PATCHING AND THE COMPLETED HOMOLOGY OF LOCALLY SYMMETRIC SPACES

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Abstract. Under an assumption on the existence of p-adic Galois representations, we carry out Taylor–Wiles patching (in the derived category) for the completed homology of the locally symmetric spaces associated to GL_n over a number field. We use our construction, and some new results in non-commutative algebra, to show that standard conjectures on completed homology imply 'big R = big $\mathbb{T}$' theorems in situations where one cannot hope to appeal to the Zariski density of classical points (in contrast to all previous results of this kind). In the case that $n = 2$ and p splits completely in the number field, we relate our construction to the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Contents

1. Introduction

In this paper we give a common generalisation of two recent extensions of the Taylor–Wiles patching method, namely the extension in [CG12] to cases where it is necessary to patch chain complexes rather than homology groups, and the idea of patching completed homology explained in [CEGGPS]. We begin by explaining why this is a useful thing to do. Our main motivations come from the p-adic Langlands program, which is well understood for $\mathrm{GL}_2/\mathbb{Q}$, but is very mysterious beyond this case; and from the problem of proving automorphy lifting theorems for p-adic automorphic forms ("big $R = \mathbb{T}$ theorems") in situations where classical automorphic forms are no longer dense (for example, $\mathrm{GL}_n/\mathbb{Q}$ for any $n > 2$).

The local p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ has been established by completely local methods (see in particular [Col10; Paš13]), and local-global compatibility for $\mathrm{GL}_2/\mathbb{Q}$ was established in [Eme10a] (which goes on to deduce

The first author was supported in part by a Leverhulme Prize, EPSRC grant EP/L025485/1, Marie Curie Career Integration Grant 303605, and by ERC Starting Grant 306326. The second author was supported by ERC Starting Grant 306326.

many cases of the Fontaine–Mazur conjecture). It has proved difficult to generalise the local constructions for $\mathrm{GL}_2/\mathbb{Q}$, and the paper [CEGGPS] proposed instead (by analogy with the original global proof of local class field theory) to construct a candidate correspondence globally, by patching the completed homology of unitary groups over CM fields.

This construction has the disadvantage that it seems to be very difficult to prove that it is independent of the global situation, and of the choices involved in Taylor–Wiles patching. However, in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$, the sequel [CEGGPS2] showed (without using the results of [Col10; Paš13]) that the patching construction is independent of global choices, and therefore uniquely determines a local correspondence.

It is natural to ask whether similar constructions can be carried out for GL_n over a number field F. Until recently it was believed that Taylor–Wiles patching only applied to groups admitting discrete series (which would limit such a construction to the case $n = 2$ and F totally real), but Calegari and Geraghty showed in [CG12] that by patching chain complexes rather than homology groups one can overcome this obstruction, provided that one admits natural conjectures on the existence and properties of Galois representations attached to torsion classes in (uncompleted) homology. For a general F these conjectures are open, but for F totally real or CM the existence of the Galois representations is known by [Sch15b], and most of the necessary properties are expected to be established in the near future (with the possible exception of local-global compatibility at places dividing p, which we discuss further below).

The patching construction in [CG12] is sometimes a little ad hoc, and it was refined in [KT14], where the patching is carried out in the derived category. The construction of [CEGGPS] was improved upon in [Sch15a], which uses ultrafilters to significantly reduce the amount of bookkeeping needed in the patching argument. We combine these two approaches, and use ultrafilters to patch complexes in the derived category. In fact, we take a different approach to [KT14], by directly patching complexes computing homology, rather than minimal resolutions of such complexes; this has the advantage that our patched complex naturally has actions of the Hecke algebras and p-adic analytic groups. The use of ultrafilters streamlines this construction, and most of our constructions are natural, resulting in cleaner statements and proofs. (We still make use of the existence of minimal resolutions to show that our ultraproduct constructions are well behaved.)

To explain our results we introduce some notation. Write $K_0 = \prod_{v|p} \mathrm{PGL}_n(\mathcal{O}_{F_v})$ and let K_1 denote a pro-p Sylow subgroup of K_0. We consider locally symmetric spaces X_U for PGL_n/F, with level $U = U_pU^p \subset \mathrm{PGL}_n(\mathbb{A}_F^\infty)$ where U^p is some fixed tame level and U_p is a compact open subgroup of K_0. Let $\mathcal{O}$ be the ring of integers in some finite extension $E/\mathbb{Q}_p$, and write k for the residue field of $\mathcal{O}$. We write $\mathcal{O}_\infty$ for a power series ring over $\mathcal{O}$ and R_∞ for a power series ring over the (completed) tensor product of the local Galois deformation rings at the places $v|p$ of F. These power series rings are in some numbers of variables which depend on the choice of Taylor–Wiles primes; these power series variables are unimportant for the present discussion. For the purposes of this introduction, we will also ignore the role of the local Galois deformation rings at places $v \nmid p$ where our residual Galois representation is ramified.

The output of our patching construction is a perfect chain complex $\widetilde{\mathcal{C}}(\infty)$ of $\mathcal{O}_\infty[[K_0]]$-modules, equipped with an $\mathcal{O}_\infty$-linear action of $\prod_{v|p} G(F_v)$ and an $\mathcal{O}_\infty$-algebra homomorphism

$$R_\infty \to \mathrm{End}_{D(\mathcal{O}_\infty)}(\widetilde{\mathcal{C}}(\infty))$$

(where $D(\mathcal{O}_\infty)$ is the unbounded derived category of $\mathcal{O}_\infty$-modules). The action of R_∞ on $\widetilde{\mathcal{C}}(\infty)$ commutes with the action of $\prod_{v|p} G(F_v)$ (and with that of $\mathcal{O}_\infty[[K_0]]$). Reducing the complex $\widetilde{\mathcal{C}}(\infty)$ modulo the ideal $\mathbf{a}$ of $\mathcal{O}_\infty$ generated by the power series variables, we obtain a complex which computes the completed homology groups

$$\widetilde{H}_*(X_{U^p}, \mathcal{O})_\mathfrak{m} := \varprojlim_{U_p} H_*(X_{U_pU^p}, \mathcal{O})_\mathfrak{m}$$

localised at a non-Eisenstein maximal ideal $\mathfrak{m}$ of a 'big' Hecke algebra $\mathbb{T}^S(U^p)$ which acts on completed homology.

Our first main result is to show that, assuming a vanishing conjecture of [CG12] (which says that homology groups vanish outside of the expected range of degrees $[q_0, q_0 + l_0]$ after localising at $\mathfrak{m}$), and a conjecture of [CE12] on the codimension of completed homology, then the homology of $\widetilde{\mathcal{C}}(\infty)$ vanishes outside of a single degree q_0, and $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is Cohen–Macaulay over both $\mathcal{O}_\infty[[K_0]]$ and $R_\infty[[K_0]]$ of the expected projective dimensions.

One novel feature of our work appears here: since we are working with finitely generated modules over the non-commutative algebras $\mathcal{O}_\infty[[K_0]]$ and $R_\infty[[K_0]]$ we are forced to establish non-commutative analogues of the commutative algebra techniques which are applied in [CG12]. The first crucial result is Lemma A.10 (a generalisation of [CG12, Lem. 6.2]) which, as in *op. cit* is used to establish vanishing of the homology of the patched complex outside degree q_0. The second is Corollary A.29, which is used to deduce the Cohen–Macaulay property for the patched module over $R_\infty[[K_0]]$ from the Cohen–Macaulay property over $\mathcal{O}_\infty[[K_0]]$.

If A is a ring and M is an A-module, then we write $\mathrm{pd}_A(M)$ for the projective dimension of M over A, and $j_A(M)$ for its grade (also known as its codimension; see Definition A.2 and Remark A.3).

Theorem A (Theorem 4.2.1)**.** *Suppose that*

(a) $H_i(X_{U^pK_1}, k)_\mathfrak{m} = 0$ *for i outside the range* $[q_0, q_0 + l_0]$.

(b) $j_{\mathcal{O}[[K_0]]}\left(\bigoplus_{i\geq 0} \widetilde{H}_i(X_{U^p}, \mathcal{O})_\mathfrak{m}\right) \geq l_0$.

Then

(1) $\widetilde{H}_i(X_{U^p}, \mathcal{O})_\mathfrak{m} = 0$ *for $i \neq q_0$ and $\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_\mathfrak{m}$ is a Cohen–Macaulay $\mathcal{O}[[K_0]]$-module with*

$$\mathrm{pd}_{\mathcal{O}[[K_0]]}(\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_\mathfrak{m}) = j_{\mathcal{O}[[K_0]]}(\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_\mathfrak{m}) = l_0.$$

(2) $H_i(\widetilde{\mathcal{C}}(\infty)) = 0$ *for $i \neq q_0$ and $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is a Cohen–Macaulay $\mathcal{O}_\infty[[K_0]]$-module with*

$$\mathrm{pd}_{\mathcal{O}_\infty[[K_0]]}\left(H_{q_0}(\widetilde{\mathcal{C}}(\infty))\right) = j_{\mathcal{O}_\infty[[K_0]]}\left(H_{q_0}(\widetilde{\mathcal{C}}(\infty))\right) = l_0.$$

(3) $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ *is a Cohen–Macaulay $R_\infty[[K_0]]$-module with*

$$\mathrm{pd}_{R_\infty[[K_0]]}\left(H_{q_0}(\widetilde{\mathcal{C}}(\infty))\right) = j_{R_\infty[[K_0]]}\left(H_{q_0}(\widetilde{\mathcal{C}}(\infty))\right) = \dim(B)$$

where $\dim(B) = (\frac{n(n+1)}{2} - 1)[F : \mathbb{Q}]$.

The conjectures of [CE12] and [CG12] are open in general, but they are known if $n = 2$ and F is imaginary quadratic.

In Section 4.3 we take this analysis further. Under a natural condition on the codimension (over $k[[K_0]]$) of the fibre of completed homology at $\mathfrak{m}$, we prove the following result, which shows that the Hecke algebra $\mathbb{T}^S(U^p)_\mathfrak{m}$ is isomorphic to a Galois deformation ring R (a 'big $R = \mathbb{T}$' theorem), making precise the heuristics discussed in [Eme14, §3.1.1] which compare the Krull dimensions of Hecke algebras and the Iwasawa theoretic dimensions of completed homology modules and their fibres.

Theorem B (Proposition 4.3.1)**.** *Suppose that the assumptions of Theorem A hold, and that we moreover have*

$$j_{k[[K_0]]}(\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_\mathfrak{m}/\mathfrak{m}\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_\mathfrak{m}) \geq \dim(B).$$

Then we have the following:

1. $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ *is a flat* R_∞*-module.*
2. *The ideal* $R_\infty \mathbf{a}$ *is generated by a regular sequence in* R_∞*.*
3. *The surjective maps*
$$R_\infty/\mathbf{a} \to R \to \mathbb{T}^S(U^p)_\mathfrak{m}$$
are all isomorphisms and $\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_\mathfrak{m}$ *is a faithfully flat* $\mathbb{T}^S(U^p)_\mathfrak{m}$*-module.*
4. *The rings* $R \cong \mathbb{T}^S(U^p)_\mathfrak{m}$ *are local complete intersections with Krull dimension equal to* $1 + \dim(B) - l_0$*.*

We note here a crucial difference between our set-up and the situation in which Taylor–Wiles patching (and its variants) is usually applied — the patched module $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is not finitely generated over R_∞. The patched module is finitely generated over $R_\infty[[K_0]]$ but is not free over this Iwasawa algebra (it has codimension $\dim(B)$). So the usual techniques to establish '$R = \mathbb{T}$' do not apply.

Moreover, even if we could establish that $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is a faithful R_∞-module, this would not be enough to conclude that the map $R \to \mathbb{T}^S(U^p)_\mathfrak{m}$ has nilpotent kernel. Instead we need to establish the stronger result that $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is a *flat* R_∞-module. The main novelty of Theorem B is that the simple codimension inequality appearing in the statement is enough to guarantee this flatness. This follows from a version of the miracle flatness criterion in commutative algebra (Prop. A.30 — again we must modify things to handle the fact that our modules are only finitely generated over a non-commutative algebra).

Establishing the codimension inequality seems to require substantial information about the mod p representations of $\prod_{v|p} \mathrm{PGL}_n(F_v)$ appearing in completed cohomology. Even in $l_0 = 0$ situations, we do not know how to establish this codimension inequality (in contrast to the assumptions made in Theorem A, which become trivial when working in an appropriate $l_0 = 0$ setup) — if we did, our methods would give a new approach to proving big $R = \mathbb{T}$ theorems in these situations. In the case $n = 2$, $F = \mathbb{Q}$, the codimension inequality follows from Emerton's p-adic local–global compatibility theorem, together with known properties of the p-adic local Langlands correspondence. In Section 5 we show that some conjectural local–global compatibility statements when $n = 2$ and p splits completely in F also imply that this codimension inequality holds.

As far as we know, our strategy for establishing big $R = \mathbb{T}$ theorems is the first to be proposed which handles the $l_0 > 0$ situation — existing results in the $l_0 = 0$ case ([GM98; Böc01; Che11; All16]) rely on establishing Zariski density of (characteristic 0) automorphic points in the unrestricted Galois deformation ring R, using generalisations of the Gouvêa–Mazur infinite fern. When $l_0 > 0$ characteristic 0 automorphic points are not expected to be Zariski dense in R, and they are not Zariski dense in the relevant eigenvarieties (see [CM09] and work of Serban described in [Per]), so this approach breaks down.

In Section 5 we specialise to the case that $n = 2$ and p splits completely in F, where we can relate our constructions to the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. We formulate a natural conjecture (Conjecture 5.1.2) saying that the patched module $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is determined by (and in fact determines) this correspondence; in the case $F = \mathbb{Q}$ this conjecture is proved in [CEGGPS2], and is essentially equivalent to the local-global compatibility result of [Eme10a]. We show that this conjecture implies a local-global compatibility result (in the derived category) for the complexes computing finite level homology modules with coefficients in an algebraic representation; this compatibility is perhaps somewhat surprising, as it is phrased in terms of crystalline deformation rings, which are not obviously well-behaved integrally.

Conversely, we show that if we assume (in addition to the assumptions made in Section 4) that this local-global compatibility holds at finite level, then Conjecture 5.1.2 holds. Our proof is an adaptation of the methods of [CEGGPS2], although some additional arguments are needed in our more general setting.

We moreover show that Conjecture 5.1.2 has as consequences an automorphy lifting theorem and a 'small $R[1/p] = \mathbb{T}[1/p]$' result (Corollary 5.1.8). Therefore, our local-global compatibility conjecture entails many new cases of the Fontaine–Mazur conjecture. The application to Fontaine–Mazur was established by [Eme10a] in the case $F = \mathbb{Q}$, and although our argument looks rather different it is closely related to that of *loc. cit.* (but see also Remark 5.1.10).

While our main results are all conditional on various natural conjectures about (completed) homology groups, in the case that $n = 2$ and F is an imaginary quadratic field in which p splits it seems that the only serious obstruction is our finite level local-global compatibility conjecture (Conjecture 5.1.12), as we explain in Section 5.4.

We end this introduction by briefly explaining the contents of the sections that we have not already described. In Section 2 we introduce the complexes that we will patch and the Hecke algebras that act on them, and prove some standard results about minimal resolutions of complexes. We also prove some basic results about ultraproducts of complexes. In Section 3 we introduce the Galois deformation rings, carry out our patching construction, and prove its basic properties (for example, we establish its compatibility with completed homology).

In Appendix A we establish analogues for Iwasawa algebras of various classical results in commutative algebra, which we apply to our patched complexes in Section 4. Finally in Appendix B we prove some basic results about tensor products and projective envelopes of pseudocompact modules that we use in Section 5.

1.1. **Acknowledgements.** We would like to thank Frank Calegari, Matt Emerton, Christian Johansson, Vytas Paškūnas, Jack Thorne and Simon Wadsley for helpful conversations.

1.2. **Notation.** Let F be a number field, and fix an algebraic closure $\overline{F}$ of F, as well as algebraic closures $\overline{F}_v$ of the completion F_v of F at v for each place v of F, and embeddings $\overline{F} \hookrightarrow \overline{F}_v$ extending the natural embeddings $F \hookrightarrow F_v$. These choices determine embeddings of absolute Galois groups $G_{F_v} \hookrightarrow G_F$. If v is a finite place of F, then we write $I_{F_v} \subset G_{F_v}$ for the inertia group, and $\mathrm{Frob}_v \in G_{F_v}/I_{F_v}$ for a geometric Frobenius element; we normalise the local Artin maps Art_{F_v} to send uniformisers to geometric Frobenius elements. We write $\mathbb{A}_F$ for the adele ring of F, and $\mathbb{A}_F^\infty$ for the finite adeles.

We fix a prime p throughout, and write $\epsilon : G_F \to \mathbb{Z}_p^\times$ for the p-adic cyclotomic character. Let $\mathcal{O}$ be the ring of integers in a finite extension $E/\mathbb{Q}_p$ with residue field k; our Galois representations will be valued in $\mathcal{O}$-algebras (but we will feel free to enlarge E where necessary). If R is a complete Noetherian local $\mathcal{O}$-algebra with residue field k, then we write CNL_R for the category of complete Noetherian local R-algebras with residue field k.

If R is a ring, we write $Ch(R)$ for the abelian category of chain complexes of R-modules. If $C_\bullet \in Ch(R)$ then we write $H_*(C_\bullet) := \oplus_{n\in\mathbb{Z}} H_n(C_\bullet)$. We write $D(R)$ for the (unbounded) derived category of R-modules — for us, the objects of $D(R)$ are *cochain* complexes of R-modules, but we regard a chain complex $C_\bullet \in Ch(R)$ as a cochain complex $C^\bullet$ by setting $C^i = C_{-i}$. We write $D^-(R)$ for the bounded-above derived category of R-modules. The objects of $D^-(R)$ are cochain complexes of R-modules with bounded-above cohomology, or (equivalently) chain complexes of R-modules with bounded-below homology. Similarly, we write $D^+(R)$ for the bounded-below derived category.

An object $C^\bullet$ of $D(R)$ is called a *perfect complex* if there is a quasi-isomorphism $P^\bullet \to C^\bullet$ where $P^\bullet$ is a bounded complex of finite projective R-modules. In fact, $C^\bullet$ is perfect if and only if it is isomorphic in $D(R)$ to a bounded complex $P^\bullet$ of finite projectives: if we have another complex $D^\bullet$ and quasi-isomorphisms $P^\bullet \to D^\bullet$, $C^\bullet \to D^\bullet$, then there is a quasi-isomorphism $P^\bullet \to C^\bullet$ ([Stacks, Tag 064E]).

If K is a compact p-adic analytic group, we have the Iwasawa algebra $\mathcal{O}[[K]] := \varprojlim_U \mathcal{O}[K/U]$, where U runs over the open normal subgroups of K. This is a (non-commutative) Noetherian ring, some of whose properties we recall in Appendix A. If R is a formally smooth (commutative) $\mathcal{O}$-algebra, then we write $R[[K]] := R\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[[K]]$; note that if R has relative dimension d over $\mathcal{O}$, then $R[[K]] \cong \mathcal{O}[[K \times \mathbb{Z}_p^d]]$, so general properties of $\mathcal{O}[[K]]$ are inherited by $R[[K]]$.

For technical reasons, we will sometimes assume that K is a uniform pro-p group in the sense explained in [Ven02, §1.2]; as explained there, this can always be achieved by replacing K with a normal open subgroup. The group $\mathbb{Z}_p^d$ is uniform pro-p, so properties of $\mathcal{O}[[K]]$ for K a uniform pro-p group are again inherited by $R[[K]]$.

If M is a pseudocompact (i.e. profinite) $\mathcal{O}$-module, we write $M^\vee := \mathrm{Hom}_{\mathcal{O}}^{cts}(M, E/\mathcal{O})$ for the Pontryagin dual of M.

2. Patching I: Completed homology complexes and ultrafilters

In this section and the following one we explain our patching construction. For the convenience of the reader, we will generally follow the notation of [KT14].

2.1. **Arithmetic quotients.** We begin by introducing the manifolds whose homology we will patch. We follow [CG12] in patching arithmetic quotients for PGL_n, rather than GL_n; this is a minor issue in practice, as the connected components of the arithmetic quotients are the same for either choice, and we are for the most part able to continue to follow [KT14], although we caution the reader that because of this change, it is sometimes the case that we use the same notation to mean something slightly different to the corresponding definition in [KT14].

Let $G = \mathrm{PGL}_{n,F}$, let $G_\infty = G(F \otimes_{\mathbb{Q}} \mathbb{R})$, and let $K_\infty \subset G_\infty$ be a maximal compact subgroup. Write $X_G := G_\infty/K_\infty$. If $U \subset G(\mathbb{A}_F^\infty)$ is an open compact subgroup, then we define

$$X_U = G(F)\backslash(G(\mathbb{A}_F^\infty)/U \times X_G),$$

If $U \subset G(\mathbb{A}_F^\infty)$ is an open compact subgroup of the form $U = \prod_v U_v$, we say that U is *good* if it satisfies the following conditions:

- For each $g \in G(\mathbb{A}_F^\infty)$, the group $\Gamma_{U,g} := gUg^{-1} \cap G(F)$ is neat, and in particular torsion-free. (By definition, $\Gamma_{U,g}$ is neat if for each $h \in \Gamma_{U,g}$, the eigenvalues of h generate a torsion-free group.)
- For each finite place v of F, $U_v \subset \mathrm{PGL}_n(\mathcal{O}_{F_v})$.

We write $U = U_p U^p$, where $U_p = \prod_{v|p} U_v$, $U^p = \prod_{v\nmid p} U_v$. If S is a finite set of finite places of F, then we say that U is *S-good* if $U_v = \mathrm{PGL}_n(\mathcal{O}_{F_v})$ for all $v \notin S$.

By the proof of [KT14, Lem. 6.1], if U is good, then X_U is a smooth manifold, and if $V \subset U$ is a normal compact open subgroup, then V is also good, and $X_V \to X_U$ is a Galois cover of smooth manifolds.

Let r_1, r_2 denote the number of real and complex places of F, respectively. Then

$$\dim X_U = \frac{r_1}{2}(n-1)(n+2) + r_2(n^2-1). \tag{2.1.1}$$

The *defect* is

$$l_0 = \operatorname{rank} G_\infty - \operatorname{rank} K_\infty = \begin{cases} r_1(\frac{n-2}{2}) + r_2(n-1) & n \text{ even;} \\ r_1(\frac{n-1}{2}) + r_2(n-1) & n \text{ odd,} \end{cases} \tag{2.1.2}$$

and we also set

$$q_0 = \frac{d-l_0}{2} = \begin{cases} r_1(\frac{n^2}{4}) + r_2\frac{n(n-1)}{2} & n \text{ even;} \\ r_1(\frac{n^2-1}{4}) + r_2\frac{n(n-1)}{2} & n \text{ odd.} \end{cases} \tag{2.1.3}$$

In particular, if F is an imaginary quadratic field and $n = 2$, then $\dim X_U = 3$, $l_0 = 1$, and $q_0 = 1$. The notation l_0, q_0 comes from [BW00], and $[q_0, q_0 + l_0]$ is the range of degrees in which tempered cuspidal automorphic representations of G contribute to the cohomology of the X_U.

Let $C_{\mathbb{A},\bullet}$ denote the complex of singular chains with $\mathbb{Z}$-coefficients which are valued in $G(\mathbb{A}_F^\infty) \times X_G$. This is naturally a $\mathbb{Z}[G(F) \times G(\mathbb{A}_F^\infty)]$-module. If U is good and M is a left $\mathbb{Z}[U]$-module, then we set

$$\mathcal{C}(U, M) := C_{\mathbb{A},\bullet} \otimes_{G(F)\times U} M.$$

Exactly as in [KT14, Prop. 6.2], there is a natural isomorphism

$$H_*(X_U, M) \cong H_*(\mathcal{C}(U, M)).$$

If $U = U_p U^p$ is good, then we have the completed homology groups in the sense of [CE12] which by definition are given by

$$\widetilde{H}_*(X_{U^p}, \mathcal{O}) := \varprojlim_{U'_p} H_*(X_{U'_p U^p}, \mathcal{O}),$$

the limit being taken over open subgroups U'_p of U_p.

2.1.4. *Hecke operators.* Our complexes have a natural Hecke action in the usual way, as described in [KT14, §6.2]. We recall some of the details. Suppose that U, V are good subgroups, that S is a finite set of places of F with $U_v = V_v$ if $v \in S$, and that M is a $\mathbb{Z}[G(\mathbb{A}_F^{\infty,S}) \times U_S]$-module. Then for each $g \in G(\mathbb{A}_F^{\infty,S})$ there is a Hecke operator

$$[UgV]_* : \mathcal{C}(V, M) \to \mathcal{C}(U, M)$$

given by the formula

$$([UgV]_*((h \times \sigma) \otimes m) = \sum_i [g_i^{-1} \cdot (h \times \sigma)] \otimes g_i^{-1} m = \sum_i (hg_i \times \sigma) \otimes g_i^{-1} m,$$

where $h \in G(\mathbb{A}_F^\infty)$, $\sigma : \Delta^j \to X_G$ is a singular simplex, and $m \in M$.

In practice, we will take $S = S_p$ to be the set of places of F lying over p, and we take M to be a finite $\mathbb{Z}_p$-module with a continuous action of $\prod_{v|p} \mathrm{PGL}_n(\mathcal{O}_{F_v})$, with the action of $G(\mathbb{A}_F^{\infty,S}) \times U_S$ on M being via projection to $U_S \subset \prod_{v|p} \mathrm{PGL}_n(\mathcal{O}_{F_v})$. (In fact, we will usually take the action on M to be the trivial action.) If $v \notin S_p$ is a finite place of F, then we choose a uniformiser ϖ_v of $\mathcal{O}_{F_v}$, and for each $1 \le i \le n$ we set $\alpha_{v,i} = \mathrm{diag}(\varpi_v, \dots, \varpi_v, 1, \dots, 1)$ (with i occurrences of ϖ_v).

Then if v is a place for which $U_v = \mathrm{PGL}_n(\mathcal{O}_{F_v})$, we set $T_v^i := [U_v \alpha_{v,i} U_v]$; these operators are independent of the choice of ϖ_v, and pairwise commute. We also consider places at which U_v is a normal subgroup of the standard Iwahori subgroup which contains the standard pro-ϖ_v-Iwahori subgroup. At these places we will set $\mathbf{U}_v^i = [U_v \alpha_{v,i} U_v]$; these operators now depend on the choice of ϖ_v, but (for the particular U_v that we use) they still pairwise commute. They also commute with the diamond operators $\langle \alpha \rangle = [U_v \alpha U_v]$, where α is an element of the standard Iwahori subgroup whose diagonal entries are all equal modulo ϖ_v.

Note that it is immediate from the definitions that the actions of the operators T_v^i and $\mathbf{U}_v^i$ are equivariant for the natural morphisms of complexes arising from shrinking the level U.

2.1.5. *Minimal resolutions.* We recall some standard material on minimal resolutions of complexes. Since we work over non-commutative rings, there don't seem to be any standard references.

Let R be a Noetherian local ring (possibly non-commutative). We denote the maximal ideal by $\mathfrak{m}$ and assume that $R/\mathfrak{m} = k$ is a field.

Definition 2.1.6. Let $\mathcal{F}_\bullet$ be a chain complex of finite free R-modules. The complex $\mathcal{F}_\bullet$ is *minimal* if for all i the boundary map $d_i : \mathcal{F}_{i+1} \to \mathcal{F}_i$ satisfies

$$d_i(\mathcal{F}_{i+1}) \subset \mathfrak{m}\mathcal{F}_i.$$

Note that if $\mathcal{F}_\bullet$ is minimal, the complex $k \otimes_R \mathcal{F}_\bullet$ has boundary maps equal to zero.

Lemma 2.1.7. *Let $\mathcal{F}_\bullet$ be a minimal complex of finite free R-modules with bounded below homology, so that thinking of $\mathcal{F}_\bullet$ as an object of the derived category $D^-(R)$, we have a well-defined object $k \otimes_R^{\mathbb{L}} \mathcal{F}_\bullet \in D^-(k)$. Then for each n we have*

$$\mathrm{rank}_R(\mathcal{F}_i) = \dim_k(H_i(k \otimes_R \mathcal{F}_\bullet)) = \dim_k(H_i(k \otimes_R^{\mathbb{L}} \mathcal{F}_\bullet)).$$

In particular, the ranks of the modules $\mathcal{F}_i$ depend only on the isomorphism class of $\mathcal{F}_\bullet$ in $D^-(R)$.

Proof. We have $\mathrm{rank}_R(\mathcal{F}_i) = \dim_k(k \otimes_R \mathcal{F}_i)$, and since $\mathcal{F}$ is minimal we have

$$k \otimes_R \mathcal{F}_i = H_i(k \otimes_R \mathcal{F}_\bullet). \qquad \square$$

Definition 2.1.8. Let $\mathcal{C}_\bullet \in Ch(R)$ with bounded below homology. If $\mathcal{F}_\bullet$ is a minimal complex (necessarily bounded below) with a quasi-isomorphism $\mathcal{F}_\bullet \to \mathcal{C}_\bullet$, we say that $\mathcal{F}_\bullet$ is a *minimal resolution* of $\mathcal{C}_\bullet$.

If $\mathcal{F}_\bullet$ is a minimal resolution of $\mathcal{C}_\bullet$, then by Lemma 2.1.7 we have

$$\mathrm{rank}_R(\mathcal{F}_i) = \dim_k(H_i(k \otimes^{\mathbb{L}}_R \mathcal{C}_\bullet)).$$

Proposition 2.1.9. *Let $\mathcal{C}_\bullet \in Ch(R)$ be a chain complex with bounded below homology, and assume further that $H_i(\mathcal{C}_\bullet)$ is a finitely generated R-module for all i. Then there exists a minimal resolution $\mathcal{F}_\bullet$ of $\mathcal{C}_\bullet$, and any two minimal resolutions of $\mathcal{C}_\bullet$ are isomorphic* (*although the isomorphism is not necessarily unique*).

Proof. By considering the canonical truncation $\tau_{\geq N}\mathcal{C}_\bullet \to \mathcal{C}_\bullet$ (which is an isomorphism for sufficiently negative N), we may assume that the complex $\mathcal{C}_\bullet$ is bounded below. The proof in the commutative case from [Rob80, §2, Theorem 2.4] applies without change (the proof in *loc. cit.* assumes that the complex has bounded homology, but this is not necessary). For the reader's convenience, we sketch the proof.

First we check the uniqueness of the minimal resolution: suppose we have two minimal resolutions $\mathcal{F}_{1,\bullet}, \mathcal{F}_{2,\bullet}$ of $\mathcal{C}_\bullet$. Then $\mathcal{F}_{1,\bullet}, \mathcal{F}_{2,\bullet}$ are isomorphic in $D(R)$. Since $\mathcal{F}_{1,\bullet}$ is a bounded below chain complex of projective modules there is a quasi-isomorphism $\mathcal{F}_{1,\bullet} \to \mathcal{F}_{2,\bullet}$ (by [Stacks, Tag 0649]). This map induces a quasi-isomorphism $k \otimes_R \mathcal{F}_{1,\bullet} \to k \otimes_R \mathcal{F}_{2,\bullet}$, and minimality implies that this quasi-isomorphism is actually an isomorphism of complexes. Nakayama's lemma now implies that $\mathcal{F}_{1,\bullet} \to \mathcal{F}_{2,\bullet}$ is an isomorphism of complexes.

Now we show existence of the minimal resolution. First, by a standard argument (see for example [Mum08, Lem. 1, pp.47–49]), there is a (not necessarily minimal) bounded below complex of finite free modules $\mathcal{G}_\bullet$ with a quasi-isomorphism $\mathcal{G}_\bullet \to \mathcal{C}_\bullet$.

We now inductively suppose that the complex $\mathcal{G}_\bullet$ satisfies $d_m(\mathcal{G}_{m+1}) \subset \mathfrak{m}\mathcal{G}_m$ for $m < i$. (Note that this is certainly true for $i \ll 0$.) We will construct a new bounded below complex $\mathcal{G}'_\bullet$ of finite free modules with $\mathcal{G}'_m = \mathcal{G}_m$ for $m < i$, together with a quasi-isomorphism $\mathcal{G}'_\bullet \to \mathcal{G}_\bullet$, such that $d_m(\mathcal{G}'_{m+1}) \subset \mathfrak{m}\mathcal{G}'_m$ for $m \leq i$. Iterating this procedure constructs the minimal resolution $\mathcal{F}_\bullet$.

So, we suppose that $d_i(\mathcal{G}_{i+1}) \not\subset \mathfrak{m}\mathcal{G}_i$. We let Y be a subset of $\mathcal{G}_{i+1}$ which lifts a linearly independent subset of $k \otimes_R \mathcal{G}_{i+1}$ mapping (injectively) to a basis for $d_i(k \otimes_R \mathcal{G}_{i+1}) \subset k \otimes_R \mathcal{G}_i$. Then the acyclic complex (with non-zero terms in degree $i+1$ and i)

$$\mathcal{C}(Y) = \oplus_{y \in Y} \left(0 \to Ry \xrightarrow{d_i} Rd_i(y) \to 0\right)$$

is a direct summand of $\mathcal{G}_\bullet$ (by Nakayama's lemma, the set Y can be extended to a free basis for $\mathcal{G}_{i+1}$ and $d_i(Y)$ can be extended to a free basis for $\mathcal{G}_i$), and we set $\mathcal{G}'_\bullet = \mathcal{G}_\bullet/\mathcal{C}(Y)$. Since $\mathcal{G}'_\bullet$ is a direct summand of $\mathcal{G}_\bullet$ we may choose a splitting $\mathcal{G}'_\bullet \to \mathcal{G}_\bullet$ of the projection map. This splitting is a quasi-isomorphism, since $\mathcal{C}(Y)$ is acyclic. It is easy to check that $\mathcal{G}'_\bullet$ has the other desired properties, so we are done. $\square$

2.1.10. *Big Hecke algebras.* Write $\mathcal{C}(U,s) := \mathcal{C}(U,\mathcal{O}/\varpi^s)$.

Definition 2.1.11. Let S be a finite set of finite places of F which contains S_p. Let $U = U_pU^p$ be an S-good subgroup, with U_p a compact open normal subgroup of K_0. We define $\mathbb{T}^S(U,s)$ to be the image of the abstract Hecke algebra $\mathbb{T}^S$ (generated over $\mathcal{O}$ by T_v^i for $v \notin S$) in $\mathrm{End}_{D(\mathcal{O}/\varpi^s[K_0/U_p])}(\mathcal{C}(U,s))$.

We let

$$\mathbb{T}^S(U^p) = \varprojlim_{U_p,s} \mathbb{T}^S(U_pU^p,s)$$

where the limit is over compact open normal subgroups U_p of K_0 and $s \in \mathbb{Z}_{\geq 1}$, and the (surjective) transition maps come from the functorial maps

$$\mathrm{End}_{D(\mathcal{O}/\varpi^{s'}[K_0/U_p'])}(\mathcal{C}(U_p'U^p,s')) \to \mathrm{End}_{D(\mathcal{O}/\varpi^s[K_0/U_p])}(\mathcal{O}/\varpi^s[K_0/U_p]\otimes_{\mathcal{O}/\varpi^{s'}[K_0/U_p']}\mathcal{C}(U_p'U^p,s'))$$

for $s' \geq s$ and $U_p' \subset U_p$ and the natural identification

$$\mathcal{O}/\varpi^s[K_0/U_p] \otimes_{\mathcal{O}/\varpi^{s'}[K_0/U_p']} \mathcal{C}(U_p'U^p,s') \cong \mathcal{C}(U_pU^p,s).$$

We equip $\mathbb{T}^S(U^p)$ with the inverse limit topology.

Remark 2.1.12. Now suppose that U_p is any compact open subgroup of K_0 (not necessarily normal) and $s \geq 1$. Let V_p be a compact open normal subgroup of U_p which is also normal in K_0. Then the natural map $\mathbb{T}^S(U^p) \to \mathrm{End}_{D(\mathcal{O}/\varpi^s[K_0/V_p])}(\mathcal{C}(V_pU^p,s))$ induces a map $\mathbb{T}^S(U^p) \to \mathrm{End}_{D(\mathcal{O}/\varpi^s[U_p/V_p])}(\mathcal{C}(V_pU^p,s))$ and therefore induces a natural map $\mathbb{T}^S(U^p) \to \mathrm{End}_{D(\mathcal{O}/\varpi^s)}(\mathcal{C}(U_pU^p,s))$, using the identification

$$\mathcal{O}/\varpi^s \otimes_{\mathcal{O}/\varpi^s[U_p/V_p]} \mathcal{C}(V_pU^p,s) \cong \mathcal{C}(U_pU^p,s).$$

For each U and s, $\mathbb{T}^S(U,s)$ is a finite $\mathcal{O}$-algebra, since $\mathcal{C}(U,s)$ is perfect as a complex of $\mathcal{O}/\varpi^s[K_0/U_p]$-modules. Moreover, the natural map $\mathbb{T}^S(U,s) \to \mathrm{End}_{\mathcal{O}}(H_*(\mathcal{C}(U,s)))$ has nilpotent kernel by [KT14, Lem. 2.5], and therefore $\mathbb{T}^S(U,s)$ is a finite ring.

Remark 2.1.13. Similarly, for each compact open normal subgroup U_p of K_0, we can define

$$\mathbb{T}^S(U_pU^p) = \varprojlim_{s} \mathbb{T}^S(U_pU^p,s).$$

Then $\mathbb{T}^S(U_pU^p)$ is a finite $\mathcal{O}$-algebra, and we have $\mathbb{T}^S(U^p) = \varprojlim_{U_p} \mathbb{T}^S(U_pU^p)$, equipped with the inverse limit topology (where each $\mathbb{T}^S(U_pU^p)$ has its natural p-adic topology).

The big Hecke algebra $\mathbb{T}^S(U^p)$ is naturally equipped with a map

$$\mathbb{T}^S(U^p) \to \mathrm{End}_{\mathcal{O}[[K_0]]}(\widetilde{H}_i(X_{U^p},\mathcal{O}))$$

which commutes with the action of $\prod_{v|p} G(F_v)$.

Lemma 2.1.14. *The profinite $\mathcal{O}$-algebra $\mathbb{T}^S(U^p)$ is semilocal. Denote its finitely many maximal ideals by $\mathfrak{m}_1,\dots,\mathfrak{m}_r$ and let $J = J(\mathbb{T}^S(U^p)) = \cap_{j=1}^r \mathfrak{m}_j$ denote the Jacobson radical. Then $\mathbb{T}^S(U^p)$ is J-adically complete and separated, and we have*

$$\mathbb{T}^S(U^p) = \mathbb{T}^S(U^p)_{\mathfrak{m}_1} \times \cdots \times \mathbb{T}^S(U^p)_{\mathfrak{m}_r}.$$

For each maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S(U^p)$, the localisation $\mathbb{T}^S(U^p)_{\mathfrak{m}}$ is an $\mathfrak{m}$-adically complete and separated local ring with residue field a finite extension of k.

Proof. First we note that if U_p is a pro-p group and V_p is a normal open subgroup of U_p, then for each $s \geq 1$ the surjective map

$$\mathbb{T}^S(V_pU^p, s) \to \mathbb{T}^S(U_pU^p, 1)$$

induces a bijection of maximal ideals. Indeed, we have (by [Wei94, Thm. 5.6.4]) a spectral sequence of $\mathbb{T}^S(V_pU^p, s)$-modules

$$E^2_{i,j} : \mathrm{Tor}_i^{\mathcal{O}/\varpi^s[U_p/V_p]}(k, H_j(\mathcal{C}(U^pV_p, s))) \Rightarrow H_{i+j}(\mathcal{C}(U^pU_p, 1)).$$

Localising at a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S(V_pU^p, s)$ and considering the largest q such that $H_q(\mathcal{C}(U^pV_p, s))_{\mathfrak{m}}$ is non-zero shows that $\mathfrak{m}$ is in the support of $H_q(\mathcal{C}(U^pU_p, 1))$, and therefore $\mathfrak{m}$ is the inverse image of a maximal ideal in $\mathbb{T}^S(U_pU^p, 1)$. (Here we have used that $\mathbb{T}^S(U, s) \to \mathrm{End}_{\mathcal{O}}(H_*(\mathcal{C}(U, s)))$ has nilpotent kernel.)

Now it is not hard to show that the maximal ideals of $\mathbb{T}^S(U^p)$ are in bijection with the maximal ideals of $\mathbb{T}^S(U_pU^p, 1)$. Indeed, we have shown that for every open $V_p \triangleleft U_p$ and $s \geq 1$ the kernel of

$$\mathbb{T}^S(V_pU^p, s) \to \mathbb{T}^S(U_pU^p, 1)$$

is contained in the Jacobson radical of $\mathbb{T}^S(V_pU^p, s)$. If $x \in \mathbb{T}^S(U^p)$ maps to a unit in $\mathbb{T}^S(V_pU^p, s)$ for every open $V_p \triangleleft U_p$ and $s \geq 1$ then x is a unit. We deduce that the kernel of

$$\mathbb{T}^S(U^p) \to \mathbb{T}^S(U_pU^p, 1)$$

is contained in the Jacobson radical J of $\mathbb{T}^S(U^p)$, and it follows that $\mathbb{T}^S(U^p)$ is semilocal.

For every open $V_p \triangleleft U_p$ and $s \geq 1$ the image of J in $\mathbb{T}^S(V_pU^p, s)$ is nilpotent. It follows that $\mathbb{T}^S(U^p)$ is J-adically complete and separated. The remainder of the lemma follows from [Mat89, Theorem 8.15]. $\square$

2.2. **Ultrafilters.** In this section we let A be a commutative finite (cardinality) local ring of characteristic p, denote the maximal ideal of A by $\mathfrak{m}_A$, and let $k = A/\mathfrak{m}_A$. We let B be a finite (but possibly non-commutative) augmented A-algebra. Denote the augmentation ideal $\ker(B \to A)$ by $\mathbf{a}$. The example we have in mind is $B = A[\Gamma]$ where Γ is a finite group.

Given an index set I, we define $A_I = \prod_{i \in I} A$, and similarly $B_I = \prod_{i \in I} B$. B_I is an augmented A_I-algebra, with augmentation ideal $\mathbf{a}_I = \prod_{i \in I} \mathbf{a} = \ker(B_I \to A_I)$. Note that $\mathbf{a}_I$ is a finitely generated ideal of B_I, so A_I is finitely presented as a B_I-module. More generally, if $\mathbf{b} \subset B$ is a two-sided ideal which contains $\mathbf{a}$, and $\mathbf{b}_I = \prod_{i \in I} \mathbf{b}$, then $B_I/\mathbf{b}_I = (B/\mathbf{b})_I$ is finitely presented as a B_I-module.

Remark 2.2.1. If $B = A[\Gamma]$ then we have $B_I = A_I[\Gamma]$.

Lemma 2.2.2. $\mathrm{Spec}(A_I)$ *can be naturally identified with the set of ultrafilters on* I*. We have* $A_{I,x} = A$ *for each* $x \in \mathrm{Spec}(A_I)$*. We also have* $A_{I,x} \otimes_{A_I} B_I = B$*.*

Proof. The bijection between ultrafilters and prime ideals is given by taking an ultrafilter $\mathfrak{F}$ to the ideal whose elements (a_i) satisfy $\{i : a_i \in \mathfrak{m}_A\} \in \mathfrak{F}$. Since the map $A_I \to k_I$ has nilpotent kernel, the fact that this gives a bijection follows from the case when A is a field [Sch15a, Lemma 8.1].

For $x \in \mathrm{Spec}(A_I)$ the associated ultrafilter $\mathfrak{F}_x$ induces a map $B_I \to B$ by sending $(b_i)_{i \in I} \mapsto b$ where $b \in B$ is the unique element with the property that $\{i : b_i = b\} \in \mathfrak{F}_x$. Since $B_I = A_I \otimes_A B$ (because B is finitely presented as an A-module), this map induces an isomorphism $A_{I,x} \otimes_{A_I} B_I \cong B$. $\square$

Given a point $x \in \operatorname{Spec}(A_I)\backslash I$ and a set of chain complexes of B-modules $\{\mathcal{C}(i)\}_{i\in I}$, we define a chain complex of B-modules

$$\mathcal{C}(\infty) := A_{I,x} \otimes_{A_I} \left(\prod_{i\in I} \mathcal{C}(i)\right).$$

Lemma 2.2.3. *Let $\{\mathcal{C}(i)\}_{i\in I}$ be a set of chain complexes of flat B-modules. Then $\prod_{i\in I}\mathcal{C}(i)$ is a chain complex of flat B_I-modules and $\mathcal{C}(\infty)$ is a chain complex of flat B-modules.*

Proof. The fact that $\prod_{i\in I}\mathcal{C}(i)$ is a chain complex of flat B_I-modules follows from [Swe82, Thm. 1.13] (condition (d) in Sweedler's Theorem is automatically satisfied because B is a finite ring). We deduce immediately that the localisation $\mathcal{C}(\infty)$ is also a chain complex of flat B-modules. $\square$

Lemma 2.2.4. *Let $\{\mathcal{C}(i)\}_{i\in I}$ be a set of chain complexes of B-modules. Let $\mathbf{b} \subset B$ be a two-sided ideal which contains $\mathbf{a}$, and let $\{\overline{\mathcal{C}}(i)\}_{i\in I} = \{(B/\mathbf{b}) \otimes_B \mathcal{C}(i)\}_{i\in I}$. Then we have a natural isomorphism*

$$(B/\mathbf{b}) \otimes_B \mathcal{C}(\infty) = \overline{\mathcal{C}}(\infty).$$

Proof. We have

$$(B/\mathbf{b}) \otimes_B \mathcal{C}(\infty) = (B/\mathbf{b})_{I,x} \otimes_{(B/\mathbf{b})_I} (B/\mathbf{b})_I \otimes_{B_I} \prod_{i\in I}\mathcal{C}(i).$$

Since $(B/\mathbf{b})_I$ is finitely presented as a (right) B_I-module, we have (by [Bou98, Ex. I.§2.9])

$$(B/\mathbf{b})_I \otimes_{B_I} \prod_{i\in I}\mathcal{C}(i) = \prod_{i\in I}\overline{\mathcal{C}}(i)$$

and we obtain the desired equality. $\square$

In the rest of this subsection, we are going to assume that B is a *local* A-algebra. The example we have in mind is $B = A[\Gamma]$ where Γ is a finite p-group.

Definition 2.2.5. Suppose B is a *local* A-algebra and fix a set $\{\mathcal{C}(i)\}_{i\in I}$ of perfect chain complexes of B-modules. For each i fix a minimal resolution $\mathcal{F}(i)$ of $\mathcal{C}(i)$. Suppose we have integers $a \le b$ and $D \ge 0$. We say that the set $\{\mathcal{C}(i)\}_{i\in I}$ has *complexity bounded by* (a,b,D) if the minimal complexes $\mathcal{F}(i)$ are all concentrated in degrees between a and b and every term in these complexes has rank $\le D$.

If there exists some a,b,D such that $\{\mathcal{C}(i)\}_{i\in I}$ has complexity bounded by (a,b,D), we say that $\{\mathcal{C}(i)\}_{i\in I}$ has *bounded complexity.*

Lemma 2.2.6. *Suppose B is a local A-algebra, and let $\{\mathcal{C}(i)\}_{i\in I}$ be a set of perfect chain complexes of B-modules with bounded complexity. Then the complex $\prod_{i\in I}\mathcal{C}(i)$ is a perfect complex of B_I-modules.*

Proof. Fix a minimal resolution $\mathcal{F}(i)$ of each perfect complex $\mathcal{C}(i)$. Since products are exact in the category of Abelian groups, it suffices to check that the complex $\prod_{i\in I}\mathcal{F}(i)$ is a bounded complex of finite projective B_I-modules. Boundedness follows immediately from the bounded complexity assumption. It remains to show that if we have a set $\{F_i\}_{i\in I}$ of finite free B-modules with ranks all $\le D$, then the product $\prod_{i\in I} F_i$ is a finite projective B_I-module.

We have a decomposition $I = \coprod_{d=0}^{D} I_d$ such that $F_i \cong B^d$ for $i \in I_d$. Then $M_d \cong \prod_{i \in I_d} B^d \cong B_{I_d}^d$ is a finite free B_{I_d}-module. Each M_d is a finite projective B_I-module (they are direct summands of finite free modules), and we have

$$\prod_{i \in I} F_i = \bigoplus_{d=0}^{D} M_d,$$

so $\prod_{i \in I} F_i$ is a finite projective B_I-module, as required. □

Corollary 2.2.7. *Let $x \in \mathrm{Spec}(A_I) \backslash I$ and suppose that $\{\mathcal{C}(i)\}_{i \in I}$ is a set of perfect chain complexes of B-modules with bounded complexity. Then $\mathcal{C}(\infty)$ is a perfect complex of B-modules.*

Proof. This follows from Lemmas 2.2.2 and 2.2.6. □

Remark 2.2.8. In fact there is another way of phrasing the proof that this complex is perfect. If we fix a, b and D then there are finitely many isomorphism classes of minimal complex with complexity bounded by (a, b, D) (since B is a finite ring). Let $x \in \mathrm{Spec}(A_I) \backslash I$, corresponding to the non-principal ultrafilter $\mathfrak{F}$ on I. Then there is an $I' \in \mathfrak{F}$ such that the minimal resolutions of $\mathcal{C}(i)$ are isomorphic for all $i \in I'$. We can therefore take a single minimal complex $\mathcal{F}(\infty)$ which is a minimal resolution of $\mathcal{C}(i)$ for all $i \in I'$. We then have a quasi-isomorphism of complexes of $B_{I'}$-modules:

$$A_{I'} \otimes_A \mathcal{F}(\infty) \to A_{I'} \otimes_{A_I} \left(\prod_{i \in I} \mathcal{C}(i) \right) = \prod_{i \in I'} \mathcal{C}(i)$$

which induces a quasi-isomorphism

$$\mathcal{F}(\infty) \to \mathcal{C}(\infty),$$

so that $\mathcal{F}(\infty)$ is a minimal resolution of $\mathcal{C}(\infty)$.

3. Patching II: Galois representations and Taylor–Wiles primes

3.1. **Deformation theory.** We fix a number field F, a prime p, a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k, and a continuous absolutely irreducible representation $\overline{\rho} : G_F \to \mathrm{GL}_n(k)$. We assume from now on that $p > n \geq 2$. Fix also a continuous character $\mu : G_F \to \mathcal{O}^\times$ lifting $\det \overline{\rho}$, and a finite set of finite places S of F, which contains the set S_p of places of F lying over p, as well as the places at which $\overline{\rho}$ or μ are ramified.

For each $v \in S$, we fix a ring $\Lambda_v \in \mathrm{CNL}_{\mathcal{O}}$, let $\mathcal{D}_v^\square : \mathrm{CNL}_{\Lambda_v} \to \mathrm{Sets}$ be the functor associating to $R \in \mathrm{CNL}_{\Lambda_v}$ the set of all continuous liftings of $\overline{\rho}|_{G_{F_v}}$ to $\mathrm{GL}_n(R)$ which have determinant $\mu|_{G_{F_v}}$. This is represented by the universal lifting ring $R_v \in \mathrm{CNL}_{\Lambda_v}$.

Then as in [KT14, §4] we have the following notions.

- For $v \in S$, a *local deformation problem* for $\overline{\rho}|_{G_{F_v}}$ is a subfunctor $\mathcal{D}_v \subset \mathcal{D}_v^\square$ which is stable under conjugation by elements of $\ker(\mathrm{GL}_n(R) \to \mathrm{GL}_n(k))$, and is represented by a quotient $\overline{R}_v$ of $R_v^\square$.
- A *global deformation problem* is a tuple

$$\mathcal{S} = (\overline{\rho}, \mu, S, \{\Lambda_v\}_{v \in S}, \{\mathcal{D}_v\}_{v \in S})$$

consisting of the objects defined above.

- If $R \in \mathrm{CNL}_\Lambda$, then a lifting of $\overline{\rho}$ to a continuous homomorphism $\rho : G_F \to \mathrm{GL}_n(R)$ is of type $\mathcal{S}$ if it is unramified outside S, has determinant μ, and for each $v \in S$, $\rho|_{G_{F_v}}$ is in $\mathcal{D}_v(R)$.
- We say that two liftings are *strictly equivalent* if they are conjugate by an element of $\ker(\mathrm{GL}_n(R) \to \mathrm{GL}_n(k))$.
- If $T \subset S$ and $R \in \mathrm{CNL}_\Lambda$, then a *$T$-framed lifting of $\overline{\rho}$ to R* is a tuple $(\rho, \{\alpha_v\}_{v\in T})$ where ρ is a lifting of $\overline{\rho}$ to a continuous homomorphism $\rho : G_F \to \mathrm{GL}_n(R)$, and each α_v is an element of $\ker(\mathrm{GL}_n(R) \to \mathrm{GL}_n(k))$. Two T-framed liftings $(\rho, \{\alpha_v\}_{v\in T}), (\rho', \{\alpha'_v\}_{v\in T})$ are *strictly equivalent* if there is an element $a \in \ker(\mathrm{GL}_n(R) \to \mathrm{GL}_n(k))$ such that $\rho' = a\rho a^{-1}$ and each $\alpha'_v = a\alpha_v$.
- The functors of liftings of type $\mathcal{S}$, strict equivalences of liftings of type $\mathcal{S}$, and strict equivalence classes of T-framed liftings of type $\mathcal{S}$, are representable by objects $R^\square_{\mathcal{S}}$, $R_{\mathcal{S}}$, $R^T_{\mathcal{S}}$ respectively of CNL_Λ. (See [KT14, Thm. 4.5].)

Write $\Lambda_T := \widehat{\otimes}_{v\in T,\mathcal{O}}\Lambda_v$. For each $v \in S$, let $\overline{R}_v \in \mathrm{CNL}_{\Lambda_v}$ denote the representing object of $\mathcal{D}_v$, and write $R^{T,\mathrm{loc}}_{\mathcal{S}} := \widehat{\otimes}_{v\in T,\mathcal{O}}\overline{R}_v$. The natural transformation $(\rho, \{\alpha_v\}_{v\in T}) \mapsto (\alpha_v^{-1}\rho|_{G_{F_v}}\alpha_v)_{v\in T}$ induces a canonical homomorphism of Λ_T-algebras $R^{T,\mathrm{loc}}_{\mathcal{S}} \to R^T_{\mathcal{S}}$.

3.2. **Enormous image.** Let $H \subset \mathrm{GL}_n(k)$ be a subgroup which acts irreducibly on the natural representation. We assume that k is chosen large enough to contain all eigenvalues of all elements of H.

Definition 3.2.1. We say that H is enormous if it satisfies the following conditions:

(1) H has no non-trivial p-power order quotient.
(2) $H^0(H, \mathrm{ad}^0) = H^1(H, \mathrm{ad}^0) = 0$ (for the adjoint action of H).
(3) For all simple $k[H]$-submodules $W \subset \mathrm{ad}^0$, there is an element $h \in H$ with n distinct eigenvalues and $\alpha \in k$ such that α is an eigenvalue of h and $\mathrm{tr}\, e_{h,\alpha}W \neq 0$, where $e_{h,\alpha} \in M_n(k) = \mathrm{ad}$ denotes the unique h-equivariant projection onto the α-eigenspace of h.

Remark 3.2.2. By definition, an enormous subgroup is big in the sense of [CHT08, Defn. 2.5.1], and thus adequate in the sense of [Tho12, Defn. 2.3]. Indeed, the only differences between these notions is that in the definition of big, the condition that h has n distinct eigenvalues is relaxed to demanding that the generalised eigenspace of α is one-dimensional, and in the definition of adequate, it is further relaxed to ask only that α is an eigenvalue of h (but the definition of $e_{h,\alpha}$ is now the projection onto the generalised eigenspace for α).

Lemma 3.2.3. *If $n = 2$, the notions of enormous, big and adequate are all equivalent. In particular, if H acts irreducibly on k^2, then H is enormous unless $p = 3$ or $p = 5$, and the image of H in $\mathrm{PGL}_2(k)$ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_p)$.*

Proof. The second statement follows from the first statement and [BLGG13, Prop. A.2.1]. By Remark 3.2.2, it is therefore enough to show that if we have a simple $k[H]$-submodule $W \subset \mathrm{ad}^0$ and an element $h \in H$ with an eigenvalue α such that $\mathrm{tr}\, e_{h,\alpha}W \neq 0$, then h necessarily has distinct eigenvalues. If not, then $e_{h,\alpha} = 1$ by definition (as $e_{h,\alpha}$ is projection onto the generalised eigenspace for α), which is a contradiction as $W \subset \mathrm{ad}^0$. $\square$

We now give two examples of classes of enormous subgroups of $\mathrm{GL}_n(k)$ when $n > 2$, following [CHT08, §2.5] (which shows that the same groups are big).

Lemma 3.2.4. *If $n > 2$ and there is a subfield $k' \subset k$ such that $k^\times \mathrm{GL}_n(k') \supset H \supset \mathrm{SL}_n(k')$, then H is enormous.*

Proof. Examining the proof of [CHT08, Lem. 2.5.6] (which shows that H is big), we see that it is enough to check that $\mathrm{SL}_n(k')$ contains an element with n distinct eigenvalues. Since we are assuming that $p > n$, we can use an element with characteristic polynomial $X^n + (-1)^n$ (for example, the matrix (a_{ij}) with $a_{i+1,i} = 1$, $a_{1,n} = (-1)^{n-1}$, and all other $a_{ij} = 0$). □

Lemma 3.2.5. *If $p > 2n+1$ and there is a subfield $k' \subset k$ such that $k^\times \mathrm{Sym}^{n-1}\mathrm{GL}_2(k') \supset H \supset \mathrm{Sym}^{n-1}\mathrm{SL}_2(k')$, then H is enormous.*

Proof. The proof of [CHT08, Cor. 2.5.4] (which shows that H is big) in fact shows that H is enormous (note that in the proof of [CHT08, Lem. 2.5.2] it is shown that the eigenspaces of the element denoted t are 1-dimensional). (Note also that as explained after [BLGGT, Prop. 2.1.2], the hypothesis that $p > 2n - 1$ in [CHT08, Cor. 2.5.4] should be $p > 2n + 1$.) □

3.3. **Taylor–Wiles primes.** Suppose that v is a finite place of F such that $\#k(v) \equiv 1 \pmod p$, that $\overline{\rho}|_{G_{F_v}}$ is unramified, and that $\overline{\rho}(\mathrm{Frob}_v)$ has n distinct eigenvalues $\gamma_{v,1}, \dots, \gamma_{v,n} \in k$. Let $\Delta_v = (k(v)^\times(p))^{n-1}$ (where $k(v)^\times(p)$ is the Sylow p-subgroup of $k(v)^\times$), and let $\Lambda_v = \mathcal{O}[\Delta_v]$.

We define $\mathcal{D}_v^{\mathrm{TW}}$ to be the functor of liftings over $R \in \mathrm{CNL}_{\Lambda_v}$ of the form

$$r \sim \chi_1 \oplus \cdots \oplus \chi_n,$$

where $\chi_1, \dots, \chi_n : G_{F_v} \to R^\times$ are continuous characters such that for each $i = 1, \dots, n-1$, we have

- $(\chi_i \pmod{\mathfrak{m}_R})(\mathrm{Frob}_v) = \gamma_{v,i}$, and
- $\chi_i|_{I_{F_v}}$ agrees, on composition with the Artin map, with the i^{th} canonical character $k(v)^\times(p) \to R^\times$.

(This definition depends on the ordering of the $\gamma_{v,i}$, but this does not affect any of our arguments.) The functor $\mathcal{D}_v^{\mathrm{TW}}$ is represented by a formally smooth Λ_v-algebra.

Suppose that $\mathcal{S} = (\overline{\rho}, \mu, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S})$ is a deformation problem. Let Q be a set of places disjoint from S of the form considered above (that is, $\#k(v) \equiv 1 \pmod p$ and $\overline{\rho}(\mathrm{Frob}_v)$ has n distinct eigenvalues). We refer to the tuple

$$(Q, (\gamma_{v,1}, \dots, \gamma_{v,n})_{v\in Q})$$

as a *Taylor–Wiles datum*, and define the augmented deformation problem

$$\mathcal{S}_Q = (\overline{\rho}, \mu, S \cup Q, \{\Lambda_v\}_{v\in S} \cup \{\mathcal{O}[\Delta_v]\}_{v\in Q}, \{\mathcal{D}_v\}_{v\in S} \cup \{\mathcal{D}_v^{\mathrm{TW}}\}_{v\in Q}).$$

Let $\Delta_Q = \prod_{v\in Q} \Delta_v = \prod_{v\in Q} k(v)^\times(p)^{n-1}$. Then $R_{\mathcal{S}_Q}$ is naturally a $\mathcal{O}[\Delta_Q]$-algebra. If $\mathbf{a}_Q \subset \mathcal{O}[\Delta_Q]$ is the augmentation ideal, then there is a canonical isomorphism $R_{\mathcal{S}_Q}/\mathbf{a}_Q \cong R_{\mathcal{S}}$.

Recall that $\overline{\rho}$ is *totally odd* if if for each complex conjugation $c \in G_F$, we have

$$\overline{\rho}(c) \sim \mathrm{diag}(\underbrace{1, \dots, 1}_{a}, \underbrace{-1, \dots, -1}_{b}),$$

with $|a-b| \le 1$. (Of course, if F is totally complex, this is a vacuous condition.) Let l_0 be the integer defined in (2.1.2) (which only depends on F and n).

Lemma 3.3.1. *Let $(\overline{\rho}, \mu, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S})$ be a global deformation problem. Suppose that:*

- *$\overline{\rho}$ is totally odd.*
- *$\overline{\rho} \not\cong \overline{\rho} \otimes \overline{\epsilon}$.*
- *$\overline{\rho}(G_{F(\zeta_p)})$ is enormous.*

Then for every $q \gg 0$ and every $N \ge 1$, there exists a Taylor–Wiles datum $(Q_N, (\gamma_{v,1}, \dots, \gamma_{v,n})_{v\in Q_N})$ satisfying the following conditions:

1. *$\# Q_N = q$.*
2. *For each $v \in Q_N$, $q_v \equiv 1 \pmod{p^N}$.*
3. *The ring $R^S_{\mathcal{S}_{Q_N}}$ is a quotient $R^{S,\mathrm{loc}}_{\mathcal{S}}$-algebra of $R_\infty := R^{S,\mathrm{loc}}_{\mathcal{S}}[\![X_1, \dots, X_g]\!]$, where*

$$g = (n-1)q - n(n-1)[F:\mathbb{Q}]/2 - l_0 - 1 + \#S.$$

Proof. This is immediate from [KT14, Cor. 4.8, Lem. 4.12]. $\square$

Fix a choice of place $v_0 \in T$ and an integer $q \gg 0$ as in Lemma 3.3.1, and set $\mathcal{T} = \mathcal{O}[[X_v^{i,j}]]_{v\in S, 1\le i,j\le n}/(X_{v_0}^{1,1})$. Set $\Delta_{Q_N} := \prod_{v\in Q_N} \Delta_v$, $\mathcal{O}_N := \mathcal{T}[\Delta_{Q_N}]$, and $\mathcal{O}_\infty := \mathcal{T}[[\Delta_\infty]]$, where $\Delta_\infty = \mathbb{Z}_p^{(n-1)q}$. For each N we fix a surjection $\Delta_\infty \twoheadrightarrow \Delta_N$, and thus a surjection of $\mathcal{T}$-algebras $\mathcal{O}_\infty \twoheadrightarrow \mathcal{O}_N$.

We now examine the behaviour of the Hecke operators at Taylor–Wiles primes. Fix U^p such that U^pK_0 is S-good. We begin by setting up some notation. Let $(Q, (\gamma_{v,1}, \dots, \gamma_{v,n})_{v\in Q})$ be a Taylor–Wiles datum. We define compact open subgroups $U_0^p(Q) = \prod_{v\nmid p} U_0(Q)_v$ and $U_1^p(Q) = \prod_{v\nmid p} U_1(Q)_v$ of $U^p = \prod_{v\nmid p} U_v$ by:

- if $v \notin Q$, then $U_0(Q)_v = U_1(Q)_v = U_v$.
- If $v \in Q$ then $U_0(Q)_v$ is the standard Iwahori subgroup of $\mathrm{PGL}_n(\mathcal{O}_{F,v})$, and $U_1(Q)_v$ is the minimal subgroup of $U_0(Q)_v$ for which $U_0(Q)_v/U_1(Q)_v$ is a p-group.

In particular $U_1(Q)_v$ contains the pro-v Iwahori subgroup of $U_0(Q)_v$, so we can and do identify $\prod_{v\in Q} U_0(Q)_v/U_1(Q)_v$ with Δ_Q. We now introduce some natural variants of the Hecke algebras that we introduced in Section 2.1.10.

For each compact open normal subgroup U_p of K_0 we define $\mathbb{T}^{S\cup Q,Q}(U_pU_0^p(Q), s)$ to be the image in $\mathrm{End}_{D(\mathcal{O}/\varpi^s[K_0/U_p])}(\mathcal{C}(U_pU_0^p(Q), s))$ of the abstract Hecke algebra $\mathbb{T}^{S\cup Q,Q}$ generated by the operators T_v^i for $v \notin S\cup Q$ and $\mathbf{U}_v^i$ for $v \in Q$, where the operators $\mathbf{U}_v^i$ act as explained in Section 2.1.4. Similarly, we let $\mathbb{T}^{S\cup Q,Q}(U_pU_1^p(Q), s)$ to be the image of $\mathbb{T}^{S\cup Q,Q}$ in $\mathrm{End}_{D(\mathcal{O}/\varpi^s[\Delta_Q\times K_0/U_p])}(\mathcal{C}(U_pU_1^p(Q), s))$ (as explained in Section 2.1.4, the operators $\mathbf{U}_v^i$ commute with the action of Δ_Q).

Note that we have a natural isomorphism of complexes

$$\mathcal{C}(U_pU_1^p(Q), s) \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \cong \mathcal{C}(U_pU_0^p(Q), s). \tag{3.3.2}$$

We then set (for each compact open normal subgroup U_p of K_0)

$$\mathbb{T}^{S\cup Q,Q}(U_pU_i^p(Q)) = \varprojlim_s \mathbb{T}^{S\cup Q,Q}(U_pU_i^p(Q), s),$$

$$\mathbb{T}^{S\cup Q,Q}(U_i^p(Q)) = \varprojlim_{U_p, s} \mathbb{T}^{S\cup Q,Q}(U_pU_i^p(Q), s),$$

for $i = 0, 1$, equipped with their inverse limit topologies. We now need to assume the existence of Galois representations associated to completed homology, as in the following conjecture.

Conjecture 3.3.3. *Let $\mathfrak{m} \subset \mathbb{T}^S(U^p)$ be a maximal ideal with residue field k.*

(1) *There exists a continuous semi-simple representation*

$$\overline{\rho}_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbb{T}^S(U^p)/\mathfrak{m})$$

satisfying the following conditions: $\overline{\rho}_{\mathfrak{m}}$ is totally odd, and for any finite place $v \notin S$ of F, $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ has characteristic polynomial

$$X^n - T_v^1 X^{n-1} + \cdots + (-1)^i q_v^{i(i-1)/2} T_v^i X^{n-i} + \cdots + (-1)^n q_v^{n(n-1)/2} T_v^n \in (\mathbb{T}^S(U^p)/\mathfrak{m})[X]$$

(2) *Suppose that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible. Then there exists a lifting of $\overline{\rho}_{\mathfrak{m}}$ to a continuous homomorphism*

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbb{T}^S(U^p)_{\mathfrak{m}})$$

satisfying the following condition: for any finite place $v \notin S$ of F, $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ has characteristic polynomial

$$X^n - T_v^1 X^{n-1} + \cdots + (-1)^i q_v^{i(i-1)/2} T_v^i X^{n-i} + \cdots + (-1)^n q_v^{n(n-1)/2} T_v^n \in \mathbb{T}^S(U^p)_{\mathfrak{m}}[X]$$

(In particular, since for each $v \notin S$ we have $T_v^n = 1$, we have $\det \rho_{\mathfrak{m}} = \epsilon^{n(1-n)/2}$.)

Remark 3.3.4. If F is a CM or totally real field, the first part of the conjecture holds by the main results of [Sch15b] and [CL14]. It also follows from Scholze's work (again with the assumption that F is CM or totally real) that there is a lifting of $\overline{\rho}_{\mathfrak{m}}$ valued in $\mathbb{T}^S(U^p)_{\mathfrak{m}}/I$ for some nilpotent ideal $I \subset \mathbb{T}^S(U^p)_{\mathfrak{m}}$, and in fact we may assume that $I^4 = 0$ by [NT15, Theorem 1.3].

Definition 3.3.5. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^S(U^p)$. For sufficiently small U_p (for example, if U_p is pro-p), $\mathfrak{m}$ is the inverse image of a maximal ideal of $\mathbb{T}^S(U_pU^p, 1)$, which we also denote by $\mathfrak{m}$. The abstract Hecke algebra $\mathbb{T}^S$ surjects onto $\mathbb{T}^S(U_pU^p, 1)$ and we again denote by $\mathfrak{m}$ the inverse image of $\mathfrak{m}$ in $\mathbb{T}^S$.

Finally, for any module M for $\mathbb{T}^S$ (or complex of such modules) we denote by $M_{\mathfrak{m}}$ the localisation $\mathbb{T}^S_{\mathfrak{m}} \otimes_{\mathbb{T}^S} M$. Note that the idea of patching singular chain complexes localised with respect to the action of the abstract Hecke algebra appears in [Han12].

We make an analogous definition for maximal ideals of the Hecke algebras $\mathbb{T}^{S\cup Q}(U_i^p(Q))$ and $\mathbb{T}^{S\cup Q, Q}(U_i^p(Q))$.

We assume Conjecture 3.3.3 from now on, and recall that we have fixed U^p such that U^pK_0 is S-good. We now fix a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S(U^p)$, and assume that

- $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible
- $\overline{\rho}_{\mathfrak{m}}(G_{F(\zeta_p)})$ is enormous, and
- $\overline{\rho}_{\mathfrak{m}} \not\cong \overline{\rho}_{\mathfrak{m}} \otimes \overline{\epsilon}$.

Enlarging our coefficient field E if necessary, we assume further that $\mathfrak{m}$ has residue field k, and that k contains the eigenvalues of all elements of the image of $\overline{\rho}_{\mathfrak{m}}$. We fix

$$\mathcal{S} = (\overline{\rho}_{\mathfrak{m}}, \epsilon^{n(1-n)/2}, S, \{\mathcal{O}\}_{v\in S}, \{\mathcal{D}_v^{\square}\}_{v\in S}).$$

The following is the analogue of [KT14, Prop. 6.26] in our context, and the proof is essentially identical.

Proposition 3.3.6. *Let $(Q,(\gamma_{v,1},\dots,\gamma_{v,n})_{v\in Q})$ be a Taylor–Wiles datum.*

1. *There are natural inclusions $\mathbb{T}^{S\cup Q}(U^p)\subset\mathbb{T}^S(U^p)$ and $\mathbb{T}^{S\cup Q}(U_0^p(Q))\subset\mathbb{T}^S(U_0^p(Q))$, and natural surjections $\mathbb{T}^{S\cup Q}(U_0^p(Q))\twoheadrightarrow\mathbb{T}^{S\cup Q}(U^p)$, $\mathbb{T}^{S\cup Q}(U_1^p(Q))\twoheadrightarrow\mathbb{T}^{S\cup Q}(U_0^p(Q))$ and $\mathbb{T}^{S\cup Q,Q}(U_1^p(Q))\twoheadrightarrow\mathbb{T}^{S\cup Q,Q}(U_0^p(Q))$.*
2. *Let $\mathfrak{m}_{Q,0}\subset\mathbb{T}^{S\cup Q,Q}(U_0^p(Q))$ denote the ideal generated by the pullback of $\mathfrak{m}$ to $\mathbb{T}^{S\cup Q}(U_0^p(Q))$ and the elements $\mathbf{U}_v^i-\prod_{j=1}^i\gamma_{v,i}$. Then $\mathfrak{m}_{Q,0}$ is a maximal ideal.*
3. *Write $\mathfrak{m}_{Q,1}$ for the pullback of $\mathfrak{m}_{Q,0}$ to $\mathbb{T}^{S\cup Q,Q}(U_1^p(Q))$, and $\mathfrak{m}'$ for the pullback of $\mathfrak{m}$ to $\mathbb{T}^{S\cup Q}(U^p)$. Then there is a quasi-isomorphism*
$$\mathcal{C}(U_pU_0^p(Q),s)_{\mathfrak{m}_{Q,0}}\to\mathcal{C}(U_pU^p,s)_{\mathfrak{m}}$$
and an isomorphism
$$\mathcal{C}(U_pU_1^p(Q),s)_{\mathfrak{m}_{Q,1}}\otimes_{\mathcal{O}[\Delta_Q]}\mathcal{O}\cong\mathcal{C}(U_pU_0^p(Q),s)_{\mathfrak{m}_{Q,0}}$$
which are both equivariant for the actions of the operators T_v^i, $v\notin S\cup Q$. Consequently, if we write $\mathbb{T}^{S\cup Q}(U_pU_1^p(Q),s)_{\mathfrak{m}_{Q,1}}$ for the $\mathcal{O}[\Delta_Q]$-subalgebra of $\mathrm{End}_{D(\mathcal{O}/\varpi^s[\Delta_Q\times K_0/U_p])}(\mathcal{C}(U_pU_1^p(Q),s)_{\mathfrak{m}_{Q,1}})$ generated by the operators T_v^i, $v\notin S\cup Q$, then there are natural maps
$$\mathbb{T}^{S\cup Q}(U_pU_1^p(Q),s)_{\mathfrak{m}_{Q,1}}\twoheadrightarrow\mathbb{T}^{S\cup Q}(U_pU^p,s)_{\mathfrak{m}'}\cong\mathbb{T}^S(U_pU^p,s)_{\mathfrak{m}}.$$

Proof. The inclusions $\mathbb{T}^{S\cup Q}(U^p)\subset\mathbb{T}^S(U^p)$ and $\mathbb{T}^{S\cup Q}(U_0^p(Q))\subset\mathbb{T}^S(U_0^p(Q))$ exist by definition. The surjections $\mathbb{T}^{S\cup Q}(U_1^p(Q))\twoheadrightarrow\mathbb{T}^{S\cup Q}(U_0^p(Q))$ and $\mathbb{T}^{S\cup Q,Q}(U_1^p(Q))\twoheadrightarrow\mathbb{T}^{S\cup Q,Q}(U_0^p(Q))$ are induced by (3.3.2), while the surjection $\mathbb{T}^{S\cup Q}(U_0^p(Q))\twoheadrightarrow\mathbb{T}^{S\cup Q}(U^p)$ comes from the splitting by the trace map of the natural map
$$\mathcal{C}(U_pU_0^p(Q),s)\to\mathcal{C}(U_pU^p,s)$$
(note that for $v\in Q$, since $p>n$ and $\#k(v)\equiv 1\pmod p$, the index of $U_0(Q)_v$ in $\mathrm{PGL}_n(\mathcal{O}_{F,v})$ is congruent to $n!$ mod p, by the Bruhat decomposition, and hence this index is prime to p).

For the second part, we need to show that $\mathfrak{m}_{Q,0}$ is in the support of $\mathcal{C}(U_pU_0^p(Q),1)$. As in the proof of [KT14, Lem. 6.25], it is enough to prove the corresponding statement for cohomology groups, which follows from [KT14, Lem. 5.3].

The isomorphism $\mathcal{C}(U_pU_1^p(Q),s)_{\mathfrak{m}_{Q,1}}\otimes_{\mathcal{O}[\Delta_Q]}\mathcal{O}\cong\mathcal{C}(U_pU_0^p(Q),s)_{\mathfrak{m}_{Q,0}}$ is induced by (3.3.2). The quasi-isomorphism is the composite of quasi-isomorphisms
$$\mathcal{C}(U_pU_0^p(Q),s)_{\mathfrak{m}_{Q,0}}\to\mathcal{C}(U_pU^p,s)_{\mathfrak{m}'}\to\mathcal{C}(U_pU^p,s)_{\mathfrak{m}}$$
which are induced by the obvious natural maps of complexes (and the morphisms of Hecke algebras from part (1)); to see that they are indeed quasi-isomorphisms, one uses respectively [KT14, Lem. 5.4] and the argument of [KT14, Lem. 6.20]. Finally the isomorphism $\mathbb{T}^{S\cup Q}(U_pU^p,s)_{\mathfrak{m}}\cong\mathbb{T}^S(U_pU^p,s)_{\mathfrak{m}}$ again follows from the argument of [KT14, Lem. 6.20] and [CHT08, Cor. 3.4.5]. □

As usual, we set $\mathbb{T}^{S\cup Q}(U_1^p(Q))_{\mathfrak{m}_{Q,1}}:=\varprojlim_{U_p,s}\mathbb{T}^{S\cup Q}(U_pU_1^p(Q),s)_{\mathfrak{m}_{Q,1}}$, $\mathbb{T}^{S\cup Q}(U_pU_1^p(Q))_{\mathfrak{m}_{Q,1}}:=\varprojlim_s\mathbb{T}^{S\cup Q}(U_pU_1^p(Q),s)_{\mathfrak{m}_{Q,1}}$, equipped with their inverse limit topologies. (These are local rings, as can easily be checked as in the proof of Lemma 2.1.14.) We will need to assume the following refinement of Conjecture 3.3.3.

Conjecture 3.3.7. *Suppose that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, and let $(Q, (\gamma_{v,1}, \dots, \gamma_{v,n})_{v\in Q})$ be a Taylor–Wiles datum. Then there exists a lifting of $\overline{\rho}_{\mathfrak{m}}$ to a continuous homomorphism*

$$\rho_{\mathfrak{m},Q} : G_{F,S\cup Q} \to \mathrm{GL}_n(\mathbb{T}^{S\cup Q}(U_1^p(Q))_{\mathfrak{m}_{Q,1}})$$

satisfying the following conditions: for any finite place $v \notin S \cup Q$ of F, $\rho_{\mathfrak{m},Q}(\mathrm{Frob}_v)$ has characteristic polynomial

$$X^n - T_v^1 X^{n-1} + \cdots + (-1)^i q_v^{i(i-1)/2} T_v^i X^{n-i} + \cdots + (-1)^n q_v^{n(n-1)/2} T_v^n \in \mathbb{T}^{S\cup Q}(U_1^p(Q))_{\mathfrak{m}_{Q,1}}[X]$$

and $\rho_{\mathfrak{m},Q}$ is of type $\mathcal{S}_Q$.

Remark 3.3.8. The requirement that $\rho_{\mathfrak{m},Q}$ be of type $\mathcal{S}_Q$ is a form of local-global compatibility at the places in Q. If F is CM or totally real, this property is expected to follow from forthcoming work of Varma (although as explained in Remark 3.3.4, the existence of the representation is only known up to a nilpotent ideal).

We assume Conjecture 3.3.7 from now on, so that in particular $\rho_{\mathfrak{m},Q}$ determines an $\mathcal{O}[\Delta_Q]$-algebra homomorphism

$$R_{\mathcal{S}_Q} \to \mathbb{T}^{S\cup Q}(U_1^p(Q))_{\mathfrak{m}_{Q,1}}, \tag{3.3.9}$$

and the choice of $\rho_{\mathfrak{m},Q}$ in its strict equivalent class determines an isomorphism

$$R^S_{\mathcal{S}_Q} \xrightarrow{\sim} \mathcal{T}\widehat{\otimes}_{\mathcal{O}} R_{\mathcal{S}_Q}. \tag{3.3.10}$$

3.4. **Patching.** For each $N \geq 1$, we let $(Q_N, (\gamma_{v,1}, \dots, \gamma_{v,n})_{v\in Q_N}$ be a choice of Taylor–Wiles datum as in Lemma 3.3.1 (for some fixed choice of $q \gg 0$). We fix a surjective $R_{\mathcal{S}}^{S,\mathrm{loc}}$-algebra map $R_\infty \to R^S_{\mathcal{S}_{Q_N}}$ for each N. We also fix a non-principal ultrafilter $\mathfrak{F}$ on the set $\mathbb{N} = \{N \geq 1\}$.

Remark 3.4.1. With the exception of Remark 3.4.17, the choice of $\mathfrak{F}$ is the only choice we make in our patching argument. This has the pleasant effect of making many of the constructions below natural, although the reader should bear in mind that they are only natural relative to our fixed choice of $\mathfrak{F}$.

Definition 3.4.2. Let U_p be a compact open subgroup of K_0, and let J be an open ideal in $\mathcal{O}_\infty$. Let I_J be the (cofinite) subset of $N \in \mathbb{N}$ such that J contains the kernel of $\mathcal{O}_\infty \to \mathcal{O}_N$. For $N \in I_J$, we define

$$\mathcal{C}(U_p, J, N) = \mathcal{O}_\infty/J \otimes_{\mathcal{O}[\Delta_{Q_N}]} \mathcal{C}(U_1^p(Q_N)U_p, \mathcal{O})_{\mathfrak{m}_{Q_N,1}}.$$

Remark 3.4.3.

(1) We have a map $R^{S_p}_{\mathcal{S}_{Q_N}} \to \mathcal{T}\widehat{\otimes}_{\mathcal{O}}\mathbb{T}^{S\cup Q_N}(U_1^p(Q_N))_{\mathfrak{m}_{Q_N,1}}$ by (3.3.9) and (3.3.10), and a map

$$\mathcal{T}\widehat{\otimes}_{\mathcal{O}}\mathbb{T}^{S\cup Q_N}(U_1^p(Q_N))_{\mathfrak{m}_{Q_N,1}} \to \mathrm{End}_{D(\mathcal{O}_\infty/J)}(\mathcal{C}(U_p, J, N))$$

by definition of $\mathbb{T}^{S\cup Q_N}(U_1^p(Q_N))_{\mathfrak{m}_{Q_N,1}}$ together with Remark 2.1.12. In particular, for all J and $N \in I_J$ we have a ring homomorphism

$$R_\infty \to \mathrm{End}_{D(\mathcal{O}_\infty/J)}(\mathcal{C}(U_p, J, N))$$

which factors through our chosen quotient map $R_\infty \to R^S_{\mathcal{S}_{Q_N}}$ and the $\mathcal{O}_N$-algebra map

$$R^S_{\mathcal{S}_{Q_N}} \to \mathcal{T}\widehat{\otimes}_{\mathcal{O}}\mathbb{T}^{S\cup Q_N}(U_1^p(Q_N))_{\mathfrak{m}_{Q_N,1}}.$$

(2) If U_p' is an open normal subgroup of U_p, $\mathcal{C}(U_p', J, N)$ is a complex of flat $\mathcal{O}_\infty/J[U_p/U_p']$-modules.

(3) Let $\mathbf{a} = \ker(\mathcal{O}_\infty \to \mathcal{O})$. Suppose that $\mathbf{a} \subset J$. Then $\mathcal{C}(U_p, J, N) = \mathcal{C}(U_0^p(Q_N)U_p, s(J))_{\mathfrak{m}_{Q_N,0}}$ where $\mathcal{O}_\infty/J \cong \mathcal{O}/\varpi^{s(J)}$ and the natural map $\mathcal{C}(U_0^p(Q_N)U_p, s(J))_{\mathfrak{m}_{Q_N,0}} \to \mathcal{C}(U^pU_p, s(J))_{\mathfrak{m}}$ is a quasi-isomorphism.

Definition 3.4.4. For $d \geq 1$, J an open ideal in $\mathcal{O}_\infty$ and $N \in I_J$, we define

$$R(d, J, N) = \left(R^S_{\mathcal{S}_{Q_N}} / \mathfrak{m}^d_{R^S_{\mathcal{S}_{Q_N}}} \right) \otimes_{\mathcal{O}_N} \mathcal{O}_\infty/J.$$

Remark 3.4.5. Each ring $R(d, J, N)$ is a finite commutative local $\mathcal{O}_\infty/J$-algebra, equipped with a surjective $\mathcal{O}$-algebra map $R_\infty \to R(d, J, N)$. As in the beginning of the proof of [KT14, Prop. 3.1], for d sufficiently large (depending on J and U_p), the map

$$R_\infty \to \mathrm{End}_{D(\mathcal{O}_\infty/J)}(\mathcal{C}(U_p, J, N))$$

factors through the quotient $R(d, J, N)$ and the map

$$R(d, J, N) \to \mathrm{End}_{D(\mathcal{O}_\infty/J)}(\mathcal{C}(U_p, J, N))$$

is an $\mathcal{O}_\infty$-algebra homomorphism. We have an isomorphism

$$R(d, J, N)/\mathbf{a} \cong R_{\mathcal{S}}/(\mathfrak{m}^d_{R_{\mathcal{S}}}, \varpi^{s(\mathbf{a}+J)})$$

induced by the canonical isomorphism $R_{\mathcal{S}_{Q_N}}/\mathbf{a}_{Q_N} \cong R_{\mathcal{S}}$.

Lemma 3.4.6. (1) *For all open ideals $J' \subset J$ and open normal subgroups $U'_p \subset U_p$ we have surjective maps of complexes*

$$\mathcal{C}(U'_p, J', N) \to \mathcal{C}(U_p, J, N)$$

inducing isomorphisms (of complexes)

$$\mathcal{O}_\infty/J \otimes_{\mathcal{O}_\infty/J'[U_p/U'_p]} \mathcal{C}(U'_p, J', N) \to \mathcal{C}(U_p, J, N).$$

(2) *Let K_1 be a pro-p Sylow subgroup of K_0 and let U_p be an open normal subgroup of K_1. Then $\{\mathcal{C}(U_p, J, N)\}_{N \in I_J}$ is a set of perfect chain complexes of $\mathcal{O}_\infty/J[K_1/U_p]$-modules with bounded complexity.*

Proof. The maps of complexes $\mathcal{C}(U'_p, J', N) \to \mathcal{C}(U_p, J, N)$ are those induced by the natural maps $\mathcal{O}_\infty/J' \to \mathcal{O}_\infty/J$ and $\mathcal{C}(U_1^p(Q_N)U'_p, \mathcal{O}) \to \mathcal{C}(U_1^p(Q_N)U_p, \mathcal{O})$.

To see that $\mathcal{C}(U_p, J, N)$ is perfect, we first observe that by part (1) we have an isomorphism $k \otimes_{\mathcal{O}_\infty/J[K_1/U_p]} \mathcal{C}(U_p, J, N) \cong \mathcal{C}(K_1, \mathfrak{m}_{\mathcal{O}_\infty}, N)$ — note that k is the residue field of the local ring $\mathcal{O}_\infty/J[K_1/U_p]$ and $\mathcal{C}(U_p, J, N)$ is a bounded-below complex of flat $\mathcal{O}_\infty/J[K_1/U_p]$-modules with finitely generated homology (finite generation follows from the existence of the Borel–Serre compactification [BS73], or the earlier work of Raghunathan [Rag67]). It follows from Proposition 2.1.9 that $\mathcal{C}(U_p, J, N)$ has a minimal resolution, and since $\mathcal{C}(K_1, \mathfrak{m}_{\mathcal{O}_\infty}, N)$ has bounded homology we deduce that $\mathcal{C}(U_p, J, N)$ is perfect.

It follows immediately from the quasi-isomorphism

$$\mathcal{C}(U_pU_1^p(Q), s)_{\mathfrak{m}_{Q,1}} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \to \mathcal{C}(U_pU^p, s)_{\mathfrak{m}}$$

(which comes from Proposition 3.3.6 (3)) that the set of complexes has bounded complexity, as required. $\square$

Definition 3.4.7. Applying the construction of section 2.2, we let $x \in \operatorname{Spec}((\mathcal{O}_\infty/J)_{I_J})$ correspond to $\mathfrak{F}$ (here we use that $\mathfrak{F}$ is non-principal, and therefore defines an ultrafilter on I_J), and define

$$\mathcal{C}(U_p, J, \infty) = (\mathcal{O}_\infty/J)_{I_J,x} \otimes_{(\mathcal{O}_\infty/J)_{I_J}} \left(\prod_{N \in I_J} \mathcal{C}(U_p, J, N) \right).$$

Remark 3.4.8. (1) It follows from Lemma 2.2.3 that if U_p' is an open normal subgroup of U_p, $\mathcal{C}(U_p', J, \infty)$ is a complex of flat $\mathcal{O}_\infty/J[U_p/U_p']$-modules.

(2) It follows from Remark 3.4.3(3) that if $\mathbf{a} \subset J$ there is a natural quasi-isomorphism $\mathcal{C}(U_p, J, \infty) \to \mathcal{C}(U^p U_p, s(J))_{\mathfrak{m}}$.

Definition 3.4.9. Similarly, we define

$$R(d, J, \infty) = (\mathcal{O}_\infty/J)_{I_J,x} \otimes_{(\mathcal{O}_\infty/J)_{I_J}} \left(\prod_{N \in I_J} R(d, J, N) \right).$$

Remark 3.4.10. For d sufficiently large (depending on J and U_p), the map

$$R_\infty \to \operatorname{End}_{D(\mathcal{O}_\infty/J)}(\mathcal{C}(U_p, J, \infty))$$

factors through $R(d, J, \infty)$ and the map

$$R(d, J, \infty) \to \operatorname{End}_{D(\mathcal{O}_\infty/J)}(\mathcal{C}(U_p, J, \infty))$$

is an $\mathcal{O}_\infty$-algebra homomorphism. By Lemma 2.2.4, we have an isomorphism

$$R(d, J, \infty)/\mathbf{a} \cong R_{\mathcal{S}}/(\mathfrak{m}_{R_{\mathcal{S}}}^d, \varpi^{s(\mathbf{a}+J)})$$

induced by the isomorphisms $R(d, J, N)/\mathbf{a} \cong R_{\mathcal{S}}/(\mathfrak{m}_{R_{\mathcal{S}}}^d, \varpi^{s(\mathbf{a}+J)})$.

Lemma 3.4.11. (1) *For all open ideals $J' \subset J$ and open normal subgroups $U_p' \subset U_p$, the natural maps of complexes*

$$\mathcal{C}(U_p', J', \infty) \to \mathcal{C}(U_p, J, \infty)$$

are surjective, and induce isomorphisms of complexes

$$\mathcal{O}_\infty/J \otimes_{\mathcal{O}_\infty/J'[U_p/U_p']} \mathcal{C}(U_p', J', \infty) \to \mathcal{C}(U_p, J, \infty).$$

(2) *Let U_p be an open normal subgroup of K_1, and let J be an open ideal in $\mathcal{O}_\infty$. Then $\mathcal{C}(U_p, J, \infty)$ is a perfect complex of $\mathcal{O}_\infty/J[K_1/U_p]$-modules. If U_p is moreover normal in K_0, then $\mathcal{C}(U_p, J, \infty)$ is a perfect complex of $\mathcal{O}_\infty/J[K_0/U_p]$-modules.*

Proof. The surjectivity claim of the first part follows immediately from Lemma 3.4.6(1), since taking the the direct product over $N \in I_J$ and localising at x preserves surjectivity. It follows from Lemma 2.2.4 and Lemma 3.4.6(1) that we obtain an isomorphism of complexes

$$\mathcal{O}_\infty/J \otimes_{\mathcal{O}_\infty/J'[U_p/U_p']} \mathcal{C}(U_p', J', \infty) \to \mathcal{C}(U_p, J, \infty).$$

For the second part, the fact that $\mathcal{C}(U_p, J, \infty)$ is a perfect complex of $\mathcal{O}_\infty/J[K_1/U_p]$-modules follows from Lemma 2.2.6 and Lemma 3.4.6(2). To get perfectness over $\mathcal{O}_\infty/J[K_0/U_p]$ we apply (an obvious variant of) Lemma 3.4.15. ☐

Definition 3.4.12. We define a complex of $\mathcal{O}_\infty[[K_0]]$-modules

$$\widetilde{\mathcal{C}}(\infty) := \varprojlim_{J, U_p} \mathcal{C}(U_p, J, \infty).$$

Remark 3.4.13. The complex $\widetilde{\mathcal{C}}(\infty)$ is naturally equipped with an $\mathcal{O}_\infty$-linear action of $\prod_{v|p} G(F_v)$ (on each term of the complex), which extends the K_0-action coming from the $\mathcal{O}_\infty[[K_0]]$-module structure. Explicitly, for $g \in \prod_{v|p} G(F_v)$, right multiplication by g gives a map of complexes

$$\cdot g : \mathcal{C}(U_p, J, N) \to \mathcal{C}(g^{-1}U_p g, J, N)$$

for each U_p, J and N. Supposing that $g^{-1}U_p g \subset K_0$, applying our (functorial) patching construction gives a map

$$\cdot g : \mathcal{C}(U_p, J, \infty) \to \mathcal{C}(g^{-1}U_p g, J, \infty)$$

As U_p runs over the cofinal subset of open subgroups of K_0 with $g^{-1}U_p g \subset K_0$, the subgroups $g^{-1}U_p g$ also run over a cofinal subset of open subgroups of K_0, so we can identify $\varprojlim_{J,U_p} \mathcal{C}(g^{-1}U_p g, J, \infty)$ with $\widetilde{\mathcal{C}}(\infty)$. Therefore, taking the inverse limit over J and U_p gives the action of g on $\widetilde{\mathcal{C}}(\infty)$.

To verify that $\widetilde{\mathcal{C}}(\infty)$ has good properties, we will need several technical Lemmas.

Lemma 3.4.14. *Let I be a countable directed poset. Let $\mathcal{C} = (\mathcal{C}(i))_{i\in I}$ be an inverse system with $\mathcal{C}(i) \in Ch(\mathcal{O})$. Suppose that the following two conditions hold:*

1. *for every $i \in I$ and $m \in \mathbb{Z}$, the homology group $H_m(\mathcal{C}(i))$ is an Artinian $\mathcal{O}$-module.*
2. *Either the entries of $\mathcal{C}(i)$ are Artinian $\mathcal{O}$-modules for every $i \in I$, or for every pair $i \le j$ in I the transition map $\mathcal{C}(j) \to \mathcal{C}(i)$ is surjective.*

Then for every $m \in \mathbb{Z}$ there are natural isomorphisms

$$H_m(\varprojlim_I \mathcal{C}) = \varprojlim_I H_m(\mathcal{C}(i)).$$

Proof. Since I is direct and countable, it has a cofinal subset which is isomorphic (as a poset) to $\mathbb{N}$ with its usual ordering. So we can assume $I = \mathbb{N}$. The proposition is then a consequence of [Wei94, Theorem 3.5.8] (as assumption (1) guarantees the Mittag-Leffler property for the $H_m(\mathcal{C}(i))$, and assumption (2) guarantees it for the $\mathcal{C}(i)$). □

Lemma 3.4.15. *Let K be a compact p-adic analytic group, and let K_1 be a pro-p Sylow subgroup of K. Let $\mathcal{C}$ be a bounded below chain complex of $\mathcal{O}[[K]]$-modules. Suppose that $\mathcal{C}|_{K_1}$ is perfect when regarded as a complex of $\mathcal{O}[[K_1]]$-modules. Then $\mathcal{C}$ is a perfect complex of $\mathcal{O}[[K]]$-modules.*

Proof. We can assume that $\mathcal{C}$ is a bounded below complex of projective $\mathcal{O}[[K]]$-modules. Let $\mathcal{F}$ be a bounded complex of finite free $\mathcal{O}[[K_1]]$-modules with a quasi-isomorphism $\alpha : \mathcal{F} \to \mathcal{C}|_{K_1}$. We have a homotopy inverse $\beta : \mathcal{C}|_{K_1} \to \mathcal{F}$ to α. We obtain maps of complexes of $\mathcal{O}[[K]]$-modules

$$\widetilde{\alpha} : \mathcal{O}[[K]] \otimes_{\mathcal{O}[[K_1]]} \mathcal{F} \to \mathcal{C}$$
$$\widetilde{\beta} : \mathcal{C} \to \mathcal{O}[[K]] \otimes_{\mathcal{O}[[K_1]]} \mathcal{F}$$

where $\widetilde{\alpha}$ is given by the usual adjunction and $\widetilde{\beta}$ is given by

$$\widetilde{\beta}(x) = \sum_{gK_1 \in K_0/K_1} [g] \otimes \beta(g^{-1}x).$$

The composite $\widetilde{\alpha} \circ \widetilde{\beta}$ is homotopic to $[K_0 : K_1]\,\mathrm{id}_{\mathcal{C}}$, and $[K_0 : K_1]$ is invertible in $\mathbb{Z}_p$, so $\mathcal{C}$ is a retract (in the homotopy category) of $\mathcal{O}[[K]] \otimes_{\mathcal{O}[[K_1]]} \mathcal{F}$. Since $\mathcal{O}[[K]] \otimes_{\mathcal{O}[[K_1]]} \mathcal{F}$ is perfect, it follows that $\mathcal{C}$ is also perfect, since perfect complexes form a thick (or épaisse) subcategory of $D(\mathcal{O}[[K]])$ (this follows from [BN93, Prop. 6.4], which identifies perfect complexes with compact objects in $D(R)$) and therefore the retraction of a perfect complex is perfect (thick subcategories of triangulated categories are closed under retraction, by definition). $\square$

As promised, we can now show that $\widetilde{\mathcal{C}}(\infty)$ has various desirable properties.

Proposition 3.4.16. (1) *For all open ideals $J \subset \mathcal{O}_\infty$ and compact open subgroups U_p of K_0 we have surjective maps of complexes* (*induced by the maps in* Lemma 3.4.11(1))

$$\widetilde{\mathcal{C}}(\infty) \to \mathcal{C}(U_p, J, \infty)$$

inducing isomorphisms of complexes

$$\mathcal{O}_\infty/J \otimes_{\mathcal{O}_\infty[[U_p]]} \widetilde{\mathcal{C}}(\infty) \to \mathcal{C}(U_p, J, \infty),$$

and $\widetilde{\mathcal{C}}(\infty)$ is a complex of flat $\mathcal{O}_\infty[[U_p]]$-modules.

(2) *$\widetilde{\mathcal{C}}(\infty)$ is a perfect complex of $\mathcal{O}_\infty[[K_0]]$-modules.*

(3) *There is a ring homomorphism $R_\infty \to \mathrm{End}_{D(\mathcal{O}_\infty)}(\widetilde{\mathcal{C}}(\infty))$ which factors as the composite of maps $R_\infty \to \varprojlim_{J,d} R(d, J, \infty)$ and $\varprojlim_{J,d} R(d, J, \infty) \to \mathrm{End}_{D(\mathcal{O}_\infty)}(\widetilde{\mathcal{C}}(\infty))$* (*the latter map is an $\mathcal{O}_\infty$-algebra map*) *given by the limit of the maps discussed in Remark 3.4.10.*

Proof. The first part follows from Lemma 3.4.11(1) and Lemma A.33. To see this, fix an open uniform pro-p subgroup U_p' of U_p, and note that if $\mathcal{J}$ is the two-sided ideal in $\mathcal{O}_\infty[[U_p]]$ generated by the maximal ideal of $\mathcal{O}_\infty[[U_p']]$, where the $\mathcal{J}$-adic topology on $\mathcal{O}_\infty[[U_p]]$ is equivalent to the canonical profinite topology. We set $K = \mathbb{Z}_p^g \times U_p$ in Lemma A.33, where g is chosen so that $\mathcal{O}[[K]] = \mathcal{O}_\infty[[U_p]]$.

For $m \geq 1$ we can define a complex of flat $\mathcal{O}_\infty[[U_p]]/\mathcal{J}^m$-modules by choosing J and $V_p \subset U_p$ sufficiently small so that $\mathcal{J}^m$ contains the kernel of the map

$$\mathcal{O}_\infty[[U_p]] \to \mathcal{O}_\infty/J[U_p/V_p]$$

and considering the complex $\mathcal{C}(V_p, J, \infty) \otimes_{\mathcal{O}_\infty/J[U_p/V_p]} \mathcal{O}_\infty[[U_p]]/\mathcal{J}^m$. This complex is independent of the choice of J and V_p, by Lemma 3.4.11(1). In particular, by choosing J and V_p sufficiently small, we get a natural surjective map

$$\mathcal{C}(V_p, J, \infty) \otimes_{\mathcal{O}_\infty/J[U_p/V_p]} \mathcal{O}_\infty[[U_p]]/\mathcal{J}^{m+1} \to \mathcal{C}(V_p, J, \infty) \otimes_{\mathcal{O}_\infty/J[U_p/V_p]} \mathcal{O}_\infty[[U_p]]/\mathcal{J}^m.$$

Taking the terms of these complexes in fixed degree as m varies gives a system of modules to which Lemma A.33 applies, and taking the inverse limit over m gives the complex $\widetilde{\mathcal{C}}(\infty)$.

For the second part, we first note that by Lemma 3.4.15 it suffices to show that $\widetilde{\mathcal{C}}(\infty)$ is perfect over $\mathcal{O}_\infty[[K_1]]$, where K_1 is a pro-p Sylow subgroup of K_0. For each J and each U_p open normal in K_1, there is by Remark 2.2.8 a minimal resolution $\mathcal{F}(U_p, J, \infty)$ of $\mathcal{C}(U_p, J, \infty)$, which is isomorphic to the minimal resolutions of $\mathcal{C}(U_p, J, N)$ for all $N \in I'$ where $I' \in \mathfrak{F}$. For each $J' \subset J$ and U_p' open normal in U_p, we choose compatible maps $\mathcal{F}(U_p', J', \infty) \to \mathcal{F}(U_p, J, \infty)$ which are also compatible with the map

$$\mathcal{C}(U_p', J', \infty) \to \mathcal{C}(U_p, J, \infty)$$

and induce isomorphisms

$$\mathcal{O}_\infty/J \otimes_{\mathcal{O}_\infty/J'[U_p/U'_p]} \mathcal{F}(U'_p, J', \infty) \cong \mathcal{F}(U_p, J, \infty).$$

In fact, rather than choosing maps for all J and U_p, it suffices to choose maps between minimal resolutions $\mathcal{F}_m$ of the complexes $\mathcal{C}(U_p, J, \infty)\otimes_{\mathcal{O}_\infty/J[K_1/U_p]}\mathcal{O}_\infty[[K_1]]/\mathcal{J}^m$ discussed in the proof of the first part. It follows from Lemma 3.4.14 that there is a quasi-isomorphism

$$\varprojlim_m \mathcal{F}_m \to \widetilde{\mathcal{C}}(\infty)$$

and $\varprojlim_m \mathcal{F}_m$ is a bounded complex of finite free $\mathcal{O}_\infty[[K_1]]$-modules by construction, as required.

The third part follows from (the proof of) [KT14, Lemma 2.13(3)]. □

Remark 3.4.17. Since the image of the map $\alpha : R_\infty \to \varprojlim_{J,d} R(d, J, \infty)$ contains (the image of) $\mathcal{O}_\infty$, $\alpha(R_\infty)$ is naturally an $\mathcal{O}_\infty$-algebra. Since $\mathcal{O}_\infty$ is formally smooth, we can choose a lift of the map $\mathcal{O}_\infty \to \alpha(R_\infty)$ to a map $\mathcal{O}_\infty \to R_\infty$. We make such a choice, and regard R_∞ as an $\mathcal{O}_\infty$-algebra and α as an $\mathcal{O}_\infty$-algebra map.

Remark 3.4.18. With some more careful bookkeeping, it should be possible to show that there is a natural map $R_\infty \to \mathrm{End}_{D(\mathcal{O}_\infty[[K_0]])}(\widetilde{\mathcal{C}}(\infty))$ lifting the map $R_\infty \to \mathrm{End}_{D(\mathcal{O}_\infty)}(\widetilde{\mathcal{C}}(\infty))$ which we have described above. However, in our applications below, the complex $\widetilde{\mathcal{C}}(\infty)$ will have homology concentrated in a single degree, so this doesn't give any additional information.

The following Proposition shows that we can think of $\widetilde{\mathcal{C}}(\infty)$ as "patched completed homology".

Proposition 3.4.19. *If we let* $\mathbf{a} = \ker(\mathcal{O}_\infty \to \mathcal{O})$*, we have natural (in particular,* $\prod_{v|p} G(F_v)$*-equivariant) isomorphisms*

$$H_i(\mathcal{O}_\infty/\mathbf{a} \otimes_{\mathcal{O}_\infty} \widetilde{\mathcal{C}}(\infty)) \cong \widetilde{H}_i(X_{U^p}, \mathcal{O})_\mathfrak{m}.$$

There are surjective maps $R_\infty/\mathbf{a} \to R_\mathcal{S} \to \mathbb{T}^S(U^p)_\mathfrak{m}$ *and the above isomorphism intertwines the action of* R_∞ *on the left hand side with the action of* $\mathbb{T}^S(U^p)_\mathfrak{m}$ *on the right.*

Proof. We have natural maps

$$\widetilde{\mathcal{C}}(\infty) = \varprojlim_{J,U_p} \mathcal{C}(U_p, J, \infty) \to \varprojlim_{\mathbf{a}\subset J,U_p} \mathcal{C}(U_p, J, \infty) \to \varprojlim_{s,U_p} \mathcal{C}(U^pU_p, s)_\mathfrak{m}.$$

It follows from Lemma 3.4.14 and Remark 3.4.8(2) that the natural map

$$\varprojlim_{\mathbf{a}\subset J,U_p} \mathcal{C}(U_p, J, \infty) \to \varprojlim_{s,U_p} \mathcal{C}(U^pU_p, s)_\mathfrak{m}$$

is a quasi-isomorphism and by Lemma 3.4.14 we have natural isomorphisms

$$H_n(\varprojlim_{s,U_p} \mathcal{C}(U^pU_p, s)_\mathfrak{m}) \cong \varprojlim_{s,U_p} H_n(X_{U^pU_p}, \mathcal{O}/\varpi^s)_\mathfrak{m}.$$

The natural map

$$\alpha : \widetilde{H}_n(X_{U^p}, \mathcal{O})_\mathfrak{m} \to \varprojlim_{s,U_p} H_n(X_{U^pU_p}, \mathcal{O}/\varpi^s)_\mathfrak{m}$$

is also an isomorphism: indeed, we have short exact sequences

$$0 \to H_n(X_{U^pU_p}, \mathcal{O})_{\mathfrak{m}}/\varpi^s \to H_n(X_{U^pU_p}, \mathcal{O}/\varpi^s)_{\mathfrak{m}} \to H_{n-1}(X_{U^pU_p}, \mathcal{O})_{\mathfrak{m}}[\varpi^s] \to 0$$

so taking the limit over (U_p, s) shows that the map α is an injection with ϖ-divisible cokernel. On the other hand, this cokernel is a finitely generated $\mathcal{O}[[K_0]]$-module, so if it is ϖ-divisible it must be zero.

To finish the proof, by Proposition 3.4.16 (1), it suffices to show that the map

$$\mathcal{O}_\infty/\mathbf{a} \otimes_{\mathcal{O}_\infty} \widetilde{\mathcal{C}}(\infty) \to \varprojlim_{\mathbf{a}\subset J, U_p} \mathcal{C}(U_p, J, \infty) = \varprojlim_{J, U_p} \mathcal{O}_\infty/(\mathbf{a} + J) \otimes_{\mathcal{O}_\infty[[U_p]]} \widetilde{\mathcal{C}}(\infty)$$

is an isomorphism of complexes. As in the proof of Proposition 3.4.16 (1), we easily reduce to the following claim, where $\mathcal{J} = \mathfrak{m}_{\mathcal{O}_\infty[[U_p]]}\mathcal{O}_\infty[[K_0]]$ for $U_p \subset K_0$ an open uniform pro-p subgroup: suppose we have flat $\mathcal{O}_\infty[[K_0]]/\mathcal{J}^m$-modules M_m for each $m \geq 1$, with $M_m = M_{m+1}/\mathcal{J}^m$. Let $M = \varprojlim_m M_m$. Then $M/\mathbf{a}M = \varprojlim_m M_m/\mathbf{a}M_m$.

This claim follows from Lemma A.33, taking $K = \mathbb{Z}_p^g \times K_0$ (where g is chosen so that $\mathcal{O}[[K]] = \mathcal{O}_\infty[[K_0]]$), and $Q = \mathcal{O}_\infty[[K_0]]/\mathbf{a}$.

The final claim of the Proposition follows from the fact that the isomorphisms $R(d, J, \infty)/\mathbf{a} \cong R_{\mathcal{S}}/(\mathfrak{m}_{R_{\mathcal{S}}}^d, \varpi^{s(\mathbf{a}+J)})$ of Remark 3.4.10 induce an isomorphism

$$\left(\varprojlim_{d,J} R(d, J, \infty)\right)/\mathbf{a} \cong R_{\mathcal{S}}. \qquad \square$$

Lemma 3.4.20. *Let $\mathfrak{m} \subset \mathbb{T}^S(U^p)$ be a maximal ideal and suppose that $\widetilde{H}_i(X_{U^p}, \mathcal{O})_{\mathfrak{m}}$ is non-zero for a single i, which we denote by q. Then the map*

$$\alpha : \mathbb{T}^S(U^p)_{\mathfrak{m}} \to \mathrm{End}_{\mathcal{O}}(\widetilde{H}_q(X_{U^p}, \mathcal{O})_{\mathfrak{m}})$$

is an injection.

Proof. The map α factors through the inclusion

$$\mathrm{End}_{\mathcal{O}[[K_0]]}(\widetilde{H}_q(X_{U^p}, \mathcal{O})_{\mathfrak{m}}) \subset \mathrm{End}_{\mathcal{O}}(\widetilde{H}_q(X_{U^p}, \mathcal{O})_{\mathfrak{m}}).$$

Suppose T is in the kernel of α. Then, as an endomorphism in $D(\mathcal{O}[[K_0]])$, T acts on $\mathcal{O}_\infty/\mathbf{a} \otimes_{\mathcal{O}_\infty} \widetilde{\mathcal{C}}(\infty)$ as 0 (by Proposition 3.4.19), and so for any $s \geq 1$ and U_p compact open normal in K_0 it acts, as an endomorphism in $D(\mathcal{O}/\varpi^s[K_0/U_p])$, as 0 on

$$\mathcal{O}/\varpi^s[K_0/U_p] \otimes_{\mathcal{O}[[K_0]]} \mathcal{O}_\infty/\mathbf{a} \otimes_{\mathcal{O}_\infty} \widetilde{\mathcal{C}}(\infty).$$

By Proposition 3.4.16 and Remark 3.4.8, we deduce that T maps to 0 in $\mathbb{T}^S(U_pU^p, s)_{\mathfrak{m}}$. Since U_p and s were arbitrary, we deduce that T is equal to 0. Of course we don't require the patched complex $\widetilde{\mathcal{C}}(\infty)$ to prove this Lemma — we can replace $\mathcal{O}_\infty/\mathbf{a} \otimes_{\mathcal{O}_\infty} \widetilde{\mathcal{C}}(\infty)$ by any suitable complex computing completed homology. $\square$

4. Applications of noncommutative algebra to patched completed homology

In this section we apply the non-commutative algebra developed in Appendix A to the output of the patching construction in Section 3.

4.1. **Formally smooth local deformation rings.** We begin by recalling some of the notation, assumptions and results of Section 3, and we then make an additional assumption.

We assume Conjectures 3.3.3 and 3.3.7. We work with a fixed U^p such that U^pK_0 is good, and we further assume that

- $p > n \geq 2$,
- $\overline{\rho}_{\mathfrak{m}}(G_{F(\zeta_p)})$ is enormous, and
- $\overline{\rho}_{\mathfrak{m}} \not\cong \overline{\rho}_{\mathfrak{m}} \otimes \overline{\epsilon}$.

We have two rings $\mathcal{O}_\infty$ and R_∞. The former is a power series ring over $\mathcal{O}$, and the latter is a power series ring over $R_{\mathcal{S}}^{S,\mathrm{loc}}$. More precisely, we have fixed an integer $q \gg 0$, and $\mathcal{O}_\infty$ is a power series ring in

$$n^2\#S - 1 + (n-1)q$$

variables over $\mathcal{O}$, while R_∞ is a power series ring in

$$(n-1)q - n(n-1)[F:\mathbb{Q}]/2 - l_0 - 1 + \#S$$

variables over $R_{\mathcal{S}}^{S,\mathrm{loc}}$.

Lemma 4.1.1. *Suppose that for each place $v|p$ of F there is no $k[G_{F_v}]$-equivariant map $\overline{\rho}|_{G_{F_v}} \to \overline{\rho}|_{G_{F_v}}(1)$. Then R_∞ is equidimensional of dimension* $\dim \mathcal{O}_\infty + (n(n+1)/2-1)[F:\mathbb{Q}] - l_0$.

Proof. For places $v|p$ we have $H^2(G_{F_v}, \mathrm{ad}^0\,\overline{\rho}) = 0$ by Tate local duality, and a standard calculation shows that R_v is formally smooth of dimension $1 + (n^2-1)[F_v : \mathbb{Q}_p] + (n^2-1)$ (see e.g. [All16, Lem. 3.3.1]). If $v \nmid p$ then R_v is equidimensional of dimension n^2 by [Sho16, Thm. 2.5]. The claim then follows immediately (using [BLGHT, Lem. 3.3] to compute the dimension of $R_{\mathcal{S}}^{S,\mathrm{loc}}$). □

Remark 4.1.2. Note that $(n(n+1)/2-1)[F:\mathbb{Q}]$ is equal to the dimension of the Borel subgroup B in G. It follows from Lemma 4.1.1 that we have

$$\dim R_\infty + \dim(G/B) = \dim \mathcal{O}_\infty[[K_0]] - l_0.$$

See [CE12, Equation (1.6)] and the surrounding discussion for the same numerology.

We now assume that

- for each place $v|p$ of F there is no $k[G_{F_v}]$-equivariant map $\overline{\rho}|_{G_{F_v}} \to \overline{\rho}|_{G_{F_v}}(1)$.

As noted in the proof of Lemma 4.1.1, under this assumption the local deformation ring R_v is formally smooth over $\mathcal{O}$. We could make a similar assumption at places $v \nmid p$, but it seems more reasonable to instead make the following more general assumption.

Hypothesis 4.1.3. For each place $v \in S$ with $v \nmid p$, we let $\overline{R}_v$ be an irreducible component of R_v which is formally smooth. Let $\mathcal{D}_v$ be the local deformation problem corresponding to $\overline{R}_v$. Let

$$\overline{\mathcal{S}} = (\overline{\rho}_{\mathfrak{m}}, \epsilon^{n(1-n)/2}, S, \{\mathcal{O}\}_{v\in S}, \{\mathcal{D}_v^{\square}\}_{v|p} \cup \{\mathcal{D}_v\}_{v\in S, v\nmid p}).$$

Then we further assume that for any set of Taylor–Wiles primes Q, the representation $\rho_{\mathfrak{m},Q}$ of Conjecture 3.3.7 is of type $\overline{\mathcal{S}}_Q$.

Remark 4.1.4. If $v \nmid p$ is such that there is no $k[G_{F_v}]$-equivariant map $\overline{\rho}|_{G_{F_v}} \to \overline{\rho}|_{G_{F_v}}(1)$, then R_v is formally smooth and we can take $\overline{R}_v = R_v$. Under the expected local-global compatibility, the question of whether $\rho_{\mathfrak{m},Q}$ is of type $\overline{\mathcal{S}}_Q$ for a given choice of components $\overline{R}_v$ is governed by the local Langlands correspondence, and therefore depends on the choices of compact open subgroups U_v.

Since our primary interest is in the behaviour at the places $v|p$, we content ourselves with mentioning one important example. For any $v \nmid p$ there is always at least one choice of irreducible component $\overline{R}_v$ which is formally smooth, namely the component corresponding to minimally ramified lifts; see [CHT08, Lem. 2.4.19]. In general we do not expect to be able to make a choice of U_v compatible with the minimally ramified lifts; this is not a problem, as instead one should be able to consider a type (in the sense of Henniart's appendix to [BM02]) at each place $v \nmid p$. Doing so would take us too far afield, so we content ourselves with noting that if $n = 2$, and v is not a vexing prime in the sense of [Dia97] (so in particular if $\#k(v) \not\equiv -1 \mod p$), then we expect to be able to take U_v to be given by the image in $\mathrm{PGL}_2(\mathcal{O}_{F_v})$ of the subgroup of matrices in $\mathrm{GL}_2(\mathcal{O}_{F_v})$ whose last row is congruent to $(0,1)$ modulo v^{n_v}, where n_v is the conductor of $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$. As in Remark 3.3.8, in the case that F is totally real or CM, this compatibility should follow from forthcoming work of Varma.

We assume Hypothesis 4.1.3 from now on. If $v|p$ then we set $\overline{R}_v = R_v$; we then write $\overline{R}_{\mathcal{S}}^{S,\mathrm{loc}} := \widehat{\otimes}_{v\in S} R_v$, and set $\overline{R}_\infty := \overline{R} \otimes_{R_{\mathcal{S}}^{S,\mathrm{loc}}} \overline{R}_{\mathcal{S}}^{S,\mathrm{loc}}$. Under our assumptions, $\overline{R}_\infty$ is a power series ring over $\mathcal{O}$, and has the same dimension as R_∞ (indeed, it is an irreducible component of R_∞).

Remark 4.1.5. If $v \nmid p$ then R_v is in fact a reduced complete intersection, and is flat over $\mathcal{O}$ ([Sho16, Thm. 2.5]). In particular R_v is Gorenstein. It seems reasonable to imagine that these properties should be sufficient to carry out our analysis below without making any assumption at the places $v \nmid p$, but this would require a substantial generalisation of the material in Appendix A (to Iwasawa algebras over more general rings than $\mathcal{O}$), so we have not pursued this. Note however that the 'miracle flatness' result used in the proof of Proposition 4.3.1 requires $\overline{R}_\infty$ to be regular — a regularity assumption is also necessary in the classical miracle flatness theorem [Mat89, Thm. 23.1], so this seems unavoidable here.

4.2. **Patched completed homology is Cohen–Macaulay.** We return to the notation and set-up of section 3, and recall that we have a perfect chain complex $\widetilde{\mathcal{C}}(\infty)$ of $\mathcal{O}_\infty[[K_0]]$-modules (see Definition 3.4.12 and Proposition 3.4.16), equipped with an $\mathcal{O}_\infty$-linear action of $\prod_{v|p} G(F_v)$ and an $\mathcal{O}_\infty$-algebra homomorphism

$$R_\infty \to \mathrm{End}_{D(\mathcal{O}_\infty)}(\widetilde{\mathcal{C}}(\infty)).$$

The action of R_∞ on $\widetilde{\mathcal{C}}(\infty)$ commutes with the action of $\prod_{v|p} G(F_v)$ (and with that of $\mathcal{O}_\infty[[K_0]]$). By Hypothesis 4.1.3 together with Remark 3.4.3 and Proposition 3.4.16 (3), this map factors through the quotient $\overline{R}_\infty$ of R_∞. Recall that $\overline{R}_\infty$ is a formal power series ring over $\mathcal{O}$. The action of $\overline{R}_\infty$ induces an $\mathcal{O}_\infty[[K_0]]$-algebra homomorphism $\overline{R}_\infty[[K_0]] \to \mathrm{End}_{D(\mathcal{O}_\infty)}(\widetilde{\mathcal{C}}(\infty))$, and in particular each homology group $H_i(\widetilde{\mathcal{C}}(\infty))$ is a finitely generated $\overline{R}_\infty[[K_0]]$-module. We refer to Definition A.5 for the notion of a Cohen–Macaulay module over $\mathcal{O}[[K_0]]$; as usual, this also gives us the definition of a Cohen–Macaulay module over $\mathcal{O}_\infty[[K]]$ or $\overline{R}_\infty[[K]]$ for any compact open $K \subset K_0$.

We have natural isomorphisms (for every $i \geq 0$)

$$H_i(\mathcal{O}_\infty/\mathbf{a} \otimes_{\mathcal{O}_\infty} \widetilde{\mathcal{C}}(\infty)) \cong \widetilde{H}_i(X_{U^p}, \mathcal{O})_\mathfrak{m},$$

where $\mathbf{a} = \ker(\mathcal{O}_\infty \to \mathcal{O})$. Recall that K_1 denotes a pro-p Sylow subgroup of K_0, and B is the Borel subgroup of G.

Proposition 4.2.1. *Suppose that*

(a) $H_i(X_{U^pK_1}, k)_\mathfrak{m} = 0$ *for* i *outside the range* $[q_0, q_0 + l_0]$.

(b) $j_{\mathcal{O}[[K_0]]}\left(\bigoplus_{i\geq 0} \widetilde{H}_i(X_{U^p}, \mathcal{O})_\mathfrak{m}\right) \geq l_0$.

Then

(1) $\widetilde{H}_i(X_{U^p}, \mathcal{O})_\mathfrak{m} = 0$ *for* $i \neq q_0$ *and* $\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_\mathfrak{m}$ *is a Cohen–Macaulay* $\mathcal{O}[[K_0]]$*-module with*

$$\mathrm{pd}_{\mathcal{O}[[K_0]]}(\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_\mathfrak{m}) = j_{\mathcal{O}[[K_0]]}(\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_\mathfrak{m}) = l_0.$$

(2) $H_i(\widetilde{\mathcal{C}}(\infty)) = 0$ *for* $i \neq q_0$ *and* $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ *is a Cohen–Macaulay* $\mathcal{O}_\infty[[K_0]]$*-module with*

$$\mathrm{pd}_{\mathcal{O}_\infty[[K_0]]}\left(H_{q_0}(\widetilde{\mathcal{C}}(\infty))\right) = j_{\mathcal{O}_\infty[[K_0]]}\left(H_{q_0}(\widetilde{\mathcal{C}}(\infty))\right) = l_0.$$

(3) $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ *is a Cohen–Macaulay* $\overline{R}_\infty[[K_0]]$*-module with*

$$\mathrm{pd}_{\overline{R}_\infty[[K_0]]}\left(H_{q_0}(\widetilde{\mathcal{C}}(\infty))\right) = j_{\overline{R}_\infty[[K_0]]}\left(H_{q_0}(\widetilde{\mathcal{C}}(\infty))\right) = \dim(B)$$

where $\dim(B) = (\frac{n(n+1)}{2} - 1)[F : \mathbb{Q}]$.

If we moreover suppose that

(c) $j_{k[[K_0]]}\left(\bigoplus_{i\geq 0} \widetilde{H}_i(X_{U^p}, k)_\mathfrak{m}\right) \geq l_0$,

then $\widetilde{H}_i(X_{U^p}, k)_\mathfrak{m} = 0$ *for* $i \neq q_0$ *and both* $\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_\mathfrak{m}$ *and* $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ *are* ϖ*-torsion free.*

Proof. We have $H_i(k \otimes^{\mathbb{L}}_{\mathcal{O}_\infty[[K_1]]} \widetilde{\mathcal{C}}(\infty)) \cong H_i(X_{U^pK_1}, k)_\mathfrak{m}$ by Proposition 3.4.16(1). So the assumption that $H_i(X_{U^pK_1}, k)_\mathfrak{m} = 0$ for i outside the range $[q_0, q_0 + l_0]$ implies (Lemma 2.1.7) that the minimal resolution $\mathcal{F}$ of $\widetilde{\mathcal{C}}(\infty)$ (viewed as a complex of $\mathcal{O}_\infty[[K_1]]$-modules) is concentrated in degrees $[q_0, q_0 + l_0]$.

Fix $H \subset K_1$ a normal compact open subgroup of K_0 which is uniform pro-p. We now apply Lemma A.10 to the shifted complex $\mathcal{O}_\infty/\mathbf{a} \otimes_{\mathcal{O}_\infty} \mathcal{F}[-q_0]$ of finite free $\mathcal{O}[[H]]$-modules to deduce that $H_i(\mathcal{O}_\infty/\mathbf{a} \otimes_{\mathcal{O}_\infty} \mathcal{F}) \cong \widetilde{H}_i(X_{U^p}, \mathcal{O})_\mathfrak{m}$ vanishes for $i \neq q_0$ and $\mathrm{pd}_{\mathcal{O}[[H]]}(\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_\mathfrak{m}) = j_{\mathcal{O}[[H]]}(\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_\mathfrak{m}) = l_0$. Lemma A.7 gives the first claim of the proposition: note that the perfect complex $\mathcal{O}_\infty/\mathbf{a} \otimes_{\mathcal{O}_\infty} \widetilde{\mathcal{C}}(\infty)$ of $\mathcal{O}[[K]]$-modules has homology equal to $\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_\mathfrak{m}$ concentrated in a single degree, so $\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_\mathfrak{m}$ has finite projective dimension as a $\mathcal{O}[[K]]$-module.

Now we move on to the second claim of the proposition. We begin by showing that $\widetilde{\mathcal{C}}(\infty)$ has non-zero homology only in degree q_0. As we will explain, this follows from the fact (which we have just established) that $\mathcal{O}_\infty/\mathbf{a} \otimes_{\mathcal{O}_\infty} \widetilde{\mathcal{C}}(\infty)$ has non-zero homology only in degree q_0. To see this, we recall that $\mathcal{O}_\infty = \mathcal{O}[[x_1, \ldots, x_g]]$, and begin by showing that $\mathcal{O}_\infty/(x_1, \ldots, x_{g-1}) \otimes_{\mathcal{O}_\infty} \widetilde{\mathcal{C}}(\infty)$ has non-zero homology only in degree q_0. For each i we have an injective map (part of a short exact sequence coming from a degenerating Tor spectral sequence)

$$H_i(\mathcal{O}_\infty/(x_1,\dots,x_{g-1})\otimes_{\mathcal{O}_\infty}\widetilde{\mathcal{C}}(\infty))/x_g \hookrightarrow H_i(\mathcal{O}_\infty/\mathbf{a}\otimes_{\mathcal{O}_\infty}\widetilde{\mathcal{C}}(\infty))$$

so it follows from Nakayama's lemma, applied to the finitely generated $\mathcal{O}_\infty[[K_1]]$-module $H_i(\mathcal{O}_\infty/(x_1,\dots,x_{g-1})\otimes_{\mathcal{O}_\infty}\widetilde{\mathcal{C}}(\infty))$, that $H_i(\mathcal{O}_\infty/(x_1,\dots,x_{g-1})\otimes_{\mathcal{O}_\infty}\widetilde{\mathcal{C}}(\infty))$ is zero for $i\neq q_0$. Repeating this argument, we eventually deduce that $H_i(\widetilde{\mathcal{C}}(\infty))$ is zero for $i\neq q_0$.

Since $\mathcal{F}$ is a chain complex whose first nonzero term has degree q_0, we have $\mathcal{O}_\infty/\mathbf{a}\otimes_{\mathcal{O}_\infty}H_{q_0}(\mathcal{F}) = H_{q_0}(\mathcal{O}_\infty/\mathbf{a}\otimes_{\mathcal{O}_\infty}\mathcal{F})\cong \widetilde{H}_{q_0}(X_{U^p},\mathcal{O})_{\mathfrak{m}}$, so Lemma A.16 implies that

$$j_{\mathcal{O}_\infty[[H]]}(H_{q_0}(\mathcal{F}))\geq j_{\mathcal{O}[[H]]}(\widetilde{H}_{q_0}(X_{U^p},\mathcal{O})_{\mathfrak{m}}) = l_0.$$

Another application of Lemmas A.10 and A.7 gives us the second claim of the proposition.

Next, we establish the third claim. It follows from Corollary A.29 and what we have established above that $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is a Cohen–Macaulay $\overline{R}_\infty[[H]]$-module, with grade and projective dimension as in the claim. Lemma A.7 and Lemma 3.4.15 establish the claim as stated once we verify that $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ has finite projective dimension over $\overline{R}_\infty[[K_1]]$. To verify this, let $\mathcal{G}$ be a minimal resolution of $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$, viewed as a complex of $\overline{R}_\infty[[K_1]]$-modules concentrated in degree q_0. The complex $\overline{R}_\infty\otimes_{\overline{R}_\infty[[K_1]]}\mathcal{G}$ (we mod out by the augmentation ideal for K_1) has bounded and finitely generated homology, since $H_i(\overline{R}_\infty\otimes_{\overline{R}_\infty[[K_1]]}\mathcal{G}) = H_i(\mathcal{O}_\infty\otimes_{\mathcal{O}_\infty[[K_1]]}\widetilde{\mathcal{C}}(\infty))$ and $\widetilde{\mathcal{C}}(\infty)$ is a perfect complex of $\mathcal{O}_\infty[[K_1]]$-modules. Since $\overline{R}_\infty$ is regular, $\overline{R}_\infty\otimes_{\overline{R}_\infty[[K_1]]}\mathcal{G}$ is therefore a perfect complex of $\overline{R}_\infty$-modules, so $k\otimes_{\overline{R}_\infty[[K_1]]}\mathcal{G}$ also has bounded homology. We deduce that the minimal complex $\mathcal{G}$ is itself bounded, so $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ has finite projective dimension over $\overline{R}_\infty[[K_1]]$.

For the last part of the Proposition, if we assume that $j_{k[[K_0]]}\left(\bigoplus_{i\geq 0}\widetilde{H}_i(X_{U^p},k)_{\mathfrak{m}}\right) = l_0$ then we may apply Lemma A.10 to the shifted complex $\mathcal{O}_\infty/\mathfrak{m}_{\mathcal{O}_\infty}\otimes_{\mathcal{O}_\infty}\mathcal{F}[-q_0]$ of finite free $k[[H]]$-modules to deduce that $\widetilde{H}_i(X_{U^p},k)_{\mathfrak{m}}=0$ for $i\neq q_0$. This shows that

$$\mathrm{Tor}_i^{\mathcal{O}}(\mathcal{O}/\varpi,\widetilde{H}_{q_0}(X_{U^p},\mathcal{O})_{\mathfrak{m}}) = \widetilde{H}_{q_0+i}(X_{U^p},k)_{\mathfrak{m}} = 0$$

for $i>0$, so $\widetilde{H}_{q_0}(X_{U^p},\mathcal{O})_{\mathfrak{m}}$ is ϖ-torsion free. Arguing as for the second part, we deduce that $H_i(\widetilde{\mathcal{C}}(\infty)/\varpi)=0$ for $i\neq q_0$ and hence $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is also ϖ-torsion free. □

Remark 4.2.2.

(1) Hypothesis (b) that $j_{\mathcal{O}[[K_0]]}\left(\bigoplus_{i\geq 0}\widetilde{H}_i(X_{U^p},\mathcal{O})_{\mathfrak{m}}\right)\geq l_0$ of the above Proposition is implied by the codimension conjecture of Calegari and Emerton [CE12, Conjecture 1.5] (indeed equality is conjectured to hold here). For PGL_2 over an imaginary quadratic field, this hypothesis holds (for example, by the argument of [CE12, Example 1.12]).

(2) Hypothesis (a), that $H_i(X_{U^pK_1},k)_{\mathfrak{m}}=0$ for n outside the range $[q_0,q_0+l_0]$, is conjectured in [CG12, Conj. B(4)(a)]. Again, for PGL_2 over an imaginary quadratic field, the hypothesis holds: we have $l_0=1$, $q_0=1$ and the dimension of $X_{U^pK_1}$ is equal to 3, so it suffices to check that $H_0(X_{U^pK_1},k)_{\mathfrak{m}} = H_3(X_{U^pK_1},k)_{\mathfrak{m}}=0$ which follows from the fact that $\mathfrak{m}$ is non-Eisenstein.

(3) In contrast to the other hypotheses, hypothesis (c) seems difficult to verify even for PGL_2 over an imaginary quadratic field. We cannot rule out (for example) $\widetilde{H}_1(X_{U^p}, \mathcal{O})_{\mathfrak{m}}$ containing a ϖ-torsion submodule which is torsion free over $k[[K_0]]$, in which case $j_{k[[K_0]]}\widetilde{H}_1(X_{U^p}, k)_{\mathfrak{m}} = 0$.

Remark 4.2.3. It follows from the second part of the Proposition that the map $R_\infty \to \mathrm{End}_{D(\mathcal{O}_\infty)}(\widetilde{\mathcal{C}}(\infty))$ (which commutes with the G action) arises from a map $R_\infty \to \mathrm{End}_{\mathcal{O}_\infty[G]}(H_{q_0}(\widetilde{\mathcal{C}}(\infty)))$. In particular, the action of R_∞ on $\widetilde{\mathcal{C}}(\infty)$ can be thought of as taking place in, for example, the derived category of $\mathcal{O}_\infty[[K_0]]$-modules with compatible G-action.

4.3. **Miracle flatness and 'big $R = \mathbb{T}$'.** We have a surjective map $R_{\overline{\mathcal{S}}} \to \mathbb{T}^S(U^p)_{\mathfrak{m}}$. If this map is an isomorphism, the global Euler characteristic formula for Galois cohomology gives an expected dimension of $1 + (\frac{n(n+1)}{2} - 1)[F : \mathbb{Q}] - l_0$ for both these rings. See [Eme14, Conj. 3.1].

The following Proposition shows that this dimension formula, as well as the isomorphism $R_{\overline{\mathcal{S}}} \cong \mathbb{T}^S(U^p)_{\mathfrak{m}}$, is implied by a natural condition on the codimension (over $k[[K_0]]$) of the fibre of the completed homology module $\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}}$ at the maximal ideal $\mathfrak{m}$ of the Hecke algebra. The method of proof is in some sense a precise version of the heuristics discussed in [Eme14, §3.1.1] which compare the Krull dimension of $\mathbb{T}^S(U^p)_{\mathfrak{m}}$ and the Iwasawa theoretic dimensions of $\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}}$ and its mod $\mathfrak{m}$ fibre. A related argument was found independently by Emerton and Paškūnas, and will appear in a forthcoming paper of theirs.

Proposition 4.3.1. *Suppose that assumptions* (a) *and* (b) *of Proposition 4.2.1 hold, and that we moreover have*

$$j_{k[[K_0]]}(\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}}/\mathfrak{m}\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}}) \geq \dim(B).$$

Then we have the following:

(1) $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ *is a flat* $\overline{R}_\infty$*-module.*
(2) *The ideal* $\overline{R}_\infty \mathbf{a}$ *is generated by a regular sequence in* $\overline{R}_\infty$.
(3) *The surjective maps*

$$\overline{R}_\infty/\mathbf{a} \to R_{\overline{\mathcal{S}}} \to \mathbb{T}^S(U^p)_{\mathfrak{m}}$$

are all isomorphisms and $\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}}$ *is a faithfully flat* $\mathbb{T}^S(U^p)_{\mathfrak{m}}$*-module.*
(4) *The rings* $R_{\overline{\mathcal{S}}} \cong \mathbb{T}^S(U^p)_{\mathfrak{m}}$ *are local complete intersections with Krull dimension equal to* $\dim(R_\infty) - \dim(\mathcal{O}_\infty) + 1 = 1 + (\frac{n(n+1)}{2} - 1)[F : \mathbb{Q}] - l_0$.
(5) *If assumption* (c) *of Proposition 4.2.1 holds, then* $\mathbb{T}^S(U^p)_{\mathfrak{m}}$ *is* ϖ*-torsion free.*

Proof. First we note that by Lemma A.16 and Proposition 4.2.1 we have

$$j_{k[[K_0]]}(\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}}/\mathfrak{m}\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}}) \leq j_{\overline{R}_\infty[[K_0]]}\Big(H_{q_0}(\widetilde{\mathcal{C}}(\infty))\Big) = \dim(B),$$

since

$$\overline{R}_\infty/\mathfrak{m}_{\overline{R}_\infty} \otimes_{\overline{R}_\infty} H_{q_0}(\widetilde{\mathcal{C}}(\infty)) = \widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}}/\mathfrak{m}\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}}.$$

So our assumption implies that we have equality of codimensions here. The first claim then follows immediately from Propositions 4.2.1 and A.30.

For the second part, write $\mathbf{a} = (x_1, \ldots, x_g)$ where $\mathcal{O}_\infty = \mathcal{O}[[x_1, \ldots, x_g]]$ (so $g = \dim(\mathcal{O}_\infty) - 1$). Note that, by Proposition 4.2.1 (which in particular says that the complexes $\widetilde{\mathcal{C}}(\infty)$ and $\mathcal{O}_\infty/\mathbf{a}\otimes_{\mathcal{O}_\infty}\widetilde{\mathcal{C}}(\infty)$ both have homology concentrated in a single degree) we have

$$\mathrm{Tor}_i^{\mathcal{O}_\infty}(\mathcal{O}_\infty/\mathbf{a}, H_{q_0}(\widetilde{\mathcal{C}}(\infty))) = \widetilde{H}_{q_0+i}(X_{U^p},\mathcal{O})_{\mathfrak{m}} = 0$$

for $i > 0$. So (by considering the Koszul complex for $(x_1,\dots,x_g)$) we see that $(x_1,\dots,x_g)$ is a regular sequence on $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$. Since $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is a flat $\overline{R}_\infty$-module and its reduction mod $\mathfrak{m}_{\overline{R}_\infty}$ is non-zero (by Nakayama, since the module is finitely generated over $\overline{R}_\infty[[K_0]]$), it follows from [Mat89, Thm. 7.2] that $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is a faithfully flat $\overline{R}_\infty$-module and we can conclude that $(x_1,\dots,x_g)$ is a regular sequence in $\overline{R}_\infty$ — this can be seen by considering the Koszul homology groups

$$H_*^{\overline{R}_\infty}((x_1,\dots,x_g), H_{q_0}(\widetilde{\mathcal{C}}(\infty))) \cong H_*^{\overline{R}_\infty}((x_1,\dots,x_g),\overline{R}_\infty)\otimes_{\overline{R}_\infty} H_{q_0}(\widetilde{\mathcal{C}}(\infty)),$$

and by faithful flatness we have $H_i^{\overline{R}_\infty}((x_1,\dots,x_g),\overline{R}_\infty) = 0$ for $i \neq 0$ and therefore $(x_1,\dots,x_g)$ is a regular sequence in $\overline{R}_\infty$. This gives the second part.

For the third part, since $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is a flat $\overline{R}_\infty$-module, $\widetilde{H}_{q_0}(X_{U^p},\mathcal{O})_{\mathfrak{m}} = \mathcal{O}_\infty/\mathbf{a}\otimes_{\mathcal{O}_\infty} H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is a flat $\overline{R}_\infty/\mathbf{a}$-module. As before, it follows from [Mat89, Thm. 7.2] that $\widetilde{H}_{q_0}(X_{U^p},\mathcal{O})_{\mathfrak{m}}$ is a faithfully flat $\overline{R}_\infty/\mathbf{a}$-module and is in particular faithful. It follows that the surjective maps appearing in the third part must also be injective, since the action of $\overline{R}_\infty/\mathbf{a}$ on $\widetilde{H}_{q_0}(X_{U^p},\mathcal{O})_{\mathfrak{m}}$ factors through these maps. This completes the proof of the third part.

The fourth part follows immediately from the second and third parts.

The fifth part follows from the fact that $\mathbb{T}^S(U^p)_{\mathfrak{m}}$ acts faithfully on $\widetilde{H}_{q_0}(X_{U^p},\mathcal{O})_{\mathfrak{m}}$ (by Lemma 3.4.20), which is ϖ-torsion free (under our additional assumption) by Proposition 4.2.1. Alternatively, one can redo the argument of part (2) of the Proposition to show that $(\varpi, x_1,\dots,x_g)$ is a regular sequence in $\overline{R}_\infty$, and so in particular ϖ is not a zero-divisor in $\overline{R}_\infty/\mathbf{a}\cong\mathbb{T}^S(U^p)_{\mathfrak{m}}$. $\square$

Remark 4.3.2. To explain the condition

$$j_{k[[K_0]]}(\widetilde{H}_{q_0}(X_{U^p},\mathcal{O})_{\mathfrak{m}}/\mathfrak{m}\widetilde{H}_{q_0}(X_{U^p},\mathcal{O})_{\mathfrak{m}}) \geq \dim(B)$$

we first note that the parabolic induction of a k-valued character from B to G has codimension $\dim(B)$ over $k[[K_0]]$. We moreover expect this to be the codimension of any 'generic' irreducible admissible smooth k-representation of G, with other irreducibles having at least this codimension. In the case $G = \mathrm{PGL}_2(\mathbb{Q}_p)$ this is true: any infinite-dimensional irreducible smooth k-representation of G has codimension $\dim(B) = 2$ [SS16, Proof of Cor. 7.5], whilst the finite-dimensional representations have codimension 3.

It seems reasonable to expect that $\widetilde{H}_{q_0}(X_{U^p},\mathcal{O})_{\mathfrak{m}}/\mathfrak{m}\widetilde{H}_{q_0}(X_{U^p},\mathcal{O})_{\mathfrak{m}}$ is a finite length representation of G, and therefore we expect it to have codimension $\geq \dim(B)$ also.

5. The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$

In this section we specialise to the case that $n = 2$ and p splits completely in F, and use the techniques of [CEGGPS2] to study the relationship of our constructions to the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

5.1. **A local-global compatibility conjecture.** We continue to make the assumptions made in Section 4, as well as assumptions (a) and (b) of Proposition 4.2.1.

In addition we assume that

- $n = 2$,
- p splits completely in F,
- if $\overline{\rho}_{\mathfrak{m}}|_{G_v}$ is ramified for some place $v \nmid p$, then v is not a vexing prime in the sense of [Dia97], and
- for each place $v|p$, $\overline{\rho}_{\mathfrak{m}}|_{G_v}$ is either absolutely irreducible, or is a non-split extension of characters, whose ratio is not the trivial character or the mod p cyclotomic character.

This last assumption allows us to use the results of [CEGGPS2]; it guarantees in particular that each $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ admits a universal deformation ring R_v^{def}. Since $n = 2$, l_0 is just equal to r_2, the number of complex places of F.

From now on in a slight abuse of notation for each place $v|p$ we write G_v for $\mathrm{PGL}_2(F_v)$ and K_v for $\mathrm{PGL}_2(\mathcal{O}_{F_v})$, and we write G for $\prod_{v|p} G_v$. Recall that $K_0 = \prod_{v|p} K_v$.

Since our interest is primarily in phenomena at places dividing p, we content ourselves with the "minimal level" situation at places not dividing p; that is, we choose $\overline{R}_\infty$ and the level U^p as in the second paragraph of Remark 4.1.4. (The reader may object that this level is not necessarily S-good; as usual in the Taylor–Wiles method, this difficulty is easily resolved by shrinking the level at an auxiliary place at which $\overline{\rho}_{\mathfrak{m}}$ admits no ramified deformations, and for simplicity of exposition we ignore this point.)

We would like to understand the action of $(\widehat{\otimes}_{v|p,\mathcal{O}} R_v^{\mathrm{def}})[G]$ on $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$. When $F = \mathbb{Q}$ it follows from the local-global compatibility theorem of [Eme10a] that this action is determined by the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, and it is natural to expect that the same applies for general number fields F.

More precisely, for each place $v|p$, we can associate an absolutely irreducible k-representation π_v of $\mathrm{GL}_2(F_v)$ to $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ via the recipe of [CEGGPS2, Lem. 2.15 (5)]; note that by [CEGGPS2, Rem. 2.17], the central character of π_v is trivial, so we can regard it as a representation of G_v.

Definition 5.1.1. If H is a p-adic analytic group and A is a complete local Noetherian $\mathcal{O}$-algebra, then we write $\mathfrak{C}_H(A)$ for the Pontryagin dual of the category of locally admissible A-representations of H (cf. Appendix B and [CEGGPS2, §4.4]).

We let $P_v \twoheadrightarrow \pi_v^\vee$ be a projective envelope of $\pi_v^\vee$ in $\mathfrak{C}_{G_v}(\mathcal{O})$. By [Paš13, Prop. 6.3, Cor. 8.7] there is a natural isomorphism $R_v^{\mathrm{def}} \to \mathrm{End}_{\mathfrak{C}_{G_v}(\mathcal{O})}(P_v)$. (This is a large part of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.)

Write $P := \widehat{\otimes}_{v|p,\mathcal{O}} P_v$ which is naturally an $R_p^{\mathrm{loc}} := \widehat{\otimes}_{v|p,\mathcal{O}} R_v^{\mathrm{def}}$-module. For each $v|p$ we make a choice (in its strict equivalence class) of the universal deformation of $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ to R_v^{def}, so that we can regard $\overline{R}_\infty$ as an R_p^{loc}-module. For some $g \geq 0$ we can and do choose an isomorphism of R_p^{loc}-algebras $\overline{R}_\infty \cong R_p^{\mathrm{loc}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[x_1, \ldots, x_g]]$.

Conjecture 5.1.2. *For some $m \geq 1$ there is an isomorphism of $\overline{R}_\infty\,[G]$-modules*

$$H_{q_0}(\widetilde{\mathcal{C}}(\infty)) \cong \overline{R}_\infty \widehat{\otimes}_{R_p^{\mathrm{loc}}} P^{\oplus m}.$$

Remark 5.1.3. We do not know what the value of m in Conjecture 5.1.2 should be in general. The natural guess is that $m = 2^{r_1}$ where r_1 is the number of real places of F, since this is the dimension of the $(\mathfrak{g}, K)$-cohomology in degree q_0 of the trivial representation for the group $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{PGL}_2$. This guess is justified by Corollary 5.1.8. Indeed, if $H_{q_0}(X_{K_0 U^p}, \sigma)_{\mathfrak{m}}$ is non-zero for some irreducible E-representation of K_0, then Corollary 5.1.8 shows that m is equal to the multiplicity of a system of Hecke eigenvalues (away from S) in $H_{q_0}(X_{K_0 U^p}, \sigma)_{\mathfrak{m}}$.

We now explain some consequences of this conjecture for completed homology and homology with coefficients. In the proof of the following result we will briefly need the notion of the *atome automorphe* κ_v associated to $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$; recall that if $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ is irreducible, then $\kappa_v = \pi_v$ is an irreducible supersingular representation of G_v, while if $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ is reducible, κ_v is a non-split extension of irreducible principal series representations with socle π_v (see for example the beginning of [Paš13, §8]).

Proposition 5.1.4. *Assume Conjecture 5.1.2. Then we have an isomorphism of local complete intersections $R_{\overline{\mathcal{S}}} \cong \mathbb{T}^S(U^p)_{\mathfrak{m}}$ with Krull dimension equal to $1 + 2[F : \mathbb{Q}] - l_0$. Furthermore, there is an isomorphism of $\mathbb{T}^S(U^p)_{\mathfrak{m}}[G]$-modules*

$$\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}} \cong \mathbb{T}^S(U^p)_{\mathfrak{m}} \widehat{\otimes}_{R_p^{\mathrm{loc}}} P^{\oplus m}.$$

If we moreover make assumption (c) *of Proposition 4.2.1 then $\mathbb{T}^S(U^p)_{\mathfrak{m}}$ is ϖ-torsion free.*

Proof. The isomorphism $R_{\overline{\mathcal{S}}} \cong \mathbb{T}^S(U^p)_{\mathfrak{m}}$ and the properties of these rings will follow immediately from Proposition 4.3.1 once we know that

$$j_{k[[K_0]]}(\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}} / \mathfrak{m} \widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}}) = 2[F : \mathbb{Q}].$$

Now, since we are assuming Conjecture 5.1.2, we have

$$\begin{aligned}
\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}} / \mathfrak{m} \widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}} &= \overline{R}_\infty / \mathfrak{m}_{\overline{R}_\infty} \otimes_{\overline{R}_\infty} H_{q_0}(\widetilde{\mathcal{C}}(\infty)) \\
&= P^{\oplus m} \otimes_{R_p^{\mathrm{loc}}} R_p^{\mathrm{loc}} / \mathfrak{m}_{R_p^{\mathrm{loc}}} \\
&= (\widehat{\otimes}_{v|p} P_v \widehat{\otimes}_{R_v^{\mathrm{def}}} k)^{\oplus m} \\
&= (\widehat{\otimes}_{v|p} \kappa_v^\vee)^{\oplus m}
\end{aligned}$$

(where in the last line we have used [Paš13, Prop. 1.12, 6.1, 8.3] and that $R_v^{\mathrm{def}} = \mathrm{End}_{\mathfrak{C}_{G_v}(\mathcal{O})}(P_v)$). By Lemma A.11 we are therefore reduced to showing that for each $v|p$,

$$j_{k[[K_v]]}(\kappa_v^\vee) = 2.$$

By the same argument as Lemma A.15 it is enough to show that $j_{k[[\mathrm{GL}_2(\mathcal{O}_{F_v})]]}(\kappa_v^\vee) = 3$ (we pass from $k[[\mathrm{GL}_2(\mathcal{O}_{F_v})]]$ to $k[[\mathrm{PGL}_2(\mathcal{O}_{F_v})]]$ by quotienting out by a central regular element which acts trivially on $\kappa_v^\vee$). By Lemma A.8 we are reduced to the same statement for irreducible principal series and supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, which is proved in [SS16, Proof of Cor. 7.5].

Finally, we have

$$\begin{aligned}
\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}} &= \overline{R}_\infty / \mathbf{a} \overline{R}_\infty \otimes_{\overline{R}_\infty} H_{q_0}(\widetilde{\mathcal{C}}(\infty)) \\
&= R_{\overline{\mathcal{S}}} \otimes_{\overline{R}_\infty} H_{q_0}(\widetilde{\mathcal{C}}(\infty)) \\
&= \mathbb{T}^S(U^p)_{\mathfrak{m}} \otimes_{\overline{R}_\infty} H_{q_0}(\widetilde{\mathcal{C}}(\infty)) \\
&\cong \mathbb{T}^S(U^p)_{\mathfrak{m}} \widehat{\otimes}_{R_p^{\mathrm{loc}}} P^{\oplus m},
\end{aligned}$$

as required. ∎

We recall from [CEGGPS2, §2] some notation for Hecke algebras and crystalline deformation rings. (In fact our setting is slightly different, as we are working with PGL_2 rather than GL_2, but this makes no difference in practice and we will not emphasise this point below.) Let σ be an irreducible E-representation of K_0. Any such representation is of the form $\otimes_{v|p}\sigma_v$, where σ_v is the representation of G_v given by $\sigma_v = \det^{a_v} \otimes \mathrm{Sym}^{b_v} E^2$ for integers a_v, b_v satisfying $b_v \geq 0$ and $2a_v + b_v = 0$. We write σ° for the K_0-stable $\mathcal{O}$-lattice $\otimes_{v|p}\sigma_v^\circ$, where $\sigma_v^\circ = \det^{a_v} \otimes \mathrm{Sym}^{b_v} \mathcal{O}^2$. We have Hecke algebras $\mathcal{H}(\sigma) := \mathrm{End}_G(\text{c-Ind}_{K_0}^G \sigma)$, $\mathcal{H}(\sigma^\circ) := \mathrm{End}_G(\text{c-Ind}_{K_0}^G \sigma^\circ)$.

A *Serre weight* is an irreducible k-representation of K_0. These are of the form $\otimes_{v|p}\overline{\sigma}_v$, where $\overline{\sigma}_v = \det^{a_v} \otimes \mathrm{Sym}^{b_v} k^2$ for integers a_v, b_v satisfying $0 \leq b_v \leq p-1$ and $2a_v + b_v = 0$. Note that for any $\overline{\sigma}$ there is a unique σ with $\sigma^\circ \otimes_{\mathcal{O}} k = \overline{\sigma}$; we say that σ° lifts $\overline{\sigma}$. As explained in the proof of Lemma B.7, we have Hecke algebras $\mathcal{H}(\overline{\sigma}) \cong \otimes_{v|p}\mathcal{H}(\overline{\sigma}_v)$, where $\mathcal{H}(\overline{\sigma}_v) := \mathrm{End}_{G_v}(\text{c-Ind}_{K_v}^{G_v} \overline{\sigma}_v) \cong k[T_v]$ is a polynomial ring in one variable by [BL94, Prop. 8].

5.1.5. *Actions of Hecke algebras.* We now describe how to define actions of the Hecke algebras $\mathcal{H}(\overline{\sigma})$ and $\mathcal{H}(\sigma^\circ)$ on objects of certain derived categories.

Let $\overline{\sigma}$ be a Serre weight. Suppose M is a pseudocompact $A[[K_0]]$-module with a compatible action of G, where A is a complete Noetherian local $\mathcal{O}$-algebra with finite residue field which is flat over $\mathcal{O}$. For example, A could be either $\mathcal{O}[\Delta_Q]$ or $\mathcal{O}_\infty$. Then the A-module $\overline{\sigma} \otimes_{\mathcal{O}[[K_0]]} M$ has a natural action of $\mathcal{H}(\overline{\sigma})$. Indeed, we have isomorphisms

$$(\overline{\sigma} \otimes_{\mathcal{O}[[K_0]]} M)^\vee \cong \mathrm{Hom}^{cts}_{\mathcal{O}[[K_0]]}(\overline{\sigma}, M^\vee) = \mathrm{Hom}_G(\text{c-Ind}_{K_0}^G \overline{\sigma}, M^\vee)$$

by Lemma B.3 and Frobenius reciprocity (note that $M^\vee \in \mathrm{Mod}_G^{sm}(A)$, where the definition of this category is recalled in Appendix B), and $\mathcal{H}(\overline{\sigma})$ naturally acts on $\mathrm{Hom}_G(\text{c-Ind}_{K_0}^G \overline{\sigma}, M^\vee)$.

We have a similar story in the derived category. If we let $M^\vee \to I^\bullet$ be an injective resolution of $M^\vee$ in $\mathrm{Mod}_G^{sm}(A)$, then each $(I^i)^\vee$ is projective as a pseudocompact $A[[K_0]]$-module (by [Eme10c, Prop. 2.1.2]), and is in particular a flat $\mathcal{O}[[K_0]]$-module, so we have a natural action of $\mathcal{H}(\overline{\sigma})$ on

$$\overline{\sigma} \otimes^{\mathbb{L}}_{\mathcal{O}[[K_0]]} M = \overline{\sigma} \otimes_{\mathcal{O}[[K_0]]} (I^\bullet)^\vee$$

in $D(A)$.

Similarly, if σ° is a lattice in σ, we have a natural action of $\mathcal{H}(\sigma^\circ)$ on

$$\mathrm{Hom}^{cts}_{\mathcal{O}[[K_0]]}(\sigma^\circ, I^i) = \varinjlim_s \mathrm{Hom}_{K_0}(\sigma^\circ/\varpi^s, I^i) = \varinjlim_s \mathrm{Hom}_G(\text{c-Ind}_{K_0}^G(\sigma^\circ/\varpi^s), I^i)$$

for each n, where the first equality uses Lemma B.2, and therefore a natural action of $\mathcal{H}(\sigma^\circ)$ on

$$\sigma^\circ \otimes^{\mathbb{L}}_{\mathcal{O}[[K_0]]} M = \sigma^\circ \otimes_{\mathcal{O}[[K_0]]} (I^\bullet)^\vee$$

in $D(A)$.

As a particular example of this construction, we get a natural action of $\mathcal{H}(\sigma^\circ)$ on $\mathcal{C}(K_0U^p, \sigma^\circ)_{\mathfrak{m}}$, in $D(\mathcal{O})$, since we have an isomorphism

$$\mathcal{C}(K_0U^p, \sigma^\circ)_{\mathfrak{m}} \cong \sigma^\circ \otimes^{\mathbb{L}}_{\mathcal{O}[[K_0]]} \widetilde{H}_{q_0}(U^p, \mathcal{O})_{\mathfrak{m}}.$$

Here we are using the part of Prop. 4.2.1 which shows that $\widetilde{H}_i(U^p, \mathcal{O})_{\mathfrak{m}} = 0$ for $i \neq q_0$. One can also define the action of $\mathcal{H}(\sigma^\circ)$ on $\mathcal{C}(K_0U^p, \sigma^\circ)$ directly, similarly to the definition of the Hecke action at places away from p, and this gives the same Hecke action.

We say that a representation $r : G_{F_v} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is crystalline of Hodge type σ_v if it is crystalline with Hodge–Tate weights $(1-a_v, -a_v-b_v)$, and we write $R_v^{\mathrm{def}}(\sigma_v)$ for the reduced, p-torsion free quotient of R_v^{def} corresponding to crystalline deformations of Hodge type σ_v. We write $R_p^{\mathrm{loc}}(\sigma) := \widehat{\otimes}_{v|p} R_v^{\mathrm{def}}(\sigma_v)$ and $\overline{R}_\infty(\sigma) := \overline{R}_\infty \otimes_{R_p^{\mathrm{loc}}} R_p^{\mathrm{loc}}(\sigma)$. By [Kis08, Thm. 3.3.8], $R_v^{\mathrm{def}}(\sigma_v)$ is equidimensional of Krull dimension 2 less than R_v^{def}, so by Lemma 4.1.1, $\overline{R}_\infty(\sigma)$ is equidimensional of dimension $\dim \mathcal{O}_\infty - l_0$.

We have a homomorphism $\mathcal{H}(\sigma) \xrightarrow{\eta} R_p^{\mathrm{loc}}(\sigma)[1/p]$, which is the tensor product over the places $v|p$ of the maps $\mathcal{H}(\sigma_v) \to R_v^{\mathrm{def}}(\sigma_v)[1/p]$ defined in [CEGGPS, Thm. 4.1], which interpolates the (unramified) local Langlands correspondence.

Proposition 5.1.6. *Assume Conjecture 5.1.2. Then for any σ, the action of R_p^{loc} on*

$$\mathcal{C}(K_0U^p, \sigma^\circ)_{\mathfrak{m}} \in D(\mathcal{O})$$

factors through $R_p^{\mathrm{loc}}(\sigma)$. Furthermore, if $h \in \mathcal{H}(\sigma^\circ)$ is such that $\eta(h) \in R_p^{\mathrm{loc}}(\sigma)$, then h acts on $\mathcal{C}(K_0U^p, \sigma^\circ)_{\mathfrak{m}}$ via $\eta(h)$.

In particular, we get the same statements for the action of R_p^{loc} and $\mathcal{H}(\sigma^\circ)$ on the homology groups $H_i(X_{K_0U^p}, \sigma^\circ)_{\mathfrak{m}}$ for any i.

Proof. As in the proof of Proposition 3.4.19, it follows from Lemma A.33 that we have a natural quasi-isomorphism (where we regard σ° as a right $\mathcal{O}_\infty[[K_0]]$-module)

$$\sigma^\circ \otimes_{\mathcal{O}_\infty[[K_0]]} \widetilde{\mathcal{C}}(\infty) \to \mathcal{C}(K_0U^p, \sigma^\circ)_{\mathfrak{m}}.$$

Conjecture 5.1.2 implies that $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is a flat $\mathcal{O}[[K_0]]$-module, so we have an isomorphism in $D(\mathcal{O})$

$$\sigma^\circ \otimes_{\mathcal{O}_\infty[[K_0]]} \widetilde{\mathcal{C}}(\infty) = \mathcal{O} \otimes^{\mathbb{L}}_{\mathcal{O}_\infty} \left(\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} H_{q_0}(\widetilde{\mathcal{C}}(\infty))\right)[+q_0]$$

Taking Conjecture 5.1.2 into account, it now suffices to show that the action of R_p^{loc} on $\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} P$ factors through $R_p^{\mathrm{loc}}(\sigma)$, and that if $h \in \mathcal{H}(\sigma^\circ)$ is such that $\eta(h) \in R_p^{\mathrm{loc}}(\sigma)$, then h acts on $\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} P$ via $\eta(h)$.

We have $\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} P = \otimes_{v|p}(\sigma_v^\circ \otimes_{\mathcal{O}[[K_v]]} P_v)$, so it suffices to show that the action of R^{def} on $\sigma_v^\circ \otimes_{\mathcal{O}[[K_v]]} P_v$ factors through $R_v^{\mathrm{def}}(\sigma_v)$, and that if $h_v \in \mathcal{H}(\sigma_v^\circ)$ is such that $\eta(h_v) \in R_v^{\mathrm{def}}(\sigma_v)$, then h_v acts on $\sigma_v^\circ \otimes_{\mathcal{O}[[K_v]]} P_v$ via $\eta(h_v)$. By Lemma B.3 we have a natural isomorphism

$$(\sigma_v^\circ \otimes_{\mathcal{O}[[K_v]]} P_v)^\vee \cong \mathrm{Hom}^{cts}_{\mathcal{O}[[K_v]]}(P_v, (\sigma_v^\circ)^\vee)$$

where we note that since σ_v° is a finitely generated $\mathcal{O}[[K_v]]$-module we do not need to take a completed tensor product. Lemma B.2 implies that this is isomorphic to $\varinjlim_s \mathrm{Hom}^{cts}_{\mathcal{O}[[K_v]]}(P_v, (\sigma_v^\circ/\varpi^s)^\vee)$ so we deduce that

$$(\sigma_v^\circ \otimes_{\mathcal{O}[[K_v]]} P_v) \cong \varprojlim_s \left(\mathrm{Hom}^{cts}_{\mathcal{O}[[K_v]]}(P_v, (\sigma_v^\circ/\varpi^s)^\vee)\right)^\vee.$$

[CEGGPS, Lem. 4.14] then shows that we have an isomorphism

$$\sigma_v^\circ \otimes_{\mathcal{O}[[K_v]]} P_v \cong \mathrm{Hom}^{cts}_{\mathcal{O}[[K_v]]}(P_v, (\sigma_v^\circ)^d)^d$$

where $(-)^d$ denotes the Schikhof dual (as defined in *loc. cit.*). The result now follows from [Paš15, Cor. 6.4, 6.5] and [CEGGPS2, Prop. 6.17]. ☐

Remark 5.1.7. It follows from the argument appearing at the end of the above proof that if P is a projective pseudocompact $\mathcal{O}[[K_0]]$-module then we have a natural isomorphism
$$\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} P \cong \mathrm{Hom}^{cts}_{\mathcal{O}[[K_0]]}(P, (\sigma^\circ)^d)^d.$$

We can also deduce the following modularity lifting theorem from Conjecture 5.1.2.

Corollary 5.1.8. *Assume (in addition to our running assumptions) Conjecture 5.1.2. Then for any σ, $H_{q_0}(X_{K_0U^p}, \sigma)_{\mathfrak{m}}$ is a free module of rank m (where m is the multiplicity in the statement of Conjecture 5.1.2) over $R_{\overline{\mathcal{S}}} \otimes_{R_p^{\mathrm{loc}}} R_p^{\mathrm{loc}}(\sigma)[1/p]$ (if this ring is non-zero).*

In particular, all characteristic 0 points of the global crystalline deformation ring $R_{\overline{\mathcal{S}}(\sigma)} := R_{\overline{\mathcal{S}}} \otimes_{R_p^{\mathrm{loc}}} R_p^{\mathrm{loc}}(\sigma)$ are automorphic, and the maximal ϖ-torsion free quotient of $R_{\overline{\mathcal{S}}(\sigma)}$ is isomorphic to a Hecke algebra acting faithfully on $H_{q_0}(X_{K_0U^p}, \sigma)_{\mathfrak{m}}$.

Moreover, the annihilator of $H_{q_0}(X_{K_0U^p}, \sigma^\circ)_{\mathfrak{m}}$ in $R_{\overline{\mathcal{S}}(\sigma)}$ is nilpotent, and $R_{\overline{\mathcal{S}}(\sigma)}$ is a finite $\mathcal{O}$-algebra.

Proof. By [Paš15, Cor. 6.5], $P(\sigma^\circ) = \sigma^\circ \otimes_{\mathcal{O}[[K_0]]} P$ is a maximal Cohen–Macaulay module with full support over $R_p^{\mathrm{loc}}(\sigma)$. Since $R_p^{\mathrm{loc}}(\sigma)[1/p]$ is regular it follows that $P(\sigma^\circ)[1/p]$ is locally free with full support over $R_p^{\mathrm{loc}}(\sigma)[1/p]$. In fact, as explained in the proof of [CEGGPS2, Prop. 6.14], it follows from [Paš15, Prop. 4.14, 2.22] that $P(\sigma^\circ)[1/p]$ is locally free of rank one over $R_p^{\mathrm{loc}}(\sigma)[1/p]$. We deduce from Conjecture 5.1.2 that $\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} H_{q_0}(\widetilde{\mathcal{C}}(\infty))[1/p]$ is locally free of rank m over $\overline{R}_\infty \otimes_{R_p^{\mathrm{loc}}} R_p^{\mathrm{loc}}(\sigma)[1/p]$. Reducing mod $\mathbf{a}$ (and noting that $\overline{R}_\infty/\mathbf{a} \cong R_{\overline{\mathcal{S}}}$ by Proposition 5.1.4) we deduce that $\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}}[1/p]$ is locally free of rank m over $R_{\overline{\mathcal{S}}} \otimes_{R_p^{\mathrm{loc}}} R_p(\sigma)[1/p]$. We complete the proof by noting that we have a natural isomorphism
$$\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}} \cong H_{q_0}(X_{K_0U^p}, \sigma^\circ)_{\mathfrak{m}}$$
so $R_{\overline{\mathcal{S}}} \otimes_{R_p^{\mathrm{loc}}} R_p(\sigma)[1/p]$ is a finite–dimensional algebra (hence semi-local) and therefore the locally free module of rank m, $H_{q_0}(X_{K_0U^p}, \sigma^\circ)_{\mathfrak{m}}$, is in fact free of rank m.

The moreover part follows from [Tay08, Lem. 2.2], since $\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is a nearly faithful $\overline{R}_\infty(\sigma)$-module so reducing mod $\mathbf{a}$ shows that $H_{q_0}(X_{K_0U^p}, \sigma^\circ)_{\mathfrak{m}}$ is a nearly faithful $R_{\overline{\mathcal{S}}(\sigma)}$-module, as well as a finite $\mathcal{O}$-module. ☐

Remark 5.1.9. As discussed in Remark 5.1.14, we could work with general potentially semistable types, and then the proof of Corollary 5.1.8 goes through unchanged to give an automorphy lifting theorem for arbitrary potentially semistable lifts of $\overline{\rho}_{\mathfrak{m}}$ with distinct Hodge–Tate weights, which satisfy the conditions imposed by $\overline{\mathcal{S}}$ at places $v \nmid p$.

Remark 5.1.10. Using Proposition 5.1.4, we can give an alternative argument to show that Conjecture 5.1.2 implies many cases of the Fontaine–Mazur conjecture, in exactly the same way that Emerton deduces [Eme10a, Corollary 1.2.2] from his local-global compatibility result. If we assume Conjecture 5.1.2 then any characteristic zero point of $R_{\overline{\mathcal{S}}}$ whose associated Galois representation is de Rham with distinct Hodge–Tate weights at each place $v|p$ is automorphic, in the sense that its associated system of Hecke eigenvalues appears in $H_{q_0}(X_{KU^p}, \sigma)_{\mathfrak{m}}$ for some compact open $K \subset K_0$ and some irreducible E-representation σ of K_0.

Moreover, again assuming Conjecture 5.1.2 and following Emerton's argument, we can show that any characteristic zero point of $R_{\overline{S}}$ whose associated Galois representation is trianguline at each place $v|p$ arises from an overconvergent p-adic automorphic form of finite slope, in the sense that its associated system of Hecke eigenvalues appears in the Emerton–Jacquet module $J_B(((\widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}})^d[\frac{1}{p}])^{an})$.

Remark 5.1.11. Assuming Conjecture 5.1.2, we obtain an action of the graded $R_{\overline{S}(\sigma)}$-algebra $\mathrm{Tor}_*^{R_p^{\mathrm{loc}}}(R_{\overline{S}}, R_p^{\mathrm{loc}}(\sigma)) = \mathrm{Tor}_*^{R_\infty}(R_\infty/\mathbf{a}, R_\infty(\sigma))$ on the graded module

$$H_*(X_{K_0U^p}, \sigma^\circ)_{\mathfrak{m}} = H_*\left(R_\infty/\mathbf{a} \otimes^{\mathbb{L}}_{R_\infty} \left(\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} H_{q_0}(\widetilde{\mathcal{C}}(\infty))\right)\right).$$

When $R_p^{\mathrm{loc}}(\sigma)$ is the representing object of a Fontaine–Laffaille moduli problem, the groups $\mathrm{Tor}_i^{R_p^{\mathrm{loc}}}(R_{\overline{S}}, R_p^{\mathrm{loc}}(\sigma))$ are the homotopy groups of a derived Galois deformation ring (since $R_{\overline{S}}$ is a complete intersection of the predicted dimension, see the discussion in [GV16, §1.3]) and the action of the graded algebra on $H_*(X_{K_0U^p}, \sigma^\circ)_{\mathfrak{m}}$ is free. This is an example of the main theorem of [GV16]. Note that it is not obvious that the action of $\mathrm{Tor}_*^{R_p^{\mathrm{loc}}}(R_{\overline{S}}, R_p^{\mathrm{loc}}(\sigma))$ on $H_*(X_{K_0U^p}, \sigma^\circ)_{\mathfrak{m}}$ is independent of the choice of non-principal ultrafilter made to carry out the patching. Under some additional hypotheses, this independence is shown in [GV16], by comparing the action of the derived Galois deformation ring with the action of a derived Hecke algebra.

Proposition 5.1.6 shows that Conjecture 5.1.2 implies a local–global compatibility statement at p. We are now going to formulate a conjectural local–global compatibility statement which will be sufficiently strong to establish Conjecture 5.1.2.

Note that for any Taylor–Wiles datum $(Q, (\gamma_{v,1}, \ldots, \gamma_{v,n})_{v\in Q})$, (3.3.9) gives an action of R_p^{loc} on the complex

$$\widetilde{\mathcal{C}}(Q) := \varprojlim_{U_p, s} \mathcal{C}(U_pU_1^p(Q), s)_{\mathfrak{m}_{Q,1}}$$

in $D(\mathcal{O}[\Delta_Q])$. For any σ, the complex $\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(Q)$ is naturally quasi-isomorphic (in particular, the quasi-isomorphism is $\mathcal{O}[\Delta_Q]$-equivariant) to $\mathcal{C}(K_0U_1^p(Q), \sigma^\circ)_{\mathfrak{m}_{Q,1}}$ Again, this is deduced from Lemma A.33. We therefore obtain an action of R_p^{loc} on $\mathcal{C}(K_0U_1^p(Q), \sigma^\circ)_{\mathfrak{m}_{Q,1}}$ in $D(\mathcal{O}[\Delta_Q])$. We also have a natural action of $\mathcal{H}(\sigma^\circ)$ on $\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(Q)$ in $D(\mathcal{O}[\Delta_Q])$, as described in section 5.1.5. To apply the construction of that section, we must note that $\widetilde{\mathcal{C}}(Q)$ has homology concentrated in degree q_0. Indeed, assumption (a) in Proposition 4.2.1 implies that the minimal resolution $\mathcal{F}$ of $\widetilde{\mathcal{C}}(Q)$ as a complex of $\mathcal{O}[\Delta_Q][[K_1]]$-modules is concentrated in degrees $[q_0, q_0 + l_0]$. We also have $j_{\mathcal{O}[[K_0]]}(H_{q_0}(\widetilde{\mathcal{C}}(Q)) \geq l_0$ because the quotient module $\mathcal{O} \otimes_{\mathcal{O}[\Delta_Q]} H_{q_0}(\widetilde{\mathcal{C}}(Q)) \cong \widetilde{H}_{q_0}(X_{U^p}, \mathcal{O})_{\mathfrak{m}}$ has grade l_0 (by Proposition 4.2.1). Applying Lemma A.10 to the complex $\mathcal{F}[-q_0]$, we deduce that $\widetilde{\mathcal{C}}(Q)$ has homology concentrated in degree q_0.

Proposition 5.1.6 motivates the following conjecture, which is a further refinement of Conjectures 3.3.3 and 3.3.7.

Conjecture 5.1.12. *For any Taylor–Wiles datum $(Q, (\gamma_{v,1}, \ldots, \gamma_{v,n})_{v\in Q})$, and any irreducible E-representation of K_0, σ, the action of R_p^{loc} on $H_*(X_{K_0U_1^p(Q)}, \sigma^\circ)_{\mathfrak{m}_{Q,1}}$*

factors through $R_p^{\mathrm{loc}}(\sigma)$*. Furthermore, if* $h \in \mathcal{H}(\sigma^\circ)$ *is such that* $\eta(h) \in R_p^{\mathrm{loc}}(\sigma)$*, then* h *acts on* $H_{q_0}(X_{K_0U_1^p(Q)}, \sigma^\circ)_{\mathfrak{m}_{Q,1}}$ *via* $\eta(h)$*.*

Remark 5.1.13. The reader may be surprised by Conjecture 5.1.12, which in particular implies that the factors at places dividing p of the Galois representations associated to torsion classes in the homology groups $H_*(X_{K_0U_1^p(Q)}, \sigma^\circ)_{\mathfrak{m}_{Q,1}}$ are controlled by the crystalline deformation rings, which are defined purely in terms of representations over p-adic fields (and are p-torsion free by fiat). Nonetheless, since we believe that Conjecture 5.1.2 is reasonable, Proposition 5.1.6 gives strong evidence for Conjecture 5.1.12; similarly, [Paš15, Cor. 6.5] shows that the crystalline deformation rings can be reconstructed from P, and this alternative construction makes it more plausible that they can also control integral phenomena. We are also optimistic that the natural analogues of Conjecture 5.1.12 should continue to hold beyond the case of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Remark 5.1.14. We have avoided the notational clutter that would result from allowing non-trivial inertial types, but the natural generalisation of Proposition 5.1.6 to more general potentially crystalline (or even potentially semistable) representations can be proved in the same way. The axioms in Section 5.2 below only refer to crystalline representations; accordingly, Corollary 5.3.2 below shows that (in conjunction with our other assumptions) Conjecture 5.1.12 implies a local-global compatibility result for general potentially semistable representations. (It is perhaps also worth remarking that rather than assuming Conjecture 5.1.12, we could instead assume a variant for arbitrary potentially Barsotti–Tate representations, or indeed any variant to which we can apply the "capture" machinery of [CDP14, §2.4].)

In the rest of this section we will explain (following [CEGGPS2]) that Conjecture 5.1.12 implies Conjecture 5.1.2.

5.2. **Arithmetic actions.** We now introduce variants of the axioms of [CEGGPS2, §3.1], and prove Proposition 5.2.2, which shows that if the axioms are satisfied for $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$, then Conjecture 5.1.2 holds. We will show in Section 5.3 that (under our various hypotheses) $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ indeed satisfies these axioms.

Fix an integer $g \geq 0$ and set $\overline{R}_\infty = R_p^{\mathrm{loc}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[x_1, \dots, x_g]]$. (Of course, in our application to $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ we will take g as in Section 5.1.)

Then an $\mathcal{O}[G]$*-module with an arithmetic action of* $\overline{R}_\infty$ is by definition a non-zero $\overline{R}_\infty[G]$-module M_∞ satisfying the following axioms (AA1)–(AA4).

(AA1) M_∞ is a finitely generated $\overline{R}_\infty[[K_0]]$-module.
(AA2) M_∞ is projective in the category of pseudocompact $\mathcal{O}[[K_0]]$-modules.

Set
$$M_\infty(\sigma^\circ) := \sigma^\circ \otimes_{\mathcal{O}[[K_0]]} M_\infty.$$
This is a finitely generated $\overline{R}_\infty$-module by (AA1). For each σ°, we have a natural action of $\mathcal{H}(\sigma^\circ)$ on $M_\infty(\sigma^\circ)$, and thus of $\mathcal{H}(\sigma)$ on $M_\infty(\sigma^\circ)[1/p]$.

(AA3) For any σ, the action of $\overline{R}_\infty$ on $M_\infty(\sigma^\circ)$ factors through $\overline{R}_\infty(\sigma)$. Furthermore, $M_\infty(\sigma^\circ)$ is maximal Cohen–Macaulay over $\overline{R}_\infty(\sigma)$.
(AA4) For any σ, the action of $\mathcal{H}(\sigma)$ on $M_\infty(\sigma^\circ)[1/p]$ is given by the composite
$$\mathcal{H}(\sigma) \xrightarrow{\eta} R_p^{\mathrm{loc}}(\sigma)[1/p] \to \overline{R}_\infty(\sigma)[1/p].$$

Remark 5.2.1. Our axioms are not quite the obvious translation of the axioms of [CEGGPS2, §3.1] to our setting. Firstly, our definition of $M_\infty(\sigma^\circ)$ is different; however, by Remark 5.1.7 it is equivalent to the definition given there. More significantly, in (AA3) we do not require that $M_\infty(\sigma^\circ)[1/p]$ is locally free of rank one over its support.

Since $R_p^{\mathrm{loc}}(\sigma)[1/p]$ is equidimensional and regular (by [Kis08, Thm. 3.3.8] and [BLGHT, Lem. 3.3]), $M_\infty(\sigma^\circ)[1/p]$ is (being maximal Cohen–Macaulay by (AA3)) locally free over its support. (This is standard, but for completeness we give an argument. Write $R = R_p^{\mathrm{loc}}(\sigma)[1/p]$, $M = M_\infty(\sigma^\circ)[1/p]$, and let $\mathfrak{p} \in \mathrm{Supp}_R(M)$. By [EGAIV$_1$, Ch. 0, Cor. 16.5.10], $M_\mathfrak{p}$ is Cohen–Macaulay over $R_\mathfrak{p}$ and we have

$$\dim_R(M) = \dim_R(M/\mathfrak{p}M) + \dim_{R_\mathfrak{p}}(M_\mathfrak{p}).$$

By [EGAIV$_1$, Ch. 0, Prop. 16.5.9] we have $\dim_R(M/\mathfrak{p}M) = \dim R/\mathfrak{p}$ and since M is maximal Cohen–Macaulay over R we have $\dim_R(M) = \dim R$. Since $\dim R_\mathfrak{p} + \dim R/\mathfrak{p} \leq \dim R$ ([EGAIV$_1$, Ch. 0, 16.1.4.1]) we deduce that $\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq \dim R_\mathfrak{p}$ and therefore $\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) = \dim R_\mathfrak{p}$. So $M_\mathfrak{p}$ is maximal Cohen–Macaulay over $R_\mathfrak{p}$. Since R is regular, and maximal Cohen–Macaulay modules over regular local rings are free [Stacks, Tag 00NT], we deduce that $M[1/p]$ is locally free over $\overline{R}(\sigma)[1/p]$.)

We do not make any prescription on the rank of $M_\infty(\sigma^\circ)[1/p]$ over its support (or even require this rank to be constant), and this is reflected in the multiplicity m in Proposition 5.2.2 below.

We now follow the approach of [CEGGPS2] to show that any $\mathcal{O}[G]$-module with an arithmetic action of $\overline{R}_\infty$ is obtained from the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. The following result shows that in order to establish Conjecture 5.1.2, it is enough to show that the action of $\overline{R}_\infty[G]$ on $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is arithmetic. We will follow the proof of [CEGGPS2, Thm. 4.30] very closely, indicating what changes are necessary to go from their G (which equals $\mathrm{GL}_2(\mathbb{Q}_p)$) to our G (which is a product of copies of $\mathrm{PGL}_2(\mathbb{Q}_p)$). We also need to make some additional adjustments due to the absence of a rank one assumption in axiom (AA3).

Proposition 5.2.2. *Let M_∞ be an $\mathcal{O}[G]$-module with an arithmetic action of $\overline{R}_\infty$. Then for some integer $m \geq 1$ there is an isomorphism of $\overline{R}_\infty\,[G]$-modules*

$$M_\infty \cong \overline{R}_\infty \widehat{\otimes}_{R_p^{\mathrm{loc}}} P^{\oplus m}.$$

Proof. As we have already remarked, we will closely follow the arguments of [CEGGPS2, §4]. To orient the reader unfamiliar with [CEGGPS2], we make some brief preliminary remarks. As a consequence of the results of [Paš13; Paš15], it is not hard to show that the natural action of $\overline{R}[G]$ on $\overline{R}_\infty \widehat{\otimes}_{R_p^{\mathrm{loc}}} P^{\oplus m}$ is an arithmetic action. We show that M_∞ is a projective object of $\mathfrak{C}_G(\mathcal{O})$, and that its cosocle only contains copies of $\pi^\vee := \widehat{\otimes}_{v|p}\pi_v^\vee$. From this we can deduce the existence of an isomorphism of $\mathcal{O}[[x_1, \ldots, x_g]]$-modules of the required kind, and we need only check that it is R_p^{loc}-linear. By a density argument, we reduce to showing that the corresponding isomorphism for $M_\infty(\sigma)$ is $R_p^{\mathrm{loc}}(\sigma)$-linear (for each σ). This in turn follows from (AA4) (and the fact that $\eta : \mathcal{H}(\sigma) \to R_p^{\mathrm{loc}}(\sigma)[1/p]$ becomes an isomorphism upon passing to completions at maximal ideals, cf. [CEGGPS2, Prop. 2.13]; this is due to the uniqueness of the Hodge filtration for crystalline representations, which is a phenomenon unique to the case of $\mathrm{GL}_2(\mathbb{Q}_p)$).

We now begin the proof proper. Set $\pi^\vee := \widehat{\otimes}_{v|p}\pi_v^\vee$; by Lemma B.8, P is a projective envelope of $\pi^\vee$ in $\mathfrak{C}_G(\mathcal{O})$. The argument of [CEGGPS2, Prop. 4.2] goes through essentially unchanged, and shows that for each Serre weight $\overline{\sigma}$ with corresponding lift σ°, we have:

(1) if $M_\infty(\sigma^\circ) \neq 0$, then it is a free $\overline{R}_\infty(\sigma)$-module of some rank m. Furthermore, the action of $\mathcal{H}(\overline{\sigma})$ on $M_\infty(\overline{\sigma})$ factors through the natural map $R_p^{\mathrm{loc}}(\sigma)/\varpi \to \overline{R}_\infty(\sigma)/\varpi$, and $M_\infty(\overline{\sigma})$ is a flat $\mathcal{H}(\overline{\sigma})$-module.
(2) If $M_\infty(\sigma^\circ) \neq 0$, then there is a homomorphism $\mathcal{H}(\overline{\sigma}) \to k$ such that $\pi \cong \text{c-Ind}_{K_0}^G \overline{\sigma} \otimes_{\mathcal{H}(\overline{\sigma})} k$. Accordingly, $\mathrm{Hom}_G(\pi, M_\infty^\vee)^\vee \cong M_\infty(\overline{\sigma}) \otimes_{\mathcal{H}(\overline{\sigma})} k$.
(3) If π' is an irreducible smooth k-representation of G then $\mathrm{Hom}_G(\pi', M_\infty^\vee) \neq 0$ if and only if $\pi' \cong \pi$.

Since $\mathcal{H}(\overline{\sigma}) = \otimes_{v|p}\mathcal{H}(\overline{\sigma}_v) \cong k[T_v]_{v|p}$, the proofs of [CEGGPS2, Lem. 4.10, Lem. 4.11, Thm. 4.15] go through with only notational changes, so that M_∞ is a projective object of $\mathfrak{C}_G(\mathcal{O})$.

Write $A = \mathcal{O}[[x_1, \ldots, x_g]]$, and choose a homomorphism $A \to \overline{R}_\infty$ inducing an isomorphism $R_p^{\mathrm{loc}}\widehat{\otimes}_{\mathcal{O}} A \cong \overline{R}_\infty$. We claim that there is an isomorphism in $\mathfrak{C}_G(A)$

$$M_\infty \cong A\widehat{\otimes}_{\mathcal{O}} P^{\oplus m}. \tag{5.2.3}$$

By (3) above, all of the irreducible subquotients of $\mathrm{cosoc}_{\mathfrak{C}_G(\mathcal{O})} M_\infty$ are isomorphic to $\pi^\vee$, so by [CEGGPS2, Prop. 4.19, Rem. 4.21] it is enough to show that $\mathrm{Hom}_G(\pi, M_\infty^\vee)^\vee$ is a free A/ϖ-module of rank m. To see this, note that by (2) above we have $\mathrm{Hom}_G(\pi, M_\infty^\vee)^\vee \cong M_\infty(\overline{\sigma}) \otimes_{\mathcal{H}(\overline{\sigma})} k$, which by (1) is a free $\overline{R}_\infty(\sigma) \otimes_{\mathcal{H}(\overline{\sigma})} k$-module of rank m. By (1) again (together with [CEGGPS2, Lem. 2.14]), the map $A \to \overline{R}_\infty$ induces an isomorphism $A/\varpi \cong \overline{R}_\infty(\sigma) \otimes_{\mathcal{H}(\overline{\sigma})} k$, as required.

It remains to show that (5.2.3) is R_p^{loc}-linear. We claim that the action of $\overline{R}_\infty$ on $A\widehat{\otimes}_{\mathcal{O}} P^{\oplus m}$ is arithmetic; admitting this claim, the proofs of [CEGGPS2, Thm. 4.30, 4.32] go over with only minor notational changes to show the required R_p^{loc}-linearity.

It is obviously enough to show that the action of R_p^{loc} on P is an arithmetic action (with $g = 0$). (AA1) holds by the topological version of Nakayama's lemma (since $\widehat{\otimes}_{v|p}\kappa_v^\vee$ is a finitely generated $k[[K_0]]$-module), while (AA2) holds by [Paš15, Cor. 5.3]. (AA3) holds by [Paš15, Cor. 6.4, 6.5], while (AA4) follows from the main result of [Paš13] exactly as in the proof of [CEGGPS2, Prop. 6.17]. □

5.3. **Local-global compatibility.** We now discuss the axioms (AA1)–(AA4) in the case $M_\infty = H_{q_0}(\widetilde{\mathcal{C}}(\infty))$.

Proposition 5.3.1. *Assume* (*in addition to our running assumptions*) *Conjecture 5.1.12. Then the action of* $\overline{R}_\infty[G]$ *on* $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ *is arithmetic.*

Proof. Certainly $H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is finitely generated over $R_\infty[[K_0]]$, by Proposition 3.4.16 (2) and Remark 3.4.17, so axiom (AA1) holds.

Next we show that the $\bar{R}_\infty$ action on $H_i(\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty))$ factors through $\bar{R}_\infty(\sigma)$ for all i. Indeed, by 3.4.14, we have natural isomorphisms

$$H_i(\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty)) \cong \varprojlim_{U_p, J} H_i(\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \mathcal{C}(U_p, J, \infty))$$

where the inverse limit is taken over pairs (J, U_p) such that U_p acts trivially on $\sigma^\circ \otimes_{\mathcal{O}} \mathcal{O}_\infty/J$. Each homology group $H_i(\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \mathcal{C}(U_p, J, \infty))$ can be obtained by applying the ultraproduct construction to the groups $H_i(\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \mathcal{C}(U_p, J, N))$, and it follows from Conjecture 5.1.12 that the action of $\bar{R}_\infty$ on all these groups factors through $\bar{R}_\infty(\sigma)$. It follows in the same way from Conjecture 5.1.12 that if $h \in \mathcal{H}(\sigma^\circ)$ is such that $\eta(h) \in R_p^{\mathrm{loc}}(\sigma)$, then h acts on $H_{q_0}(\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty))$ via $\eta(h)$, so axiom (AA4) holds.

We can now apply Lemma A.10 (or [CG12, Lem. 6.2]) to the complex of $\mathcal{O}_\infty$-modules $\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty)$ (more precisely, we replace $\widetilde{\mathcal{C}}(\infty)$ by a quasi-isomorphic complex of finite projective modules in degrees $[q_0, q_0 + l_0]$, which we can do by Proposition 4.2.1(2)). As in the proof of [CG12, Thm. 6.3], since the action of $\mathcal{O}_\infty$ on $H_*(\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty))$ factors through $\overline{R}_\infty(\sigma)$, and $\dim \overline{R}_\infty(\sigma) = \dim \mathcal{O}_\infty - l_0$, we have $j_{\mathcal{O}_\infty}(H_*(\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty))) \geq l_0$. We deduce that the complex $\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty)$ has non-zero homology only in degree q_0, and that $H_{q_0}(\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty)) = \sigma^\circ \otimes_{\mathcal{O}[[K_0]]} H_{q_0}(\widetilde{\mathcal{C}}(\infty))$ is maximal Cohen–Macaulay over $\bar{R}_\infty(\sigma)$. We have now established that axiom (AA3) holds.

Finally, it remains to check (AA2). By [Bru66, Prop. 3.1], it is enough to show that for each Serre weight $\overline{\sigma}$ we have $\mathrm{Tor}_1^{\mathcal{O}[[K_0]]}(\overline{\sigma}, H_{q_0}(\widetilde{\mathcal{C}}(\infty))) = 0$. Once again, we apply Lemma A.10 (or [CG12, Lem. 6.2]) — this time to the complex of $\mathcal{O}_\infty/\varpi$-modules $\overline{\sigma} \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty)$. We see that it suffices to prove that $j_{\mathcal{O}_\infty/\varpi}(H_*(\overline{\sigma} \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty))) \geq l_0$. We let σ° be the lift of $\overline{\sigma}$. From what we have already shown about the complex $\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty)$ we deduce that we have

$$H_{q_0}(\overline{\sigma} \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty)) = \mathcal{O}/\varpi \otimes_{\mathcal{O}} H_{q_0}(\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty))$$

and

$$H_{q_0+1}(\overline{\sigma} \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty)) = \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{O}/\varpi, H_{q_0}(\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty)))$$

with all other homology groups vanishing.

The action of $\mathcal{O}_\infty/\varpi$ on these two groups factors through $\overline{R}_\infty(\sigma)/\varpi$, since the action on $H_{q_0}(\sigma^\circ \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty))$ factors through $\overline{R}_\infty(\sigma)$, and $\dim \overline{R}_\infty(\sigma)/\varpi = \dim \mathcal{O}_\infty/\varpi - l_0$, so we deduce the desired inequality for $j_{\mathcal{O}_\infty/\varpi}(H_*(\overline{\sigma} \otimes_{\mathcal{O}[[K_0]]} \widetilde{\mathcal{C}}(\infty)))$. □

Corollary 5.3.2. *Assume (in addition to our running assumptions) Conjecture 5.1.12. Then Conjecture 5.1.2 holds. In particular, we obtain as consequences the 'big $R = \mathbb{T}$' result of Proposition 5.1.4 and the automorphy lifting result of Corollary 5.1.8.*

Proof. This is immediate from Propositions 5.2.2 and 5.3.1. □

5.4. **The totally real and imaginary quadratic cases.** We conclude by discussing the cases in which unconditional results seem most in reach. If F is totally real, then $l_0 = 0$, and the existence of Galois representations is known; the only assumption that is not established is assumption (a) of Proposition 4.2.1, that the homology groups $H_i(X_{U^pK_1}, k)_{\mathfrak{m}}$ vanish for $i \neq q_0$. It might be hoped that a generalisation of the results of [CS15] to non-compact Shimura varieties could establish this. Of course the totally real cases where $l_0 = 0$ are less interesting from the point of view of this paper, as they could already have been studied using the methods of [CEGGPS].

If F is imaginary quadratic, then the biggest obstacle to unconditional results is Conjecture 5.1.12; indeed, as explained in Remarks 3.3.4 and 3.3.8, the other hypotheses on the Galois representations seem to be close to being known, and as explained in Remark 4.2.2, assumptions (a) and (b) of Proposition 4.2.1 are known in this case.

Appendix A. Non-commutative algebra

In this section we make some definitions and establish some results for non-commutative Iwasawa algebras which generalise standard facts about complete regular local rings. Section A.1 contains the basic definitions which will be needed for discussing our results on patching completed homology.

A.1. Depth and dimension.

Definition A.2. *Let A be a ring and let M be a left or right A-module. We denote the* projective dimension *of M over A by* $\mathrm{pd}_A(M)$*. We define the* grade $j_A(M)$ *of M over A by*

$$j_A(M) = \inf\{i : \mathrm{Ext}^i_A(M, A) \neq 0\}.$$

If A is local with maximal ideal $\mathfrak{m}_A$, then we define the depth $\mathrm{depth}_A(M)$ *of M by*

$$\mathrm{depth}_A(M) = \inf\{i : \mathrm{Ext}^i_A(A/\mathfrak{m}_A, M) \neq 0\}.$$

A Noetherian ring A is called Auslander–Gorenstein *if it has finite left and right injective dimension and if for any finitely generated left or right A-module M, any integer m, and any submodule $N \subset \mathrm{Ext}^m_A(M, A)$, we have $j_A(N) \geq m$.*

An Auslander–Gorenstein ring is called Auslander regular *if it has finite global dimension.*

Finally, let A be an Auslander regular ring and let M be a finitely generated left A-module. We define the dimension $\delta_A(M)$ *of M over A by*

$$\delta_A(M) = \mathrm{gld}(A) - j_A(M),$$

where $\mathrm{gld}(A)$ *is the global dimension of A.*

Let K be a compact p-adic analytic group. We are going to apply the above definitions for $A = \mathcal{O}[[K]]$, the Iwasawa algebra of K with coefficients in $\mathcal{O}$. Note that taking inverses of group elements induces an isomorphism between $\mathcal{O}[[K]]$ and its opposite ring, so there is an equivalence between the categories of left and right $\mathcal{O}[[K]]$-modules.

$\mathcal{O}[[K]]$ is Noetherian, and when K is moreover a pro-p group, $\mathcal{O}[[K]]$ is a local ring with $\mathcal{O}[[K]]/\mathfrak{m}_{\mathcal{O}[[K]]} = k$.

Remark A.3. If M is an $\mathcal{O}[[K]]$-module, then $j_{\mathcal{O}[[K]]}(M)$ is sometimes referred to as the *codimension* of M (cf. [CE12, §1.2]).

When K is pro-p and torsion-free, Venjakob [Ven02] has established that $\mathcal{O}[[K]]$ has nice homological properties, which are summarised in the next proposition.

Proposition A.4 (Venjakob)**.** *Let K be compact p-adic analytic group which is torsion free and pro-p. Let $\Lambda = \mathcal{O}[[K]]$ and let M be a finitely generated Λ-module.*

1. *Λ is Auslander regular with global dimension* $\mathrm{gld}(\Lambda)$ *and depth* $\mathrm{depth}_\Lambda(\Lambda)$ *both equal to* $1 + \dim(K)$*.*
2. *The Auslander–Buchsbaum equality holds for M:*

$$\mathrm{pd}_\Lambda(M) + \mathrm{depth}_\Lambda(M) = \mathrm{depth}_\Lambda(\Lambda) = 1 + \dim(K).$$

(3) *We have*

$$\mathrm{pd}_\Lambda(M) = \max\{i : \mathrm{Ext}^i_\Lambda(M, \Lambda) \neq 0\}.$$

In particular, we have $\mathrm{pd}_\Lambda(M) \geq j_\Lambda(M)$*.*

Proof. All these statements are contained in [Ven02]. For the first part of the proposition, Auslander regularity is [Ven02, Theorem 3.26]. The depth of Λ is equal to its global dimension by [Ven02, Lemma 5.5 (iii)]. The computation of the global dimension of Λ follows from results of Brumer [Bru66, Theorem 4.1], Lazard [Laz65, Théorème V.2.2.8] and Serre [Ser65].

The Auslander–Buchsbaum equality is [Ven02, Theorem 6.2]. Finally, the formula for $\mathrm{pd}_\Lambda(M)$ is [Ven02, Corollary 6.3]. □

Definition A.5. *If K is a compact p-adic analytic group, then a finitely generated $\mathcal{O}[[K]]$-module M is* Cohen–Macaulay *if* $\mathrm{Ext}^i_{\mathcal{O}[[K]]}(M, \mathcal{O}[[K]])$ *is non-zero for just one degree i.*

Remark A.6. If K is furthermore torsion-free and pro-p, then by Proposition A.4, a finitely generated $\mathcal{O}[[K]]$-module M is Cohen–Macaulay if and only if $\mathrm{depth}_{\mathcal{O}[[K]]}(M) = \delta_{\mathcal{O}[[K]]}(M)$.

If K is an arbitrary compact p-adic analytic group, then $\mathcal{O}[[K]]$ is not necessarily local (although it is semilocal), and is not necessarily Auslander regular. But the notions of grade and projective dimension are still well-behaved, because we can apply the following lemma with H a normal compact open subgroup of K which is torsion free and pro-p.

Lemma A.7. *Suppose K is a compact p-adic analytic group and let $H \subset K$ be a normal compact open subgroup. Let M be a $\mathcal{O}[[K]]$-module.*

- *For all $i \geq 0$ we have an isomorphism of $\mathcal{O}[[H]]$-modules*

$$\mathrm{Ext}^i_{\mathcal{O}[[H]]}(M, \mathcal{O}[[H]]) \cong \mathrm{Ext}^i_{\mathcal{O}[[K]]}(M, \mathcal{O}[[K]]).$$

  In particular, we have $j_{\mathcal{O}[[K]]}(M) = j_{\mathcal{O}[[H]]}(M)$*.*
- *Suppose M is finitely generated and of finite projective dimension over $\mathcal{O}[[K]]$. Suppose that H is torsion free and pro-p. Then*

$$\mathrm{pd}_{\mathcal{O}[[K]]}(M) = \mathrm{pd}_{\mathcal{O}[[H]]}(M).$$

Proof. The first item follows from [AB07, Lemma 5.4]. The second item is a combination of the first with the fact that we have

$$\mathrm{pd}_\Lambda(M) = \max\{i : \mathrm{Ext}^i_\Lambda(M, \Lambda) \neq 0\}$$

for $\Lambda = \mathcal{O}[[H]]$ by Proposition A.4 and we also have the same equality for $\Lambda = \mathcal{O}[[K]]$ by [Ven02, Remark 6.4]. □

From now on in this subsection we fix a compact p-adic analytic group K and assume that K is torsion free and pro-p. We let $\Lambda = \mathcal{O}[[K]]$, and let $d = 1 + \dim(K)$, so d is the global dimension of Λ.

We use the following fundamental fact (again due to Venjakob) in this section:

Lemma A.8. *If we have a short exact sequence of finitely generated Λ-modules $0 \to L \to M \to N \to 0$ then* $j_\Lambda(M) = \min(j_\Lambda(L), j_\Lambda(N))$*.*

Proof. This is [Ven02, Proposition 3.6]. □

The next two lemmas are generalisations of [CG12, Lemmas 6.1, 6.2]:

Lemma A.9. *If N is a finitely generated Λ-module with projective dimension j, and $0 \neq M \subseteq N$, then $j_\Lambda(M) \leq j$.*

Proof. Since Λ is Auslander regular, this follows immediately from [Ven02, Proposition 3.10]. □

Lemma A.10. *Suppose l_0 is an integer with $0 \leq l_0 \leq d$. Let $P_\bullet$ be a chain complex of finite free Λ-modules, concentrated in degrees $0, \dots, l_0$. Assume that $H_*(P_\bullet) \neq 0$. Then $j_\Lambda(H_*(P_\bullet)) \leq l_0$ and if equality occurs then:*

1. *$P_\bullet$ is a projective resolution of $H_0(P_\bullet)$.*
2. *We have $\mathrm{pd}_\Lambda(H_0(P_\bullet)) = j_\Lambda(H_0(P_\bullet)) = l_0$.*

We have the same statements if we replace Λ with $\Omega := \Lambda/\varpi = k[[K]]$.

Proof. Let $m \geq 0$ be the largest integer such that $H_m(P_\bullet) \neq 0$. Consider the complex

$$P_{l_0} \to \cdots \to P_{m+1} \overset{d_{m+1}}{\to} P_m.$$

By the definition of m, this complex is a projective resolution of $K_m := P_m/\operatorname{im}(d_{m+1})$. It follows that $\mathrm{pd}_\Lambda(K_m) \leq l_0 - m$.

Since $H_m(P_\bullet)$ is a non-trivial submodule of K_m, by Lemmas A.8 and A.9 we have

$$j_\Lambda(H_*(P_\bullet)) \leq j_\Lambda(H_m(P_\bullet)) \leq \mathrm{pd}_\Lambda(K_m) \leq l_0 - m \leq l_0,$$

as claimed.

If we have the equality $j_\Lambda(H_*(P_\bullet)) = l_0$, then equality holds in all the above inequalities, so that in particular $m = 0$, $K_m = H_0(P_\bullet)$, and the other claims follow immediately.

The proof with Λ replaced by Ω is identical, using the fact that the relevant lemmas all hold with Λ replaced by Ω (which is again Auslander regular). □

We finish this subsection with a Lemma computing the codimension of a tensor product of two modules.

Lemma A.11. *Let G, H be compact p-adic analytic groups. Let M, N be finitely generated $k[[G]]$- and $k[[H]]$-modules. Then $j_{k[[G\times H]]}(M\widehat{\otimes}_k N) = j_{k[[G]]}(M) + j_{k[[H]]}(N)$.*

Proof. By Lemma A.7 we can assume that G and H are torsion free pro-p.

Set $\Omega = k[[G \times H]]$, $\Omega_1 = k[[G]]$ and $\Omega_2 = k[[H]]$. Let $P_\bullet \to M$ and $Q_\bullet \to N$ be finite free resolutions of M and N respectively (they exist since Ω_1 and Ω_2 have finite global dimension).

The complex $P_\bullet \widehat{\otimes}_k Q_\bullet$ is a finite free resolution of $M\widehat{\otimes}_k N$. We have natural isomorphisms

$$\mathrm{Hom}_\Omega(P_\bullet \widehat{\otimes}_k Q_\bullet, \Omega) = \mathrm{Hom}_{\Omega_1}(P_\bullet, \Omega_1) \widehat{\otimes}_k \mathrm{Hom}_{\Omega_2}(Q_\bullet, \Omega_2).$$

The equality $j_{k[[G\times H]]}(M\widehat{\otimes}_k N) = j_{k[[G]]}(M) + j_{k[[H]]}(N)$ follows immediately. Indeed, we have a spectral sequence

$$\mathrm{Ext}^i_{\Omega_1}(M, \Omega_1) \widehat{\otimes}_k \mathrm{Ext}^j_{\Omega_2}(N, \Omega_2) \Rightarrow \mathrm{Ext}^{i+j}_\Omega(M\widehat{\otimes}_k N, \Omega).$$ □

A.12. **Gelfand–Kirillov dimension.** In this section we assume that K is a compact p-adic analytic group which is uniform pro-p. (Note that any compact p-adic analytic group contains a normal open subgroup which is uniform pro-p, so this will not be a problematic assumption in our applications.) We again let $\Lambda = \mathcal{O}[[K]]$, and set $d = 1 + \dim(K)$. We let $\Omega = \Lambda/\varpi\Lambda$. We denote by J_Ω the Jacobson radical of Ω. The ring Ω is again Auslander regular, and for finitely generated Ω modules the dimension δ_Ω (or equivalently the grade j_Ω) can be computed as a Gelfand–Kirillov dimension:

Proposition A.13. *Let M be a finitely generated Ω-module. We have*

$$\delta_\Omega(M) = \limsup \log_n \dim_k M/J_\Omega^n M.$$

Proof. This is [AB06, Prop. 5.4 (3)]. □

A.14. **Comparing dimensions.** We again assume that K is uniform pro-p and let $\Lambda = \mathcal{O}[[K]]$. Fix a topological generating set $a_1, \ldots, a_m$ for K. We consider two more Auslander regular rings $A = \Lambda\hat{\otimes}_{\mathcal{O}}\mathcal{O}[[x_1, \ldots, x_r]]$ and $B = \Lambda\hat{\otimes}_{\mathcal{O}}\mathcal{O}[[y_1, \ldots, y_s]]$ together with a map $A \to B$ induced from a (local $\mathcal{O}$-algebra) map $\mathcal{O}[[x_1, \ldots, x_r]] \to \mathcal{O}[[y_1, \ldots, y_s]]$.

Note that we can think of A and B as the Iwasawa algebras $\Lambda_{\mathcal{O}}[[K \times \mathbb{Z}_p^r]]$ for appropriate r, and $K \times \mathbb{Z}_p^r$ is uniform pro-p, so we can apply the results of the previous subsections to A and B.

We set $\overline{A} = A/\varpi A$ and $\overline{B} = B/\varpi B$. The goal of this subsection is Lemma A.19 which shows that if M is a finitely generated B-module which is also finitely generated as an A-module, then $\delta_A(M) = \delta_B(M)$. This generalises a well-known fact in commutative algebra [EGAIV$_1$, Ch. 0, Prop. 16.1.9].

Lemma A.15. *Suppose M is a finitely generated A-module, and let x be one of $\varpi, x_1, \ldots, x_r$. Then*

- *if M is killed by x, $\delta_A(M) = \delta_{A/x}(M)$.*
- *if M is x-torsion free, $\delta_A(M) = 1 + \delta_{A/x}(M/xM)$.*

Proof. First we assume that M is killed by x. The base change spectral sequence [Wei94, Ex. 5.6.3] for Ext is

$$E_2^{i,j} : \mathrm{Ext}^i_{A/x}(M, \mathrm{Ext}^j_A(A/x, A)) \implies \mathrm{Ext}^{i+j}_A(M, A)$$

and $\mathrm{Ext}^j_A(A/x, A)$ is zero unless $j = 1$, when we have $\mathrm{Ext}^1_A(A/x, A) = A/x$. Since M is killed by x, $j_A(M) > 0$, and we have

$$\mathrm{Ext}^i_{A/x}(M, A/x) = \mathrm{Ext}^{i+1}_A(M, A)$$

for $i \geq 0$. We deduce that $j_A(M) = 1 + j_{A/x}(M)$, and therefore $\delta_A(M) = \delta_{A/x}(M)$.

Now we assume that M is x-torsion free. [Lev92, Thm. 4.3] implies that $j_A(M/xM) \geq 1 + j_A(M)$, so we have an exact sequence

$$0 \to \mathrm{Ext}^{j_A(M)}_A(M, A) \xrightarrow{\times x} \mathrm{Ext}^{j_A(M)}_A(M, A) \to \mathrm{Ext}^{1+j_A(M)}_A(M/xM, A)$$

and $\mathrm{Ext}^{j_A(M)}_A(M, A)$ is a non-zero finitely generated A-module. By Nakayama's lemma we see that $\mathrm{Ext}^{j_A(M)}_A(M, A)/x\,\mathrm{Ext}^{j_A(M)}_A(M, A)$ is non-zero, and so $\mathrm{Ext}^{1+j_A(M)}_A(M/xM)$ is also non-zero. This implies that $j_A(M/xM) = 1 + j_A(M)$. The first part of the lemma then gives $j_{A/x}(M/xM) = j_A(M)$ and so $\delta_A(M) = 1 + \delta_{A/x}(M/xM)$. □

Lemma A.16. *Suppose M is a finitely generated A-module and let x be one of $\varpi, x_1, \dots, x_r$. Then*
$$j_A(M) \geq j_{A/x}(M/xM).$$
In particular,
$$j_A(M) \geq j_\Lambda(M/(x_1, \dots, x_r)M)$$
and
$$j_A(M) \geq j_\Omega(M/(\varpi, x_1, \dots, x_r)M).$$

Proof. The 'in particular' part of the lemma follows from the first part by induction.

Applying Lemma A.15, we see that if M is x-torsionfree, then $j_A(M) = j_{A/x}(M/xM)$. In general, we have an exact sequence
$$0 \to M[x^\infty] \to M \to M/M[x^\infty] \to 0$$
where $M/M[x^\infty]$ is x-torsionfree, so we have a short exact sequence
$$0 \to A/x \otimes_A M[x^\infty] \to A/x \otimes_A M \to A/x \otimes_A (M/M[x^\infty]) \to 0.$$
By Lemma A.8, it now suffices to show that if M is killed by x^N for some $N \geq 1$, then $j_A(M) \geq j_{A/x}(M/xM)$. Consider the filtration $\{0\} = x^N M \subset x^{N-1}M \subset \cdots \subset xM \subset M$. We have $j_A(M) = \min_i(j_A(x^iM/x^{i+1}M))$ by a repeated application of Lemma A.8 and we therefore have $j_A(M) = 1 + \min_i(j_{A/x}(x^iM/x^{i+1}M))$ by another application of Lemma A.15. Multiplication by x^i gives a surjective A-linear map $M/xM \to x^iM/x^{i+1}M$, so $j_{A/x}(M/xM) \leq j_{A/x}(x^iM/x^{i+1}M)$ for all i (by Lemma A.8 again). In particular we have $j_A(M) = 1 + j_{A/x}(M/xM)$ which gives the desired conclusion. □

Lemma A.17. *We have $J_{\overline{A}}\overline{B} = \overline{B}J_{\overline{A}}$ and $J_{\overline{A}}J_{\overline{B}} = J_{\overline{B}}J_{\overline{A}}$.*

Proof. $J_{\overline{A}}$ is the (right, left, two-sided) ideal of $\overline{A}$ generated by $a_1 - 1, \dots, a_m - 1, x_1, \dots, x_r$ and $J_{\overline{B}}$ is the (right, left, two-sided) ideal of $\overline{B}$ generated by $a_1 - 1, \dots, a_m - 1, y_1, \dots, y_s$. The lemma is now easy, since the x_i map to central elements in $\overline{B}$. □

The next lemma is a mild variation on [Wad07, Lemma 3.1].

Lemma A.18. *Suppose M is a finitely generated $\overline{B}$-module, which is also finitely generated as an $\overline{A}$-module. Then $\delta_{\overline{A}}(M) = \delta_{\overline{B}}(M)$.*

Proof. We show the lemma by comparing Gelfand–Kirillov dimensions. Since M is a finitely generated $\overline{A}$-module, $M/J_{\overline{A}}M$ is a finite dimensional k-vector space. By Lemma A.17, $J_{\overline{A}}M$ is a $\overline{B}$-submodule of M. So $M/J_{\overline{A}}M$ is an Artinian $\overline{B}$-module. Therefore $J_{\overline{B}}^k(M/J_{\overline{A}}M) = 0$ for some positive integer k. So $J_{\overline{B}}^kM \subset J_{\overline{A}}M \subset J_{\overline{B}}M$. Using the fact that $J_{\overline{A}}J_{\overline{B}} = J_{\overline{B}}J_{\overline{A}}$ (Lemma A.17) an induction shows that
$$J_{\overline{B}}^{kN}M \subset J_{\overline{A}}^NM \subset J_{\overline{B}}^NM$$
for all $N \geq 1$. Using Proposition A.13 we conclude that $\delta_{\overline{A}}(M) = \delta_{\overline{B}}(M)$. □

Lemma A.19. *Suppose M is a finitely generated B-module, which is also finitely generated as an A-module. Then $\delta_A(M) = \delta_B(M)$.*

Proof. M has a finite filtration by B-submodules $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_l = M$ such that each M_i/M_{i-1} is either ϖ-torsionfree or killed by ϖ. Each M_i is also a finitely generated A-module. By Lemma A.8, we have $\delta_A(M) = \max_i(\delta_A(M_i/M_{i-1}))$ and $\delta_B(M) = \max_i(\delta_B(M_i/M_{i-1}))$, so we may assume that M is either ϖ-torsionfree or killed by ϖ. Applying Lemma A.15 and Lemma A.18 gives $\delta_A(M) = \delta_B(M)$. □

A.20. **Comparing depths.** We retain the assumptions and notation of the previous subsection. Recall that we have two Λ-algebras $A = \Lambda\hat{\otimes}_{\mathcal{O}}\mathcal{O}[[x_1, \ldots, x_r]]$ and $B = \Lambda\hat{\otimes}_{\mathcal{O}}\mathcal{O}[[y_1, \ldots, y_s]]$. The goal of this subsection is Lemma A.28, which shows that if M is a finitely generated B-module which is also finitely generated as an A-module, then $\mathrm{depth}_A(M) \le \mathrm{depth}_B(M)$. In fact, we can show that $\mathrm{depth}_A(M) = \mathrm{depth}_B(M)$ (which again generalises a well-known result in commutative algebra [EGAIV$_1$, Ch. 0, Prop. 16.4.8] but proving the inequality suffices for our applications and is already sufficiently painful.

We set $\overline{R} = k[[x_1, \ldots, x_r]] = A/J_\Lambda A$ and $\overline{S} = k[[y_1, \ldots, y_s]] = B/J_\Lambda B$. We have a map of local k-algebras $\overline{R} \to \overline{S}$.

Lemma A.21. *Suppose I is an injective left B-module. Then I is injective as a left Λ-module.*

Proof. Suppose $0 \to L \to M \to N \to 0$ is a short exact sequence of left Λ-modules. Since B is a flat right Λ-module, we have an exact sequence of left B-modules

$$0 \to B \otimes_\Lambda L \to B \otimes_\Lambda M \to B \otimes_\Lambda N \to 0$$

and hence an exact sequence

$$0 \to \mathrm{Hom}_B(B \otimes_\Lambda N, I) \to \mathrm{Hom}_B(B \otimes_\Lambda M, I) \to \mathrm{Hom}_B(B \otimes_\Lambda L, I) \to 0.$$

Finally, the tensor-hom adjunction implies that

$$0 \to \mathrm{Hom}_\Lambda(N, I) \to \mathrm{Hom}_\Lambda(M, I) \to \mathrm{Hom}_\Lambda(L, I) \to 0$$

is exact. □

For any left B-module M, note that $\mathrm{Hom}_\Lambda(k, M) = \{m \in M : J_\Lambda m = 0\}$ is naturally a left $\overline{S}$-module. We denote by $\mathrm{RHom}_\Lambda^{\overline{S}}(k, M)$ the object of $D^+(\overline{S})$ given by taking an injective A-module resolution of M and applying $\mathrm{Hom}_\Lambda(k, -)$ to get a complex of $\overline{S}$-modules. By Lemma A.21, we have natural isomorphisms of Abelian groups $H^i(\mathrm{RHom}_\Lambda^{\overline{S}}(k, M)) = \mathrm{Ext}_\Lambda^i(k, M)$.

Remark A.22. Note that the natural $\overline{S}$-module structure on $\mathrm{Ext}_\Lambda^i(k, M)$ can also be defined using the facts that $\mathrm{Ext}_\Lambda^i(k, M) = \mathrm{Ext}_B^i(B \otimes_\Lambda k, M)$ (extension of scalars) and that $B \otimes_\Lambda k$ is a $(B, \overline{S})$-bimodule.

Remark A.23. For an A-module M, we can similarly define $\mathrm{RHom}_\Lambda^{\overline{R}}(k, M)$.

Lemma A.24. *For a B-module M, there is a natural isomorphism*

$$\mathrm{RHom}_\Lambda^{\overline{R}}(k, M) = \iota_{\overline{R}}^{\overline{S}}\mathrm{RHom}_\Lambda^{\overline{S}}(k, M),$$

where $\iota_{\overline{R}}^{\overline{S}}$ is the derived functor of the (exact) forgetful functor from $\overline{S}$-modules to $\overline{R}$-modules.

Proof. We can compute $\mathrm{RHom}_\Lambda^{\overline{R}}(k,M)$ using an injective B-module resolution of M, since an injective B-module is acyclic for the functor $\mathrm{Hom}_\Lambda(k,-)$ from A-modules to $\overline{R}$-modules. Computing $\mathrm{RHom}_\Lambda^{\overline{S}}(k,M)$ using the same injective resolution gives the desired isomorphism. □

Lemma A.25. *For B-modules M, we have natural isomorphisms*

$$\mathrm{RHom}_B(k,M)=\mathrm{RHom}_{\overline{S}}(k,\mathrm{RHom}_\Lambda^{\overline{S}}(k,M))$$

and

$$\mathrm{RHom}_A(k,M)=\mathrm{RHom}_{\overline{R}}(k,\iota_{\overline{R}}^{\overline{S}}\mathrm{RHom}_\Lambda^{\overline{S}}(k,M)).$$

Proof. Consider the functor $\mathrm{Hom}_\Lambda(k,-)$ from B-modules to $\overline{S}$-modules. This takes injectives to injectives, since for an $\overline{S}$-module X we have $\mathrm{Hom}_{\overline{S}}(X,\mathrm{Hom}_\Lambda(k,M))=\mathrm{Hom}_B(X,M)$.

The functor $\mathrm{Hom}_{\overline{S}}(k,\mathrm{Hom}_\Lambda(k,-))$ from B-modules to Abelian groups is naturally equivalent to the functor $\mathrm{Hom}_B(k,M)$. The derived functor of the composition of functors is given by $\mathrm{RHom}_{\overline{S}}(k,\mathrm{RHom}_\Lambda^{\overline{S}}(k,-))$, and this gives the first collection of natural isomorphisms.

Applying the same argument to A-modules, together with Lemma A.24, we get the second collection of natural isomorphisms. □

At this point we recall that for a commutative Noetherian local ring X with maximal ideal $\mathfrak{m}_X$ there is a good notion of depth for objects in $D^+(X)$ [Iye99][1].

Definition A.26. *For $M\in D^+(X)$ we define*

$$\mathrm{depth}_X(M)=\inf\{i:\mathrm{Ext}^i_X(X/\mathfrak{m}_X,M)\neq 0\}.$$

Lemma A.27. *Let $M\in D^+(\overline{S})$. We have*

$$\mathrm{depth}_{\overline{R}}(\iota_{\overline{R}}^{\overline{S}}M)\le \mathrm{depth}_{\overline{S}}(M).$$

Proof. Combine [Iye99, Thm. 6.1] (which shows that our definition of depth coincides with the definition given in [Iye99, §2]) with [Iye99, Prop. 5.2 (2)]. □

Lemma A.28. *Let M be a B-module. We have*

$$\mathrm{depth}_A(M)\le\mathrm{depth}_B(M).$$

Proof. By Lemma A.25 and Lemma A.27 we have

$$\begin{aligned}\mathrm{depth}_A(M)&=\mathrm{depth}_{\overline{R}}(\iota_{\overline{R}}^{\overline{S}}\mathrm{RHom}_\Lambda^{\overline{S}}(k,M))\\&\le\mathrm{depth}_{\overline{S}}(\mathrm{RHom}_\Lambda^{\overline{S}}(k,M))=\mathrm{depth}_B(M).\end{aligned}$$ □

Corollary A.29. *Suppose M is a finitely generated B-module, which is also finitely generated as an A-module. Moreover, suppose that M is a Cohen–Macaulay A-module. Then M is a Cohen–Macaulay B-module, with* $\mathrm{depth}_B(M)=\delta_B(M)=\delta_A(M)$.

Proof. By Lemma A.28 we have $\delta_A(M)=\mathrm{depth}_A(M)\le\mathrm{depth}_B(M)$. We also have $\mathrm{depth}_B(M)\le\delta_B(M)$, by parts (2) and (3) of Proposition A.4 (or by local duality). Since $\delta_A(M)=\delta_B(M)$ (by Lemma A.19), all these inequalities are equalities. □

[1] In fact one needn't restrict to bounded complexes, see [FI03]

Proposition A.30 (Miracle Flatness). *Let M be a finitely generated Cohen–Macaulay A-module.*

Then M is a flat $\mathcal{O}[[x_1,\ldots,x_r]]$-module if and only if

$$j_A(M) = j_\Omega(M/(\varpi, x_1, \ldots, x_r)M).$$

Proof. We let $R = \mathcal{O}[[x_1,\ldots,x_r]]$ and $\mathfrak{m}_R = (\varpi, x_1, \ldots, x_r) \subset R$. First suppose M is a flat $\mathcal{O}[[x_1,\ldots,x_r]]$-module. Then $(\varpi, x_1, \ldots, x_r)$ is an M-regular sequence (using Nakayama's lemma for finitely generated A-modules to see that $M/(\varpi, x_1, \ldots, x_r) \neq 0$). It follows from Lemma A.15 that we have the desired equality of codimensions.

Conversely, suppose that $j_A(M) = j_\Omega(M/(\varpi, x_1, \ldots, x_r)M)$. We claim that $(\varpi, x_1, \ldots, x_r)$ is an M-regular sequence. To prove the claim, it suffices (by induction on r) to show that for $x \in \{\varpi, x_1, \ldots, x_r\}$ we have the following

(1) $j_A(M) = j_{A/x}(M/xM)$.
(2) x is M-regular.
(3) M/xM is a Cohen–Macaulay A/x-module.

By Lemma A.16, we have $j_A(M) \geq j_{A/x}(M/xM) \geq j_\Omega(M/\mathfrak{m}_R M)$, so our assumption implies that (1) holds.

Next we check that x is M-regular. As in the proof of Lemma A.16, we have a short exact sequence

$$0 \to M[x^\infty] \to M \to M/M[x^\infty] \to 0$$

where $M/M[x^\infty]$ is x-torsion free. Suppose for a contradiction that $M[x^\infty]$ is nonzero. By [Ven02, Prop. 3.9, Prop. 3.5(v)] M has pure δ-dimension $\dim_A(M)$. By [Ven02, Prop. 3.5(vi)(b)] we therefore have $j_A(M[x^\infty]) = j_A(M)$ (if a module has pure δ-dimension, all its non-zero submodules have the same dimension). As in the proof of Lemma A.16 we also have $j_A(M[x^\infty]) = 1 + j_{A/x}(M[x^\infty]/xM[x^\infty])$. Combining the two equalities, we get $j_{A/x}(M[x^\infty]/xM[x^\infty]) = j_A(M) - 1$, which (by Lemma A.8) contradicts (1), since $M[x^\infty]/xM[x^\infty]$ is a submodule of M/xM. This completes the proof that (2) holds.

Now we must show that M/xM is a Cohen–Macaulay A/x-module. By Lemma A.15 we have $j_A(M/xM) = 1 + j_{A/x}(M/xM) = 1 + j_A(M)$. By (2), we have a short exact sequence

$$0 \to M \xrightarrow{\times x} M \to M/xM \to 0.$$

Considering the long exact sequence for $\mathrm{Hom}_A(-, A)$ we see that $\mathrm{Ext}^i_A(M/xM, A) = 0$ for all $i \neq 1 + j_A(M)$. The argument of the first paragraph of the proof of Lemma A.15 now implies that $\mathrm{Ext}^i_{A/x}(M/xM, A/x) = 0$ for all $i \neq j_A(M)$, and this shows that M/xM is Cohen–Macaulay (by Remark A.6).

Finally, we have established the claim that $(\varpi, x_1, \ldots, x_r)$ is an M-regular sequence. It follows that $\mathrm{Tor}_1^R(R/\mathfrak{m}_R, M) = 0$. If I is an ideal in R then $I \otimes_R M$ is naturally a finitely generated A-module and is therefore separated for the $\mathfrak{m}_R$-adic topology. Now [Mat89, Theorem 22.3] implies that M is a flat R-module (the previous sentence shows that M is $\mathfrak{m}_R$-adically ideal-separated, in Matsumura's terminology). □

A.31. **An application of the Artin–Rees lemma.** We now recall a version of the Artin–Rees Lemma.

Lemma A.32. *Let K be a compact p-adic analytic group, and let M be a $\mathcal{O}[[K]]$-submodule of $\mathcal{O}[[K]]^{\oplus t}$, for some $t \geq 1$. Let K' be an open uniform pro-p subgroup of K, and let $\mathcal{J}$ denote the two-sided ideal of $\mathcal{O}[[K]]$ generated by the maximal ideal $\mathfrak{m}$ of the local ring $\Lambda_{\mathcal{O}}(K')$. Then there is a constant $c \geq 0$ such that $M \cap (\mathcal{J}^{m+c})^{\oplus t} \subset \mathcal{J}^m M$ for all $m \geq 0$.*

Proof. The associated graded of $\mathcal{O}[[K]]$ for the $\mathcal{J}$-adic filtration is finite over the Noetherian ring $\mathrm{gr}_{\mathfrak{m}}\Lambda_{\mathcal{O}}(K')$, so it is itself Noetherian. Now we can apply [LO96, Prop. II.2.2.1, Thm. II.2.1.2(2)]. This shows that the $\mathcal{J}$-adic filtration on $\mathcal{O}[[K]]$ has the Artin–Rees property (defined in [LO96, Defn. II.1.1.1]), and the statement of the Lemma is a special case of this property. □

Lemma A.33. *Keep the same notation as in the previous Lemma. Suppose we have flat $\mathcal{O}[[K]]/\mathcal{J}^m$-modules M_m for each $n \geq 1$, with $M_m = M_{m+1}/\mathcal{J}^m M_{m+1}$. Then $M := \varprojlim_m M_m$ is a flat $\mathcal{O}[[K]]$-module and*

$$Q \otimes_{\mathcal{O}[[K]]} M = \varprojlim Q \otimes_{\mathcal{O}[[K]]} M_m$$

for every finitely generated (right) $\mathcal{O}[[K]]$-module Q.

In particular, we have $M/\mathcal{J}^m M = M_m$.

Proof. This follows from [Stacks, Tag 0912]. The reference assumes that the rings in question are commutative, so we will write out the proof in our setting. Set $A = \mathcal{O}[[K]]$ to abbreviate our notation.

We first show that $Q \otimes_A M = \varprojlim Q \otimes_A M_m$ for every finitely generated (right) A-module Q. Since A is Noetherian, we may choose a resolution $F_2 \to F_1 \to F_0 \to Q \to 0$ by finite free A-modules F_i. Then

$$F_2 \otimes_A M_m \to F_1 \otimes_A M_m \to F_0 \otimes_A M_m$$

is a chain complex whose homology in degree 0 is $Q \otimes_A M_m$ and whose homology in degree 1 is

$$\mathrm{Tor}_1^A(Q, M_m) = \mathrm{Tor}_1^A(Q, A/\mathcal{J}^m) \otimes_{A/\mathcal{J}^m} M_m$$

as M_m is flat over $A/\mathcal{J}^m$. Set $K = \ker(F_0 \to Q)$. We have

$$\mathrm{Tor}_1^A(Q, A/\mathcal{J}^m) = (K \cap (\mathcal{J}^m F_0))/\mathcal{J}^m K$$

so Lemma A.32 implies that there exists a $c \geq 0$ such that the map

$$\mathrm{Tor}_1^A(Q, A/\mathcal{J}^{n+c}) \to \mathrm{Tor}_1^A(Q, A/\mathcal{J}^m)$$

is zero for all m.

It follows from [Stacks, Tag 070E] that $\varprojlim Q \otimes_A M_m = \mathrm{coker}(\varprojlim F_1 \otimes_A M_m \to \varprojlim F_0 \otimes_A M_m)$. Since the F_i are finite free this equals $\mathrm{coker}(F_1 \otimes_A M \to F_0 \otimes_A M) = Q \otimes_A M$, as claimed. Taking $Q = A/\mathcal{J}^m$, we obtain $M/\mathcal{J}^m M = M_m$.

It remains to show that M is flat. Let $Q \to Q'$ be an injective map of finitely generated right A-modules; we must show that $Q \otimes_A M \to Q' \otimes_A M$ is injective. By the above we see

$$\ker(Q \otimes_A M \to Q' \otimes_A M) = \ker(\varprojlim Q \otimes_A M_m \to \varprojlim Q' \otimes_A M_m).$$

For each m we have an exact sequence

$$\mathrm{Tor}_1^A(Q', M_m) \to \mathrm{Tor}_1^A(Q'', M_m) \to Q \otimes_A M_m \to Q' \otimes_A M_m$$

where $Q'' = \mathrm{coker}(Q \to Q')$. Above we have seen that the inverse systems of Tor's are essentially constant with value 0. It follows from [Stacks, Tag 070E] that the inverse limit of the right most maps is injective, as required. □

Appendix B. Tensor products and projective covers

B.1. **Tensor products.** We recall from [Bru66, §2] that if R is a pseudocompact ring and M, N are pseudocompact (right, resp. left) R-modules, then the completed tensor product $M\widehat{\otimes}_R N$ is a pseudocompact R-module, which satisfies the usual universal property for the tensor product in the category of pseudocompact R-modules. $M\widehat{\otimes}_R N$ is the completion of $M \otimes_R N$ in the topology induced by taking $\mathrm{Im}(M \otimes_R V + U \otimes_R N)$ as a fundamental system of open neighborhoods of 0, where U (resp. V) runs through the open submodules of M (resp. N).

If A and B are pseudocompact R-algebras, and M, N (respectively) are pseudocompact A and B-modules, then $M\widehat{\otimes}_R N$ is naturally a pseudocompact $A\widehat{\otimes}_R B$-module.

Lemma B.2. *Let M, N be pseudocompact $\mathcal{O}$-modules. Suppose $M = \varprojlim_i M_i$ and $N = \varprojlim_j N_j$, where M_i and N_j are also pseudocompact $\mathcal{O}$-modules. Suppose that the transition maps $M_j \to M_i$ and $N_j \to M_i$ are surjective. Then the natural map*

$$\varinjlim_{i,j} \mathrm{Hom}^{cts}_{\mathcal{O}}(M_i, N_j^\vee) \to \mathrm{Hom}^{cts}_{\mathcal{O}}(M, N^\vee)$$

is an isomorphism.

The natural map

$$M\widehat{\otimes}_{\mathcal{O}} N \to \varprojlim_{i,j} M_i \widehat{\otimes}_{\mathcal{O}} N_j$$

is also an isomorphism.

Proof. The first claim is (a special case of) [Bru66, Lem. A.3]. The second claim is a special case of [Bru66, Lem. A.4]. □

Lemma B.3. *Let M, N be pseudocompact $\mathcal{O}$-modules. There is a natural isomorphism*

$$\left(M\widehat{\otimes}_{\mathcal{O}} N\right)^\vee \cong \mathrm{Hom}^{cts}_{\mathcal{O}}(M, N^\vee)$$

where $N^\vee$ has the discrete topology.

Proof. By Lemma B.2 we may assume that M and N are finite length $\mathcal{O}$-modules. By the universal property of the tensor product, we have

$$(M \otimes_{\mathcal{O}} N)^\vee = \mathrm{Hom}_{\mathcal{O}}(M, N^\vee).$$ □

We now recall some terminology about categories of smooth representations of p-adic analytic groups from [Eme10b]. Let G be a p-adic analytic group, with a compact open subgroup K_0 (all the notions recalled below will be independent of the choice of K_0). We let A denote a complete Noetherian local $\mathcal{O}$-algebra with finite residue field and maximal ideal $\mathfrak{m}_A$. In particular, A is a pseudocompact $\mathcal{O}$-algebra. $\mathrm{Mod}^{sm}_G(A)$ denotes the abelian category of smooth G-representations with coefficients in A [Eme10b, Defn. 2.2.5]. Pontryagin duality gives an anti-equivalence of categories between $\mathrm{Mod}^{sm}_G(A)$ and the category of pseudocompact $A[[K_0]]$-modules with a compatible G-action [Eme10b, (2.2.8)]. Here we write $A[[K_0]]$ for $A\widehat{\otimes}_{\mathcal{O}}[[K_0]]$.

An object $V \in \mathrm{Mod}^{sm}_G(A)$ is admissible if $V^\vee$ is a finitely generated $A[[K_0]]$-module (we take this as the definition, but see [Eme10b, Lem. 2.2.11]). An element $v \in V$ is called locally admissible if the G-subrepresentation of V generated by v is admissible, and V is called locally admissible if every element of V is locally admissible.

Similarly, an element $v \in V$ is called locally finite if the G-subrepresentation of V generated by v is a finite length object in $\mathrm{Mod}_G^{sm}(A)$, and V is called locally finite if every element of V is locally finite.

Lemma B.4. *Let G, H be p-adic analytic groups and suppose that $V \in \mathrm{Mod}_G^{sm}(\mathcal{O})$ and $W \in \mathrm{Mod}_H^{sm}(\mathcal{O})$. Suppose that V and W are locally admissible. Then $(V^\vee \widehat{\otimes}_{\mathcal{O}} W^\vee)^\vee = \mathrm{Hom}_{\mathcal{O}}^{cts}(V^\vee, W)$ is a locally admissible object of $\mathrm{Mod}_{G\times H}^{sm}(\mathcal{O})$.*

Proof. Let $M = V^\vee$ and $N = W^\vee$. Since V and W are locally admissible, we can write $M = \varprojlim_i M_i$ and $N = \varprojlim_j N_j$ where the $M_i^\vee$ and $N_j^\vee$ are admissible and the transition maps in the inverse systems are surjective. It follows from Lemma B.2 that it suffices to prove the Lemma under the additional assumption that V and W are admissible.

Let K_1 and K_2 be compact open subgroups of G and H respectively. We may assume that M and N are finitely generated $\mathcal{O}[[K_1]]$- and $\mathcal{O}[[K_2]]$-modules respectively. In particular, we have (continuous) surjections $\mathcal{O}[[K_1]]^{\oplus a} \to M$ and $\mathcal{O}[[K_2]]^{\oplus b} \to N$. Therefore, we have a surjective map of $\mathcal{O}[[K_1]]\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[[K_2]] = \mathcal{O}[[K_1 \times K_2]]$-modules:

$$\mathcal{O}[[K_1]]^{\oplus a}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[[K_2]]^{\oplus b} = \mathcal{O}[[K_1 \times K_2]]^{\oplus ab} \to M\widehat{\otimes}_{\mathcal{O}}N.$$

In particular, $(M\widehat{\otimes}_{\mathcal{O}}N)^\vee$ is admissible. □

Lemma B.5. *Let G, H be p-adic analytic groups and suppose that $V \in \mathrm{Mod}_G^{sm}(\mathcal{O})$ and $W \in \mathrm{Mod}_H^{sm}(\mathcal{O})$. Suppose that V and W are locally finite, and that (at least) one of V, W is moreover locally admissible. Then $(V^\vee \widehat{\otimes}_{\mathcal{O}} W^\vee)^\vee = \mathrm{Hom}_{\mathcal{O}}^{cts}(V^\vee, W)$ is a locally finite object of $\mathrm{Mod}_{G\times H}^{sm}(\mathcal{O})$.*

If V and W are irreducible, with at least one of them admissible, then $(V^\vee \widehat{\otimes}_{\mathcal{O}} W^\vee)^\vee = V \otimes_k W$ is also irreducible.

Proof. Let $M = V^\vee$ and $N = W^\vee$. Since V and W are locally finite, we can write $M = \varprojlim_i M_i$ and $N = \varprojlim_j N_j$ where the $M_i^\vee$ and $N_j^\vee$ are finite length and the transition maps in the inverse systems are surjective. It follows from Lemma B.2 that it suffices to prove the Lemma under the additional assumption that V and W are finite length. By induction on the length, we can assume that V and W are irreducible (and one of them is admissible). In this case (since V and W are killed by ϖ), $\mathrm{Hom}_{\mathcal{O}}^{cts}(M, N^\vee) = \varinjlim \mathrm{Hom}_k(M/U, N^\vee) = V \otimes_k W$, where U runs over open submodules over M, and the first equality follows from Lemma B.2.

It remains to show that $V \otimes_k W$ is an irreducible representation of $G \times H$. Without loss of generality, we assume that V is admissible. By extending scalars if necessary, we may assume that V is absolutely irreducible (we just do this for simplicity of exposition). We have

$$\mathrm{Hom}_G(V, V \otimes_k W) = \mathrm{Hom}_G(V, V) \otimes_k W$$

since V has finite length. By Schur's lemma (which applies to *admissible* absolutely irreducible representations) we can identify $\mathrm{Hom}_G(V, V \otimes_k W)$ with W.

Suppose $U \subset V \otimes_k W$ is a nonzero $G \times H$-subrepresentation. Then $\mathrm{Hom}_G(V, U)$ is an H-subrepresentation of $\mathrm{Hom}_G(V, V \otimes_k W) = W$. Since $V \otimes_k W$ is locally finite as a G-representation, with every simple submodule isomorphic to V, we have $\mathrm{Hom}_G(V, U) \neq 0$ and therefore $\mathrm{Hom}_G(V, U) = W$. This says that for all

$w \in W$, the map $v \mapsto v \otimes w$ lies in $\mathrm{Hom}_G(V, U)$. In other words, $v \otimes w \in U$ for all $v \in V, w \in W$. So $U = V \otimes W$. □

Lemma B.6. *Let G, H be p-adic analytic groups. Suppose that both G and H have the property that locally admissible representations are locally finite. Let X be an admissible simple object of $\mathrm{Mod}^{sm}_{G\times H}(\mathcal{O})$. Then X is isomorphic to $V \otimes W$, for some admissible simple $V \in \mathrm{Mod}^{sm}_G(\mathcal{O})$ and $W \in \mathrm{Mod}^{sm}_H(\mathcal{O})$.*

Proof. Since X is admissible as a $G \times H$-representation, it is locally admissible as a G-representation. Indeed for every $x \in X$ there is a compact open subgroup $K_2 \subset H$ such that $x \in X^{K_2}$, and X^{K_2} is a locally admissible G-representation. It follows from our assumptions that X is a locally finite G-representation.

So, there is a simple admissible $V \in \mathrm{Mod}^{sm}_G(\mathcal{O})$ with $\mathrm{Hom}_G(V, X) \neq 0$. The H-representation $\mathrm{Hom}_G(V, X)$ is admissible, and hence locally finite. Indeed, if $K_2 \subset H$ is compact open, then X^{K_2} is an admissible G-representation and $\mathrm{Hom}_G(V, X)^{K_2} = \mathrm{Hom}_G(V, X^{K_2})$ is a finitely generated $\mathcal{O}$-module by [Eme10b, Lem. 2.3.10]. We conclude that there is a simple admissible $W \in \mathrm{Mod}^{sm}_H(\mathcal{O})$ with a injective H-linear map $W \to \mathrm{Hom}_G(V, X)$. It follows that we have a non-zero $G \times H$-linear map $V \otimes_k W \to X$. By Lemma B.5, this map is an isomorphism. □

Lemma B.7. *Let $G = \prod_{i=1}^m G_i$, where $G_i = \mathrm{PGL}_2(\mathbb{Q}_p)$. Let $V \in \mathrm{Mod}^{sm}_G(\mathcal{O})$ be admissible and finitely generated over $\mathcal{O}[G]$. Then V is of finite length. In particular, locally admissible G-representations are locally finite.*

If V is irreducible as a G-representation, then V is isomorphic to $\otimes_{i=1}^m V_i$, where the V_i are irreducible G_i-representations.

Proof. Let $K_0 = \prod_{i=1}^m \mathrm{PGL}_2(\mathbb{Z}_p)$. Following the argument of [Eme10b, Thm. 2.3.8], it suffices to show that every admissible quotient V of $\text{c-Ind}_{K_0}^G W$ is of finite length, where W is a finite dimensional absolutely irreducible representation of K_0 over k. W decomposes as a tensor product $W = \otimes_{i=1}^m W_i$ of representations of $PGL_2(\mathbb{Z}_p)$. As in *loc. cit.* we consider $\mathrm{Hom}_{k[G]}(\text{c-Ind}_{K_0}^G W, V)$ which is a finite dimensional k-vector space and a module over $\mathcal{H}(W) := \mathrm{End}_{k[G]}(\text{c-Ind}_{K_0}^G W)$. We have a surjective map

$$\mathrm{Hom}_{k[G]}(\text{c-Ind}_{K_0}^G W, V) \otimes_{\mathcal{H}(W)} \text{c-Ind}_{K_0}^G W \to V.$$

The Hecke algebra $\mathcal{H}(W)$ is isomorphic to the convolution algebra of compactly supported functions $f : G \to \mathrm{End}_k(W)$ such that $f(h_1 g h_2) = h_1 \circ f(g) \circ h_2$ for all $h_1, h_2 \in K_0$ and $g \in G$. With this description, one can show that

$$\mathcal{H}(W) \cong \otimes_{i=1}^m \mathcal{H}_i(W_i)$$

where $\mathcal{H}_i(W_i) = \mathrm{End}_{k[G_i]}(\text{c-Ind}_{\mathrm{PGL}_2(\mathbb{Z}_p)}^{G_i} W_i)$. By [BL94, Prop. 8], we have $\mathcal{H}_i(W_i) \cong k[T_i]$ and therefore we have $\mathcal{H}(W) \cong k[T_1, \dots, T_m]$.

Now it suffices to show that

$$X \otimes_{\mathcal{H}(W)} \text{c-Ind}_{K_0}^G W$$

is of finite length, where X is a finite dimensional $\mathcal{H}(W)$-module. By induction on the dimension of X, extending scalars if necessary, we may assume that $X \cong \mathcal{H}(W)/(T_1 - \lambda_1, \dots, T_m - \lambda_m)$, with $\lambda_i \in k$.

Since $\text{c-Ind}_{K_0}^G W \cong \otimes_{i=1}^m \text{c-Ind}_{\mathrm{PGL}_2(\mathbb{Z}_p)}^{G_i} W_i$ we need to show that

$$\otimes_{i=1}^m \text{c-Ind}_{\mathrm{PGL}_2(\mathbb{Z}_p)}^{G_i} W_i/(T_i - \lambda_i)$$

has finite length, which follows from Lemma B.5 and the results of [BL94; Bre03].

Finally, we repeatedly apply Lemma B.6 to show that if V is irreducible it factors as a tensor product. □

Lemma B.8. *Let $G = \prod_{i=1}^m G_i$, where $G_i = \mathrm{PGL}_2(\mathbb{Q}_p)$. Let $V = \otimes_{i=1}^m V_i$ be an irreducible admissible representation of G. Let $V_i \hookrightarrow I_i$, $i = 1, \ldots m$ be injective envelopes of V_i in $\mathrm{Mod}_{G_i}^{\mathrm{loc\ adm}}(\mathcal{O})$ (the category of locally admissible representations). Dually, set $M_i = V_i^\vee$ and $P_i = I_i^\vee$.*

Then $\widehat{\otimes}_{i=1}^m P_i \to \widehat{\otimes}_{i=1}^m M_i$ is a projective envelope in $\mathfrak{C}_G(\mathcal{O})$ (see Definition 5.1.1).

Proof. First we show that $P := \widehat{\otimes}_{i=1}^m P_i$ is projective in $\mathfrak{C}_G(\mathcal{O})$. Note that it follows from Lemma B.4 and Lemma B.7 that $P^\vee$ is locally admissible and locally finite. Let $M = \widehat{\otimes}_{i=1}^m M_i \in \mathfrak{C}_G(\mathcal{O})$. We induct on m. Let $P' = \widehat{\otimes}_{i=2}^m P_i$ and $G' = \prod_{i=2}^m G_i$. By the universal property of the completed tensor product we have

$$\mathrm{Hom}^{cts}_{G_1 \times G'}(P_1 \widehat{\otimes} P', M) = \mathrm{Hom}^{cts}_{G_1}(P_1, \mathrm{Hom}^{cts}_{G'}(P', M)), \tag{B.9}$$

so projectivity of P follows from projectivity of P' and P_1.

Now we prove that $P \to M$ is an essential surjection. Since $P^\vee$ is locally finite, it suffices to show that $M = \mathrm{cosoc}(P)$ (see [CEGGPS2, Lem. 4.6]). Again we proceed by induction on m. So we assume that $\mathrm{cosoc}(P') = \widehat{\otimes}_{i=2}^m M_i$. Let N be a simple object of $\mathfrak{C}_G(\mathcal{O})$. It follows from Lemma B.7 that $N \cong \widehat{\otimes}_{i=1}^m N_i$ where the N_i are simple in $\mathfrak{C}_{G_i}(\mathcal{O})$. Let $N' = \widehat{\otimes}_{i=2}^m N_i$. By (B.9), we have

$$\mathrm{Hom}^{cts}_G(P, N) = \mathrm{Hom}^{cts}_{G_1}(P_1, \mathrm{Hom}^{cts}_{G'}(P', N)).$$

As an object of $\mathfrak{C}_{G'}(\mathcal{O})$, we have $N = N_1 \widehat{\otimes}_{\mathcal{O}} N' = (\varprojlim N_1/U) \widehat{\otimes}_{\mathcal{O}} N'$ where the limit runs over open submodules of N_1 and so N_1/U is a finite length $\mathcal{O}$-module. In fact, since N_1 is simple, N_1/U is just a finite dimensional k-vector space. It follows from Lemma B.2 that, in $\mathfrak{C}_{G'}(\mathcal{O})$, we have an isomorphism $N \cong \varprojlim(N_1/U \otimes_{\mathcal{O}} N')$ and so we obtain isomorphisms

$$\mathrm{Hom}^{cts}_{G'}(P', N) \cong \varprojlim \mathrm{Hom}^{cts}_{G'}(P', N_1/U \otimes_{\mathcal{O}} N') = \varprojlim \mathrm{Hom}^{cts}_{G'}(P', N') \otimes_{\mathcal{O}} N_1/U$$

$$\mathrm{Hom}^{cts}_G(P, N) \cong \varprojlim \mathrm{Hom}^{cts}\, \mathrm{Hom}^{cts}_{G'}(P', N_1/U \otimes_{\mathcal{O}} N') = \varprojlim \mathrm{Hom}^{cts}_{G'}(P', N') \otimes_{\mathcal{O}} N_1/U.$$

Applying a similar argument, we conclude that

$$\mathrm{Hom}^{cts}_G(P, N) \cong \mathrm{Hom}^{cts}_{G_1}(P_1, N_1) \widehat{\otimes}_{\mathcal{O}} \mathrm{Hom}^{cts}_{G'}(P', N').$$

We immediately deduce (from our inductive hypothesis) that if $N \ncong M$ then $\mathrm{Hom}^{cts}_G(P, N) = 0$ whilst we have

$$\mathrm{Hom}^{cts}_G(P, M) = \mathrm{Hom}^{cts}_{G_1}(M_1, M_1) \widehat{\otimes}_{\mathcal{O}} \mathrm{Hom}^{cts}_{G'}(M', M') = \mathrm{Hom}^{cts}_G(M, M).$$

We deduce that $\mathrm{cosoc}(P) = M$, as desired. □

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