LOCAL-GLOBAL COMPATIBILITY FOR $l = p$, II.

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Abstract. We prove the compatibility at places dividing l of the local and global Langlands correspondences for the l-adic Galois representations associated to regular algebraic essentially (conjugate) self-dual cuspidal automorphic representations of GL_n over an imaginary CM or totally real field. We prove this compatibility up to semisimplification in all cases, and up to Frobenius semisimplification in the case of Shin-regular weight.

Résumé. **(Compatibilité entre les correspondances de Langlands locale aux places divisant l, II.)** Nous prouvons la compatibilité entre les correspondances de Langlands locale et globale aux places divisant l pour les représentations galoisiennes l-adiques associèes à des représentations automorphes cuspidales algébriques régulières de GL_n sur un corps CM ou totalement réel qui sont duales de leur conjuguée complexe à un twist près. Nous prouvons cette compatibilité à semi-simplification près dans tous les cas, et à semi-simplification de Frobenius près lorsque le poids est régulier au sens de Shin.

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Introduction.

Thanks to the work of (among others) Chenevier, Clozel, Harris, Kottwitz, Labesse, Shin and R.T., given F an imaginary CM field or totally real field, and (Π, χ) a regular, algebraic, essentially (conjugate) self-dual automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$, if l is prime and we fix some $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, then there is a semisimple l-adic Galois representation $r_{l,\imath}(\Pi) : G_F \to \mathrm{GL}_m(\overline{\mathbb{Q}}_l)$, where G_F is the absolute Galois group of F. This representation is uniquely determined by the requirement that it satisfies local-global compatibility at the unramified places. It is also expected to satisfy local-global compatibility at all finite places; this has been established for the places not dividing l by Caraiani ([Car10]), building on the work of Harris–Taylor, Taylor–Yoshida, Shin and Chenevier–Harris.

It is important in some applications to have this compatibility at places dividing l; for example, our original motivation for considering this problem was to improve the applicability of the main results of [BLGGT10]; in that paper a variety of automorphy lifting theorems are proved via making highly ramified base changes, and one loses control of the level of the automorphic representations under consideration. This control can be recovered if one knows local-global compatibility at primes dividing l, and this is important in applications to the weight part of Serre's conjecture (cf. [BLGG11a], [BLGG11b]).

Our main result is as follows (see Theorem 1.1 and Corollary 1.2).

Theorem A. *Let F be an imaginary CM field or totally real field, let (Π, χ) be a regular, algebraic, essentially (conjugate) self-dual automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ and let $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. If $v|l$ is a place of F, then*

$$\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\mathrm{ss}} \cong \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2})^{\mathrm{ss}}.$$

Furthermore, if Π has Shin-regular weight, then

$$\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

Here $\mathrm{WD}(r)$ denotes the Weil–Deligne representation attached to a de Rham l-adic representation r of the absolute Galois group of an l-adic field; and rec denotes the local Langlands correspondence; and F-ss denotes Frobenius semi-simplification. (See Section 1 for details on the terminology.) In fact, we prove a slight refinement of this result which gives some information about the monodromy operator in the case where Π does not have Shin-regular weight; see Section 1 for the details of this.

The proof of Theorem A is surprisingly simple, and relies on a generalisation of a base change trick that we learned from the papers [Kis08] and [Ski09] (see the proof of Theorem 4.3 of [Kis08] and Section 2.2 of [Ski09]). The idea is as follows. Suppose that Π has Shin-regular weight. We wish to determine the Weil–Deligne representation $\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{F-ss}}$. The monodromy may be computed after any finite base change, and in particular we may make a base change so that Π has Iwahori-fixed vectors, which is the situation covered by [BLGGT11]; so it suffices to compute the representation of the Weil group W_{F_v}. It is straightforward to check that in order to do so it is enough to compute the traces of the elements $\sigma \in W_{F_v}$ of nonzero valuation (that is, those elements which map to a nonzero power of the Frobenius element in the Galois group of the residue field). This trace is then computed as follows: one makes a global base change to a CM field E/F such that there is a place w of E lying over v such that $\mathrm{BC}_{E/F}(\Pi)_w$ has Iwahori-fixed vectors, and σ is an element of $W_{E_w} \leq W_{F_v}$. By the compatibility of base change with the local Langlands correspondence, the trace of σ on $\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{F-ss}}$ may then be computed over E, where the result follows from [BLGGT11].

The subtlety in this argument is that the field E/F need not be Galois, so one cannot immediately appeal to solvable base change. However, it will have solvable normal closure, so that by a standard descent argument due to Harris, together with local-global compatibility for the p-adic Galois representations with $p \neq l$, it is enough to know that for some prime l', the global Galois representation $r_{l',\imath'}(\Pi)$ is irreducible. Under the additional assumption that Π has extremely regular weight, the existence of such an l' is established in [BLGGT10]. Having thus established Theorem A in the case that Π has extremely regular and Shin-regular weight, we then pass to the general case by means of an l-adic interpolation argument of Chenevier and Harris, [CH09] and [Che09]. The details are in Section 3.

Notation and terminology. We write all matrix transposes on the left; so ${}^tA$ is the transpose of A. We let $B_m \subset \mathrm{GL}_m$ denote the Borel subgroup of upper triangular matrices and $T_m \subset \mathrm{GL}_m$ the diagonal torus. We let I_m denote the identity matrix in GL_m.

If M is a field, we let $\overline{M}$ denote an algebraic closure of M and G_M the absolute Galois group $\mathrm{Gal}(\overline{M}/M)$. Let ϵ_l denote the l-adic cyclotomic character.

Let p be a rational prime and $K/\mathbb{Q}_p$ a finite extension. We let $\mathcal{O}_K$ denote the ring of integers of K, $\wp_K$ the maximal ideal of $\mathcal{O}_K$, $k(\nu_K)$ the residue field $\mathcal{O}_K/\wp_K$, $\nu_K : K^\times \twoheadrightarrow \mathbb{Z}$ the canonical valuation and $|\ |_K : K^\times \to \mathbb{Q}^\times$ the absolute value given by $|x|_K = \#(k(\nu_K))^{-\nu_K(x)}$. We let $|\ |_K^{1/2} : K^\times \to \mathbb{R}^\times_{>0}$ denote the unique positive unramified square root of $|\ |_K$. If K is clear from the context, we will sometimes write $|\ |$ for $|\ |_K$. We let Frob_K denote the geometric Frobenius element of $G_{k(\nu_K)}$ and I_K the kernel of the natural surjection $G_K \twoheadrightarrow G_{k(\nu_K)}$. We will sometimes abbreviate $\mathrm{Frob}_{\mathbb{Q}_p}$ by Frob_p. We let W_K denote the preimage of $\mathrm{Frob}_K^{\mathbb{Z}}$ under the map $G_K \twoheadrightarrow G_{k(\nu(K))}$, endowed with a topology by decreeing that $I_K \subset W_K$ with its usual topology is an open subgroup of W_K. We let $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ denote the local Artin map, normalized to take uniformizers to lifts of Frob_K.

Let Ω be an algebraically closed field of characteristic 0. A Weil–Deligne representation of W_K over Ω is a triple (V, r, N) where V is a finite dimensional vector space over Ω, $r : W_K \to \mathrm{GL}(V)$ is a representation with open kernel and $N : V \to V$ is an endomorphism with $r(\sigma)Nr(\sigma)^{-1} = |\mathrm{Art}_K^{-1}(\sigma)|_K N$. We say that (V, r, N) is Frobenius semisimple if r is semisimple. We let $(V, r, N)^{\text{F-ss}}$ denote the Frobenius semisimplification of (V, r, N) (see for instance Section 1 of [TY07]) and we let $(V, r, N)^{\mathrm{ss}}$ denote $(V, r^{\mathrm{ss}}, 0)$. If Ω has the same cardinality as $\mathbb{C}$, we have the notions of a Weil–Deligne representation being *pure* or *pure of weight* k – see the paragraph before Lemma 1.4 of [TY07]. (If $N = 0$ then the representation is pure if the eigenvalues of Frobenius are Weil numbers of the same weight, but if N is nonzero then the definition is more involved.)

We will let rec_K be the local Langlands correspondence of [HT01], so that if π is an irreducible complex admissible representation of $\mathrm{GL}_n(K)$, then $\mathrm{rec}_K(\pi)$ is a Weil–Deligne representation of the Weil group W_K. We will write rec for rec_K when the choice of K is clear. If ρ is a continuous representation of G_K over $\overline{\mathbb{Q}}_l$ with $l \neq p$ then we will write $\mathrm{WD}(\rho)$ for the corresponding Weil–Deligne representation of W_K. (See for instance Section 1 of [TY07].)

If $m \geq 1$ is an integer, we let $\mathrm{Iw}_{m,K} \subset \mathrm{GL}_m(\mathcal{O}_K)$ denote the subgroup of matrices which map to an upper triangular matrix in $\mathrm{GL}_m(k(\nu_K))$. If π is an irreducible admissible supercuspidal representation of $\mathrm{GL}_m(K)$ and $s \geq 1$ is an integer we let $\mathrm{Sp}_s(\pi)$ be the square integrable representation of $\mathrm{GL}_{ms}(K)$ defined for instance in Section I.3 of [HT01]. Similarly, if $r : W_K \to \mathrm{GL}_m(\Omega)$ is an irreducible representation with open kernel and π is the supercuspidal representation $\mathrm{rec}_K^{-1}(r)$, we let $\mathrm{Sp}_s(r) = \mathrm{rec}_K(\mathrm{Sp}_s(\pi))$. If K'/K is a finite extension and if π is an irreducible smooth representation of $\mathrm{GL}_n(K)$ we will write $\mathrm{BC}_{K'/K}(\pi)$ for the base change of π to K' which is characterized by $\mathrm{rec}_{K'}(\pi_{K'}) = \mathrm{rec}_K(\pi)|_{W_{K'}}$.

If ρ is a continuous de Rham representation of G_K over $\overline{\mathbb{Q}}_p$ then we will write $\mathrm{WD}(\rho)$ for the corresponding Weil–Deligne representation of W_K (its construction, which is due to Fontaine, is recalled in Section 1 of [TY07]), and if $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$ is a continuous embedding of fields then we will write $\mathrm{HT}_\tau(\rho)$ for the multiset of Hodge–Tate numbers of ρ with respect to τ. Thus $\mathrm{HT}_\tau(\rho)$ is a multiset of $\dim \rho$ integers. In fact, if W is a de Rham representation of G_K over $\overline{\mathbb{Q}}_p$ and if $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$ then the multiset $\mathrm{HT}_\tau(W)$ contains i with multiplicity $\dim_{\overline{\mathbb{Q}}_p}(W \otimes_{\tau,K} \widehat{\overline{K}}(i))^{G_K}$. Thus for example $\mathrm{HT}_\tau(\epsilon_l) = \{-1\}$.

If F is a number field and v a prime of F, we will often denote Frob_{F_v}, $k(\nu_{F_v})$ and Iw_{m,F_v} by Frob_v, $k(v)$ and $\mathrm{Iw}_{m,v}$. If $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_p$ or $\mathbb{C}$ is an embedding of fields, then we will write F_σ for the closure of the image of σ. If F'/F is a soluble, finite Galois extension and if π is a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ we will write $\mathrm{BC}_{F'/F}(\pi)$ for its base change to F', an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{K'})$. If $R : G_F \to \mathrm{GL}_m(\overline{\mathbb{Q}}_l)$ is a continuous representation, we say that R is *pure of weight* w if for all but finitely many primes v of F, R is unramified at v and every eigenvalue of $R(\mathrm{Frob}_v)$ is a Weil $(\#k(v))^w$-number. (See Section 1 of [TY07].) If F is an imaginary CM field, we will denote its maximal totally real subfield by F^+ and let c denote the non-trivial element of $\mathrm{Gal}\,(F/F^+)$.

1. Automorphic Galois representations

We recall some now-standard notation and terminology. Let F be an imaginary CM field or totally real field. Let F^+ denote the maximal totally real subfield of F. By a *RAECSDC* (if F is imaginary) or *RAESDC* (if F is totally real) (regular, algebraic, essentially (conjugate) self dual, cuspidal) automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ we mean a pair (Π, χ) where

– Π is a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ such that Π_∞ has the same infinitesimal character as some irreducible algebraic representation of the restriction of scalars from F to $\mathbb{Q}$ of GL_m,
– $\chi : \mathbb{A}_{F^+}^\times/(F^+)^\times \to \mathbb{C}^\times$ is a continuous character such that $\chi_v(-1)$ is independent of $v|\infty$,
– and $\Pi^c \cong \Pi^\vee \otimes (\chi \circ \mathbf{N}_{F/F^+} \circ \det)$.

If χ is the trivial character we will often drop it from the notation and refer to Π as a RACSDC or RASDC (regular, algebraic, (conjugate) self dual, cuspidal) automorphic representation. We will say that (Π, χ) has *level prime to* l (resp. *level potentially prime to* l) if for all $v|l$ the representation Π_v is unramified (resp. becomes unramified after a finite base change).

If Ω is an algebraically closed field of characteristic 0 we will write $(\mathbb{Z}^m)^{\mathrm{Hom}\,(F,\Omega),+}$ for the set of $a = (a_{\tau,i}) \in (\mathbb{Z}^m)^{\mathrm{Hom}\,(F,\Omega)}$ satisfying

$$a_{\tau,1} \geq \cdots \geq a_{\tau,m}.$$

Let $w \in \mathbb{Z}$. If F is totally real or imaginary CM (resp. if $\Omega = \mathbb{C}$) we will write $(\mathbb{Z}^m)_w^{\mathrm{Hom}\,(F,\Omega)}$ for the subset of elements $a \in (\mathbb{Z}^m)^{\mathrm{Hom}\,(F,\Omega)}$ with

$$a_{\tau,i} + a_{\tau\circ c,m+1-i} = w$$

(resp.

$$a_{\tau,i} + a_{c\circ\tau,m+1-i} = w.)$$

(These definitions are consistent when F is totally real or imaginary CM and $\Omega = \mathbb{C}$.) If F'/F is a finite extension we define $a_{F'} \in (\mathbb{Z}^m)^{\mathrm{Hom}\,(F',\Omega),+}$ by

$$(a_{F'})_{\tau,i} = a_{\tau|_F,i}.$$

Following [Shi10] we will be interested, *inter alia*, in the case that either m is odd; or that m is even and for some $\tau \in \mathrm{Hom}\,(F,\Omega)$ and for some odd integer i we have $a_{\tau,i} > a_{\tau,i+1}$. If either of these conditions hold then we will say that a is *Shin-regular*. (This is often referred to as 'slightly regular' in the literature. However as this notion is strictly stronger than 'regularity' we prefer the terminology 'Shin-regular'.) Following [BLGGT10], we say that a is *extremely regular* if for some τ the $a_{\tau,i}$ have the following property: for any subsets H and H' of $\{a_{\tau,i} + n - i\}_{i=1}^n$ of the same cardinality, if $\sum_{h\in H} h = \sum_{h\in H'} h$ then $H = H'$. (The condition of extreme regularity will be used in order to apply Theorem 5.5.2 of [BLGGT10], in order to guarantee that a Galois representation associated to an automorphic representation is irreducible.)

If $a \in (\mathbb{Z}^m)^{\mathrm{Hom}\,(F,\mathbb{C}),+}$, let Ξ_a denote the irreducible algebraic representation of $\mathrm{GL}_m^{\mathrm{Hom}\,(F,\mathbb{C})}$ which is the tensor product over τ of the irreducible representations of GL_n with highest weights a_τ. We will say that a RAECSDC automorphic representation Π of $\mathrm{GL}_m(\mathbb{A}_F)$ has *weight* a if Π_∞ has the same infinitesimal character as $\Xi_a^\vee$. Note that in this case a must lie in $(\mathbb{Z}^m)_w^{\mathrm{Hom}\,(F,\mathbb{C})}$ for some $w \in \mathbb{Z}$.

We recall (see for example Theorem 1.2 of [BLGHT09]) that to a RAECSDC or RAESDC automorphic representation (Π, χ) of $\mathrm{GL}_m(\mathbb{A}_F)$ and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ we can associate a continuous semisimple representation

$$r_{l,\imath}(\Pi) : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow \mathrm{GL}_m(\overline{\mathbb{Q}}_l)$$

This representation satisfies

$$r_{l,\imath}(\Pi)^c \cong r_{l,\imath}(\Pi)^\vee \otimes \epsilon_l^{1-m} r_{l,\imath}(\chi)|_{G_F},$$

where $r_{l,\imath}(\chi) : G_{F^+} \to \overline{\mathbb{Q}}_l^\times$ is the de Rham character with the property that

$$\imath\left((r_{l,\imath}(\chi) \circ \mathrm{Art}_{F^+})(x) \prod_{\tau\in\mathrm{Hom}\,(F^+,\mathbb{C})} (\imath^{-1}\tau)(x_l)^{-a_\tau}\right) = \chi(x) \prod_{\tau\in\mathrm{Hom}\,(F^+,\mathbb{C})} (\tau x)^{-a_\tau},$$

where $a \in \mathbb{Z}^{\mathrm{Hom}\,(F^+,\mathbb{C})}$ is determined by the property that

$$\chi|_{((F^+)_\infty^\times)^0} : x \longmapsto \prod_{\tau\in\mathrm{Hom}\,(F^+,\mathbb{C})} (\tau x)^{a_\tau}.$$

For $v|l$ a place of F, the representation $r_{l,\imath}(\Pi)|_{G_{F_v}}$ is de Rham and if $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ then

$$\mathrm{HT}_\tau(r_{l,\imath}(\pi)) = \{a_{\imath\tau,1} + m - 1, a_{\imath\tau,2} + m - 2, ..., a_{\imath\tau,m}\}.$$

If $v \not| l$, then the main result of [Car10] states that

$$\imath \mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

Let p be a prime number, $K/\mathbb{Q}_p$ be a finite extension and let Ω be an algebraically closed field of characteristic 0. Let $\mathcal{J}$ denote the set of equivalence classes of irreducible representations of W_K over Ω with open kernel, where $s \sim s'$ if $s \cong s' \otimes \chi \circ \det$ for some unramified character $\chi : K^\times \to \Omega^\times$. Let $\rho = (V, r, N)$ be a Weil–Deligne representation of W_K over Ω. We decompose

$$V \cong \bigoplus_{\sigma \in \mathcal{J}} V[\sigma]$$

where $V[\sigma]$ is the largest W_K-submodule of V with all its irreducible subquotients lying in σ. Then each $V[\sigma]$ is stable by N and $\rho[\sigma] := (V[\sigma], r|_{V[\sigma]}, N|_{V[\sigma]})$ is a Weil–Deligne subrepresentation of (V, r, N). For each $\sigma \in \mathcal{J}$ with $V[\sigma] \neq (0)$, there is a unique decreasing sequence of integers $m_1(\rho, \sigma) \geq \cdots \geq m_{n(\rho,\sigma)}(\rho, \sigma) \geq 1$ with

$$\rho[\sigma]^{\text{F-ss}} \cong \bigoplus_{i=1}^{n(\rho,\sigma)} \mathrm{Sp}_{m_i(\rho,\sigma)}(s_i)$$

$s_i \in \sigma$ for each i. If ρ' is another Weil–Deligne representation of W_K over Ω, we say that

$$\rho \prec \rho'$$

if $\rho^{\mathrm{ss}} \cong (\rho')^{\mathrm{ss}}$ and if for each $\sigma \in \mathcal{J}$ we have

$$m_1(\rho, \sigma) + \cdots + m_i(\rho, \sigma) \leq m_1(\rho', \sigma) + \cdots + m_i(\rho', \sigma)$$

for each $i \geq 1$. The goal of this paper is to establish the following local-global compatibility result at places dividing l, our main theorem.

Theorem 1.1. *Let (Π, χ) be a RAECSDC automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ and let $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. If $v|l$ is a place of F, then*

$$\imath \mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \prec \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

Furthermore, if Π has Shin-regular weight, then

$$\imath \mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

The following corollary follows immediately using base change as in Proposition 4.3.1 of [CHT08].

Corollary 1.2. *Let (Π, χ) be a RAESDC automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ and let $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. If $v|l$ is a place of F, then*

$$\imath \mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \prec \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

Furthermore, if Π has Shin-regular weight, then

$$\imath \mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

2. The extremely regular, Shin-regular case

We start by treating the special case where, thanks to the irreducibility results of [BLGGT10], we can give a direct argument. We use an analogue of the trick of [Kis08] and [Ski09] (see the proof of Theorem 4.3 of [Kis08] and Section 2.2 of [Ski09]), but in a situation where we need to use a non-abelian, indeed non-Galois, base change. Because of this the argument makes essential use of the irreducibility results of [BLGGT10], and hence at present can only be made in the extremely regular case.

Theorem 2.1. *Let $m \geq 2$ be an integer, l a rational prime and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Let F be an imaginary CM field and (Π, χ) a RAECSDC automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$. If Π has extremely regular and Shin-regular weight and $v|l$ is a place of F, then*

$$\imath \mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

Proof. We first reduce to the RACSDC case: using Lemma 4.1.4 of [CHT08] we choose an algebraic Hecke character $\psi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ such that $\psi \cdot (\psi \circ c) = \chi_F^{-1} \circ \mathbf{N}_{F/F^+}$. Then $\Pi \otimes \psi \circ \det$ is RACSDC and the theorem holds for Π if and only if it holds for $\Pi \otimes \psi \circ \det$. We may therefore assume that Π is RACSDC.

To prove the theorem, it suffices to establish the weaker result that

$$\imath \mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{ss}} \cong \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2})^{\text{ss}}.$$

(Suppose this weaker result holds. By Proposition 1.1 of [BLGGT11], it suffices to prove that $\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})$ is pure. This is established in Corollary 1.3 of [BLGGT11].)

To establish the weaker result, it suffices to show that

$$\mathrm{tr}\,(\sigma | \imath \mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})) = \mathrm{tr}\,(\sigma | \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2}))$$

for every $\sigma \in W_{F_v}$ mapping to a non-zero power of $\mathrm{Frob}_v \in G_{k(v)}$. (This follows from the proof of Lemma 1 of [Sai97].) Fix such an element $\sigma \in W_{F_v}$. We can and do choose a finite extension E_v/F_v inside $\overline{F}_v$ such that

– $\sigma \in W_{E_v} \subset W_{F_v}$ and
– $\mathrm{BC}_{E_v/F_v}(\Pi_v)^{\mathrm{Iw}_{m,E_v}} \neq \{0\}$.

(If we write $\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}}) = (V, r, N)$, we could take E_v to be the fixed field of the subgroup of W_{F_v} generated by σ and the kernel of $r|_{I_{F_v}}$.) Let E'_v/E_v denote the normal closure of E_v/F_v. Choose a finite CM soluble Galois extension F'/F such that for each place $w|v$ of F', $F'_w/F_v \cong E'_v/F_v$. Let $\Pi_{F'} = \mathrm{BC}_{F'/F}(\Pi)$. By Theorem 5.5.2 of [BLGGT10] we can and do choose a rational prime l' and $\imath' : \mathbb{Q}_{l'} \xrightarrow{\sim} \mathbb{C}$ such that $r_{l',\imath'}(\Pi_{F'})$ is irreducible. Choose a prime $w|v$ of F' and an F_v-embedding $F'_w \hookrightarrow \overline{F}_v$. Let $E = F' \cap E_v \subset F'_w$ be the fixed field of $\mathrm{Gal}\,(F'_w/E_v) \subset \mathrm{Gal}\,(F'/F)$. The inclusion $E \hookrightarrow E_v$ determines a prime u of E. By Lemma 1.4 of [BLGHT09] (which we can apply because $r_{l',\imath'}(\Pi_{F'})$ is irreducible), there exists a RACSDC automorphic representation Π_E of $\mathrm{GL}_m(\mathbb{A}_E)$ with $r_{l',\imath'}(\Pi_E) \cong r_{l',\imath'}(\Pi)|_{G_E}$ and hence $r_{l,\imath}(\Pi_E) \cong r_{l,\imath}(\Pi)|_{G_E}$. Local-global compatibility for $r_{l',\imath'}(\Pi_E)|_{G_{E_u}}$ and $r_{l',\imath'}(\Pi)|_{G_{F_v}}$ (which is part of the main theorem of [Shi10]) shows that $\Pi_{E,u} = \mathrm{BC}_{E_v/F_v}(\Pi_v)$. Then Theorem 1.2 of [BLGGT11] (which we can apply by our assumption above that $\mathrm{BC}_{E_v/F_v}(\Pi_v)^{\mathrm{Iw}_{m,E_v}} \neq \{0\}$)

implies that

$$\begin{aligned}\operatorname{tr}(\sigma|\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})) &= \operatorname{tr}(\sigma|\imath\mathrm{WD}(r_{l,\imath}(\Pi_E)|_{G_{E_u}}))\\ &= \operatorname{tr}(\sigma|\mathrm{rec}(\Pi_{E,u}\otimes|\det|^{(1-m)/2}))\\ &= \operatorname{tr}(\sigma|\mathrm{rec}(\Pi_v\otimes|\det|^{(1-m)/2})),\end{aligned}$$

and the result follows. □

3. The general case

We will prove the next result using Theorem 2.1 and the methods of [Che09] and [BC09a]. It establishes the first statement of Theorem 1.1.

Theorem 3.1. *Let $m \geq 2$ be an integer, l a rational prime and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Let F be an imaginary CM field and (Π, χ) a RAECSDC automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$. If $v|l$ is a place of F, then*

$$\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \prec \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

Before giving the proof, we first deduce the second statement of Theorem 1.1 as a corollary.

Corollary 3.2. *Let $m \geq 2$ be an integer, l a rational prime and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Let F be an imaginary CM field and (Π, χ) a RAECSDC automorphic representation of $\mathrm{GL}_m(\mathbb{A})$. If Π has Shin-regular weight and $v|l$ is a place of F, then*

$$\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}(\Pi_{F,v} \otimes |\det|^{(1-m)/2}).$$

Proof. This follows immediately from Theorem 3.1 together with Corollary 1.3 of [BLGGT11] and Proposition 1.1 of [BLGGT11]. □

Let p be a prime number, $K/\mathbb{Q}_p$ be a finite extension and let Ω be an algebraically closed field of characteristic 0. In Section 1, we introduced a relation $\rho \prec \rho'$ on Weil–Deligne representations of W_K over Ω. Following [Che09, §3.10], we now introduce another such relation $\prec_I$ which will play a role in the proof below. See [Che09, Lemme 3.14] for the relationship between $\prec$ and $\prec_I$. Let $\mathcal{J}_I$ denote the set of equivalence classes of irreducible representations of I_K over Ω with open kernel. Let $\rho = (V, r, N)$ be a Weil–Deligne representation of W_K over Ω. We decompose

$$V \cong \bigoplus_{\sigma \in \mathcal{J}_I} V[\sigma]$$

where $V[\sigma]$ is the σ-isotypical component of $V|_{I_K}$. Then each $V[\sigma]$ is stable by N and I_K. For each $\sigma \in \mathcal{J}_I$ we let $p_I(\rho, \sigma)$ denote the partition of the integer $\dim V[\sigma]/\dim \sigma$ which determines the conjugacy class of the operator N on $V[\sigma]$. (See [BC09b, §7.8.1].) If $\rho' = (V', r', N')$ is another Weil–Deligne representation of W_K over Ω, we say that

$$\rho \prec_I \rho'$$

if $V|_{I_K} \cong V'|_{I_K}$ and if for each $\sigma \in \mathcal{J}_I$ we have $p_I(\rho, \sigma) \prec p_I(\rho', \sigma)$. (If $p = (m_1 \geq m_2 \geq \dots)$ and $p' = (m'_1 \geq m'_2 \geq \dots)$ are partitions of some integer d, we say $p \prec p'$ if $m_1 + \dots + m_i \leq m'_1 + \dots + m'_i$ for all $i \geq 1$.)

Proof of Theorem 3.1. As in the proof of Theorem 2.1, we may assume that Π is RACSDC. Replacing F by a suitable finite soluble CM Galois extension in which v splits we may also assume that:

– $[F^+ : \mathbb{Q}]$ is even;
– F/F^+ is unramified at all finite places;
– all places of F^+ dividing l are split in F;
– if Π_w is ramified, then $w|_{F^+}$ is split in F;
– if $w \neq v$ then $\Pi_w^{\mathrm{Iw}_{m,w}} \neq \{0\}$.

Since $[F^+ : \mathbb{Q}]$ is even,we can and do choose a unitary group U/F^+ such that:
– $U \times_{F^+} F \cong \mathrm{GL}_m / F$;
– $U \times_{F^+} F_u^+$ is quasi-split for each prime u of F^+;
– $U(F_\sigma^+)$ is compact for each $\sigma : F^+ \hookrightarrow \mathbb{R}$.

(We write F_σ^+ for the completion of F^+ with respect to the absolute value induced by σ.) For each place u of F^+ which splits in F and $w|u$ a prime of F , we fix an isomorphism $\imath_w : U(F_u^+) \xrightarrow{\sim} \mathrm{GL}_m(F_w)$ such that $\imath_{w^c} = {}^t\imath_w^{-c}$. If $\sigma : F^+ \hookrightarrow \mathbb{R}$ and $\widetilde{\sigma} : F \hookrightarrow \mathbb{C}$ extends σ, we fix an embedding $\imath_{\widetilde{\sigma}} : U(F_\sigma^+) \hookrightarrow \mathrm{GL}_n(F_{\widetilde{\sigma}})$ which identifies $U(F_\sigma^+)$ with the set of all g with ${}^tg^c \cdot g = 1_m$. By Corollaire 5.3 and Théorème 5.4 of [Lab09], there exists an automorphic representation π_0 of $U(\mathbb{A}_{F^+})$ such that:
– if u is a prime of F^+ which splits as ww^c in F, then $\pi_{0,u} \cong \Pi_w \circ \imath_w$;
– if u is a prime of F^+ which is inert in F, then Π_u is given by the local base change of $\pi_{0,u}$ (see [Lab09]);
– if $\sigma : F^+ \hookrightarrow \mathbb{R}$ and $\widetilde{\sigma} : F \hookrightarrow \mathbb{C}$ extends σ, then there is an irreducible algebraic representation $W_{\widetilde{\sigma}}$ of $\mathrm{GL}_m(F_{\widetilde{\sigma}})$ such that $\pi_{0,\sigma} \cong W_{\widetilde{\sigma}}^\vee \circ \imath_{\widetilde{\sigma}}$. Moreover, if $W_{\widetilde{\sigma}}$ has highest weight $a_{\widetilde{\sigma}} = (a_{\widetilde{\sigma},1}, \ldots, a_{\widetilde{\sigma},m})$, then Π has weight $a = (a_{\widetilde{\sigma}})_{\widetilde{\sigma}:F\hookrightarrow\mathbb{C}}$.

We now follow the arguments of [Che09]. We have chosen to closely follow [Che09] even when we could somewhat simplify the argument in the case of interest to us, in order to ease comparison with that paper. We note however that we take the prime p of [Che09] to be the prime l of this paper. Make the following definitions: let $\widetilde{S}_l$ (resp. $\widetilde{S}_v$) denote the set of primes of F dividing l but not equal to v or v^c (resp. $\widetilde{S}_v = \{v, v^c\}$). Let $\widetilde{R}$ denote the set of primes w of F not dividing l and with Π_w ramified. Set $\widetilde{S} = \widetilde{S}_v \cup \widetilde{R}$. Let S_l, S_v, R and S denote the sets of primes of F^+ lying under $\widetilde{S}_l$, $\widetilde{S}_v$, $\widetilde{R}$ and $\widetilde{S}$ respectively. For each $u \in S_l \cup S$, fix a prime $\widetilde{u}$ of F dividing u such that $\widetilde{u} = v$ when $u = v|_{F^+}$. We will henceforth identify $U(F_u^+)$ and $\mathrm{GL}_m(F_{\widetilde{u}})$ via $\imath_{\widetilde{u}}$ for $u \in S_l \cup S$.

Fix embeddings $\imath_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\imath_l : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$ such that $\imath \circ \imath_l = \imath_\infty$. For $u|l$ a prime of F^+, following [Che09], we let $\Sigma(u) \subset \mathrm{Hom}\,(F^+, \overline{\mathbb{Q}}_l)$ denote the set of embeddings inducing u and let $\Sigma_\infty(u) = \imath\Sigma(u) \subset \mathrm{Hom}\,(F^+, \mathbb{R})$. Let W_∞ denote the representation $\otimes_{\sigma\in\Sigma_\infty(v|_{F^+})}\pi_{0,\sigma}$ of $\prod_{\sigma\in\Sigma_\infty(v|_{F^+})} U(F_\sigma^+)$.

Let $K^S = \prod_{u\notin S} K_u \subset U(\mathbb{A}_{F^+}^{\infty,S})$ be a compact open subgroup with
– $K_u = \mathrm{Iw}_{m,\widetilde{u}}$ if $u \in S_l$;
– K_u a hyperspecial maximal compact subgroup of $U(F_u^+)$ otherwise.

Let $\mathcal{H}^{S\cup S_l} = \mathbb{Z}[U(\mathbb{A}_{F^+}^{\infty,S\cup S_l})//K^{S\cup S_p}]$ denote the commutative spherical Hecke algebra. For u a finite place of F^+, let $\mathcal{H}(U(F_u^+))$ denote the Hecke algebra consisting of smooth, compactly supported functions on $U(F_u^+)$ with values in $\mathbb{Z}$. For $u \notin S\cup S_l$, let $e_u = \mathbb{1}_{K_u} \in \mathcal{H}(U(F_u^+))$ be the idempotent corresponding to K_u.

Choose a finite Galois extension $E/\mathbb{Q}$ in $\overline{\mathbb{Q}}$ such that $\Pi_{\widetilde{u}}$ can be defined over E for each $u \in S$. For $u \in S$, let $\mathcal{B}_u$ denote the subcategory of the category of smooth E-representations of $\mathrm{GL}_m(F_{\widetilde{u}})$ determined by the supercuspidal support of $\Pi_{\widetilde{u}}$ (see Proposition-définition 2.8 of [Ber84]). Let $\mathfrak{z}_u$ denote the center of the category $\mathcal{B}_u$. For $u \in R$, let $e_u = \mathbb{1}_{\mathrm{Iw}_{m,\widetilde{u}}}$ denote the idempotent in $\mathcal{H}(\mathrm{GL}_m(F_{\widetilde{u}}))$ corresponding to $\mathrm{Iw}_{m,\widetilde{u}}$. For $u \in S_v = \{v|_{F^+}\}$, choose an idempotent e_u in $\mathcal{H}(\mathrm{GL}_m(F_v))$ such that

- $b_u e_u = e_u$ where $b_u \in \mathcal{H}(\mathrm{GL}_m(F_v))$ is the projector to $\mathcal{B}_u$;
- $e_u \Pi_v \neq \{0\}$;
- for every irreducible $\pi \in \mathcal{B}_u \otimes_{E,\imath_\infty} \mathbb{C}$, if $e_u \pi \neq \{0\}$, then
$$\mathrm{rec}(\pi) \prec_I \mathrm{rec}(\Pi_v).$$
(We refer to Section 3.6 of [Che09] for the fact that one can choose such an idempotent e_u; the definition of the relation $\prec_I$ is recalled in the discussion preceding this proof.)

Extending E if necessary, we may assume that e_u is defined over E for each $u \in S$ and we set $e = \otimes'_{u \notin S_l} e_u$, an idempotent in the algebra
$$\mathcal{H} := \mathcal{H}^{S \cup S_l} \otimes_{\mathbb{Z}} (\bigotimes_{u \in S} {}_E\ \mathfrak{z}_u).$$

Let L_E denote the Galois closure (over $\mathbb{Q}_l$) of the closure of $\imath_l(EF)$ in $\overline{\mathbb{Q}}_l$. Let T denote the diagonal maximal torus in $\prod_{u \in S_l} \mathrm{GL}_m(F_{\widetilde{u}})$ and let $\mathcal{T} = \mathrm{Hom}\,(T, \mathbb{G}_m^{\mathrm{rig}})$ denote the rigid analytic space over $\mathbb{Q}_l$ parametrizing continuous l-adic characters of T.

Let $\mathcal{A}$ denote the set of automorphic representations π of $U(\mathbb{A}_{F^+})$ for which $e(\pi^\infty)^{K_{S_l}} \neq \{0\}$ and $\otimes_{\sigma \in \Sigma_\infty(v|_{F^+})} \pi_\sigma \cong W_\infty$. If $\pi \in \mathcal{A}$, then $\mathcal{H}$ acts on $e(\pi^{\infty, S_l})$ through an E-algebra homomorphism $\psi_{\mathbb{C}}(\pi) : \mathcal{H} \to \mathbb{C}$ (this follows from the fact that $\pi_u^{K_u}$ is 1-dimensional for $u \notin S \cup S_l$ while $\mathfrak{z}_u$ acts on π_u through a character for $u \in S$). We define $\psi(\pi) : \mathcal{H} \to \overline{\mathbb{Q}}_l$ to be $\imath^{-1} \circ \psi_{\mathbb{C}}(\pi)$.

If $\pi \in \mathcal{A}$, we now associate to it an algebraic character $\kappa(\pi) \in \mathcal{T}(L_E)$ as in Section 1.4 of [Che09]; this character records the highest weights of the representations π_σ for $\sigma \in \Sigma_\infty(u)$ and $u \in S_l$. If $u \in S_l$ and $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$ is an embedding inducing $\widetilde{u}$, let κ_σ denote the highest weight of the representation $W_{\imath\sigma}^\vee$. Thus $\kappa_\sigma = (\kappa_{\sigma,1}, \ldots, \kappa_{\sigma_m})$ with $\kappa_{\sigma_1} \geq \cdots \geq \kappa_{\sigma_m}$. We regard κ_σ as an $L_E^\times$-valued character of T as follows:
$$\kappa_\sigma : t = (t_{u'})_{u' \in S_l} \mapsto \prod_{i=1}^m (\sigma t_{u,i})^{\kappa_{\sigma,i}}.$$
(Here we denote the extension of σ to an embedding $F_{\widetilde{u}} \hookrightarrow \overline{\mathbb{Q}}_l$ again by σ.) We then take $\kappa(\pi) = \prod \kappa_\sigma$ where the product is over all $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$ inducing $\widetilde{u}$ for some $u \in S_l$. By definition, we may regard $\kappa(\pi)$ as an element of $\mathcal{T}(L_E)$.

If $u \in S_l$ and $\pi_{\widetilde{u}}$ is an irreducible smooth representation of $\mathrm{GL}_m(F_{\widetilde{u}})$ with $\pi_{\widetilde{u}}^{\mathrm{Iw}_{m,\widetilde{u}}} \neq \{0\}$, an *accessible refinement* of $\pi_{\widetilde{u}}$ is an unramified character $\chi_{\widetilde{u}} : T_m(F_{\widetilde{u}}) \to \mathbb{C}^\times$such that $\pi_{\widetilde{u}}$ embeds as a subrepresentation of $\text{n-Ind}_{B_m(F_{\widetilde{u}})}^{\mathrm{GL}_m(F_{\widetilde{u}})} \chi_{\widetilde{u}}$. (Such a character always exists.) If $\pi \in \mathcal{A}$, then an accessible refinement of π is a character $\chi = \prod_{u \in S_l} \chi_{\widetilde{u}} : T = \prod_{u \in S_l} T_m(F_{\widetilde{u}}) \to \overline{\mathbb{Q}}_l^\times$ where each $\chi_{\widetilde{u}} : T_m(F_{\widetilde{u}}) \to \overline{\mathbb{Q}}_l^\times$ is unramified and $\imath\chi_{\widetilde{u}}$ is an accessible refinement of $\pi_{\widetilde{u}} \otimes |\det|^{(1-m)/2}$. Given such a pair (π, χ), we associate to it the character
$$\nu(\pi, \chi) := \kappa(\pi) \chi \delta_{B_m}^{-1/2} |\det|^{\frac{m-1}{2}} \in \mathcal{T}(\overline{\mathbb{Q}}_l)$$
as in Section 1.4 of [Che09].

We let
$$\mathcal{Z} \subset \mathrm{Hom}_E(\mathcal{H}, \overline{\mathbb{Q}}_l) \times \mathcal{T}(\overline{\mathbb{Q}}_l)$$

denote the set of all pairs $(\psi(\pi), \nu(\pi, \chi))$ where $\pi \in \mathcal{A}$ and χ is an accessible refinement of π.

By Théorème 1.6 of [Che09], the data $(S_l, W_\infty, \mathcal{H}, e)$ determines a four-tuple (X, ψ, ν, Z) where:

- X is a reduced rigid analytic space over L_E which is equidimensional of dimension $m \sum_{u \in S_l} [F_u^+ : \mathbb{Q}_l]$;
- $\psi : \mathcal{H} \to \mathcal{O}(X)$ is a ring homomorphism with $\psi(\mathcal{H}^{S \cup S^l}) \subset \mathcal{O}(X)^{\leq 1}$;
- $\nu : X \to \mathcal{T}$ is a finite analytic morphism;
- $Z \subset X(\overline{\mathbb{Q}}_l)$ is a Zariski-dense accumulation subset of $X(\overline{\mathbb{Q}}_l)$ such that the map
$$X(\overline{\mathbb{Q}}_l) \to \mathrm{Hom}_E(\mathcal{H}, \overline{\mathbb{Q}}_l) \times \mathcal{T}(\overline{\mathbb{Q}}_l)$$
which sends $x \mapsto (h \mapsto \psi(h)(x), \nu(x))$ induces a bijection $Z \xrightarrow{\sim} \mathcal{Z}$. (A subset $Z \subset X(\overline{\mathbb{Q}}_l)$ is said to be an accumulation subset if for each $z \in Z$ and each open affinoid neighbourhood U in X of z, there exists an open affinoid $V \subset U$ containing z such that $Z \cap V$ is Zariski dense in V. (See [Che09, §1.5].)) We henceforth identify Z and $\mathcal{Z}$.

If $\pi \in \mathcal{A}$, then by Corollaire 5.3 of [Lab09] there exists a partition $m = m_1 + \ldots + m_r$ of m and conjugate self-dual discrete automorphic representations Π_i of $\mathrm{GL}_{m_i}(\mathbb{A}_F)$ such that $\widetilde{\Pi} := \Pi_1 \boxplus \cdots \boxplus \Pi_r$ is a strong base change of π. Let $\Sigma = \widetilde{S} \cup \widetilde{S}_l$ and let F_Σ denote the maximal extension of F which is unramified outside Σ. Let $G_{F,\Sigma} = \mathrm{Gal}(F_\Sigma/F)$. By Theorem 3.2.5 of [CH09] and the argument of Theorem 2.3 of [Gue09], there is a continuous semisimple representation $r_{l,\imath}(\pi) : G_{F,\Sigma} \to \mathrm{GL}_m(\overline{\mathbb{Q}}_l)$ with
$$\imath \mathrm{WD}(r_{l,\imath}(\pi)|_{G_{F_w}})^{\mathrm{ss}} \cong \mathrm{rec}(\widetilde{\Pi}_w \otimes |\det|^{(1-m)/2})^{\mathrm{ss}}$$
for each prime $w \nmid l$ of F. Moreover, there is a unique continuous m-dimensional pseudo-representation $\mathrm{T} : G_{F,\Sigma} \to \mathcal{O}(X)$ such that $\mathrm{T}_z = \mathrm{tr}\,(r_{l,\imath}(\pi))$ for each $z = (\psi(\pi), \nu(\pi, \chi)) \in Z$. (Here, for any $x \in X(\overline{\mathbb{Q}}_l)$, T_x denotes the composition of T with the evaluation map $\mathcal{O}(X) \to \overline{\mathbb{Q}}_l; g \mapsto g(x)$.) The existence of T follows from the proof of Proposition 7.1.1 of [Che04] together with Proposition 7.2.11 of [BC09a] (which shows that $\mathcal{O}(X)^{\leq 1}$ is compact, as $\mathcal{T}$ is nested and ν is finite) and the fact that $\psi(\mathcal{H}^{S \cup S^l}) \subset \mathcal{O}(X)^{\leq 1}$. By Theorem 1 of [Tay91], for any $x \in X(\overline{\mathbb{Q}}_l)$, there is a unique continuous semisimple representation $r_x : G_{F,\Sigma} \to \mathrm{GL}_m(\overline{\mathbb{Q}}_l)$ with $\mathrm{T}_x = \mathrm{tr}\,(r_x)$.

Now, let $u = v|_{F^+}$ and recall that $\widetilde{u} = v$. By Proposition 3.11 of [Che09], there is a unique m-dimensional pseudo-character
$$\mathrm{T}^{\mathcal{B}_u} : W_{F_{\widetilde{u}}} \to \mathfrak{z}_u$$
such that for each irreducible smooth representation $\pi_{\widetilde{u}}$ of $\mathrm{GL}_m(F_{\widetilde{u}})$ in $\mathcal{B}_u \otimes_{E,\imath_\infty} \mathbb{C}$, if $\mathrm{T}^{\mathcal{B}_u}_{\pi_{\widetilde{u}}}$ denotes the composition of $\mathrm{T}^{\mathcal{B}_u}$ with the character $\mathfrak{z}_u \to \mathbb{C}$ giving the action of $\mathfrak{z}_u$ on $\pi_{\widetilde{u}}$, then
$$\mathrm{T}^{\mathcal{B}_u}_{\pi_{\widetilde{u}}} = \mathrm{tr}\,(\mathrm{rec}(\pi_{\widetilde{u}} \otimes |\det|^{(1-m)/2})).$$

Let $z_0 \in Z$ be a point corresponding to π_0 together with the choice of some accessible refinement. Let $Z^{\mathrm{reg}} \subset Z$ denote the subset associated to pairs (π, χ) where π_∞ is Shin-regular and extremely regular. (If $\widetilde{\sigma} : F \hookrightarrow \mathbb{C}$ and $\sigma := \widetilde{\sigma}|_{F^+}$, then $\pi_\sigma \circ \imath_{\widetilde{\sigma}}$ is the restriction of an irreducible algebraic representation of $\mathrm{GL}_m(F_{\widetilde{\sigma}})$ of highest weight $b_{\widetilde{\sigma}}$, say. We say π_∞ is Shin-regular or extremely regular if $b := (b_{\widetilde{\sigma}})_{\widetilde{\sigma}}$ has the corresponding property.) Then Z^{reg} is a Zariski-dense accumulation subset of $X(\overline{\mathbb{Q}}_l)$. Choose an open affinoid $\Omega \subset X$ such that $z_0 \in \Omega$ and $Z^{\mathrm{reg}} \cap \Omega$ is Zariski-dense in Ω. Let T_Ω denote the restriction of T to Ω. By Lemme 7.8.11 of [BC09a], there exists a reduced, separated, quasi-compact rigid analytic space Y and a proper, generically finite, surjective morphism $f : Y \to \Omega$ such that there exists an $\mathcal{O}_Y$-module M which is locally free of rank n and carries a continuous action of $G_{F,\Sigma}$ whose trace is given by $f^* \mathrm{T}_\Omega$.

By Proposition 3.16 of [Che09] (a result of Sen), the (generalized) Hodge–Tate weights of $M_y|_{G_{F_{\widetilde{u}}}}$ are independent of $y \in Y(\overline{\mathbb{Q}}_l)$. (This follows from the quoted result and the fact that the Hodge–Tate weights of $r_z|_{G_{F_{\widetilde{u}}}}$ are independent of $z \in Z$.) Moreover, by the improvement to Theorem C of [BC08] made in Corollary 3.19 of [Che09], there exists a finite Galois extension $F'_{\widetilde{u}}/F_{\widetilde{u}}$ such that if $F'_{\widetilde{u},0} \subset F'_{\widetilde{u}}$ denotes the maximal subfield which is unramified over $\mathbb{Q}_l$, then the $\mathcal{O}_Y \otimes_{\mathbb{Q}_l} F'_{\widetilde{u},0}$-module

$$D_{\mathrm{st}}^{F'_{\widetilde{u}}}(M) := (M \otimes_{\mathbb{Q}_l} B_{\mathrm{st}})^{G_{F'_{\widetilde{u}}}}$$

is locally free of rank m and satisfies the following: if $y \in Y(\overline{\mathbb{Q}}_l)$, then the natural map $D_{\mathrm{st}}^{F'_{\widetilde{u}}}(M)_y \to D_{\mathrm{st}}^{F'_{\widetilde{u}}}(M_y)$ is an isomorphism (and hence $M_y|_{G_{F'_{\widetilde{u}}}}$ is semistable).

The diagonal action of $G_{F_{\widetilde{u}}}$ on $M \otimes_{\mathbb{Q}_l} B_{\mathrm{st}}$ induces an $\mathcal{O}_Y$-linear, $F'_{\widetilde{u},0}$-semilinear action of $G_{F_{\widetilde{u}}}$ on $D_{\mathrm{st}}^{F'_{\widetilde{u}}}(M)$. We define an $\mathcal{O}_Y \otimes_{\mathbb{Q}_l} F'_{\widetilde{u},0}$-linear action $r_{\widetilde{u}}$ of $W_{F_{\widetilde{u}}} \subset G_{F_{\widetilde{u}}}$ on $D_{\mathrm{st}}^{F'_{\widetilde{u}}}(M)$ by letting $g \in W_{F_{\widetilde{u}}}$ act as $g \circ \varphi^{w(g)}$ where $w(g) \in \mathbb{Z}$ is the power of Frob_l to which g maps in $G_{F_{\widetilde{u}}}/I_{F_{\widetilde{u}}}$. We have that $N \circ r_{\widetilde{u}}(g) = l^{w(g)} r_{\widetilde{u}}(g) \circ N$ on $D_{\mathrm{st}}^{F'_{\widetilde{u}}}(M)$. For each continuous embedding $\tau : F'_{\widetilde{u},0} \hookrightarrow L_E$, we let

$$\mathrm{WD}_{\widetilde{u},\tau} = D_{\mathrm{st}}^{F'_{\widetilde{u}}}(M) \otimes_{\mathcal{O}_Y \otimes_{\mathbb{Q}_p} F'_{\widetilde{u}},1\otimes\tau} \mathcal{O}_Y.$$

Then $\mathrm{WD}_{\widetilde{u},\tau}$ is locally free of rank m as an $\mathcal{O}_Y$-module and $N \circ r_{\widetilde{u}}(g) = l^{w(g)} r_{\widetilde{u}}(g) \circ N$ on $\mathrm{WD}_{\widetilde{u},\tau}$. Moreover, φ induces an isomorphism $\mathrm{WD}_{\widetilde{u},\tau\circ\mathrm{Frob}_l} \xrightarrow{\sim} \mathrm{WD}_{\widetilde{u},\tau}$ compatible with $r_{\widetilde{u}}$ and N. We let $\mathrm{WD}_{\widetilde{u}}$ denote $\mathrm{WD}_{\widetilde{u},\tau}$ for some choice of τ, regarded as a $W_{F_{\widetilde{u}}}$-module with an operator N. We note that for each $y \in Y(\overline{\mathbb{Q}}_l)$, $\mathrm{WD}_{\widetilde{u},y}$ is the Weil–Deligne representation associated to $M_y|_{G_{F_{\widetilde{u}}}}$. It follows that $N^m = 0$ on $\mathrm{WD}_{\widetilde{u}}$. Let

$$\mathrm{T}^{Y,\widetilde{u}} = \mathrm{tr}\,(r_{\widetilde{u}}(\cdot)|\mathrm{WD}_{\widetilde{u}}) : W_{F_{\widetilde{u}}} \to \mathcal{O}_Y.$$

We claim that

$$\mathrm{T}^{Y,\widetilde{u}} = f^* \circ \psi \circ \mathrm{T}^{\mathcal{B}_u}.$$

This is proved as follows: let $y \in f^{-1}(Z^{\mathrm{reg}} \cap \Omega)$ and let $z = f(y)$. Then z corresponds to a pair (π, χ) where $\pi \in \mathcal{A}$ is Shin-regular and extremely regular (and χ is an accessible refinement of π). Theorem 2.1 together with the regularity conditions satisfied by π and the construction of the representation $r_{l,\imath}(\pi)$ in the proof of Theorem 2.3 of [Gue09] show that

$$\mathrm{WD}(r_{l,\imath}(\pi)|_{G_{F_{\widetilde{u}}}})^{\text{F-ss}} \cong \imath^{-1}\mathrm{rec}(\pi_u \circ \imath_{\widetilde{u}}^{-1} \otimes |\det|^{(1-m)/2}).$$

Since $M_y^{\mathrm{ss}} \cong r_z = r_{l,\imath}(\pi)$, we deduce that $\mathrm{T}^{Y,\widetilde{u}}(g)$ and $f^*(\psi(\mathrm{T}^{\mathcal{B}_u}(g)))$ agree on $y \in Y(\overline{\mathbb{Q}}_l)$ for each $g \in W_{F_{\widetilde{u}}}$. The claimed result now follows from the Zariski-density of $f^{-1}(Z^{\mathrm{reg}} \cap \Omega)$ in Y.

We now choose some $y_0 \in Y(\overline{\mathbb{Q}}_l)$ with $f(y_0) = z_0$. Since $r_{l,\imath}(\Pi) = r_{l,\imath}(\pi_0) = r_{z_0} \cong M_{y_0}^{\mathrm{ss}}$, the result just proved shows that

$$\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_{\widetilde{u}}}})^{\mathrm{ss}} \cong \mathrm{rec}(\Pi_{\widetilde{u}} \otimes |\det|^{(1-m)/2}).$$

We deduce from this that

$$\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_{\widetilde{u}}}})^{\text{F-ss}} \prec \mathrm{rec}(\Pi_{\widetilde{u}} \otimes |\det|^{(1-m)/2}),$$

as follows: By Lemma 3.14(ii) of [Che09], it suffices to show that

$$\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_{\widetilde{u}}}})^{\text{F-ss}} \prec_I \mathrm{rec}(\Pi_{\widetilde{u}} \otimes |\det|^{(1-m)/2}).$$

For each $y \in f^{-1}(Z^{\mathrm{reg}} \cap \Omega)$ with $f(y)$ corresponding to a pair (π, χ), we have

$$\imath\mathrm{WD}(M_y^{\mathrm{ss}}|_{G_{F_{\widetilde{u}}}})^{\text{F-ss}} \cong \mathrm{rec}(\pi_u \circ \imath_{\widetilde{u}}^{-1} \otimes |\det|^{(1-m)/2}) \prec_I \mathrm{rec}(\Pi_{\widetilde{u}} \otimes |\det|^{(1-m)/2})$$

(where the last relation follows from the choice of idempotent e_u). By the proof of Proposition 7.8.19(iii) of [BC09a] and the Zariski-density of $f^{-1}(Z^{\mathrm{reg}} \cap \Omega)$ in Y, we have $\imath\mathrm{WD}(M_y^{\mathrm{ss}}|_{G_{F_{\widetilde{u}}}})^{\text{F-ss}} \prec_I \mathrm{rec}(\Pi_{\widetilde{u}} \otimes |\det|^{(1-m)/2})$ for all $y \in Y(\overline{\mathbb{Q}}_l)$. Taking y above z_0 gives the required result. □

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