements whose reduction modulo v_0 is unipotent;

- $U_v = \iota_v^{-1}\,\mathrm{Iw}(v)$ if $v \nmid l$ and $\widetilde{\pi}_v$ is Steinberg.

The subgroup U then satisfies hypothesis 3.1.2 of [Kis07c] (see the remarks following (2.1.2) of [Kis09]).

Let R denote the set of primes $v \nmid l$ of F_1^+ where $U_v \neq \mathcal{O}_{B,v}^\times$. For each $v|l$ with $\widetilde{\pi}_v$ unramified, let τ_v denote the trivial representation of $\mathcal{O}_{B,v}^\times$ on $\mathcal{O}$. For each $v|l$ with $\widetilde{\pi}_v$ Steinberg, let τ_v denote the $\mathcal{O}$-dual of the representation $\mathrm{Ind}_{\iota_v^{-1}\mathrm{Iw}(v)}^{\mathcal{O}_{B,v}^\times}\mathcal{O}$ modulo the constants. Let $\chi : G_{F_1^+}^{\mathrm{ab}} \to \overline{\mathbb{Q}}_l^\times$ denote the character $\epsilon \det r_{l,\iota}(\pi)|_{G_{F_1^+}}$ and let $\psi = \chi \circ \mathrm{Art}_{F_1^+} : \mathbb{A}_{F_1^+}^\times/\overline{(F_{1,\infty}^+)_{>0}^\times (F_1^+)^\times} \to \overline{\mathbb{Q}}_l$. Note that χ is totally even and hence we may regard ψ as a character of $(\mathbb{A}_{F_1^+}^\infty)^\times/(F_1^+)^\times \xrightarrow{\sim} \mathbb{A}_{F_1^+}^\times/\overline{(F_{1,\infty}^+)^\times (F_1^+)^\times}$. Extending K if necessary, we can and do assume that ψ is valued in $\mathcal{O}^\times$. Note that for each $v|l$, $\mathrm{Hom}_{\mathcal{O}_{B,v}^\times}(\tau_v^\vee \otimes_{\mathcal{O},\iota} \mathbb{C}, \widetilde{\pi}_v) \neq \{0\}$, while for $v \nmid l$ we have $\widetilde{\pi}_v^{U_v} \neq \{0\}$. It follows that the subspace of $S_{0,\tau,\psi}(U_l,\mathcal{O})\otimes_{\mathcal{O},\iota}\mathbb{C}$ corresponding to $\widetilde{\pi}$ under (6.1.1) has non-zero intersection with $S_{0,\tau,\psi}(U,\mathcal{O})\otimes_{\mathcal{O},\iota}\mathbb{C}$. Further extending K if necessary, we can and do choose a $\mathbb{T}^R$-eigenform f in $S_{0,\tau,\psi}(U,\mathcal{O})$ which lies in this intersection. The $\mathbb{T}^R$-eigenvalues on f give rise to an $\mathcal{O}$-algebra homomorphism $\mathbb{T}^R \to \mathcal{O}$ and reducing this modulo $\mathfrak{m}_\mathcal{O}$ gives rise to a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^R$.

By Corollary 3.1.6 of [Kis07c], after extending K, we can and do choose a collection $\{\tau_v'\}_{v|l}$ where each τ_v' is a cuspidal $F_{1,v}^+$-type such that the action of $\mathcal{O}_{F_1^+,v}^\times$ on τ_v' is given by $\psi^{-1}|_{\mathcal{O}_{F_{1,v}^+}^\times}$ (the trivial character) and such that $S_{0,\tau',\psi}(U,\mathcal{O})_\mathfrak{m} \neq \{0\}$ where $\tau' = \otimes_{v|l}\tau_v'$.

At each place $v \in R$ choose a non-trivial character $\chi_v : \mathcal{O}_{F_{1,v}^+}^\times \to \mathcal{O}^\times$ which factors through $k(v)^\times$ (where $k(v)$ is the residue field of v) and reduces to the trivial character modulo $\mathfrak{m}_\mathcal{O}$. Here we use the assumption that $\mathbf{N}v \equiv 1 \mod l$. Let $\chi = \{\chi_v\}_{v\in R}$. For $v \in R$ let W_{χ_v} denote the free rank 1 $\mathcal{O}$-module with action of $\mathrm{Iw}(v)$ given by the character $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi_v(ad^{-1})$. Let $W_\chi = \otimes_{v\in R} W_{\chi_v}$. Let $S_{0,\tau',\psi,\chi}(U,\mathcal{O})$ denote the space of functions

$$f : B^\times \backslash (B \otimes_\mathbb{Q} \mathbb{A}^\infty)^\times \to W_\tau \otimes_\mathcal{O} W_\chi$$

with $f(gu) = (\tau \otimes \chi)(u_{l,R})^{-1} f(g)$ and $f(gz) = \psi(z)f(g)$ for all $u \in U$, $z \in (\mathbb{A}_{F_1^+}^\infty)^\times$ and $g \in (B \otimes_\mathbb{Q} \mathbb{A}^\infty)^\times$. Since $\overline{\chi}_v = 1$ for each $v \in R$, we have $S_{0,\tau',\psi,\chi}(U,\mathcal{O})_\mathfrak{m} \neq \{0\}$ by Lemma 3.1.4 of [Kis07c]. Extending $\mathcal{O}$ we can choose a $\mathbb{T}^R$-eigenform $g \in S_{0,\tau',\psi,\chi}(U,\mathcal{O})_\mathfrak{m} \neq \{0\}$. Applying the Jacquet-Langlands correspondence, g gives rise to a regular algebraic cuspidal automorphic representation π_1 of $\mathrm{GL}_2(\mathbb{A}_{F_1^+})$ of weight 0 such that $\bar{r}_{l,\iota}(\pi_1) \cong \bar{r}|_{G_{F_1^+}}$ and $\pi_{1,v}$ is supercuspidal for all $v|l$, a ramified principal series representation for all $v \in R$ and unramified otherwise. The result follows easily by replacing F_1^+ with an appropriate totally real solvable extension L^+/F_1^+. □

Lemma 6.1.2. *Let F be an imaginary CM field with maximal totally real subfield F^+. Let π be a RAECSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ which is ι-ordinary at all places dividing l. Let $\chi : \mathbb{A}_{F^+}^\times/(F^+)^\times \to \mathbb{C}^\times$ be an algebraic character with $\chi_v(-1)$ independent of $v|\infty$ and $\pi^c \cong \pi^\vee \otimes (\chi \circ \mathbf{N}_{F/F^+} \circ \det)$. Let*

$\widetilde{\chi} : \mathbb{A}_{F^+}^\times/(F^+)^\times \to \mathbb{C}^\times$ *be the finite order algebraic character with* $r_{l,\iota}(\widetilde{\chi})$ *equal to the Teichmüller lift of* $\bar{r}_{l,\iota}(\chi)$*. Suppose that* $\bar{r}_{l,\iota}(\pi)$ *is irreducible.*

Let $\lambda' \in (\mathbb{Z}_+^n)_c^{\mathrm{Hom}(F,\mathbb{C})}$ *be a (conjugate-self-dual) weight. Let* F'/F *be a finite extension. We can find a finite solvable CM extension* L *of* F*, linearly disjoint from* F' *over* F *and a RAECSDC automorphic representation* π' *of* $\mathrm{GL}_n(\mathbb{A}_L)$ *such that:*

(1) $(\pi')^c \cong (\pi')^\vee \circ (\widetilde{\chi} \circ \mathbf{N}_{L/F^+} \circ \det)$.
(2) π' *is of weight* λ'_L.
(3) π' *is* ι*-ordinary at all places dividing* l *and unramified at all finite places.*
(4) $\bar{r}_{l,\iota}(\pi') \cong \bar{r}_{l,\iota}(\pi)|_{G_L}$.

Proof. Using Lemma 4.1.4 of [CHT08], choose an algebraic character $\psi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ such that $\psi\psi^c = \chi^{-1} \circ \mathbf{N}_{F/F^+}$. After replacing F by a solvable CM extension, linearly disjoint from $F'\overline{F}^{\ker \bar{r}_{l,\iota}(\pi)}$ over F, we may assume that ψ is unramified at all finite places. (Lemma 5.1.6 of [Ger09] shows that the ordinarity of π is preserved under such a base change.) Then $\pi \otimes (\psi \circ \det)$ is RACSDC. Applying Lemma 5.1.7 of [Ger09], we can find a solvable CM extension L of F, linearly disjoint from F' over F, and a RACSDC automorphic representation π'' of $\mathrm{GL}_n(\mathbb{A}_L)$ of weight λ'_L which is unramified at all finite places, ι-ordinary at all places dividing l and satisfies $\bar{r}_{l,\iota}(\pi') \cong \bar{r}_{l,\iota}(\pi)|_{G_L} \otimes \bar{r}_{l,\iota}(\psi)|_{G_L}$. Let $\widetilde{r}_{l,\iota}(\psi) : G_F \to \mathcal{O}_{\overline{\mathbb{Q}}_l}^\times$ be the Teichmüller lift of $\bar{r}_{l,\iota}(\psi)$ and let $\widetilde{\psi} : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ be the algebraic character with $r_{l,\iota}(\widetilde{\psi}) = \widetilde{r}_{l,\iota}(\psi)$. We now take $\pi' = \pi'' \otimes (\widetilde{\psi} \circ \mathbf{N}_{L/F} \circ \det)$. □

The following proposition is the main technical result of this paper.

Proposition 6.1.3. *Let* $\bar{r} : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ *be an irreducible modular representation. Assume further that*

(1) $l \geq 5$.
(2) $\bar{r}(G_{F^+(\zeta_l)})$ *is 2-big.*
(3) $[\overline{F^+}^{\ker \mathrm{ad}\, \bar{r}}(\zeta_l) : \overline{F^+}^{\ker \mathrm{ad}\, \bar{r}}] > 2$.

Then there exists a finite solvable extension of totally real fields L^+/F^+ *such that*

- L^+ *is linearly disjoint from* $\overline{F^+}^{\ker \bar{r}}(\zeta_l)$ *over* F^+.
- $\bar{r}|_{G_{L^+_v}}$ *is trivial for all places* $v|l$ *of* L^+.
- *There is a regular algebraic cuspidal automorphic representation* π *of* $\mathrm{GL}_2(\mathbb{A}_{L^+})$ *of weight* 0 *such that*
  - $\bar{r}_{l,\iota}(\pi) \cong \bar{r}|_{G_{L^+}}$.
  - π *is unramified at all finite places.*
  - *For all places* $v|l$ *of* L^+, $r_{l,\iota}(\pi)|_{G_{L^+_v}}$ *is ordinary.*

Proof. We firstly apply Lemma 6.1.1, to deduce that there is a finite solvable extension of totally real fields F_2^+/F^+ such that

- F_2^+ is linearly disjoint from $\overline{F^+}^{\ker \bar{r}}(\zeta_l)$ over F^+.
- $\bar{r}|_{G_{F^+_{2,v}}}$ is trivial for all places $v|l$ of F_2^+.
- For each place $v|l$, $[F^+_{2,v} : \mathbb{Q}_l] \geq 2$.
- There is a regular algebraic cuspidal automorphic representation π_2 of $\mathrm{GL}_2(\mathbb{A}_{F_2^+})$ of weight 0 such that

– $\bar{r}_{l,\iota}(\pi_2) \cong \bar{r}|_{G_{F_2^+}}$.
– π_2 is unramified at all finite places.
– For all places $v|l$ of F_2^+, $r_{l,\iota}(\pi_2)|_{G_{F_{2,v}^+}}$ is non-ordinary.

We now employ Lemma 2.1.2 in the following setting:

- F^+ is the present F_2^+.
- l is as in the present setting.
- $m = n = 2$.
- $F^{(\mathrm{avoid})} = F_2^+ \overline{F^+}^{\ker \bar{r}}(\zeta_l)$.
- $T = \emptyset$.
- $\eta = \eta' = 1$.
- $\{h_{1,\tau}, h_{2,\tau}\} = \{0, 1\}$ for each τ. Furthermore, for each place $v|l$, there is at least one τ corresponding to v with $h_{1,\tau} = 0$, and at least one τ corresponding to v with $h_{1,\tau} = 1$.
- $h'_{1,\tau} = 0$ and $h'_{2,\tau} = 2$ for all τ.
- $w = 1$
- $w' = 2$

We obtain a quadratic CM extension M/F_2^+ together with two continuous characters

$$\theta, \theta' : G_M \to \overline{\mathbb{Z}}_l^\times$$

such that

(1) θ, θ' are congruent modulo l.
(2) $(\bar{r}|_{G_{F_2^+}} \otimes \mathrm{Ind}_{G_M}^{G_{F_2^+}} \bar{\theta})(G_{F_2^+(\zeta_l)})$ is big.
(3) $\zeta_l \notin \overline{F_2^+}^{\mathrm{ad}(\bar{r}|_{G_{F_2^+}} \otimes \mathrm{Ind}_{G_M}^{G_{F_2^+}} \bar{\theta})}$.
(4) $(\mathrm{Ind}_{G_M}^{G_{F_2^+}} \theta) \cong (\mathrm{Ind}_{G_M}^{G_{F_2^+}} \theta)^\vee \otimes \epsilon^{-1}$.
(5) $(\mathrm{Ind}_{G_M}^{G_{F_2^+}} \theta') \cong (\mathrm{Ind}_{G_M}^{G_{F_2^+}} \theta')^\vee \otimes \epsilon^{-2}\tilde{\omega}$.
(6) For each v above l, the representation $(\mathrm{Ind}_{G_M}^{G_{F_2^+}} \theta)|_{G_{F_{2,v}^+}}$ is conjugate to a representation which breaks up as a direct sum of characters:

$$(\mathrm{Ind}_{G_M}^{G_{F_2^+}} \theta)|_{G_{F_{2,v}^+}} \cong \chi_1^{(v)} \oplus \chi_2^{(v)}$$

where, for each $i = 1, 2$, and each embedding $\tau : F_{2,v}^+ \to \overline{\mathbb{Q}}_l$ we have that:

$$\mathrm{HT}_\tau(\chi_i^{(v)}) = h_{i,\tau}.$$

In particular, the representation $(\mathrm{Ind}_{G_M}^{G_{F_2^+}} \theta)|_{G_{F_{2,v}^+}}$ is Barsotti-Tate and non-ordinary. Similarly, the representation $(\mathrm{Ind}_{G_M}^{G_{F_2^+}} \theta')|_{G_{F_{2,v}^+}}$ is conjugate to a representation which breaks up as a direct sum of characters:

$$(\mathrm{Ind}_{G_M}^{G_{F_2^+}} \theta')|_{G_{F_{2,v}^+}} \cong \chi_1'^{(v)} \oplus \chi_2'^{(v)}$$

where, for each $i = 1, 2$, and each embedding $\tau : F_{2,v}^+ \to \overline{\mathbb{Q}}_l$ we have that:

$$\mathrm{HT}_\tau(\chi_i'^{(v)}) = h'_{i,\tau}.$$

Thus by the choice of the $h'_{i,\tau}$, $\chi_1'^{(v)}|_{I_{F^+_{2,v}}}$ has finite order, and $\chi_2'^{(v)}|_{I_{F^+_{2,v}}}$ is a finite order character times ϵ^{-2}, so that $(\mathrm{Ind}_{G_M}^{G_{F_2^+}}\theta')|_{G_{F^+_{2,v}}}$ is ordinary.

Let F_3^+/F_2^+ be a solvable extension of totally real fields such that

- F_3^+ is linearly disjoint from $\overline{F^+}^{\ker\bar{r}}(\zeta_l)$ over F^+, and
- $(\mathrm{Ind}_{G_M}^{G_{F_2^+}}\theta)|_{G_{F_3^+}}$ and $(\mathrm{Ind}_{G_M}^{G_{F_2^+}}\theta')|_{G_{F_3^+}}$ are both unramified at all places of F_3^+ not lying over l, and crystalline at all places dividing l.
- If v is a place of F_3^+ lying over l, then $(\mathrm{Ind}_{G_M}^{G_{F_2^+}}\bar{\theta})|_{G_{F_3^+,v}}$ is trivial.
- If v is a place of F_3^+ lying over l, then $F_{3,v}^+$ contains a primitive l-th root of unity.

Let F_3/F_3^+ be a quadratic CM extension which is linearly disjoint from $M\overline{F^+}^{\ker\bar{r}}(\zeta_l)$ over F^+, and in which all places of F_3^+ lying over l split completely. We choose $K\subset\overline{\mathbb{Q}}_l$ a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}$ and residue field k. Assume that K is sufficiently large that it contains the image of every embedding $F_3\hookrightarrow\overline{\mathbb{Q}}_l$, and the images of θ and θ' and that $r_{l,\iota}(\pi_2)$ can be defined over K. Choosing stable lattices, we now regard $r_{l,\iota}(\pi_2)$, $(\mathrm{Ind}_{G_M}^{G_{F_2^+}}\theta)|_{G_{F_3^+}}$ and $(\mathrm{Ind}_{G_M}^{G_{F_2^+}}\theta')|_{G_{F_3^+}}$ as representations to $\mathrm{GL}_2(\mathcal{O})$, and (after conjugating if necessary) we can and do suppose that $(\mathrm{Ind}_{G_M}^{G_{F_2^+}}\bar{\theta})|_{G_{F_3^+}}=(\mathrm{Ind}_{G_M}^{G_{F_2^+}}\bar{\theta}')|_{G_{F_3^+}}$. Note that there is a character $\chi_{\pi_2}:G_{F_2^+}\to\mathcal{O}^\times$ such that

$$r_{l,\iota}(\pi_2)\cong r_{l,\iota}(\pi_2)^\vee\chi_{\pi_2}.$$

We now apply Theorem 3.5.1 in the following setting:

- $m=n=2$, and l is as it has been throughout this section.
- $F=F'=F_3$.
- $\bar{r}:=\bar{r}|_{G_{F_3}}$.
- $\chi=\epsilon^{-1}\tilde{\omega}\chi_{\pi_2}|_{G_{F_3^+}}$.
- $\chi'=\epsilon^{-1}$.
- $\chi''=\epsilon^{-2}\tilde{\omega}\chi_{\pi_2}|_{G_{F_3^+}}\delta_{F_3/F_3^+}$.
- $\tilde{S}_l$ is any set of places of F_3 consisting of exactly one place above each place in S_l.
- For each place $\tilde{v}\in\tilde{S}_l$, the element $a_{\tilde{v}}\in(\mathbb{Z}^2_+)^{\mathrm{Hom}(F_{3,\tilde{v}},\overline{\mathbb{Q}}_l)}$ is given by setting $a_{\tau,1}=1$ and $a_{\tau,2}=0$ for each $\tau:F_{3,\tilde{v}}\hookrightarrow\overline{\mathbb{Q}}_l$.
- $R_{\tilde{v}}$ is the unique ordinary component of $R^{\mathbf{v}_{a_{\tilde{v}}},cr}_{\bar{r}|_{G_{F_{3,\tilde{v}}}}}$ (note that $\bar{r}|_{G_{F_{3,\tilde{v}}}}$ is trivial, and $\bar{\epsilon}|_{G_{F_{3,\tilde{v}}}}$ is trivial, so there is a unique ordinary component by Lemma 3.4.3 of [Ger09]).
- $r'=(\mathrm{Ind}_{G_M}^{G_{F_2^+}}\theta)|_{G_{F_3}}$.
- $r''=r_{l,\iota}(\pi_2)|_{G_{F_3}}\otimes(\mathrm{Ind}_{G_M}^{G_{F_2^+}}\theta')|_{G_{F_3}}$.

We now check that the hypotheses of Theorem 3.5.1 hold.

(1),(3): That $\bar{r}^c \cong \bar{r}^\vee \overline{\chi}|_{G_{F_3}}$, and $\bar{r}$ is odd, follow directly from the definitions and the fact that $\bar{r}$, being modular, is odd (see the discussion of section 3.1.3). That $(r')^c \cong (r')^\vee \chi'|_{G_{F_3}}$ follows from (4) above.

(2),(4) That $\bar{r}, r', r'', \chi, \chi'$ and χ'' are unramified away from l and r' is crystalline above l follows from the choice of F_3 and properties of π_2.

(5) r'' is automorphic of level prime to l by Proposition 5.1.3 of [BLGG09]. It satisfies $(r'')^c \cong (r'')^\vee \chi''|_{G_{F_3}}$ by construction.

(6) $\bar{r}'' = \bar{r} \otimes \bar{r}'$ because $\bar{\theta} = \bar{\theta}'$ and $r_{l,\iota}(\pi_2)|_{G_{F_3}} = \bar{r}$. By definition we have $\chi''|_{G_{F_3}} = \chi|_{G_{F_3}} \chi'|_{G_{F_3}}$.

(7), (8) That $\bar{r}''(G_{F_3(\zeta_l)})$ is big and $\bar{F}^{\ker \operatorname{ad} \bar{r}''}$ does not contain ζ_l follow from property (2) of Lemma 2.1.2, the assumptions on $\bar{r}$ and the choice of F_3.

(9) For all places $v \in \tilde{S}_l$, if $\rho_v : G_{F_{3,\tilde{v}}} \to \mathrm{GL}_2(\mathcal{O})$ is a lifting of $\bar{r}|_{G_{F_{3,\tilde{v}}}}$ corresponding to a closed point on $R_{\tilde{v}}$, then ρ_v is ordinary (by the definition of $R_{\tilde{v}}$), so $\rho_v \sim (\operatorname{Ind}_{G_M}^{G_{F_2^+}} \theta')|_{G_{F_{3,\tilde{v}}}}$ by Lemma 3.4.3 of [Ger09] (because both representations are crystalline and ordinary with the same Hodge–Tate weights, and trivial reduction). Similarly, $r'|_{G_{F_{3,\tilde{v}}}} = (\operatorname{Ind}_{G_M}^{G_{F_2^+}} \theta)|_{G_{F_{3,\tilde{v}}}} \sim r_{l,\iota}(\pi_2)|_{G_{F_{3,\tilde{v}}}}$ by Proposition 2.3 of [Gee06b] (because both representations are Barsotti-Tate and non-ordinary with trivial reduction). By the results of sections 3.3 and 3.4 of [BLGG09], it follows that $\rho_v \otimes r'|_{G_{F_{3,\tilde{v}}}} \sim (\operatorname{Ind}_{G_M}^{G_{F_2^+}} \theta')|_{G_{F_{3,\tilde{v}}}} \otimes r_{l,\iota}(\pi_2)|_{G_{F_{3,\tilde{v}}}} \sim r_{l,\iota}(\pi_2)|_{G_{F_{3,\tilde{v}}}} \otimes (\operatorname{Ind}_{G_M}^{G_{F_2^+}} \theta')|_{G_{F_{3,\tilde{v}}}} = r''|_{G_{F_{3,\tilde{v}}}}$.

We conclude that, after possibly extending $\mathcal{O}$, there is a continuous lifting $r : G_{F_3} \to \mathrm{GL}_2(\mathcal{O})$ of $\bar{r}|_{G_{F_3}}$ such that:

- r is unramified at all places of F not dividing l.
- $r|_{G_{F_{3,\tilde{v}}}}$ is crystalline and ordinary at each place $\tilde{v} \in \tilde{S}_l$, with Hodge–Tate weights 0 and 2.
- $r^c \cong r^\vee \chi|_{G_F}$.
- $r \otimes (\operatorname{Ind}_{G_M}^{G_{F_2^+}} \theta)|_{G_{F_3}}$ is automorphic of level prime to l.

Applying Proposition 5.1.1 (noting that $r|_{G_{MF_3}}$ is certainly irreducible, because $\bar{r}|_{G_{MF_3}}$ is irreducible), we deduce that in fact

- r is automorphic of level prime to l.

Since, in addition, $r|_{G_{F_{3,\tilde{v}}}}$ is ordinary for all $\tilde{v} \in \tilde{S}_l$, it follows from Lemma 5.2.1 of [Ger09] that in fact

- r is ι-ordinarily automorphic of level prime to l.

By Lemma 6.1.2 we can and do choose a solvable extension F_4^+/F_3^+ of totally real fields together with a RAECSDC automorphic representation π_4 of $\mathrm{GL}_2(\mathbb{A}_{F_4})$ of weight 0 where $F_4 = F_4^+ F_3$ such that

- F_4^+ is linearly disjoint from $F_3 \overline{F^+}^{\ker \bar{r}}(\zeta_l)$ over F_3^+.
- π_4 is unramified at all finite places and ι-ordinary.
- $\bar{r}_{l,\iota}(\pi) \cong \bar{r}|_{G_{F_4}}$.
- $r_{l,\iota}(\pi_4)^c \cong r_{l,\iota}(\pi_4)^\vee \epsilon^{-1} \widetilde{\omega} \widetilde{\chi}|_{G_{F_4}}$ where $\widetilde{\chi}$ is the Teichmüller lift of $\overline{\chi}$.

Enlarging $\mathcal{O}$ if necessary, we may assume that $r_{l,\iota}(\pi_4)$ is valued in $\mathrm{GL}_2(\mathcal{O})$. Let $\chi_{\pi_4} = \epsilon^{-1} \widetilde{\omega} \widetilde{\chi}|_{G_{F_4^+}}$.

We now consider two deformation problems, one for $\bar{r}|_{G_{F_4^+}}$ and one for $\bar{r}|_{G_{F_4}}$. Let $R_{\bar{r}|_{G_{F_4^+}}}$ be the universal deformation ring for ordinary Barsotti-Tate deformations of $\bar{r}|_{G_{F_4^+}}$ which have determinant χ_{π_4} and which are unramified outside l. Let S_l' be the set of places of F_4^+ above l, and let $\tilde{S}_l'$ be the set of places of F_4 which lie above places in $\tilde{S}_l$. We can and do extend $\bar{r}|_{G_{F_4}}$ to a representation $\bar{r}_4 : G_{F_4^+} \to \mathcal{G}_2(k)$. For each place $\tilde{v} \in \tilde{S}_l'$, let R_v be the unique ordinary component of the Barsotti-Tate lifting ring for $\bar{r}|_{G_{F_{4,\tilde{v}}}}$. Let $\mathcal{S}$ be the deformation problem (in the sense of section 3.1.5)

$$(F_4/F_4^+, S_l', \tilde{S}_l', \mathcal{O}, \bar{r}_4, \chi_{\pi_4}, \{R_v\}_{v \in S_l'}).$$

Then by Proposition 3.6.3 of [BLGG09], the corresponding universal deformation ring $R_{\mathcal{S}}$ is a finite $\mathcal{O}$-algebra. Furthermore, $R_{\bar{r}|_{G_{F_4^+}}}$ is a finite $R_{\mathcal{S}}$-algebra in a natural fashion (cf. section 7.4 of [GG09]). In addition, by Proposition 3.1.4 of [Gee06a], $\dim R_{\bar{r}|_{G_{F_4^+}}} \geq 1$. Thus $R_{\bar{r}|_{G_{F_4^+}}}$ is a finite $\mathcal{O}$-algebra of rank at least one (cf. the proof of Theorem 4.2.8 of [Kis07b]), and so it has a $\overline{\mathbb{Q}}_l$-point, which corresponds to a deformation $r_4 : G_{F_4^+} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ of $\bar{r}|_{G_{F_4^+}}$ which is Barsotti-Tate and ordinary at each place dividing l, and which is unramified outside l. Furthermore, by Theorem 5.4.2 of [Ger09], $r_4|_{G_{F_4}}$ is automorphic, so that r_4 is automorphic by Lemma 1.5 of [BLGHT09], as required. □

We can prove a similar result more directly in the case that $\bar{r}$ is induced from a CM extension.

Proposition 6.1.4. *Suppose that $l \geq 3$ is a prime, and that M/F^+ is a quadratic extension, with M a CM field. Let $\bar{\theta} : G_M \to \overline{\mathbb{F}}_l^\times$ be a continuous character that does not extend to G_{F^+}, so that $\bar{r} := \mathrm{Ind}_{G_M}^{G_{F^+}} \bar{\theta}$ is an irreducible modular representation. Then there is a finite solvable extension L^+/F^+ such that*

- *L^+ is linearly disjoint from $M\overline{F^+}^{\ker \bar{r}}(\zeta_l)$ over F^+.*
- *There is an ι-ordinary regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_{L^+})$ of weight 0 and level prime to l such that*
  - *$\bar{r}_{l,\iota}(\pi) \cong \bar{r}|_{G_{L^+}}$.*
  - *π is unramified at all finite places.*

Proof. We construct π as an automorphic induction. Choose F_1^+/F^+ a solvable extension of totally real fields such that

- F_1^+ is linearly disjoint from $M\overline{F^+}^{\ker \bar{r}}(\zeta_l)$ over F^+.
- Every place $v|l$ of F_1^+ splits in F_1^+M.

We now apply Lemma 4.1.6 of [CHT08] with

- $F = F_1^+M$.
- S the set of places of F dividing l.
- $\bar{\theta}$ equal to our $\bar{\theta}|_{G_F}$.
- $\chi = \epsilon^{-1}(\widetilde{\det \bar{r}|_{G_{F_1^+}} \bar{\epsilon}})$, where a tilde denotes the Teichmüller lift.
- For each place $v|l$ of F_1^+, write the places of F lying over v as $\tilde{v}$ and $\tilde{v}^c$. Then

$$\psi_{\tilde{v}} = \widetilde{\bar{\theta}|_{G_{F_{\tilde{v}}}}},$$

and

$$\psi_{\tilde{v}^c} = \epsilon^{-1}\widetilde{\bar{\epsilon}\bar{\theta}|_{G_{F_{\tilde{v}}^c}}}.$$

We conclude that there is an algebraic character $\theta : G_{F_1^+M} \to \overline{\mathbb{Z}}_l^\times$ lifting $\bar{\theta}$, such that $\mathrm{Ind}_{G_{F_1^+M}}^{G_{F_1^+}} \theta$ is ordinary and potentially Barsotti-Tate. Choose F_2^+/F_1^+ a solvable extension of totally real fields such that

- F_2^+ is linearly disjoint from $M\overline{F^+}^{\ker \bar{r}}(\zeta_l)$ over F^+.
- $\theta|_{G_{F_2^+M}}$ is crystalline at all places dividing l, and unramified at all other places.
- F_2^+M/F_2^+ is unramified at all finite places.

Then by Theorem 4.2 of [AC89], there is a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_{F_2^+})$ of weight 0 and level prime to l such that

$$r_{l,\iota}(\pi) \cong \mathrm{Ind}_{G_{F_2^+M}}^{G_{F_2^+}} \theta|_{G_{F_2^+M}}.$$

The result follows, as π is ι-ordinary by Lemma 5.2.1 of [Ger09]. □

We now deduce the main results, essentially by combining the previous results with those of [Gee06a]. For the terminology of inertial types, see section 3 of [Gee06a] (although note the slightly different conventions for Hodge–Tate weights in force there).

Theorem 6.1.5. *Let $l \geq 3$ be prime, and let F^+ be a totally real field. Let $\bar{r} : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ be irreducible and modular. Fix a character $\psi : G_{F^+} \to \overline{\mathbb{Q}}_l^\times$ such that $\epsilon\psi$ has finite order, and $\bar{\psi} = \det \bar{r}$. Let S denote a finite set of finite places of F^+ containing all places at which $\bar{r}$ or ψ is ramified, and all places dividing l.*

For each place $v \in S$, fix an inertial type τ_v of $I_{F_v^+}$ on a $\overline{\mathbb{Q}}_l$-vector space, of determinant $(\psi\epsilon)|_{I_{F_v^+}}$. Assume that for each place $v \in S$, $v \nmid l$, $\bar{r}|_{G_{F_v^+}}$ has a lift of type τ_v and determinant $\psi|_{G_{F_v^+}}$, and for each place $v \in S$, $v|l$, $\bar{r}|_{G_{F_v^+}}$ has a potentially Barsotti-Tate lift of type τ_v and determinant $\psi|_{G_{F_v^+}}$. For each place $v \in S$, we let R_v denote an irreducible component of the corresponding lifting ring $R^{\square,\psi,\tau_v}[1/l]$ for (potentially Barsotti-Tate) lifts of type τ_v and determinant $\psi|_{G_{F_v^+}}$.

Assume further that

- *Either*
  - ∗ $l \geq 5$.
    ∗ $\bar{r}(G_{F^+(\zeta_l)})$ *is 2-big.*
    ∗ $[\overline{F^+}^{\ker \mathrm{ad}\, \bar{r}}(\zeta_l) : \overline{F^+}^{\ker \mathrm{ad}\, \bar{r}}] > 2$.

  Or:
  - ∗ *There is a quadratic CM extension M/F^+, with M not equal to the quadratic extension of F^+ in $F^+(\zeta_l)$, and a continuous character $\bar{\theta} : G_M \to \overline{\mathbb{F}}_l^\times$ such that $\bar{r} = \mathrm{Ind}_{G_M}^{G_{F^+}} \bar{\theta}$.*

Then there is a continuous representation $r : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{Z}}_l)$ lifting $\bar{r}$ of determinant ψ such that

- *r is modular.*
- *r is unramified at all places $v \notin S$.*

- *For each place $v|l$ of F^+, $r|_{G_{F_v^+}}$ is potentially Barsotti-Tate of type τ_v (and indeed corresponds to a point of R_v).*
- *For each place $v \in S$, $v \nmid l$, $r|_{G_{F_v^+}}$ has type τ_v (and indeed corresponds to a point of R_v).*

Proof. We argue as in the proofs of Proposition 3.1.5 and Corollary 3.1.7 of [Gee06a]. Indeed, examining the arguments of *loc. cit.*, we see that it is enough to demonstrate that there is a finite solvable extension of totally real fields F_2^+/F^+ such that:

- There is an ι-ordinary regular algebraic cuspidal automorphic representation π_2 of $\mathrm{GL}_2(\mathbb{A}_{F_2^+})$ of weight 0 which is unramified at every finite place, with $\bar{r}_{l,\iota}(\pi_2) \cong \bar{r}|_{G_{F_2^+}}$, and $\det r_{l,\iota}(\pi_2) = \psi|_{G_{F_2^+}}$.
- If w is a place of F_2^+ lying over a place $v \in S$, then $\tau_v|_{G_{F_{2,w}^+}}$ is trivial.
- $\bar{r}|_{G_{F_{2,w}^+}}$ is trivial for all places $w|l$ of F_2^+.
- $\bar{r}|_{G_{F_2^+(\zeta_l)}}$ is irreducible, $[F_2^+ : \mathbb{Q}]$ is even, and $[F_2^+(\zeta_l) : F_2^+] = [F^+(\zeta_l) : F^+]$.

To see that we can do this, note that by Propositions 6.1.3 and 6.1.4, there is a finite solvable extension of totally real fields F_1^+/F^+ and a regular algebraic ι-ordinary cuspidal automorphic representation π_1 of $\mathrm{GL}_2(\mathbb{A}_{F_1^+})$ of weight 0 such that

- π_1 is unramified at all finite places.
- $\bar{r}_{l,\iota}(\pi_2) \cong \bar{r}|_{G_{F_1^+}}$.
- F_1^+ is linearly disjoint from $\overline{F^+}^{\ker \bar{r}}(\zeta_l)$ over F^+.

Choose a solvable extension of totally real fields F_2^+/F_1^+ such that

- $\psi|_{G_{F_2^+}}$ is crystalline at all places dividing l, and unramified at all other finite places.
- If w is a place of F_2^+ lying over a place $v \in S$, then $\tau_v|_{G_{F_{2,w}^+}}$ is trivial.
- F_2^+ is linearly disjoint from $\overline{F^+}^{\ker \bar{r}}(\zeta_l)$ over F^+.
- $\bar{r}|_{G_{F_{2,w}^+}}$ is trivial for all places $w|l$ of F_2^+.
- $[F_2^+ : \mathbb{Q}]$ is even.

Let π_2 be the base change of π_1 to F_2^+. Note that $\psi|_{G_{F_2^+}}$ and $\det r_{l,\iota}(\pi_2)$ are crystalline characters with the same Hodge–Tate weights, and are both unramified at all places not dividing l, so they differ by a finite order totally even character which is unramified at all finite places. Thus we may replace π_2 by a twist by a finite order character to ensure that $\det r_{l,\iota}(\pi_2) = \psi$, without affecting any of the other properties of π_2 noted above. The result follows (since the hypotheses of the theorem imply that $\bar{r}|_{G_{F^+(\zeta_l)}}$ is irreducible). □

We thank Brian Conrad for showing us the argument for case 2 in the proof of the following lemma.

Lemma 6.1.6. *Suppose l is an odd prime. Let K be a finite extension of $\mathbb{Q}_l$, and $\bar{r} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ a continuous representation, with*

$$\bar{r} \cong \begin{pmatrix} \overline{\psi}_1 & * \\ 0 & \overline{\psi}_2\bar{\epsilon}^{-1} \end{pmatrix}$$

for some characters $\overline{\psi}_1$, $\overline{\psi}_2$. Let $\chi : G_K \to \overline{\mathbb{Z}}_l^\times$ be a finite order character lifting $\overline{\psi}_1\overline{\psi}_2$. Then there is a continuous potentially Barsotti-Tate representation $r : G_K \to \mathrm{GL}_2(\overline{\mathbb{Z}}_l)$ of determinant $\chi\epsilon^{-1}$ lifting $\bar{r}$ with

$$r \cong \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2\epsilon^{-1} \end{pmatrix}$$

for some potentially unramified characters ψ_1, ψ_2 lifting $\overline{\psi}_1$, $\overline{\psi}_2$.

Proof. We first remark that any lift of the form

$$r \cong \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2\epsilon^{-1} \end{pmatrix}$$

is automatically potentially crystalline if ψ_1 and ψ_2 are finitely ramified $\psi_1 \neq \psi_2$.

Case 1: $\overline{\psi}_1\overline{\psi}_2^{-1} \neq 1$: Choose an arbitrary finite-order lift ψ_1 of $\overline{\psi}_1$, and set $\psi_2 = \chi\psi_1^{-1}$. Let $E \subset \overline{\mathbb{Q}}_l$ be a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$ such that ψ_1 and ψ_2 are valued in $\mathcal{O}^\times$. By the remark above, it suffices to show that the natural map $H^1(G_K, \mathcal{O}(\psi_1\psi_2^{-1}\epsilon)) \to H^1(G_K, \mathbb{F}(\overline{\psi}_1\overline{\psi}_2^{-1}\overline{\epsilon}))$ is surjective. However, the cokernel of this map is a submodule of $H^2(G_K, \mathcal{O}(\psi_1\psi_2^{-1}\epsilon))$ which, by Tate-duality, is Pontryagin dual to $H^0(G_K, (E/\mathcal{O})(\psi_2\psi_1^{-1})) = \{0\}$.

Case 2: $\overline{\psi}_1\overline{\psi}_2^{-1} = 1$: Choose finitely ramified lifts ψ_1 and ψ_2 of $\overline{\psi}_1$ and $\overline{\psi}_2$ such that $\chi = \psi_1\psi_2$. Let $\psi = \psi_1\psi_2^{-1}$. Choose $E \subset \overline{\mathbb{Q}}_l$, a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$ such that ψ_1 and ψ_2 are valued in $\mathcal{O}^\times$. Let L be the line in $H^1(G_K, \mathbb{F}(\overline{\epsilon}))$ determined $\bar{r}$ (an extension of $\overline{\psi}_2\overline{\epsilon}^{-1}$ by $\overline{\psi}_1$) and let H be the hyperplane in $H^1(G_K, \mathbb{F})$ which annihilates L under the Tate pairing. Let $\varpi \in \mathcal{O}$ be a uniformizer. Let $\delta_1 : H^1(G_K, \mathbb{F}(\overline{\epsilon})) \to H^2(G_K, \mathcal{O}(\psi\epsilon))$ be the map coming from the exact sequence $0 \to \mathcal{O}(\psi\epsilon) \stackrel{\varpi}{\to} \mathcal{O}(\psi\epsilon) \to \mathbb{F}(\overline{\epsilon}) \to 0$ of G_K-modules. We need to show that $\delta_1(L) = 0$.

Let δ_0 be the map $H^0(G_K, (E/\mathcal{O})(\psi^{-1})) \to H^1(G_K, \mathbb{F})$ coming from the exact sequence $0 \to \mathbb{F} \to (E/\mathcal{O})(\psi^{-1}) \stackrel{\varpi}{\to} (E/\mathcal{O})(\psi^{-1}) \to 0$ of G_K-modules. By Tate-duality, the condition that L vanishes under the map δ_1 is equivalent to the condition that the image of the map δ_0 is contained in H. Let $n \geq 1$ be the largest integer with the property that $\psi^{-1} \equiv 1 \mod \varpi^n$. Then we can write $\psi^{-1}(x) = 1 + \varpi^n\alpha(x)$ for some function $\alpha : G_K \to \mathcal{O}$. Let $\overline{\alpha}$ denote $\alpha \mod \varpi : G_K \to \mathbb{F}$. Then $\overline{\alpha}$ is additive and the choice of n ensures that it is non-trivial. It is straightforward to check that the image of the map δ_0 is the line spanned by $\overline{\alpha}$. If $\overline{\alpha}$ is in H, we are done. Suppose this is not the case. We break the rest of the proof into two cases.

Case 2a: $\overline{\psi}_1\overline{\psi}_2^{-1} = 1$, and $\bar{r}$ is trés ramifié: We can and do suppose that we have chosen ψ_1 and ψ_2 so that ψ is ramified and further, that $\overline{\alpha}$ is ramified. The fact that $\bar{r}$ is trés ramifié implies that H does not contain the unramified line in $H^1(G_K, \mathbb{F})$. Thus there is a unique $\overline{x} \in \mathbb{F}^\times$ such that $\overline{\alpha} + u_{\overline{x}} \in H$ where $u_{\overline{x}} : G_K \to \mathbb{F}$ is the unramified homomorphism sending Frob_K to $\overline{x}$. Let y be an element of $\mathcal{O}^\times$ with $2\overline{y} = \overline{x}$. Replacing ψ_1 with ψ_1 times the unramified character sending Frob_K to $(1 + \varpi^n y)^{-1}$ and ψ_2 with ψ_2 times the unramified character sending Frob_K to $1 + \varpi^n y$, we are done.

Case 2b: $\overline{\psi}_1\overline{\psi}_2^{-1} = 1$, and $\bar{r}$ is peu ramifié: Making a ramified extension of $\mathcal{O}$ if necessary, we can and do assume that $n \geq 2$. The fact that $\bar{r}$ is peu ramifié implies that H contains the unramified line. It follows that if we replace ψ_1 with

ψ_1 times the unramified character sending Frob_K to $1+\varpi$ and ψ_2 with ψ_2 times the unramified character sending Frob_K to $(1+\varpi)^{-1}$, then we are done (as the new $\overline{\alpha}$ will be unramified). □

Using the previous two results, we can now give our main result on the existence of ordinary modular lifts.

Theorem 6.1.7. *Let $l \geq 3$ be prime, and let F^+ be a totally real field. Let $\bar{r} : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ be irreducible and modular. Fix a character $\psi : G_{F^+} \to \overline{\mathbb{Q}}_l^\times$ such that $\epsilon\psi$ has finite order, and $\bar{\psi} = \det \bar{r}$. Assume further that*

- *For every place $v|l$ of F^+, $\bar{r}|_{G_{F_v^+}}$ is reducible.*
- *Either*
  - * $l \geq 5$.
    * $\bar{r}(G_{F^+(\zeta_l)})$ *is 2-big.*
    * $[\overline{F^+}^{\ker \mathrm{ad}\, \bar{r}}(\zeta_l) : \overline{F^+}^{\ker \mathrm{ad}\, \bar{r}}] > 2$.

  Or:
  - * *There is a quadratic CM extension M/F^+, with M not equal to the quadratic extension of F^+ in $F^+(\zeta_l)$, and a continuous character $\bar{\theta} : G_M \to \overline{\mathbb{F}}_l^\times$ such that $\bar{r} = \mathrm{Ind}_{G_M}^{G_{F^+}} \bar{\theta}$.*

Then there is a continuous representation $r : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{Z}}_l)$ lifting $\bar{r}$ such that

- *r is modular.*
- *For each place $v|l$ of F^+, $r|_{G_{F_v^+}}$ is potentially Barsotti-Tate and ordinary.*
- $\det r = \psi$.

Proof. For each place $v|l$ of F^+, using Lemma 6.1.6, choose a potentially Barsotti-Tate lift $r^{(v)} : G_{F_v^+} \to \mathrm{GL}_2(\overline{\mathbb{Z}}_l)$ of $\bar{r}|_{G_{F_v^+}}$ with

$$r^{(v)} \cong \begin{pmatrix} \psi_1^{(v)} & * \\ 0 & \psi_2^{(v)}\epsilon^{-1} \end{pmatrix}$$

for some finitely ramified characters $\psi_1^{(v)}$, $\psi_2^{(v)}$ with $\psi_1^{(v)}\psi_2^{(v)} = \epsilon\psi|_{G_{F_v}}$. Let S denote the set of primes of F^+ not dividing l at which $\bar{r}$ or ψ is ramified. For each $v \in S$, choose a lift $r^{(v)}$ of $\bar{r}|_{G_{F_v^+}}$ of determinant $\psi|_{G_{F_v^+}}$ (that this is possible follows easily from Lemma 3.1.4 of [Kis07c], which shows that $\bar{r}$ has a (global) modular lift of determinant ψ). For each $v \in S \cup \{v|l\}$, let τ_v be the inertial type of $r^{(v)}$ and let R_v denote an irreducible component of the lifting ring $R^{\square,\psi,\tau_v}_{\bar{r}|_{G_{F_v^+}}}[1/l]$ containing $r^{(v)}$. (The closed points of this ring correspond to lifts which have type τ_v, determinant ψ and are potentially Barsotti-Tate if $v|l$.) We note that if $v|l$, then R_v is unique and moreover any other lift of $\bar{r}|_{G_{F_v^+}}$ corresponding to a closed point of $R_v[1/l]$ is ordinary. The result now follows from Theorem 6.1.5. □

Combining this result with the improvements made in [Gee06b] to the modularity lifting theorem of [Kis07c], we obtain the following theorem, where in contrast to previous results in the area we do not need to assume that $\bar{r}$ has a modular lift which is ordinary at a specified set of places.

Theorem 6.1.8. *Let $l \geq 3$ be a prime, F^+ a totally real field, and $r : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ a continuous representation unramified outside of a finite set of primes,* *with determinant a finite order character times the inverse of the cyclotomic character. Suppose further that*

1. *r is potentially Barsotti-Tate for each $v|l$.*
2. *$\bar{r}$ is modular and irreducible.*
3. *Either*
   - – $l \geq 5$.
     – $\bar{r}(G_{F^+(\zeta_l)})$ *is 2-big.*
     – $[\overline{F^+}^{\ker \mathrm{ad}\, \bar{r}}(\zeta_l) : \overline{F^+}^{\ker \mathrm{ad}\, \bar{r}}] > 2$.

   Or:
   - – *There is a quadratic CM extension M/F^+, with M not equal to the quadratic extension of F^+ in $F^+(\zeta_l)$, and a continuous character $\bar{\theta} : G_M \to \overline{\mathbb{F}}_l^\times$ such that $\bar{r} = \mathrm{Ind}_{G_M}^{G_{F^+}} \bar{\theta}$.*

Then r is modular.

Proof. This follows from Theorem 6.1.5, together with Theorem 1.1 of [Gee09]. More precisely, let S be the set of places of F^+ for which $r|_{G_{F_v^+}}$ is ramified (so that S automatically contains all the places dividing l). Let $\psi = \det r$. After conjugating, we may assume that r takes values in $\mathrm{GL}_2(\overline{\mathbb{Z}}_l)$. For each $v \in S$, let τ_v be the type of $r|_{G_{F_v^+}}$, and let R_v be a component of the lifting ring $R^{\square,\psi,\tau_v}[1/l]$ corresponding to $r|_{G_{F_v^+}}$. (The closed points of this ring correspond to lifts which have type τ_v, determinant ψ and are potentially Barsotti-Tate if $v|l$.) Applying Theorem 6.1.5, we deduce that there is a modular lift r' of $\bar{r}$ which is potentially Barsotti-Tate for all places $v|l$, with $r'|_{G_{F_v^+}}$ potentially ordinary for precisely the places at which $r|_{G_{F_v^+}}$ is potentially ordinary. The modularity of r then follows immediately from Theorem 1.1 of [Gee09]. □

We can use the results of section 4 to deduce corollaries of these theorems in which the conditions on the image of $\bar{r}$ are more explicit. For example, we have the following result.

Theorem 6.1.9. *Let $l \geq 7$ be prime, and let F^+ be a totally real field. Let $\bar{r} : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ be irreducible and modular. Fix a character $\psi : G_{F^+} \to \overline{\mathbb{Q}}_l^\times$ such that $\epsilon\psi$ has finite order, and $\bar{\psi} = \det \bar{r}$. Let S denote a finite set of finite places of F^+ containing all places at which $\bar{r}$ or ψ is ramified, and all places dividing l.*

For each place $v \in S$, fix an inertial type τ_v of $I_{F_v^+}$ on a $\overline{\mathbb{Q}}_l$-vector space, of determinant $(\psi\epsilon)|_{I_{F_v^+}}$. Assume that for each place $v \in S$, $v \nmid l$, $\bar{r}|_{G_{F_v^+}}$ has a lift of type τ_v and determinant $\psi|_{G_{F_v^+}}$, and for each place $v \in S$, $v|l$, $\bar{r}|_{G_{F_v^+}}$ has a potentially Barsotti-Tate lift of type τ_v and determinant $\psi|_{G_{F_v^+}}$. For each place $v \in S$, we let R_v denote an irreducible component of the corresponding lifting ring $R^{\square,\psi,\tau_v}[1/l]$ for (potentially Barsotti-Tate) lifts of type τ_v and determinant $\psi|_{G_{F_v^+}}$.

Assume further that

- $[F^+(\zeta_l) : F^+] > 4$.
- $\bar{r}|_{G_{F^+(\zeta_l)}}$ *is irreducible.*

Then there is a continuous representation $r : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{Z}}_l)$ lifting $\bar{r}$ of determinant ψ such that

- *r is modular.*

- *r is unramified at all places $v \notin S$.*
- *For each place $v|l$ of F^+, $r|_{G_{F_v^+}}$ is potentially Barsotti-Tate of type τ_v (and indeed corresponds to a point of R_v).*
- *For each place $v \in S$, $v \nmid l$, $r|_{G_{F_v^+}}$ has type τ_v (and indeed corresponds to a point of R_v).*

Proof. By Lemma 4.4.2 and its proof, we see that if the projective image of $\bar{r}$ is not dihedral or A_4, then $\bar{r}(G_{F^+(\zeta_l)})$ is 2-big. Furthermore, in these cases the image of $\operatorname{ad}\bar{r}$ is either of the form $\mathrm{PGL}_2(k)$, $\mathrm{PSL}_2(k)$, S_4 or A_5, so that $\operatorname{ad}\bar{r}(G_{F^+})$ has no cyclic quotients of order greater than 2. By the assumption that $[F^+(\zeta_l) : F^+] > 4$, we deduce that $[\overline{F^+}^{\ker \operatorname{ad}\bar{r}}(\zeta_l) : \overline{F^+}^{\ker \operatorname{ad}\bar{r}}] > 2$, so in this case the result follows directly from Theorem 6.1.5.

In the remaining cases, we see from Lemma 4.4.2 that there are extensions $K_2/K_1/F^+$, with K_2/K_1 quadratic and K_1/F^+ cyclic of degree 1 or 3, together with a continuous character $\bar{\theta} : K_2 \to \overline{\mathbb{F}}_l^\times$ such that $\bar{r}|_{G_{K_1}} \cong \operatorname{Ind}_{G_{K_2}}^{G_{K_1}} \bar{\theta}$. Then K_1 is totally real (as it is an extension of F^+ of odd degree), and K_2 is CM (because $\bar{r}$, being modular, is totally odd). Furthermore, $\bar{r}|_{G_{K_1^+(\zeta_l)}}$ is irreducible because $\bar{r}|_{G_{F^+(\zeta_l)}}$ is irreducible, by assumption, and K_1/F^+ is cyclic of degree 1 or 3. Thus we can apply Proposition 6.1.4 to $\bar{r}|_{G_{K_1}}$, and the result follows as in the proof of Theorem 6.1.5. □

From this, we immediately deduce the following versions of Theorems 6.1.7 and 6.1.8.

Theorem 6.1.10. *Let $l \geq 7$ be prime, and let F^+ be a totally real field. Let $\bar{r} : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ be a modular representation such that $\bar{r}|_{G_{F^+(\zeta_l)}}$ is irreducible. Assume that $[F^+(\zeta_l) : F^+] > 4$. Assume further that*

- *For every place $v|l$ of F^+, $\bar{r}|_{G_{F_v^+}}$ is reducible.*

Then there is a continuous representation $r : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{Z}}_l)$ lifting $\bar{r}$ such that

- *r is modular.*
- *For each place $v|l$ of F^+, $r|_{G_{F_v^+}}$ is potentially Barsotti-Tate and ordinary.*

Theorem 6.1.11. *Let $l \geq 7$ be a prime, F^+ a totally real field, and $r : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ a continuous representation unramified outside of a finite set of primes, with determinant a finite order character times the inverse of the cyclotomic character. Suppose further that*

(1) r is potentially Barsotti-Tate for each $v|l$.
(2) $\bar{r}$ is modular.
(3) $\bar{r}|_{G_{F^+(\zeta_l)}}$ is irreducible.
(4) $[F^+(\zeta_l) : F^+] > 4$.

Then r is modular.

We also deduce improvements such as the following to the main theorem of [Gee06c], to which we refer the reader for the definitions of the terminology used.

Theorem 6.1.12. *Let $l \geq 7$ be prime which is unramified in a totally real field F^+, and let $\bar{r} : G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ be an irreducible modular representation. Assume that $\bar{r}|_{G_{F^+(\zeta_l)}}$ is irreducible.*

Let σ be a regular weight. Then $\bar{r}$ is modular of weight σ if and only if $\sigma \in W(\bar{r})$.

Proof. Note that since $l \geq 7$ and l is unramified in F^+, $[F^+(\zeta_l) : F^+] > 4$. The result now follows by replacing the use of Corollary 3.1.7 of [Gee06a] with Theorem 6.1.9 above. □

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