

G -VALUED LOCAL DEFORMATION RINGS AND GLOBAL LIFTS

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ABSTRACT. We study G -valued Galois deformation rings with prescribed properties, where G is an arbitrary (not necessarily connected) reductive group over an extension of \mathbf{Z}_ℓ for some prime ℓ . In particular, for the Galois groups of p -adic local fields (with p possibly equal to ℓ) we prove that these rings are generically smooth, compute their dimensions, and show that functorial operations on Galois representations give rise to well-defined maps between the sets of irreducible components of the corresponding deformation rings. We use these local results to prove lower bounds on the dimension of global deformation rings with prescribed local properties. Applying our results to unitary groups, we improve results in the literature on the existence of lifts of mod ℓ Galois representations, and on the weight part of Serre's conjecture.

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1. INTRODUCTION

The study of Galois deformation rings was initiated in [Maz89], and was crucial to the proof of Fermat's Last Theorem in [Wil95], and in particular to the modularity lifting theorems proved in [Wil95, TW95]. Many generalisations of these modularity lifting theorems have been proved over the last 25 years, and it has become increasingly important to consider Galois representations valued in reductive groups other than GL_n . From the point of view of the Langlands program, it is particularly important to be able to use disconnected groups, as the L -groups of non-split groups are always disconnected. In particular, it is important to study the structure of local deformation rings for general reductive groups, and to prove lifting results for global deformation rings. We briefly review the history of such results in Section 1.1, but we firstly explain the main theorems of this paper.

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We begin with a result about local deformation rings. Let K/\mathbf{Q}_p be a finite extension, let \mathcal{O} be the ring of integers in a finite extension of \mathbf{Q}_ℓ , where ℓ is possibly equal to p , and let G be a (not necessarily connected) reductive group over \mathcal{O} . Given a representation $\bar{\rho} : G_K \rightarrow G(\mathcal{O}/\mathfrak{m}_{\mathcal{O}})$, we consider liftings of $\bar{\rho}$ of some inertial type τ , and in the case $\ell = p$, some p -adic Hodge type \mathbf{v} . There is a corresponding universal framed deformation ring $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}$, and we prove the following result (as well as a variant for “fixed determinant” deformations).

Theorem A (Thm. 3.3.3). *Fix an inertial type τ , and if $\ell = p$ then fix a p -adic Hodge type \mathbf{v} . Then $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/\ell]$ is generically smooth. In addition, $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}$ is equidimensional of dimension $1 + \dim_E G + \delta_{\ell=p} \dim_E(\mathrm{Res}_{E \otimes K/E} G)/P_{\mathbf{v}}$, and $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}, \psi}$ is equidimensional of dimension $1 + \dim_E G^{\mathrm{der}} + \delta_{\ell=p} \dim_E(\mathrm{Res}_{E \otimes K/E} G)/P_{\mathbf{v}}$.*

(We are abusing notation here; $P_{\mathbf{v}}$ is a $(\mathrm{Res}_{E \otimes K/E} G)_{\bar{E}}^{\circ}$ -conjugacy class of parabolic subgroups of $\mathrm{Res}_{E \otimes K/E} G$, and we choose a representative defined over E to compute the dimension of the quotient.) We are also able to describe the smooth locus of $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/\ell]$ precisely in terms of the corresponding Weil–Deligne representations; see Corollary 3.3.4. In the case that $G = \mathrm{GL}_n$ and $\ell = p$ this is a theorem of Kisin [Kis08], and results for general groups (but with more restrictive hypotheses than those of Theorem A) were previously proved by Balaji [Bal12] and R.B. [Bel16].

Combining Theorem A with results of Balaji [Bal12], we obtain the following result (see Section 4 for any unfamiliar notation or terminology); in the case of potentially crystalline representations, this is the main result of [Bal12].

Theorem B (Prop. 4.2.6). *Let F be totally real, assume that $\ell > 2$, let S be a finite set of places of F containing all places dividing $\ell\infty$, and let $\bar{\rho} : \mathrm{Gal}_{F,S} \rightarrow G(\bar{\mathbf{F}}_\ell)$ be a representation admitting a universal deformation ring. Fix inertial types at all places $v \in S$, and Hodge types at all places $v|\ell$, and let R^{univ} denote the corresponding fixed determinant universal deformation ring for $\bar{\rho}$.*

Assume that $\bar{\rho}$ is odd, and that $H^0(\mathrm{Gal}_{F,S}, (\mathfrak{g}_{\mathbf{F}}^0)^(1)) = 0$. Suppose also that for each place $v|\ell$ the corresponding Hodge type is regular. Then if R^{univ} is non-zero, it has Krull dimension at least one.*

We use this result to improve on some results about automorphic forms on unitary groups proved using the methods of [BLGGT14]. Beginning with the paper [CHT08], Galois deformations were considered for representations valued in a certain disconnected group \mathcal{G}_n , whose connected component is $\mathrm{GL}_n \times \mathrm{GL}_1$ (this group is related to the L -group of a unitary group, see [BG14, §8]). In the case that $G = \mathcal{G}_n$, Theorem B generalises [BLGGT14, Prop. 1.5.1], removing restrictions on the places in S (which were chosen to split in the splitting field of the corresponding unitary group, in order to reduce the local deformation theory to the GL_n case).

We deduce corresponding improvements to a number of results proved using the methods of [BLGGT14], such as the following general result about Serre weights for rank two unitary groups, which removes a “split ramification” hypothesis on the ramification of $\bar{\rho}$ at places away from ℓ .

Theorem C (Theorem 5.2.2). *Let F be an imaginary CM field with maximal totally real subfield F^+ , and suppose that F/F^+ is unramified at all finite places, that each*

place of F^+ above l splits in F , and that $[F^+ : \mathbf{Q}]$ is even. Suppose that l is odd, that $\bar{\tau} : G_{F^+} \rightarrow \mathcal{G}_2(\overline{\mathbf{F}}_\ell)$ is irreducible and modular, and that $\bar{\tau}(G_{F(\zeta_l)})$ is adequate.

Then the set of Serre weights for which $\bar{\tau}$ is modular is exactly the set of weights given by the sets $W(\bar{\tau}|_{G_{F_v}})$, $v|l$.

(See Remark 5.2.3 for a discussion of further improvements to this result that could be made by techniques orthogonal to those of this paper.) These results are also crucially applied in the forthcoming paper [CEG], where they are used to construct lifts of representations valued in \mathcal{G}_n which have prescribed ramification at certain inert places.

1.1. A brief historical overview. We now give a very brief overview of some of the developments in the deformation theory of Galois representations, which was introduced for representations valued in GL_n by Mazur in the paper [Maz89]; we apologise for the many important papers that we do not discuss here for reasons of space. The abstract parts of this deformation theory were generalised to arbitrary reductive groups in [Til96]. However, for applications to the Langlands program (and in particular to proving automorphy lifting theorems), one needs to study conditions on Galois deformations coming from p -adic Hodge theory.

This was initially done in a somewhat ad-hoc fashion, mostly for the group GL_2 and mostly for conditions coming from p -divisible groups, culminating in the paper [BCDT01], which used a detailed study of some particular such deformation rings to complete the proof of the Taniyama–Shimura–Weil conjecture. This situation changed with paper [Kis08], which proved the existence of local deformation rings for GL_n corresponding to general p -adic Hodge theoretic conditions (namely being potentially crystalline or semi-stable of a given inertial type), and determined the structure of their generic fibres, in particular showing that they are generically smooth, and computing their dimensions.

The results of [Kis08] were generalised in [Bal12] to the case of general reductive groups G under the hypothesis of being potentially crystalline, and in [Bel16] to the case that G is connected, and the inertial type is totally ramified. In the potentially crystalline case the generic fibres of the deformation rings can easily be shown to be smooth, whereas in the potentially semistable case, one has to gain some control of the singularities, which is why there are additional restrictions in the theorems of [Bel16]. Our Theorem A is a common generalisation of these results to the case that G is possibly disconnected, and the representation is potentially semistable with no condition on the inertial type. (We also simultaneously handle the case that $p \neq l$.)

Another important application of Galois deformation theory to the Langlands program is to prove results showing that mod l representations of the Galois groups of number fields admit lifts to characteristic zero with prescribed local properties; for example, such results were an important part of Khare–Wintenberger’s proof of Serre’s conjecture. The first such results were proved by Ramakrishna for GL_2 [Ram02], and this method has now been generalised to a wide class of reductive groups; see in particular [Pat16] and [Boo16]. However, it has two disadvantages: it loses control of the local properties at a finite set of places, and it only applies in cases where smooth deformation rings exist.

A different approach was found in the paper [KW09], which observed that in conjunction with the theory of potential modularity, such lifting results can be

deduced from a lower bound on the Krull dimension of a global deformation ring, which was provided by the results of [Böc99]. In the paper [Kis07], Kisin improved on the results of [Böc99], proving a result about presentations of global deformation rings over local ones for GL_n , and deducing a lower bound on the dimensions of global deformation rings. These results were generalised to general reductive groups by Balaji [Bal12], and given our Theorem A, results such as Theorem B are essentially immediate from Balaji's.

1.2. Some details. We now explain our local results (and their proofs) in more detail. Theorem A is a generalisation of [Kis08, Thm. 3.3.4], which proves the result in the case $l = p$ and $G = \mathrm{GL}_n$. It was previously adapted to the (much easier) case $G = \mathrm{GL}_n$ and $l \neq p$ in [Gee11] by using Weil–Deligne representations in place of the filtered (φ, N) -modules employed in [Kis08]. It was also generalised in [Bel16] to the case that G is connected, $l = p$, and τ is totally ramified. Our approach is in some sense a synthesis of the approaches of [Gee11, Bel16], in that we treat the cases $l \neq p$ and $l = p$ essentially simultaneously, by using Weil–Deligne representations.

We briefly explain our approach, which in broad outline follows that of [Kis08]. It is relatively straightforward (by passing from Galois representations to Weil–Deligne representations using Fontaine's constructions in the case $l = p$, and Grothendieck's monodromy theorem if $l \neq p$) to reduce Theorem A to analogous statements about moduli spaces of Weil–Deligne representations over l -adic fields. These moduli spaces admit an explicit tangent-obstruction theory given by an analogue of Herr's complex computing Galois cohomology in terms of (φ, Γ) -modules, and the key problem is to prove that the H^2 of this complex generically vanishes. We can think of this H^2 as a coherent sheaf over the moduli space, so by considering its support, we can reduce to the problem of exhibiting sufficiently many points at which the H^2 vanishes (which turn out to be precisely the smooth points).

Our approach to exhibiting these points is related to that taken in [Bel16], in that it makes use of the theory of associated cocharacters (see Section 2.3), but it is more streamlined and conceptual (for example, we do not need to consider the case $N = 0$ separately, as was done in [Bel16]). Surprisingly (at least to us), it is possible to construct all the smooth points that we need by considering the single Weil–Deligne representation $W_K \rightarrow \mathrm{SL}_2(\overline{\mathbf{Q}}_\ell)$ which is trivial on I_K , takes an arithmetic Frobenius element of W_K to

$$\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}$$

where q is the order of the residue field of K , has

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easy to check that this gives a smooth point of the moduli space of Weil–Deligne representations (while the point with the same representation of W_K but with $N = 0$ is not smooth).

Returning to the case of general G , suppose that the inertial type τ is trivial. If we consider a nilpotent element $N \in \mathrm{Lie} G$, the theory of associated cocharacters allows us to construct a particular homomorphism $\mathrm{SL}_2 \rightarrow G$ taking $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to N , and an elementary calculation using the representation theory of \mathfrak{sl}_2 shows that the

pushout of our fixed representation $W_K \rightarrow \mathrm{SL}_2(\overline{\mathbf{Q}}_\ell)$ by this homomorphism defines a smooth point. We obtain further smooth points by multiplication by elements of $G(\overline{\mathbf{Q}}_\ell)$ of finite order, and this turns out to give us all the smooth points we need (even when G is not connected). (See Remark 2.3.11 for an interpretation of this construction in terms of the SL_2 version of the Weil–Deligne group.)

In the case of general τ we reduce to the same situation by replacing G by the normaliser in G of τ , which is also a reductive group. This use of Weil–Deligne representations is what allows us to remove the assumption made in [Bel16] that the inertial type is totally ramified, which was used in order to choose coordinates so that the inertial type τ was invariant under scalar Frobenius. (Similarly, it clarifies the calculations made for GL_n in [Kis08], as the semilinear algebra becomes linear algebra.) Under this assumption, when studying the structure of the moduli space of G -valued (φ, N, τ) -modules one could exploit the fact that Φ was in the centralizer $Z_G(\tau)$ and N was in $\mathrm{Lie} Z_G(\tau)$. Passing to Weil–Deligne representations r lets us argue similarly for general τ : a generator Φ of the unramified quotient of the Weil group normalizes the inertial type and N is centralized by the inertial type. Since $Z_G(r|_{I_{L/K}})$ has finite index in the normalizer $N_G(r|_{I_{L/K}})$, we see that N is again in the Lie algebra of the algebraic group containing Φ .

In view of the functorial nature of our construction of smooth points, we are able to produce points on each irreducible component of the generic fiber of the deformation ring which are furthermore “very smooth” in the sense that they give rise to smooth points after restriction to any finite extension K'/K (these points were called “robustly smooth” in [BLGGT14] when $p \neq l$). In particular, the images of such points on the corresponding deformation rings for $G_{K'}$ lie on only one irreducible component, so that we obtain a well-defined “base change” map between irreducible components. We prove a similar result for the maps between deformation rings induced by morphisms of algebraic groups $G \rightarrow G'$ (see Theorem 3.5 for this, and for the case of base change). In particular, this allows one to talk about taking tensor products of components of deformation rings, which is frequently convenient when applying the Harris tensor product trick; see for example [CEG].

We end this introduction by explaining the structure of the paper. In Section 2, we prove our main results about the structure of the moduli spaces of Weil–Deligne representations; we explain the tangent-obstruction theory and exhibit smooth points, and study the relationship with Galois representations. In doing so we remove the connectedness hypothesis on G made in [Bel16], by studying exact tensor-filtrations on fiber functors for disconnected reductive groups. We do this via a functor of points approach, using the dynamic approach to parabolic subgroups discussed in [CGP15, §I.2.1]. In Section 3 we deduce our results on the local structure of Galois deformation rings, which we then combine with the results of [Bal12] to prove our lower bound on the dimension of a global deformation ring in Section 4. Finally, in Section 5 we specialise these results to the case of unitary groups.

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1.4. Notation and conventions. All representations considered in this paper are assumed to be continuous with respect to the natural topologies, and we will never draw attention to this.

If K is a field then we write $\mathrm{Gal}_K := \mathrm{Gal}(\overline{K}/K)$ for its absolute Galois group, where \overline{K} is a fixed choice of algebraic closure; we will regard all algebraic extensions of K as subfields of \overline{K} without further comment, so that in particular we can take the compositum of any two such extensions. If L/K is a Galois extension then we write $\mathrm{Gal}_{L/K} := \mathrm{Gal}(L/K)$, a quotient of Gal_K . If K is a number field and v is a place of K then we fix an embedding $\overline{K} \hookrightarrow \overline{K}_v$, so that we have a homomorphism $\mathrm{Gal}_{K_v} \rightarrow \mathrm{Gal}_K$. If S is a finite set of places of a number field K , then we let $K(S)$ be the maximal extension of K (inside \overline{K}) which is unramified outside S , and write $\mathrm{Gal}_{K,S} := \mathrm{Gal}(K(S)/K)$.

If K/\mathbf{Q}_p is a finite extension for some prime p then we write I_K for the inertia subgroup of Gal_K , W_K for the Weil group, and f_K for the inertial degree of K/\mathbf{Q}_p . We let φ denote the arithmetic Frobenius on $\overline{\mathbf{F}}_p$, so that we have an exact sequence

$$1 \rightarrow I_K \rightarrow W_K \rightarrow \langle \varphi^{f_K} \rangle \rightarrow 1,$$

and we let $v : W_K \rightarrow \mathbf{Z}$ be the function such that $v(g) = i$ if the image of g modulo I_K is φ^{if_K} . Recall that a Weil–Deligne representation of W_K is a pair (r, N) consisting of a finite-dimensional representation $r : W_K \rightarrow \mathrm{End}(V)$ and a (necessarily nilpotent) endomorphism $N \in \mathrm{End}(V)$ satisfying

$$\rho(g)N = p^{v(g)f_K} N \rho(g)$$

for all $g \in W_K$.

1.4.1. Parabolic subgroups. If G is a finite-type affine group scheme over a ring A , and $\lambda : \mathbf{G}_m \rightarrow G$ is a cocharacter of G , then there is a subgroup $P_G(\lambda)$ of G associated to λ as follows. Following [CGP15, §I.2.1], for any \mathcal{O} -algebra A we define the functors

$$P_G(\lambda)(A) = \{g \in G(A) \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\},$$

and

$$U_G(\lambda)(A) = \{g \in P_G(\lambda)(A) \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} = 1\}.$$

We also let $Z_G(\lambda)$ denote the scheme-theoretic centralizer of λ . All of these functors are representable by subgroup schemes of G , and they are smooth if G is smooth. By construction, the formation of $P_G(\lambda)$, $U_G(\lambda)$, and $Z_G(\lambda)$ commutes with base change on A .

The cocharacter λ induces a grading on the Lie algebra $\mathfrak{g} := \mathrm{Lie} G$. Let $\mathfrak{g}_n := \{v \in \mathfrak{g} : \mathrm{Ad}(\lambda(t))(v) = t^n v\}$ and let $\mathfrak{g}_{\geq 0} := \bigoplus_{n \geq 0} \mathfrak{g}_n$. Then $\mathrm{Lie} P_G(\lambda) = \mathfrak{g}_{\geq 0}$, $\mathrm{Lie} U_G(\lambda) = \mathfrak{g}_{\geq 1}$, and $\mathrm{Lie} Z_G(\lambda) = \mathfrak{g}_0$.

The multiplication map $Z_G(\lambda) \times U_G(\lambda) \rightarrow P_G(\lambda)$ is an isomorphism. Furthermore, the fibers of $U_G(\lambda)$ are unipotent and connected. If the morphism $G \rightarrow \mathrm{Spec} A$ has connected reductive fibers, then $P_G(\lambda)$ is a parabolic subgroup scheme with connected fibers, $U_G(\lambda)$ is its unipotent radical, and $Z_G(\lambda)$ is connected and reductive.

1.4.2. *Deformation rings.* Let l be prime, and let \mathcal{O} be the ring of integers in a finite extension E/\mathbf{Q}_ℓ with residue field \mathbf{F} . Write $\text{CNL}_{\mathcal{O}}$ for the category of complete local noetherian \mathcal{O} -algebras with residue field \mathbf{F} .

Let Γ be either the absolute Galois group Gal_K of a finite extension K of \mathbf{Q}_ℓ for some p (possibly equal to l), or a group $\text{Gal}_{K,S}$ where S is a finite set of places of a number field K .

Let G be a (not necessarily connected) reductive group over \mathcal{O} , and fix a homomorphism $\bar{\rho} : \Gamma \rightarrow G(\mathbf{F})$. A *framed deformation* of $\bar{\rho}$ to a ring $A \in \text{CNL}_{\mathcal{O}}$ is a homomorphism $\rho : \Gamma \rightarrow G(A)$ whose reduction modulo \mathfrak{m}_A is equal to $\bar{\rho}$. The functor of framed deformations is represented by the universal framed deformation \mathcal{O} -algebra $R_{\bar{\rho}}^{\square}$, an object of $\text{CNL}_{\mathcal{O}}$ ([Bal12, Thm. 1.2.2]).

Suppose from now on for the rest of the paper that the centre Z_G of G is smooth over \mathcal{O} . Write $\mathfrak{g}_{\mathbf{F}}$ and $\mathfrak{z}_{\mathbf{F}}$ for the \mathbf{F} -points of the Lie algebras of G and Z_G respectively. A *deformation* of $\bar{\rho}$ to A is a $(\ker(G(A) \rightarrow G(\mathbf{F})))$ -conjugacy class of framed deformations of $\bar{\rho}$ to A . If $H^0(\Gamma, \mathfrak{g}_{\mathbf{F}}) = \mathfrak{z}_{\mathbf{F}}$, then the functor of deformations is represented by the universal framed deformation \mathcal{O} -algebra $R_{\bar{\rho}}$, an object of $\text{CNL}_{\mathcal{O}}$ (see [Bal12, Thm. 1.2.2] or [Til96, Thm. 3.3], together with Comment (2) following [Til96, Thm. 3.3]).

We will also consider “fixed determinant” versions of these (framed) deformation rings. Let G^{ab} and G^{der} respectively denote the abelianisation and derived subgroup of G , and write $\text{ab} : G \rightarrow G^{\text{ab}}$ for the natural map. Write $\mathfrak{g}_{\mathbf{F}}^0$ for the \mathbf{F} -points of the Lie algebra of G^{der} . Fix a homomorphism $\psi : \Gamma \rightarrow G^{\text{ab}}(\mathcal{O})$ such that $\text{ab} \circ \bar{\rho} = \bar{\psi}$. We let $R_{\bar{\rho}}^{\square, \psi}$ (resp. $R_{\bar{\rho}}^{\psi}$) denote the quotient of $R_{\bar{\rho}}^{\square}$ (resp. $R_{\bar{\rho}}$) corresponding to (framed) deformations ρ with $\text{ab} \circ \rho = \psi$.

We write G° for the connected component of G containing the identity. We will always consider representations up to G° -conjugacy, rather than G -conjugacy; note that this is compatible with our definition of deformations, as an element of $(\ker(G(A) \rightarrow G(\mathbf{F})))$ is necessarily contained in $G^\circ(A)$.

We for the most part allow any coefficient field E , although for some constructions in p -adic Hodge theory we need to allow it to be sufficiently large; we will comment when we do this. The effect of replacing E with a finite extension E' with ring of integers \mathcal{O}' is simply to replace $R_{\bar{\rho}}^{\square}$ and $R_{\bar{\rho}}$ with $R_{\bar{\rho}}^{\square} \otimes_{\mathcal{O}} \mathcal{O}'$ and $R_{\bar{\rho}} \otimes_{\mathcal{O}} \mathcal{O}'$ respectively.

2. MODULI OF WEIL–DELIGNE REPRESENTATIONS

Let K/\mathbf{Q}_p be a finite extension, and let l be a prime, possibly equal to p . In this section we prove analogues for l -adic Weil–Deligne representations of some results on moduli spaces of weakly admissible modules from [Kis08, Bel16], and remove some hypotheses imposed in those papers; in particular, we allow our groups to be disconnected, and we work with arbitrary inertial types (rather than totally ramified types). In the case that $l = p$ we relate our moduli spaces to those for weakly admissible modules. In Section 3 we will use these results to study the generic fibers of deformation rings in both the case $l = p$ and the case $l \neq p$.

2.1. Moduli of Weil–Deligne representations. Let K/\mathbf{Q}_p be a finite extension, and let L/K be a finite Galois extension. As in Section 1.4, we let E/\mathbf{Q}_ℓ be a finite extension for some prime l , with ring of integers \mathcal{O} . We also continue to let G be a (not necessarily connected) reductive group over \mathcal{O} ; in fact, throughout this section

we will be working with p inverted, and we will write G for G_E without further comment. We write \mathfrak{g}_E for the Lie algebra of G .

A morphism of G -torsors $f : D \rightarrow D'$ is a morphism of the underlying schemes which is equivariant for the action of G . Such a morphism is necessarily an isomorphism. The G -equivariant automorphisms of D , which we denote by $\text{Aut}_G(D)$, form a group, and it makes sense to talk about homomorphisms $r : W_K \rightarrow \text{Aut}_G(D)$.

Definition 2.1.1. Let $G - \text{WD}_E(L/K)$ be the groupoid whose fiber over an E -algebra A is a G -torsor D over A together with a pair (r, N) , where now $r : W_K \rightarrow \text{Aut}_G(D)$ is a representation of the Weil group such that $r|_{I_L}$ is trivial, $N \in \text{Lie } \text{Aut}_G(D)$, and $N = p^{-v(g)f\kappa} \text{Ad}(r(g))(N)$ for all $g \in W_K$.

We define a cover of this groupoid by a scheme as follows. The exact sequence

$$0 \rightarrow I_K \rightarrow W_K \rightarrow \langle \varphi^{f\kappa} \rangle \cong \mathbf{Z} \rightarrow 0$$

is non-canonically split, and choosing a splitting is the same as choosing a lift $g_0 \in W_K$ of $\varphi^{f\kappa}$. Thus, to specify a representation $r : W_K \rightarrow \text{Aut}_G(D)$, it suffices to specify $r|_{I_K}$ and $r(g_0)$ (which we denote Φ). Since we are interested in representations which are trivial on I_L , we may replace $r|_{I_K}$ with $r|_{I_{L/K}}$. For an E -algebra A , we let $\text{Rep}_A I_{L/K}$ denote the set of A -linear representations of $I_{L/K}$ on $G(A)$.

Definition 2.1.2. Choose $g_0 \in W_K$ lifting $\varphi^{f\kappa}$. We let $Y_{L/K, \varphi, N}$ be the functor on the category of E -algebras whose A -points are triples

$$(\Phi, N, \tau) \in G(A) \times \mathfrak{g}_E(A) \times \text{Rep}_A I_{L/K}$$

which satisfy

- $N = p^{-f\kappa} \text{Ad}(\Phi)(N)$,
- $\Phi \circ \tau(g) \circ \Phi^{-1} = \tau(g_0 g g_0^{-1})$ for all $g \in I_{L/K}$, and
- $N = \text{Ad}(\tau(g))(N)$ for all $g \in \text{Gal}_{L/K}$.

To go from $Y_{L/K, \varphi, N}$ to $G - \text{WD}_E(L/K)$, we need to forget the trivializing section and also forget g_0 ; the representation associated to (Φ, N, τ) is given by

$$r(g_0^n i) = \Phi^n \tau(h)$$

where $n \in \mathbf{Z}$ and $h \in I_K$.

The functor $Y_{L/K, \varphi, N}$ is visibly represented by a finite-type affine scheme over E , and there is an action of G on $Y_{L/K, \varphi, N}$ given by

$$a \cdot (\Phi, N, \{r(g)\}_{g \in I_{L/K}}) := (a\Phi a^{-1}, \text{Ad}(a)(N), \{ar(g)a^{-1}\}_{g \in I_{L/K}}).$$

It is easy to show (see the proof of Lemma 2.6.3 below for a very similar argument) that the quotient stack $[Y_{L/K, \varphi, N}/G]$ is equivalent to the groupoid $G - \text{WD}_E(L/K)$.

Similarly, we let $Y_{L/K, N}$ denote the functor on the category of E -algebras parametrizing pairs

$$(N, \tau) \in \mathfrak{g}_E(A) \times \text{Rep}_A I_{L/K}$$

such that $N = \text{Ad}(\tau(g))(N)$ for all $g \in I_{L/K}$; and we let $Y_{L/K}$ be the functor on the category of E -algebras, whose A -points are $\text{Rep}_A I_{L/K}$.

Let K'/K be a finite extension, and write L'/K' for the compositum of K' and L . Then L'/K' is Galois, with Galois group $\text{Gal}_{L'/K'} \subset \text{Gal}_{L/K}$. There are versions of the above functors for L'/K' which we write $Y_{L'/K', \varphi, N}$, $Y_{L'/K', N}$, and $Y_{L'/K'}$. Restriction of Weil–Deligne representations from W_K to $W_{K'}$ induces morphisms $Y_{L/K, \varphi, N} \rightarrow Y_{L'/K', \varphi, N}$, $Y_{L/K, N} \rightarrow Y_{L'/K', N}$ and $Y_{L/K} \rightarrow Y_{L'/K'}$.

2.2. A tangent-obstruction theory for $\mathrm{WD}_E(L/K)$. Choose an object $D_A \in G - \mathrm{WD}_E(L/K)$ with coefficients in an E -algebra A . Choose $g_0 \in W_K$ which lifts φ^{f_K} and write $\Phi := r(g_0)$, let $\mathrm{Ad}(\Phi)$ denote the action on $\mathrm{ad}D_A$ given by differentiating the homomorphism $G \rightarrow G$ given by $g \mapsto \Phi g \Phi^{-1}$, and let ad_N act by $x \mapsto [N, x]$. If $G = \mathrm{GL}_n$, these actions become $x \mapsto \Phi \circ x \circ \Phi^{-1}$ and $x \mapsto N \circ x - x \circ N$, respectively. Then we have an anti-commutative diagram

$$\begin{array}{ccc} (\mathrm{ad}D_A)^{I_{L/K}} & \xrightarrow{1-\mathrm{Ad}(\Phi)} & (\mathrm{ad}D_A)^{I_{L/K}} \\ \downarrow \mathrm{ad}_N & & \downarrow \mathrm{ad}_N \\ (\mathrm{ad}D_A)^{I_{L/K}} & \xrightarrow{p^{-f_K}\mathrm{Ad}(\Phi)-1} & (\mathrm{ad}D_A)^{I_{L/K}} \end{array}$$

Here $g \in I_{L/K}$ acts on $\mathrm{ad}D_A$ via $\mathrm{Ad}(\tau(g))$; note that the minus sign in p^{-f_K} arises because g_0 is a lift of arithmetic Frobenius. This diagram does not depend on our choice of g_0 , because any two lifts of φ^{f_K} differ by an element of $I_{L/K}$, which acts trivially on $(\mathrm{ad}D_A)^{I_{L/K}}$.

The total complex $C^\bullet(D_A)$ of this double complex controls the deformation theory of objects of $G - \mathrm{WD}_E(L/K)$. We write $H^i(\mathrm{ad}D_A)$ for the cohomology groups of $C^\bullet(D_A)$. The following result will be proved in a very similar way to [Kis08, Proposition 3.1.2], which is an analogous result for semilinear representations in the case $G = \mathrm{GL}_n$.

Proposition 2.2.1. *Let A be a local E -algebra with maximal ideal \mathfrak{m}_A and let $I \subset A$ be an ideal with $\mathrm{Im}_A = (0)$. Let $D_{A/I}$ be an object of $G - \mathrm{WD}_E(L/K)$ with coefficients in A/I , with Weil–Deligne representation (\bar{r}, \bar{N}) . Then*

- (1) *if $H^2(\mathrm{ad}D_{A/\mathfrak{m}_A}) = 0$, then there exists an object D_A in $G - \mathrm{WD}_E(L/K)$ with coefficients in A , such that $(A/I) \otimes_A D_A \cong D_{A/I}$, and*
- (2) *the set of isomorphism classes of liftings of $D_{A/I}$ to D_A is either empty or a torsor under $I \otimes_{A/\mathfrak{m}_A} H^1(\mathrm{ad}D_{A/\mathfrak{m}_A})$.*

We begin by proving a preliminary lemma.

Lemma 2.2.2. *Let D_A be a G -torsor over A , and suppose there is a representation $\bar{r} : W_K \rightarrow \mathrm{Aut}_G(D_{A/I})$ such that $\bar{r}|_{I_L}$ is trivial. Then there is a representation $r : W_K \rightarrow \mathrm{Aut}_G(D_A)$ such that $r|_{I_L}$ is trivial and r lifts \bar{r} . Moreover, the set of infinitesimal automorphisms of r (as a lift of \bar{r}) is a torsor under $H^0(W_K/I_L, I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}^{I_L}) = I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}^{W_K}$, and the set of lifts of \bar{r} is a torsor under $H^1(W_K/I_K, I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}^{I_L})$.*

Proof. An isomorphism $\bar{f} : D_{A/I} \rightarrow D_{A/I}$ lifts to an isomorphism $f : D_A \rightarrow D_A$, and the set of such lifts is a torsor under either a left- or right-action of $H^0(A, I \otimes_{A/\mathfrak{m}_A} D_{A/\mathfrak{m}_A})$ by [Bel16, Lemma 3.5]. Thus, for each $g \in W_K$, we can lift the map $\bar{r}(g) : D_{A/I} \rightarrow D_{A/I}$ to an isomorphism $r(g) : D_A \rightarrow D_A$.

The assignment

$$(g_1, g_2) \mapsto r(g_1)r(g_2)r(g_1g_2)^{-1}$$

is a 2-cocycle of W_K/I_L valued in $I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}$. Since we are in characteristic 0, and $I_{L/K}$ is a finite group, the Hochschild–Serre spectral sequence implies that for each $i > 0$, we have an isomorphism

$$H^i(W_K/I_K, I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}^{I_{L/K}}) \xrightarrow{\sim} H^i(W_K/I_L, I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}).$$

In particular,

$$H^2(W_K/I_L, I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}) \cong H^2(\mathbf{Z}, I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}^{I_{L/K}}) = 0,$$

so \bar{r} lifts to a representation $r : W_K \rightarrow \mathrm{Aut}_G(D)$ with $r|_{I_L} = 0$, as claimed.

An isomorphism $f : D_A \rightarrow D_A$ is an automorphism of r if and only if it is the identity modulo I and $r(g) \circ f = f \circ r(g)$ for all $g \in W_K$. Equivalently, $f \in I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_A^{W_K}$, as desired.

Finally, if $r' : W_K \rightarrow \mathrm{Aut}_G(D)$ is another such lift, then $g \mapsto r'(g)r(g)^{-1}$ is a 1-cocycle of W_K/I_L valued in $I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}$. But $H^1(W_K/I_L, I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}) \cong H^1(W_K/I_K, I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}^{I_{L/K}})$, so we are done. \square

Proof of Proposition 2.2.1. By [Bel16, Lemma 3.4], the underlying G -torsor $D_{A/I}$ lifts to a G -torsor D_A over $\mathrm{Spec} A$, and D_A is unique up to isomorphism, and by Lemma 2.2.2, \bar{r} lifts to a representation $r : W_K \rightarrow \mathrm{Aut}_G(D_A)$. Moreover, by [Bel16, Lemma 3.7], $\bar{N} \in \mathrm{ad}D_{A/I}$ lifts to some $N \in \mathrm{ad}D_A$ such that $\mathrm{Ad}(r(g))(N) = N$ for all $g \in I_{L/K}$, and any two lifts differ by an element of $I \otimes_{A/\mathfrak{m}_A} (\mathrm{ad}D_{A/\mathfrak{m}_A})^{I_{L/K}}$.

Now D_A , together with r and N , is an object of $G - \mathrm{WD}_E(L/K)$ if and only if $N = p^{-f\kappa} \mathrm{Ad}(\Phi)(N)$, where $\Phi := r(\varphi^{f\kappa})$. We define

$$h := N - p^{-f\kappa} \mathrm{Ad}(\Phi)(N) \in I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}^{I_{L/K}}.$$

If $H^2(D_{A/\mathfrak{m}_A}) = 0$, then there exist $f, g \in I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}^{I_{L/K}}$ such that $h = \mathrm{ad}_{\bar{N}}(f) + (p^{-f\kappa} \mathrm{Ad}(\Phi) - 1)(g)$. Then we claim that if we define $\tilde{N} := N + g$ and $\tilde{\Phi} := f^{-1} \circ \Phi$, then $\tilde{N} = p^{-f\kappa} \mathrm{Ad}(\tilde{\Phi})(\tilde{N})$. Indeed,

$$\begin{aligned} \tilde{N} - p^{-f\kappa} \mathrm{Ad}(\tilde{\Phi})(\tilde{N}) &= N + g - p^{-f\kappa} (\mathrm{Ad}(1 - f) \circ \mathrm{Ad}(\Phi))(N + g) \\ &= N + g - p^{-f\kappa} \mathrm{Ad}(\Phi)(N) - p^{-f\kappa} \mathrm{Ad}(\Phi)(g) \\ &\quad + p^{-f\kappa} [f, \mathrm{Ad}(\Phi)(N)] + p^{-f\kappa} [f, \mathrm{Ad}(\Phi)(g)] \\ &= \mathrm{ad}_{\bar{N}}(f) + p^{-f\kappa} [f, \mathrm{Ad}(\Phi)(N)] \\ &= [h, f] = 0 \end{aligned}$$

Here we have used that $f, g, h \in I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}^{\mathrm{Gal} L/K}$ and $I \cdot I \subset \mathrm{Im} \mathfrak{m}_A = 0$, so the Lie brackets $[f, \mathrm{Ad}(\Phi)(g)]$ and $[h, f]$ vanish. This proves part (1).

Now suppose that $\tilde{N} = p^{-f\kappa} \mathrm{Ad}(\tilde{\Phi})(\tilde{N})$, and let $f, g \in I \otimes_{A/\mathfrak{m}_A} \mathrm{ad}D_{A/\mathfrak{m}_A}^{I_{L/K}}$. Define $\tilde{N}' := N + g$ and define $\tilde{\Phi}' := f^{-1} \circ \tilde{\Phi}$. Then

$$\begin{aligned} \tilde{N}' - p^{-f\kappa} \mathrm{Ad}(\tilde{\Phi}')(\tilde{N}') &= \tilde{N} + g - p^{-f\kappa} \mathrm{Ad}(\tilde{\Phi})(\tilde{N}) - p^{-f\kappa} \mathrm{Ad}(\tilde{\Phi})(g) \\ &\quad + p^{-f\kappa} [f, \mathrm{Ad}(\tilde{\Phi})(\tilde{N})] + p^{-f\kappa} [f, \mathrm{Ad}(\tilde{\Phi})(g)] \\ &= (1 - p^{-f\kappa} \mathrm{Ad}(\tilde{\Phi}))(g) + [f, \tilde{N}] \\ &= -(p^{-f\kappa} \mathrm{Ad}(\Phi) - 1)(g) - \mathrm{ad}_N(f). \end{aligned}$$

Thus, $\tilde{\Phi}', \tilde{N}'$ give another lift if and only if $(f, g) \in \ker(d^1)$.

Moreover, if $(\tilde{\Phi}', \tilde{N}')$ is another lift, it is isomorphic to $(\tilde{\Phi}, \tilde{N})$ if and only if there is some $j \in I \otimes_{A/\mathfrak{m}_A} D_{A/\mathfrak{m}_A}^{I_{L/K}}$ such that $\tilde{N}' = \mathrm{Ad}(1 + j)(\tilde{N})$ and $(1 + j)\tilde{\Phi} = \tilde{\Phi}'(1 + j)$. This is equivalent to $\tilde{N} - \tilde{N}' = \mathrm{ad}_N(j)$ and $\tilde{\Phi}(\tilde{\Phi}')^{-1} = 1 - (1 - \mathrm{Ad}(\Phi))(j)$. In other words, $(\tilde{\Phi}, \tilde{N})$ and $(\tilde{\Phi}', \tilde{N}')$ differ by an element of $\mathrm{im}(d^0)$, as required. \square

2.3. Construction of smooth points. We wish to show that “most” points of $Y_{L/K,\varphi,N}$ are smooth, and so are their images in $Y_{L'/K',\varphi,N}$ for any finite extension K'/K . In this section we will consider a single fixed extension K'/K , and in section 2.4 below we will deduce a result for all extensions K'/K simultaneously.

We begin by fixing an inertial type $\tau : I_{L/K} \rightarrow G(E)$. This amounts to considering the fiber of $Y_{L/K,\varphi,N} \rightarrow Y_{L/K}$ over the point corresponding to τ . Next, we observe that if we can find $r : W_K \rightarrow G(E)$ such that $r|_{I_K} = \tau$, then $\Phi := r(g_0)$ is an element of the algebraic group defined over E

$$N_G(\tau) := \{h \in G : hr(g)h^{-1} \in r(I_{L/K}) \text{ for all } g \in I_{L/K}\}.$$

Note that Φ is not necessarily an element of the centraliser

$$Z_G(\tau) := \{h \in G : hr(g)h^{-1} = r(g) \text{ for all } g \in I_{L/K}\}.$$

However, since $I_{L/K}$ is finite (and in particular has only finitely many automorphisms), $Z_G(\tau) \subset N_G(\tau)$ has finite index; so we have $Z_G(\tau)^\circ = N_G(\tau)^\circ$ and $\text{Lie } Z_G(\tau) = \text{Lie } N_G(\tau)$. In particular, this implies that $N_G(\tau)$ and $Z_G(\tau)$ are reductive:

Theorem 2.3.1. *The normalizer $N_G(\tau) := \{h \in G : hr(g)h^{-1} \in r(I_{L/K}) \text{ for all } g \in I_{L/K}\}$ of $\tau(I_{L/K})$ is a reductive group.*

Proof. Since we are working over a field of characteristic 0, it is enough to prove that the connected component of the identity $N_G(\tau)^\circ = Z_G(\tau)^\circ$ is reductive. The conjugation action of $I_{L/K}$ on G preserves connected components, and $Z_G(\tau)^\circ = Z_{G^\circ}(\tau)^\circ$. But reductivity for the latter group follows from [PY02, Theorem 2.1], which states that when a finite group acts on a connected reductive group, the connected component of the identity of the fixed points is reductive. \square

Remark 2.3.2. Prasad and Yu prove their result under the assumption that the characteristic of the ground field does not divide the order of the group. Conrad, Gabber, and Prasad prove a more general result [CGP15, Proposition A.8.12], assuming only that the algebraic group acting is geometrically linearly reductive.

Our hypotheses imply that $N \in \text{Lie } Z_G(\tau)$ and $\Phi \in N_G(\tau)$. However, if (r, N) exists and has the correct inertial type, the set of $\Phi \in G(E)$ compatible with $r|_{I_{L/K}}$ and N is a torsor under $Z_G(\tau) \cap Z_G(N)$.

We now briefly recall the theory of associated cocharacters over a field of characteristic 0; we refer the reader to [Jan04] (in particular section 5) for further details and proofs. We will not draw attention to the assumption that our ground field has characteristic 0 below (but we will frequently use it); on the other hand, we do explain why the results that we are recalling hold over arbitrary fields of characteristic 0, which are not necessarily algebraically closed.

If $N \in \mathfrak{g}$ is nilpotent, a cocharacter $\lambda : \mathbf{G}_m \rightarrow G$ is said to be *associated* to N if

- $\text{Ad}(\lambda(t))(N) = t^2N$, and
- λ takes values in the derived subgroup of a Levi subgroup $L \subset G$ for which $N \in \mathfrak{l} := \text{Lie } L$ is distinguished (that is, every torus contained in $Z_L(N)$ is contained in the center of L).

By [McN04, Thm. 26], for any N there exists a cocharacter associated to N which is defined over the same field as N . Any two cocharacters associated to N are conjugate under the action of $Z_G(N)^\circ$.

An \mathfrak{sl}_2 -triple is as usual a non-zero triple (X, H, Y) of elements of \mathfrak{g} such that $[H, X] = 2X$, $[H, Y] = -2Y$, and $[X, Y] = H$. The Jacobson–Morozov theorem [Bou05, Ch. VIII §11 Prop. 2] states that for a non-zero nilpotent element N in a semisimple Lie algebra, an \mathfrak{sl}_2 -triple (N, H, Y) always exists, and any two such triples (N, H, Y) and (N, H', Y') are conjugate under the action of $Z_G(N)^\circ$ [Bou05, Ch. VIII §11 Prop. 1]. Given a pair (N, H) such that $[H, N] = 2N$ and $H \in [N, \mathfrak{g}]$, it is possible to construct an \mathfrak{sl}_2 -triple (N, H, Y) [Bou05, Ch. VIII §11 Lem. 6] (or the zero triple if $N = H = 0$). Since SL_2 is simply connected, this implies that there is a homomorphism $\mathrm{SL}_2 \rightarrow G$ which sends the “standard” basis for \mathfrak{sl}_2 to (N, H, Y) .

If we let $\lambda : \mathbf{G}_m \rightarrow \mathrm{SL}_2 \rightarrow G$ be the composition of the cocharacter $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ with this homomorphism $\mathrm{SL}_2 \rightarrow G$, then λ is associated to N . Moreover, the association $\lambda \mapsto d\lambda(1)$ sends cocharacters associated to N to elements H such that $[H, N] = 2N$ and $H \in [N, \mathfrak{g}]$, and this is an injective map [Jan04, Prop. 5.5] (this reference assumes that the ground field is algebraically closed, but this hypothesis is not used). Thus (in characteristic 0) associated cocharacters are a group-theoretic analogue of the Jacobson–Morozov theorem.

We use the following properties of associated cocharacters; the given reference assumes the ground field is algebraically closed, but these statements can all be checked after extension of the ground field.

Proposition 2.3.3 ([Jan04, 5.9,5.10,5.11]).

- (1) *The associated parabolic $P_G(\lambda)$ depends only on N , not on the choice of associated cocharacter.*
- (2) *We have $Z_G(N) \subset P_G(\lambda)$. In particular, $Z_G(N) = Z_{P_G(\lambda)}(N)$.*
- (3) *$Z_G(N) = (U_G(\lambda) \cap Z_G(N)) \rtimes (Z_G(\lambda) \cap Z_G(N))$*
- (4) *$Z_G(\lambda) \cap Z_G(N)$ is reductive.*

In particular, by Proposition 2.3.3 (3), the disconnectedness of $Z_G(N)$ is entirely accounted for by the disconnectedness of $Z_G(\lambda) \cap Z_G(N)$. Since the fibers of the map $Y_{L/K, \varphi, N} \rightarrow Y_{L/K, N}$ are torsors under groups of the form $Z_G(\tau) \cap Z_G(\lambda) \cap Z_G(N)$, this will permit us to pick out the connected components of these fibers. We will use the following lemma in the proof of Theorem 2.3.6 below.

Lemma 2.3.4. *If λ is an associated cocharacter of N , then the weight-2 part of \mathfrak{g} for the adjoint action of λ is in the image of ad_N .*

Proof. If $N = 0$, then λ is the constant cocharacter and the corresponding weight-2 subspace is trivial. Otherwise, we may find an \mathfrak{sl}_2 -triple of the form $(N, d\lambda(1), Y)$ and view \mathfrak{g} as a representation of \mathfrak{sl}_2 . If $T \in \mathfrak{g}$ is in the weight-2 part, then $\frac{1}{2}[Y, T]$ is in the weight-0 part and

$$[N, \frac{1}{2}[Y, T]] = \frac{1}{2}[[N, Y], T] = \frac{1}{2}[d\lambda(1), T] = T$$

so T is in the image of ad_N . □

Let $f : G \rightarrow G'$ be a morphism of reductive groups over E , inducing a morphism $\mathfrak{g} \rightarrow \mathfrak{g}'$ on Lie algebras, which we also denote by f . We use the following lemma in the proof of Theorem 2.3.9 below.

Lemma 2.3.5. *If λ is an associated cocharacter for $N \in \mathfrak{g}$, then $f \circ \lambda$ is an associated cocharacter for $f(N)$.*

Proof. It is clear that $d\lambda(1)$ is semisimple. Then there exists some $Y \in \mathfrak{g}$ such that $(N, d\lambda(1), Y)$ is an \mathfrak{sl}_2 -triple, and therefore there is a homomorphism $\mathrm{SL}_2 \rightarrow G$ such that the precomposition with the diagonal is λ . The composition $\mathbf{G}_m \rightarrow \mathrm{SL}_2 \rightarrow G \rightarrow G'$ is $f \circ \lambda$. Moreover, if we consider the composition $\mathrm{SL}_2 \rightarrow G \rightarrow G'$ and differentiate, we get a map $\mathfrak{sl}_2 \rightarrow \mathfrak{g}'$ sending the “standard” basis of \mathfrak{sl}_2 to $(f(N), f(d\lambda(1)), f(Y))$. This shows that $[f(d\lambda(1)), f(N)] = 2f(N)$ and $f(d\lambda(1))$ is in the image of $\mathrm{ad}_{f(N)}$. Since $f(d\lambda(1)) = d(f \circ \lambda)(1)$, this shows that $f \circ \lambda$ is associated to $f(N)$, by [Jan04, Prop. 5.5]. \square

If K'/K is a finite extension, we write $H_{L'/K'}^2$ for the coherent sheaf on $Y_{L'/K', \varphi, N}$ given by the cokernel of

$$(\mathrm{ad}D)^{I_{L'/K'}} \oplus (\mathrm{ad}D)^{I_{L'/K'}} \xrightarrow{\mathrm{ad}_{N_{L'}} - (p^{-f_{K'}} \mathrm{Ad}(\Phi^{f_{K'}/f_K}) - 1)} (\mathrm{ad}D)^{I_{L'/K'}}$$

where (D, Φ, N, τ) is the universal object over $Y_{L/K, \varphi, N}$; so by Proposition 2.2.1, the specialisation of $H_{L'/K'}^2$ at a closed point controls the obstruction theory of the restriction to $W_{K'}$ of the corresponding Weil–Deligne representation.

Theorem 2.3.6. *Let K'/K be a finite extension. Then there is a dense open subset $U \subset Y_{L/K, \varphi, N}$ (possibly depending on K') such that $H_{L'/K'}^2|_U = 0$.*

Proof. Since the support of $H_{L'/K'}^2$ is closed, it suffices to show that if we consider the map $Y_{L/K, \varphi, N} \rightarrow Y_{L/K, N}$, then each component of the fiber over some point $N \in Y_{L/K, N}$ contains a point (Φ, N) whose corresponding H^2 vanishes (when viewed as a point of $Y_{L'/K', \varphi, N}$).

To do this, we consider a new moduli problem $\tilde{Y}_{L/K, \varphi, N}$, which by definition is the functor on the category of E -algebras whose A -points are triples

$$(\Phi, N, \tau) \in N_G(\tau) \times \mathrm{Lie} Z_G(\tau) \times \mathrm{Rep}_A I_{L/K}$$

which satisfy $N = p^{-f_K} \mathrm{Ad}(\Phi)(N)$.

This is representable by an affine scheme which we also write as $\tilde{Y}_{L/K, \varphi, N}$, and there is a natural morphism $\tilde{Y}_{L/K, \varphi, N} \rightarrow Y_{L/K, N}$. Indeed, the map $Y_{L/K, \varphi, N} \rightarrow Y_{L/K, N}$ factors through the natural inclusion $Y_{L/K, \varphi, N} \hookrightarrow \tilde{Y}_{L/K, \varphi, N}$, and the fibers of $Y_{L/K, \varphi, N} \rightarrow Y_{L/K, N}$ are closed and open in the fibers of $\tilde{Y}_{L/K, \varphi, N} \rightarrow Y_{L/K, N}$. Thus, it suffices to study the fibers of the map $\tilde{Y}_{L/K, \varphi, N} \rightarrow Y_{L/K, N}$. (Note that the tangent-obstruction complex for objects of $G - \mathrm{WD}_E(L/K)$ makes sense over $\tilde{Y}_{L/K, \varphi, N}$ as well.)

Choose an associated cocharacter $\lambda : \mathbf{G}_m \rightarrow Z_G(\tau)^\circ$ for N , so that in particular $\mathrm{Ad}(\lambda(t))(N) = t^2 N$, and let $\Phi := \lambda(p^{f_K/2})$. Then (Φ, N, τ) is a point of $\tilde{Y}_{L/K, \varphi, N}$, and we wish to study the restriction $(\Phi^{f_{K'}/f_K}, N_{L'}, \tau|_{I_{L'/K'}})$.

If D denotes the underlying G -torsor for (Φ, N, τ) , and $\mathrm{ad}D$ denotes its pushout via the adjoint representation, then $\mathrm{Ad}(\Phi)$ and $\mathrm{Ad}(\Phi^{f_{K'}/f_K})$ are semi-simple operators on $(\mathrm{ad}D)^{I_{L/K}}$ and $(\mathrm{ad}D)^{I_{L'/K'}}$, respectively. Therefore, $p^{-f_K} \mathrm{Ad}(\Phi) - 1$ and $p^{f_{K'}} \mathrm{Ad}(\Phi^{f_{K'}/f_K}) - 1$ are semi-simple as well (since they are the difference of commuting semi-simple operators).

Thus, to compute the cokernel of $p^{-f_{K'}} \mathrm{Ad}(\Phi^{f_{K'}/f_K}) - 1$, it suffices to compute its kernel. Now $(\mathrm{ad}D)^{I_{L'/K'}}$ is graded by the adjoint action of $\lambda : \mathbf{G}_m \rightarrow Z_G(\tau) \subset Z_G(\tau|_{I_{L'/K'}})$, and if $(\mathrm{ad}D)_k^{I_{L'/K'}}$ denotes the weight- k subspace, then

$p^{-f_{K'}} \text{Ad}(\Phi^{f_{K'}/f_K}) - 1$ preserves it, so it suffices to compute

$$\ker(p^{-f_{K'}} \text{Ad}(\Phi^{f_{K'}/f_K}) - 1)|_{(\text{ad}D)_k^{I_{L'/K'}}}$$

for each k . But $p^{-f_{K'}} \text{Ad}(\Phi^{f_{K'}/f_K}) - 1$ acts invertibly unless $k = 2$ (in which case it acts by 0), so the cokernel of $p^{-f_{K'}} \text{Ad}(\Phi^{f_{K'}/f_K}) - 1$ is exactly $(\text{ad}D)_2^{I_{L'/K'}}$. By Lemma 2.3.4, the weight-2 part of $\mathfrak{g}^{I_{L'/K'}}$ is in the image of ad_N , so we conclude that $H_{L'/K'}^2$ vanishes at (Φ, N) , and at its image in $Y_{L'/K', \varphi, N}$.

We need to find similar points on every connected component of the fiber of $\tilde{Y}_{L/K, \varphi, N} \rightarrow Y_{L/K, N}$ over $N \in Y_{L/K, N}$. This fiber is a torsor under $N_G(\tau) \cap Z_G(N)$, and the disconnectedness of $N_G(\tau) \cap Z_G(N)$ is entirely accounted for by the disconnectedness of $N_G(\tau) \cap Z_G(\lambda) \cap Z_G(N)$, by [Bel16, Prop. 4.9] (applied with $G' = N_G(\tau)$). On each component of $N_G(\tau) \cap Z_G(N)$, we may therefore by [Bel16, Lem. 5.3] choose a finite-order element $c \in N_G(\tau) \cap Z_G(\lambda) \cap Z_G(N)$. (Note that $N_G(\tau) \cap Z_G(\lambda) \cap Z_G(N) = Z_{N_G(\tau)}(N) \cap Z_{N_G(\tau)}(\lambda)$ is reductive by Proposition 2.3.3.)

We now check that that $H_{L/K}^2$ and $H_{L'/K'}^2$ vanish at the points of $\tilde{Y}_{L/K, \varphi, N}$ and $\tilde{Y}_{L'/K', \varphi, N}$, respectively, corresponding to $(\Phi \cdot c, N)$.

Firstly, we claim that $p^{-f_{K'}} \text{Ad}((\Phi \cdot c)^{f_{K'}/f_K}) - 1$ is semi-simple, or equivalently, that $\text{Ad}((\Phi \cdot c)^{f_{K'}/f_K})$ is semi-simple. For this, it suffices to check that some iterate of $\text{Ad}((\Phi \cdot c)^{f_{K'}/f_K})$ is semi-simple (since we are in characteristic 0). Let n be the order of c . Since c and $\Phi = \lambda(p^{f_K/2})$ commute, $\text{Ad}(\Phi^{f_{K'}/f_K} \cdot c)^n = \text{Ad}(\Phi^{nf_{K'}/f_K} \cdot c^n) = \text{Ad}(\Phi^{nf_{K'}/f_K})$. But since $\text{Ad}(\Phi)$ is semi-simple by construction, so is $\text{Ad}(\Phi^{nf_{K'}/f_K})$, as claimed.

Thus, to compute the cokernel of $p^{-f_{K'}} \text{Ad}((\Phi \cdot c)^{f_{K'}/f_K}) - 1$, it suffices to compute its kernel, which is contained in the kernel of $p^{-nf_{K'}} \text{Ad}(\Phi^{nf_{K'}/f_K}) - 1$. Since $p^{-nf_{K'}} \text{Ad}(\Phi^{nf_{K'}/f_K}) - 1$ acts invertibly on each weight space $(\text{ad}D)_k^{I_{L'/K'}}$ unless $k = 2$, the cokernel of $p^{-f_{K'}} \text{Ad}(\Phi^{f_{K'}/f_K} \cdot c) - 1$ is contained in $(\text{ad}D)_2^{I_{L'/K'}}$. Since $(\text{ad}D)_2^{I_{L'/K'}}$ is again in the image of ad_N by Lemma 2.3.4, we are done. \square

Remark 2.3.7. If N and N' are defined over E' and conjugate by an element of $G(E')$, then $N_G(\tau) \cap Z_G(N)$ and $N_G(\tau) \cap Z_G(N')$ are also conjugate over E' . We may therefore take the associated cocharacters λ and λ' of N and N' , and the finite-order points in $N_G(\tau) \cap Z_G(\lambda) \cap Z_G(N)$ and $N_G(\tau) \cap Z_G(\lambda') \cap Z_G(N)$ to be conjugate over E' . Since there are only finitely many E' -rational orbits in $Y_{L/K, N}$, it follows that if $Y_{L/K, N}(E') \subset Y_{L/K, N}(\bar{E})$ is Zariski dense, there is a finite extension E''/E' and a set $Z \subset Y_{L/K, \varphi, N}(E'')$ such that any Zariski open subscheme $U \subset Y_{L/K, \varphi, N}$ containing Z is Zariski dense.

Corollary 2.3.8. *The stack $G - \text{WD}_E(L/K)$ is generically smooth, and is equidimensional of dimension 0; equivalently, the scheme $Y_{L/K, \varphi, N}$ is generically smooth, and is equidimensional of dimension $\dim G$. The nonsmooth locus is precisely the locus of Weil–Deligne representations D with $H^2(\text{ad}D) \neq 0$.*

Proof. It is enough to prove the statement for $Y_{L/K, \varphi, N}$. Let $U \subset Y_{L/K, \varphi, N}$ be the dense open subscheme provided by Theorem 2.3.6 (with $K' = K$). Then at each closed point x of U , it follows from Lemma 2.2.2 and Proposition 2.2.1 that $Y_{L/K, \varphi, N}$ is formally smooth at x . Furthermore, for any closed point x of $Y_{L/K, \varphi, N}$ with corresponding Weil–Deligne representation D_x , the dimension of the tangent

space at x is $\dim G + \dim H^0(D_x) - \dim H^1(D_x)$. Since the Euler characteristic of $C^\bullet(D_x)$ is zero, this is equal to $\dim G + \dim H^2(D_x)$, and the remaining claims follow immediately. \square

If $G \rightarrow G'$ is a morphism of reductive groups over E , then for any family of G -torsors D over $\text{Spec } A$, we can push out to a family D' of G' -torsors. Therefore, the moduli space $Y_{L/K, \varphi, N}$ of (framed) G -valued $(\varphi, N, \text{Gal}_{L/K})$ -modules carries a family D' of G' -torsors, and $\text{ad} D' := \text{Lie Aut}_{G'}(D')$ is a coherent sheaf on $Y_{L/K, \varphi, N}$. Since D is a trivial G -torsor, D' is a trivial G' -torsor. Since pushing out G -torsors to G' -torsors is functorial, D' is a family of G' -valued $(\varphi, N, \text{Gal}_{L/K})$ -modules and we can construct the complex $C^\bullet(D')$. We let $H_{G'}^2$ denote its cohomology in degree 2.

Theorem 2.3.9. *Let $f : G \rightarrow G'$ be a morphism of reductive groups over E . Then there is a dense open subset $U \subset Y_{L/K, \varphi, N}$ (possibly depending on G') such that $H_{G'}^2|_U = 0$.*

Proof. As in the proof of Theorem 2.3.6, it suffices to construct a point on each connected component of each fiber of the map $Y_{L/K, \varphi, N} \rightarrow Y_{L/K, N}$ where $H_{G'}^2$ vanishes. In fact, the same points work: by Lemma 2.3.5 the composition $f \circ \lambda$ is an associated cocharacter for $f_*(N)$. Therefore, $H_{G'}^2$ vanishes at the point corresponding to $(\lambda(p^{f\kappa/2}), N)$. Similarly, if $c \in N_G(\tau) \cap Z_G(\lambda) \cap Z_G(N)$ is a finite order point, then $H_{G'}^2$ vanishes at the point corresponding to $(\lambda(p^{f\kappa/2}) \cdot c, N)$. \square

Remark 2.3.10. The proofs of Theorems 2.3.6 and 2.3.9 justify the claim we made in the introduction, that all of the smooth points that we explicitly construct arise from pushing out a single “standard” smooth point for SL_2 . Indeed, as discussed above, given an associated cocharacter λ for N , the map $\lambda \mapsto d\lambda(1)$ allows us to determine a homomorphism $\text{SL}_2 \rightarrow G$, and we see that the choice of Φ, N made in the proof of Theorem 2.3.6 is the image under this homomorphism of the elements Φ, N for SL_2 discussed in the introduction.

Remark 2.3.11. The Jacobson–Morosov theorem allows one to think of semisimple Weil–Deligne representations as representations of $W_K \times \text{SL}_2$; see [GR10, Prop. 2.2] for a precise statement. From this perspective, our construction of smooth points from associated cocharacters can be summarised as follows: given a nilpotent $N \in \text{Lie } G$, we obtain a map $\text{SL}_2 \rightarrow G$, and the corresponding Weil–Deligne representation is obtained by composing with the map

$$W_K \times \text{SL}_2 \rightarrow \text{SL}_2$$

which on the first factor is unramified and takes an arithmetic Frobenius to the matrix $\begin{pmatrix} p^{f\kappa} & 0 \\ 0 & p^{-f\kappa} \end{pmatrix}$, and is the identity on the second factor.

2.4. Tate local duality for Weil–Deligne representations. If D is a G -valued Weil–Deligne representation over a field E , we can also prove an analogue of Tate local duality for the complex $C^\bullet(D)$. In addition to allowing us to compute with either kernels or cokernels, this pairing allows us to give an explicit characterisation of the smooth locus (see Corollary 2.4.2). It is presumably possible to work at the level of the derived category, but for our purposes there is no need to do so, and we work directly with the cohomology groups instead. In fact, since we only need the pairing between H^0 and H^2 , we have not worked out the details of the pairing on H^1 s, which for reasons of space we leave to the interested reader.

To construct pairings $H^i((\mathrm{ad}D)^*(1)) \times H^{2-i}(\mathrm{ad}D) \rightarrow E(1)$, we use the evaluation pairing $\mathrm{ev} : (\mathrm{ad}D)^* \times \mathrm{ad}D \rightarrow E$. Here the “(1)” means that we multiply the action of $\mathrm{Ad}(\Phi)$ by $p^{f\kappa}$. Note that if $X \in (\mathrm{ad}D)^*$, $Y \in \mathrm{ad}D$, then $\mathrm{ev}(\mathrm{Ad}(\Phi)(X), \mathrm{Ad}(\Phi)(Y)) = \mathrm{ev}(X, Y)$, and if $X \in (\mathrm{ad}D)^*(1)$, $Y \in \mathrm{ad}D$, then $\mathrm{ev}(\mathrm{Ad}(\Phi)(X), \mathrm{Ad}(\Phi)(Y)) = p^{f\kappa} \mathrm{ev}(X, Y) = \mathrm{Ad}(\Phi)(\mathrm{ev}(X, Y))$.

Proposition 2.4.1. *Let D be as above. Then the evaluation pairing induces a perfect pairing $H^0((\mathrm{ad}D)^*(1)) \times H^2(\mathrm{ad}D) \rightarrow E(1)$.*

Proof. We first check that the pairing $\mathrm{ev} : (\mathrm{ad}D)^*(1) \times \mathrm{ad}D \rightarrow E(1)$ descends to a well-defined pairing $H^0((\mathrm{ad}D)^*(1)) \times H^2(\mathrm{ad}D) \rightarrow E(1)$. If $X \in (\mathrm{ad}D)^*(1)$ is in the kernel of ad_N and the kernel of $1 - \mathrm{Ad}(\Phi)$, and $Y \in \mathrm{ad}D$, then

$$\begin{aligned} \mathrm{ev}(X, Y + \mathrm{ad}_N(Z)) &= \mathrm{ev}(X, Y) + \mathrm{ev}(X, \mathrm{ad}_N(Z)) \\ &= \mathrm{ev}(X, Y) - \mathrm{ev}(\mathrm{ad}_N(X), Z) \\ &= \mathrm{ev}(X, Y), \end{aligned}$$

and

$$\begin{aligned} \mathrm{ev}(X, Y + (p^{-f\kappa} \mathrm{Ad}(\Phi) - 1)(Z)) &= \mathrm{ev}(X, Y) + \mathrm{ev}(X, p^{-f\kappa} \mathrm{Ad}(\Phi)(Z)) - \mathrm{ev}(X, Z) \\ &= \mathrm{ev}(X, Y) + p^{-f\kappa} \mathrm{ev}(\mathrm{Ad}(\Phi)(X), \mathrm{Ad}(\Phi)(Z)) - \mathrm{ev}(X, Z) \\ &= \mathrm{ev}(X, Y) + \mathrm{ev}(X, Z) - \mathrm{ev}(X, Z) \\ &= \mathrm{ev}(X, Y), \end{aligned}$$

so the pairing is indeed well-defined.

Next, we need to check that this pairing is perfect. Suppose $X \in H^0((\mathrm{ad}D)^*(1))$ and $\mathrm{ev}(X, Y) = 0$ for all $Y \in H^2(\mathrm{ad}D)$. Then $\mathrm{ev}(X, Y) = 0$ for all $Y \in (\mathrm{ad}D)^{I_{L/K}}$, so $X = 0$. This implies that the natural map $H^0((\mathrm{ad}D)^*(1)) \rightarrow (H^2(\mathrm{ad}D)^*(1))$ is injective.

On the other hand, let $f : H^2(\mathrm{ad}D) \rightarrow E(1)$ be an element of $(H^2(\mathrm{ad}D)^*(1))$. By composition, we have a linear functional

$$f : (\mathrm{ad}D)^{I_{L/K}} \rightarrow H^2(\mathrm{ad}D) \rightarrow E(1)$$

This is an element of $(\mathrm{ad}D)^*(1)$; we need to show that $\mathrm{ad}_N(f) = (1 - \mathrm{Ad}(\Phi))(f) = 0$. But for any $Y \in (\mathrm{ad}D)^{I_{L/K}}$,

$$\mathrm{ev}(\mathrm{ad}_N(f), Y) = \mathrm{ev}(f, -\mathrm{ad}_N(Y)) = \mathrm{ev}(f, 0) = 0$$

since f factors through $H^2(\mathrm{ad}D)$. Similarly, for any $Y \in (\mathrm{ad}D)^{I_{L/K}}$,

$$\begin{aligned} \mathrm{ev}((1 - \mathrm{Ad}(\Phi))(f), \mathrm{Ad}(\Phi)(Y)) &= \mathrm{ev}(f, Y) - \mathrm{ev}(\mathrm{Ad}(\Phi)(f), \mathrm{Ad}(\Phi)(Y)) \\ &= \mathrm{ev}(f, \mathrm{Ad}(\Phi)(Y)) - p^{f\kappa} \mathrm{ev}(f, Y) = \mathrm{ev}(f, (\mathrm{Ad}(\Phi) - p^{f\kappa})(Y)) \\ &= p^{f\kappa} \mathrm{ev}(f, (p^{-f\kappa} \mathrm{Ad}(\Phi) - 1)(Y)) \\ &= p^{f\kappa} \mathrm{ev}(f, 0) = 0. \end{aligned}$$

Since $\mathrm{Ad}(\Phi) : (\mathrm{ad}D)^{I_{L/K}} \rightarrow (\mathrm{ad}D)^{I_{L/K}}$ is an isomorphism, this suffices. \square

Corollary 2.4.2. *The nonsmooth locus of the stack $G - \mathrm{WD}_E(L/K)$ is precisely the locus of Weil–Deligne representations D with $H^0((\mathrm{ad}D)^*(1)) \neq 0$.*

Proof. This is immediate from Corollary 2.3.8 and Proposition 2.4.1. \square

We now use Corollary 2.4.2 to deduce that there are a dense set of points of $Y_{L/K, \varphi, N}$ which give smooth points for every finite extension K'/K .

Definition 2.4.3. A point $x \in Y_{L/K,\varphi,N}$ is *very smooth* if its image in $Y_{L'/K',\varphi,N}$ is smooth for every finite extension K'/K .

Lemma 2.4.4. Fix a finite extension E'/E . There is a finite extension K'/K (which depends only on E') such that H^2 vanishes at the image of $x \in Y_{L/K,\varphi,N}(E')$ in $Y_{L'/K',\varphi,N}$ if and only if x is very smooth.

Proof. Suppose (D, Φ, N, τ) corresponds to a point of $Y_{L/K,\varphi,N}$ such that H^2 does not vanish at its image in $Y_{L''/K'',\varphi,N}$. By Corollary 2.4.2, this holds if and only if $H^0(\text{ad}D)^*(1)$ does not vanish.

Thus, it suffices to consider the injectivity of $1 - p^{f_{K''}} \text{Ad}(\Phi^{f_{K''}/f_K})^* : (\text{ad}D)^{f_{L''/K''}} \rightarrow (\text{ad}D)^{f_{L''/K''}}$ on $\ker(\text{ad}_N)$, where $\text{Ad}(\Phi^{f_{K''}/f_K})^*$ denotes the dual of $\text{Ad}(\Phi^{f_{K''}/f_K})$. If this map is not injective, this implies that $p^{f_K} \text{Ad}(\Phi)^*$ has a generalized eigenvalue λ satisfying $\lambda^{f_{K''}/f_K} = 1$. But the characteristic polynomial of $\text{Ad}(\Phi)$ acting on $\text{ad}D$ has degree $\dim \text{ad}D = \dim G$ and there are only finitely many roots of unity with minimal polynomial of bounded degree. There are also finitely many roots of unity in E' or its finite extensions. It follows that there are only a finite number of possibilities for λ .

In other words, to check whether $1 - p^{f_{K''}} \text{Ad}(\Phi^{f_{K''}/f_K})^*$ has a non-trivial kernel for any finite extension K''/K , it suffices to consider some fixed K' such that $f_{K'}/f_K$ is divisible by all n such that $\phi(n) \leq \dim G$ and such that $\tau|_{I_{L'/K'}}$ is trivial (where $\phi(n)$ denotes Euler's totient function), as required. \square

Corollary 2.4.5. The set of closed points of $G\text{-WD}_E(L/K)$ which are very smooth is Zariski dense.

Proof. Let E'/E be a finite extension such that $Y_{L/K,N}(E')$ is Zariski-dense in $Y_{L/K,N}$ and $Y_{L/K,\varphi,N}(E')$ is Zariski-dense in $Y_{L/K,\varphi,N}$, and let E'' and $Z \subset Y_{L/K,\varphi,N}(E'')$ be as in Remark 2.3.7. By Lemma 2.4.4, there is a finite extension K'/K such that $x \in Y_{L/K,\varphi,N}(E'')$ is very smooth if $H^2_{L'/K'}$ vanishes at x . But there is a Zariski-dense open subscheme $U \subset Y_{L/K,\varphi,N}$ containing Z where $H^2_{L'/K'}$ vanishes, and therefore $U \cap Y_{L/K,\varphi,N}(E'')$ is a Zariski-dense set of very smooth points in $Y_{L/K,\varphi,N}$. Since $Y_{L/K,\varphi,N}$ is a cover of $G - \text{WD}_E(L/K)$, the result follows. \square

2.5. l -adic Hodge theory. We suppose in this subsection that $l \neq p$. We briefly recall some results from [Fon94], which will allow us to relate l -adic representations of G_K to Weil–Deligne representations.

Recall that by a theorem of Grothendieck, a continuous representation $\rho : G_K \rightarrow \text{GL}_d(E)$ is automatically potentially semi-stable, in the sense that there is a finite extension L/K such that $\rho|_{I_L}$ is unipotent. After making a choice of a compatible system of l -power roots of unity in \overline{K} , we see from [Fon94, Prop. 1.3.3, 2.3.4] that there is an equivalence of Tannakian categories between the category of E -linear representations of G_K which become semi-stable over L , and the full subcategory of Weil–Deligne representations (r, N) of W_K over E with the properties that $r|_{I_L}$ is trivial and the roots of the characteristic polynomial of any arithmetic Frobenius element of W_L are l -adic units (such an equivalence is given by the functor \widehat{WD}_{pst} of [Fon94, §2.3.7]).

2.6. The case $l = p$: (φ, N) -modules. In this section we let $l = p$, and we explain the relationship between Weil–Deligne representations and (φ, N) -modules. Let K_0, L_0 be the maximal unramified subfields of K, L respectively, of respective degrees

f_K, f_L over \mathbf{Q}_p . Let E/\mathbf{Q}_p be a finite extension, which is large enough that it contains the image of all embeddings $L_0 \hookrightarrow E$, so that we may identify $E \otimes_{\mathbf{Q}_p} L_0$ with $\bigoplus_{L_0 \hookrightarrow E} E$. Let φ denote the arithmetic Frobenius.

For any E -algebra A , let $\text{Rep}_{A \otimes_{\mathbf{Q}_p} L_0} \text{Gal}_{L/K}$ denote the set of A -linear, L_0 -semilinear representations of $\text{Gal}_{L/K}$ on $\text{Res}_{E \otimes_{\mathbf{Q}_p} L_0/E} G(A) = G(A \otimes_{\mathbf{Q}_p} L_0)$, and let $\text{Rep}_A \text{Gal}_{L/K}$ denote the set of A -linear representations of $\text{Gal}_{L/K}$ on $G(A)$.

Here $\underline{\text{Ad}}\Phi$ and $\underline{\text{Ad}}\tau(g)$ are ‘‘twisted adjoint’’ actions on $\text{Lie Aut}_G D$; after pushing out Y by a representation $\sigma \in \text{Rep}_E(G)$, they are given by $M \mapsto \Phi_\sigma \circ M \circ \Phi_\sigma^{-1}$ and $M \mapsto \tau(g)_\sigma \circ M \circ \tau(g)_\sigma^{-1}$, respectively. Note that the action of $\text{Gal}_{L/K}$ on scalars factors through the abelian quotient $\langle \varphi^{f_K} \rangle$, which also commutes with φ , so $(g_1 g_2)^* = g_1^* \circ g_2^*$ and $g^* \varphi^* = \varphi^* g^*$.

This motivates us to define the following groupoid on E -algebras.

Definition 2.6.1. The category of G -valued $(\varphi, N, \text{Gal}_{L/K})$ -modules, which we denote $G - \text{Mod}_{L/K, \varphi, N}$, is the groupoid whose fiber over an E -algebra A consists of a $\text{Res}_{E \otimes_{\mathbf{Q}_p} L_0/E} G$ -torsor D over A , equipped with:

- an isomorphism $\Phi : \varphi^* D \xrightarrow{\sim} D$,
- a nilpotent element $N \in \text{Lie Aut}_G D$, and
- for each $g \in \text{Gal}_{L/K}$, an isomorphism $\tau(g) : g^* D \xrightarrow{\sim} D$.

These are required to satisfy the following compatibilities:

- (1) $\underline{\text{Ad}}\Phi(N) = \frac{1}{p}N$.
- (2) $\underline{\text{Ad}}\tau(g)(N) = N$ for all $g \in \text{Gal}_{L/K}$.
- (3) $\tau(g_1 g_2) = \tau(g_1) \circ g_1^* \tau(g_2)$ for all $g_1, g_2 \in \text{Gal}_{L/K}$.
- (4) $\tau(g) \circ g^* \Phi = \Phi \circ \varphi^* \tau(g)$ for all $g \in \text{Gal}_{L/K}$.

Requiring D to be a trivial $\text{Res}_{E \otimes_{\mathbf{Q}_p} L_0/E}$ -torsor equipped with a trivializing section lets us define a representable functor which covers $G - \text{Mod}_{L/K, \varphi, N, \tau}$, as follows.

Definition 2.6.2. Let $X_{L/K, \varphi, N}$ denote the functor on the category of E -algebras whose A -points are triples

$$(\Phi, N, \tau) \in (\text{Res}_{E \otimes_{\mathbf{Q}_p} L_0/E} G)(A) \times (\text{Res}_{E \otimes_{\mathbf{Q}_p} L_0/E} \mathfrak{g}_E)(A) \times \text{Rep}_{A \otimes_{\mathbf{Q}_p} L_0} \text{Gal}_{L/K}$$

which satisfy

- $N = p \underline{\text{Ad}}(\Phi)(N)$,
- $\tau(g) \circ \Phi = \Phi \circ \tau(g)$, and
- $\underline{\text{Ad}}(\tau(g))(N) = N$ for all $g \in \text{Gal}_{L/K}$.

This functor is visibly representable by a finite-type affine scheme over E , which we also denote by $X_{L/K, \varphi, N}$. Moreover, there is a left action of $\text{Res}_{E \otimes_{\mathbf{Q}_p} L_0/E} G$ on $X_{L/K, \varphi, N}$ coming from changing the choice of trivializing section. Explicitly,

$$a \cdot (\Phi, N, \{\tau(g)\}_{g \in \text{Gal}_{L/K}}) = (a \Phi \varphi(a)^{-1}, \text{Ad}(a)(N), \{a \tau(g)(g \cdot a)^{-1}\}_{g \in \text{Gal}_{L/K}}).$$

Recall that if Z is an E -scheme equipped with a left-action of an algebraic group H over E , then for any E -scheme S , the groupoid $[Z/H](S)$ over S is the category

$$[Z/H](S) := \{\text{Left } H\text{-bundle } D \rightarrow S \text{ and } H\text{-equivariant morphism } D \rightarrow Z\}.$$

A morphism $f : D \rightarrow D'$ in this fiber category is a morphism of H -torsors over S .

Lemma 2.6.3. *The stack quotient $[X_{L/K, \varphi, N} / \text{Res}_{E \otimes_{\mathbf{Q}_p} L_0/E} G]$ is isomorphic to $G - \text{Mod}_{L/K, \varphi, N}$.*

Proof. Given an A -valued point of $G - \text{Mod}_{L/K, \varphi, N}$ with underlying $\text{Res}_{E \otimes L_0/E} G$ -torsor D , the base change $D \otimes_A D \rightarrow D$ (which is projection on the first factor) is a trivial $\text{Res}_{E \otimes L_0/E} G$ -torsor (with $\text{Res}_{E \otimes L_0/E} G$ acting on the second factor). The identity morphism $D \xrightarrow{\sim} D$ induces a canonical trivializing section $D \rightarrow D \times_A D$, namely the diagonal. Pulling back Φ , N , and $\{\tau(g)\}_{g \in G}$ to $D \otimes_A D$ and writing them in coordinates (with respect to the trivializing section) gives us a morphism $D \rightarrow X_{L/K, \varphi, N}$.

We need to check that the morphism $D \rightarrow X_{L/K, \varphi, N}$ is $\text{Res}_{E \otimes L_0/E} G$ -equivariant. If A' is an A -algebra, the morphism $D \rightarrow X_{L/K, \varphi, N}$ carries $x \in D(A')$ to the fiber of $(\Phi, N, \{\tau(g)\})$ over x . The fiber of $D \otimes_A D \rightarrow D$ over x is a copy of $D_{A'}$, together with a section (defined by taking the fiber of the diagonal over x). If $g \in (\text{Res}_{E \otimes L_0/E} G)(A')$, the fiber of $D \otimes_A D \rightarrow D$ over $g \cdot x$ is also a copy of $D_{A'}$, but the section has been multiplied by g . Thus, our “change-of-basis” formula for triples $(\Phi, N, \{\tau(g)\}_{g \in \text{Gal}_{L/K}})$ implies that the morphism $D \rightarrow X_{L/K, \varphi, N}$ is $\text{Res}_{E \otimes L_0/E} G$ -equivariant, as required. \square

Given a $(\varphi, N, \text{Gal}_{L/K})$ -module, there is a standard recipe due to Fontaine for constructing a Weil–Deligne representation, and there is an analogous construction for $\text{Res}_{E \otimes L_0/E} G$ -torsors. Indeed, let A be an E -algebra. Given a $\text{Res}_{E \otimes L_0/E} G$ -torsor D over A , and an embedding $\sigma : L_0 \hookrightarrow E$, the σ -isotypic part is a G -torsor over A which we denote D_σ . Moreover, if N_σ denotes the σ -isotypic component of N , then $N_\sigma \in \text{Lie Aut}_G(D_\sigma)$ is nilpotent.

Given an isomorphism $\Phi : \varphi^* D \xrightarrow{\sim} D$, $\Phi^{f_L} := \Phi \circ \varphi^*(\Phi) \circ \dots \circ (\varphi^{f_L-1})^*(\Phi)$ restricts to an isomorphism $D_\sigma \rightarrow D_\sigma$ for each σ .

Lemma 2.6.4. *For any σ and any E -algebra A , the association $(D, \Phi) \rightsquigarrow (D_\sigma, \Phi^{f_L})$ defines an equivalence of categories between $\text{Res}_{E \otimes L_0/E} G$ -torsors D over A equipped with an isomorphism $\Phi : \varphi^* D \xrightarrow{\sim} D$, and G -torsors D_σ over A equipped with an isomorphism $\Phi'_\sigma : D_\sigma \xrightarrow{\sim} D_\sigma$.*

Proof. Write the embeddings $\sigma_i : L_0 \hookrightarrow E$, $i \in \mathbf{Z}/f_L \mathbf{Z}$, with the numbering chosen so that $\sigma_1 = \sigma$, and Φ induces isomorphisms $\sigma_i : D_{i+1} \xrightarrow{\sim} D_i$ for each i (where we write D_i for D_{σ_i}).

Let $A \rightarrow A'$ be an fpqc cover trivializing D , so that $D_{A'}$ is a trivial torsor and we may choose a section. Then we can write $\Phi = (\Phi_1, \dots, \Phi_{f_L})$.

We define

$$\underline{a} := (1, (\Phi_2 \cdots \Phi_{f_L})^{-1}, (\Phi_3 \cdots \Phi_{f_L})^{-1}, \dots, \Phi_{f_L}^{-1}).$$

Then if we multiply our choice of trivializing section by \underline{a} , we replace Φ by

$$\underline{a} \Phi \varphi(\underline{a})^{-1} = (\Phi_1 \cdots \Phi_{f_L}, 1, \dots, 1)$$

Thus, we can recover $(D_{A'}, \Phi)$ from $((D_\sigma)_{A'}, \Phi^{f_L})$.

Furthermore, $D_{A'}$ is equipped with a descent datum, since it is the base change of D . Therefore, $(D_i)_{A'}$ has a descent datum, and since $(D_i)_{A'} \rightarrow \text{Spec } A'$ is affine, it is effective.

Now suppose that $f = (f_1, \dots, f_{f_L}) : D \xrightarrow{\sim} D'$ is an isomorphism of $\text{Res}_{E \otimes L_0/E} G$ -torsors equipped with isomorphisms $\Phi : \varphi^* D \xrightarrow{\sim} D$, $\Phi' : \varphi^* D' \xrightarrow{\sim} D'$. We obtain a corresponding isomorphism $f_{A'} : D_{A'} \xrightarrow{\sim} D'_{A'}$, together with a covering datum. Then each $f_i : D_i \xrightarrow{\sim} D'_i$ is an isomorphism of G -torsors, and we have

$$f_i \circ \Phi_i = \Phi'_i \circ f_{i+1} : D_{i+1} \rightarrow D'_i.$$

Multiplying the trivializing section of $D_{A'}$ by \underline{a} and multiplying the trivializing section of $D_{A'}$ by \underline{a}' has the effect of replacing \underline{f} with $\underline{a}' \circ \underline{f} \circ \underline{a}^{-1}$. Then if we let \underline{a} and \underline{a}' be as above, \underline{f} becomes (f_1, \dots, f_1) . Thus, we can also recover morphisms of pairs $(D, \Phi) \rightarrow (D', \Phi')$ from the associated morphisms of pairs $(D_i, \Phi^{f_L}) \rightarrow (D'_i, (\Phi')^{f_L})$, as required. \square

Now suppose that D is a $\text{Res}_{E \otimes L_0/E} G$ -torsor equipped with an isomorphism $\Phi : \varphi^* D \xrightarrow{\sim} D$, and suppose in addition that D is equipped with a semi-linear action τ of $\text{Gal}_{L/K}$, compatible with Φ in the sense that $\Phi \circ \varphi^* \tau(g) = \tau(g) \circ g^*(\Phi)$ for all $g \in \text{Gal}_{L/K}$. For each σ , we will construct a Weil–Deligne representation on D_σ which is trivial on I_L .

There is a surjective map $W_K \rightarrow \text{Gal}_{L/K}$ which restricts to a surjection $I_K \rightarrow I_{L/K}$. If $g \in W_K$, we write \bar{g} for its image in $\text{Gal}_{L/K}$. For $g \in W_K$, we have an isomorphism

$$\tau(\bar{g}) : g^* D \xrightarrow{\sim} D$$

and we have an isomorphism

$$\Phi^{-v(g)f_K} := D \xrightarrow{\Phi^{-1}} \varphi^* D \xrightarrow{\varphi^* \Phi^{-1}} \dots \xrightarrow{(g\varphi^{-1})^* \Phi^{-1}} g^* D.$$

Accordingly, we define $r(g) : D_\sigma \xrightarrow{\sim} D_\sigma$ to be the restriction of

$$r(g) := \tau(\bar{g}) \circ \Phi^{-v(g)f_K} : D \xrightarrow{\sim} D.$$

Note that $r|_{I_L}$ is trivial.

Lemma 2.6.5. *Let D be a G -torsor and let $r : W_K \rightarrow \text{Aut}_G(D)$ be a homomorphism such that $r|_{I_L}$ is trivial. Then $r(W_L)$ centralizes $r(W_K)$.*

Proof. Let $g \in W_K$ and let $h \in W_L$. Then $v(ghg^{-1}h^{-1}) = 0$, so $ghg^{-1}h^{-1} \in I_K$. Moreover, $W_L \subset W_K$ is a normal subgroup, so that $ghg^{-1}h^{-1} \in W_L$. But $I_K \cap W_L = I_L$, so $r(ghg^{-1}h^{-1}) = 1$, as required. \square

We now prove the equivalence between Weil–Deligne representations and (φ, N) -modules. In the case that $G = \text{GL}_n$ the following lemma is [BS07, Prop. 4.1].

Lemma 2.6.6. *The map $r : W_K \rightarrow \text{Aut}_G(D_\sigma)$ is a homomorphism, and $(D, \Phi, N, \tau) \rightsquigarrow (D_\sigma, r, N_\sigma)$ is an equivalence of categories between $G\text{-Mod}_{L/K, \varphi, N}$ and $G\text{-WD}_E(L/K)$.*

Proof. Since $\tau(\bar{g}) \circ g^*(\Phi) = \Phi \circ \varphi^*(\tau(\bar{g}))$, we have $\Phi^{-1} \circ \tau(\bar{g}) = \varphi^*(\tau(\bar{g})) \circ g^*(\Phi^{-1})$ as isomorphisms $g^* D \xrightarrow{\sim} \varphi^* D$. It follows that

$$\begin{aligned} r(g_1)r(g_2) &= \left(\tau(\bar{g}_1) \circ \Phi^{-v(g_1)f_K} \right) \circ \left(\tau(\bar{g}_2) \circ \Phi^{-v(g_2)f_K} \right) \\ &= \tau(\bar{g}_1) \circ (\varphi^{v(g_1)f_K})^* \left(\tau(\bar{g}_2) \circ \Phi^{-v(g_1 g_2)f_K} \right) \\ &= \tau(\bar{g}_1 \bar{g}_2) \circ \Phi^{-v(g_1 g_2)f_K} = r(g_1 g_2) \end{aligned}$$

and r is a homomorphism. Another short computation shows that

$$N_\sigma = p^{-v(g)f_K} \text{Ad}(r(g))(N_\sigma),$$

so that (E_σ, r, N_σ) is a G -valued Weil–Deligne representation.

The association $(D, \Phi, N, \tau) \rightsquigarrow (D_\sigma, r, N_\sigma)$ is clearly functorial. Moreover, if $f : D \rightarrow D'$ is a morphism of G -valued $(\varphi, N, \text{Gal}_{L/K})$ -modules, then $\Phi' \circ \varphi^*(f) = f \circ \Phi$. This implies that f is determined by its restriction $f|_{D_\sigma}$ to the σ -isotypic piece, and therefore, the functor is fully faithful.

We need to check that this functor is essentially surjective. In other words, we need to check that we can construct (D, Φ, N, τ) from (D_σ, r, N_σ) . To do so, we number the embeddings as σ_i , as in the proof of Lemma 2.6.4. For each element $h \in I_{L/K}$, we fix a lift to an element $\tilde{h} \in I_K$; note that since $r|_{I_L}$ is trivial, $r(\tilde{h})$ is independent of the choice of \tilde{h} .

To construct $\Phi^{f_L}|_{D_i}$ from r , we observe that if $g_0 \in W_K$ lifts φ^{f_K} and (D_i, r, N_i) is in the essential image of our functor, then

$$r(g_0^{f_L/f_K}) = \tau(\overline{g_0^{f_L/f_K}}) \Phi^{-f_L}.$$

But $\overline{g_0^{f_L/f_K}} \in I_{L/K}$, so we can define $\Phi^{f_L}|_{D_i} := r(g_0^{f_L/f_K})^{-1} r(\overline{g_0^{f_L/f_K}})$.

We need to check that $\Phi^{f_L}|_{D_i}$ does not depend on our choice of g_0 . Indeed, if $h \in I_K$, then $(g_0 h)^{f_L/f_K} = h_1 \cdots h_{f_L/f_K-1} g_0^{f_L/f_K}$, where $h_i := g_0^i h g_0^{-i} \in I_K$, so we may write $(g_0 h)^{f_L/f_K} = h' g_0^{f_L/f_K}$ for some $h' \in I_K$. Then $r(\tilde{h}') = r(h')$, so

$$\begin{aligned} r((g_0 h)^{f_L/f_K})^{-1} r(\overline{(g_0 h)^{f_L/f_K}}) &= r(g_0^{f_L/f_K})^{-1} r(h')^{-1} r(\tilde{h}') r(\overline{g_0^{f_L/f_K}}) \\ &= r(g_0^{f_L/f_K})^{-1} r(\overline{g_0^{f_L/f_K}}), \end{aligned}$$

as required.

Lemma 2.6.4 now implies that we can construct (D, Φ) from $(D_i, \Phi^{f_L}|_{D_i})$. Since $W_K \rightarrow \text{Gal}_{L/K}$ is surjective, we define for $g \in \text{Gal}_{L/K}$

$$\tau(g) := r(\tilde{g}) \circ \Phi^{v(\tilde{g})f_K} = r(\tilde{g}) \circ (\Phi \circ \cdots \circ (\varphi^{-1})^* g^* \Phi)$$

as a map $D_{i+v(\tilde{g})f_K} \rightarrow D_i$. We need to check that this is well-defined. Note that the kernel of $W_K \rightarrow \text{Gal}_{L/K}$ is W_L , and if $h \in W_L$, then $v(h) = (f_L/f_K) \cdot i$ for some $i \in \mathbf{Z}$. Thus, for any $h \in W_L$,

$$r(\tilde{g}h) \circ \Phi^{v(\tilde{g}h)f_K} = r(\tilde{g})r(h) \circ \Phi^{i \cdot f_L} \circ \Phi^{v(\tilde{g})f_K},$$

so it suffices to show that $r(h) \circ \Phi^{i \cdot f_L} = 1$. Since $r|_{I_L}$ is trivial, it suffices to consider the case $i = 1$, i.e., h generates the unramified quotient of W_L . But then $r(h) \circ \Phi^{f_L} = r(h)r(g_0^{f_L/f_K})^{-1} r(\overline{g_0^{f_L/f_K}})$; on the one hand $h g_0^{-f_L/f_K} \in I_K$ and $\overline{g_0^{f_L/f_K}} \in I_K$, and on the other hand $g_0^{-f_L/f_K} \overline{g_0^{f_L/f_K}} \in W_L$. It follows that

$$h g_0^{-f_L/f_K} \overline{g_0^{f_L/f_K}} \in I_K \cap W_L = I_L$$

and the result follows.

We can also construct $\tau(g) : D_{j+v(\tilde{g})f_K} \xrightarrow{\sim} D_j$ for the remaining σ_j -isotypic factors. Indeed, the desired compatibility between Φ and τ forces us to set $\varphi^* \tau(g) := \Phi^{-1} \circ \tau(g) \circ g^* \Phi : D_{i+v(\tilde{g})f_K+1} \xrightarrow{\sim} D_{i+1}$ (and we proceed inductively).

We need to check that this is well-defined. More precisely, we need to check that $(\varphi^{f_L})^* \tau(g) = \tau(g)$ for all $g \in \text{Gal}_{L/K}$. In other words, we need to check that

$$\tau(g) \circ (g^* \Phi \circ \varphi^* g^* \Phi \circ \cdots \circ (\varphi^{f_L-1})^* g^* \Phi) = (\Phi \circ \varphi^* \Phi \circ \cdots \circ (\varphi^{f_L-1})^* \Phi) \circ \tau(g)$$

as isomorphisms $D_{i+v(\tilde{g})f_K} \xrightarrow{\sim} D_i$, or equivalently that

$$\tau(g) \circ g^* \Phi^{f_L} = \Phi^{f_L} \circ \tau(g).$$

But

$$\begin{aligned}
\tau(g) \circ g^* \Phi^{f_L} &= \left(r(\tilde{g}) \circ \Phi^{v(\tilde{g})f_K} \right) \circ g^*(\Phi^{f_L}) \\
&= r(\tilde{g}) \circ \Phi^{f_L} \circ \Phi^{v(\tilde{g})f_K} \\
&= r(\tilde{g}) \cdot r(g_0^{-f_L/f_K} \widetilde{g_0^{f_L/f_K}}) \circ \Phi^{v(\tilde{g})f_K} \\
&= r(g_0^{-f_L/f_K} \widetilde{g_0^{f_L/f_K}}) \cdot r(\tilde{g}) \circ \Phi^{v(\tilde{g})f_K} \\
&= \Phi^{f_L} \circ \tau(g).
\end{aligned}$$

Here we used Lemma 2.6.5 and the fact that $g_0^{-f_L/f_K} \widetilde{g_0^{f_L/f_K}} \in W_L$.

It remains to show that τ is a semi-linear representation, or more precisely, that $\tau(g_1 g_2) = \tau(g_1) \circ g_1^* \tau(g_2)$ for all $g_1, g_2 \in \text{Gal}_{L/K}$. Now since by definition we have $\varphi^* \tau(g) := \Phi^{-1} \circ \tau(g) \circ g^* \Phi : D_{i+v(\tilde{g})f_{K+1}} \xrightarrow{\sim} D_{i+1}$, we see that

$$\begin{aligned}
\tau(g_1) \circ g_1^* \tau(g_2) &= \tau(g_1) \circ \left((g_1 \varphi^{-1})^* \Phi^{-1} \circ \dots \circ \Phi^{-1} \right) \circ \tau(g_2) \circ (g_2^* \Phi \circ \dots \circ (g_1 \varphi^{-1})^* g_2^* \Phi) \\
&= \tau(g_1) \circ \left((g_1 \varphi^{-1})^* \Phi^{-1} \circ \dots \circ \Phi^{-1} \right) \circ \tau(g_2) \circ g_2^* (\Phi \circ \dots \circ (g_1 \varphi^{-1})^* \Phi) \\
&= r(\tilde{g}_1) \circ r(\tilde{g}_2) \circ \Phi^{v(\tilde{g}_2)f_K} \circ g_2^* \Phi^{v(\tilde{g}_1)f_K} \\
&= r(\tilde{g}_1) r(\tilde{g}_2) \circ \Phi^{v(\tilde{g}_1 \tilde{g}_2)f_K} \\
&= \tau(g_1 g_2),
\end{aligned}$$

as required.

Finally, we construct N . We have N_i , and we use the desired relation $N = p\text{Ad}(\Phi)(N)$ to construct the Frobenius-conjugates of N_i . It then follows that for any $g \in \text{Gal}_{L/K}$

$$\begin{aligned}
\underline{\text{Ad}}(\tau(g))(N) &= \underline{\text{Ad}}(r(\tilde{g}) \circ \Phi^{v(g)f_K})(N) \\
&= \text{Ad}(r(\tilde{g}) \circ \Phi^{v(g)f_K})(p^{-v(g)f_K} \text{Ad}(\Phi^{-v(g)f_K})(N)) \\
&= \text{Ad}(r(\tilde{g}))(N) = N
\end{aligned}$$

so we are done.

The assignment $(D_i, r, N_i) \rightsquigarrow (D, \Phi, N, \tau)$ is clearly functorial and quasi-inverse to $(D, \Phi, N, \tau) \rightsquigarrow (D_i, r, N_i)$. \square

2.7. Exact \otimes -filtrations for disconnected groups. In this section we prove some results on tensor filtrations that we will apply to the Hodge filtration in p -adic Hodge theory.

Let G be an affine group scheme over a field k of characteristic zero, let A be a k -algebra, and let η be a fiber functor from $\text{Rep}_k(G)$ to Proj_A which is equipped with an exact \otimes -filtration. More precisely, $\text{Rep}_k(G)$ is the category of k -linear finite-dimensional representations of G , Proj_A is the category of finite projective A -modules (which we will also think of as being vector bundles on $\text{Spec } A$), and by a ‘‘fiber functor’’ we mean that

- (1) η is k -linear, exact, and faithful.
- (2) η is a tensor functor, that is, $\eta(V_1 \otimes_k V_2) = \eta(V_1) \otimes_A \eta(V_2)$.
- (3) If $\mathbf{1}$ denotes the trivial representation of G , then $\eta(\mathbf{1})$ is the trivial A -module of rank 1.

By an “exact \otimes -filtration”, we mean that for each $V \in \text{Rep}_k(G)$, we have a decreasing filtration $\mathcal{F}^\bullet(\eta(V))$ of vector sub-bundles on each $\eta(V)$ such that

- (1) the specified filtrations are functorial in V .
- (2) the specified filtrations are tensor-compatible, in the sense that

$$\mathcal{F}^n \eta(V \otimes_k V') = \sum_{p+q=n} \mathcal{F}^p \eta(V) \otimes_A \mathcal{F}^q \eta(V') \subset \eta(V) \otimes_A \eta(V').$$

- (3) $\mathcal{F}^n(\eta(\mathbf{1})) = \eta(\mathbf{1})$ if $n \leq 0$ and $\mathcal{F}^n(\eta(\mathbf{1})) = 0$ if $n \geq 1$.
- (4) the associated functor from $\text{Rep}_k(G)$ to the category of graded projective A -modules is exact.

Equivalently, an exact \otimes -filtration of η is the same as a factorization of η through the category of filtered vector bundles over $\text{Spec } A$.

Definition 2.7.1. Let $\omega, \eta : \text{Rep}_k(G) \rightrightarrows \text{Proj}_A$ be fiber functors. Then $\underline{\text{Hom}}^\otimes(\omega, \eta)$ is the functor on A -algebras given by

$$\underline{\text{Hom}}^\otimes(\omega, \eta)(A') := \text{Hom}^\otimes(\varphi_{A'} \circ \omega, \varphi_{A'} \circ \eta).$$

Here $\underline{\text{Hom}}^\otimes$ refers to natural transformations of functors which preserve tensor products.

Given a fiber functor $\eta : \text{Rep}_k(G) \rightarrow \text{Proj}_A$ and an A -algebra A' , there is a natural fiber functor $\eta' : \text{Rep}_k(G) \rightarrow \text{Proj}_{A'}$ given by composing η with the natural base extension functor $\varphi_{A'} : \text{Proj}_A \rightarrow \text{Proj}_{A'}$ sending M to $M \otimes_A A'$.

Theorem 2.7.2 ([DM82, Thm. 3.2]). *Let $\omega : \text{Rep}_k(G) \rightarrow \text{Vec}_k$ be the natural forgetful functor.*

- (1) *For any fiber functor $\eta : \text{Rep}_k(G) \rightarrow \text{Proj}_A$, $\underline{\text{Hom}}^\otimes(\varphi_A \circ \omega, \eta)$ is representable by an affine scheme faithfully flat over $\text{Spec } A$; it is therefore a G -torsor.*
- (2) *The functor $\eta \rightsquigarrow \underline{\text{Hom}}^\otimes(\varphi_A \circ \omega, \eta)$ is an equivalence between the category of fiber functors $\eta : \text{Rep}_k(G) \rightarrow \text{Proj}_A$ and the category of G -torsors over $\text{Spec } A$. The quasi-inverse assigns to any G -torsor X over A the functor η sending any $\rho : G \rightarrow \text{GL}(V)$ to the $M \in \text{Proj}_A$ associated to the push-out of X over A .*

Corollary 2.7.3. *Let $\eta : \text{Rep}_k(G) \rightarrow \text{Proj}_A$ be a fiber functor, corresponding to a G -torsor $X \rightarrow \text{Spec } A$. Then the functor $\underline{\text{Aut}}^\otimes(\eta)$ is representable by the A -group scheme $\text{Aut}_G(X)$. This is a form of G_A .*

We define two auxiliary subfunctors of $\underline{\text{Aut}}^\otimes(\eta)$.

- $P_{\mathcal{F}} = \underline{\text{Aut}}_{\mathcal{F}}^\otimes(\eta)$ is the functor on A -algebras such that

$$P_{\mathcal{F}}(A') = \{\lambda \in \text{Aut}_G(\eta)(A') \mid \lambda(\mathcal{F}^n \eta(V)) \subset \mathcal{F}^n \eta(V) \text{ for all } V \in \text{Rep}_k(G) \text{ and } n \in \mathbf{Z}\}.$$

- $U_{\mathcal{F}} = \underline{\text{Aut}}_{\mathcal{F}}^{\otimes!}(\eta)$ is the functor on A -algebras such that

$$U_{\mathcal{F}}(A') = \{\lambda \in \text{Aut}_G(\eta)(A') \mid (\lambda - \text{id})(\mathcal{F}^n \eta(V)) \subset \mathcal{F}^{n+1} \eta(V) \text{ for all } V \in \text{Rep}_k(G) \text{ and } n \in \mathbf{Z}\}.$$

By [SR72, Chapter IV, 2.1.4.1], these functors are both representable by closed subgroup schemes of $\text{Aut}_G(X)$, and they are smooth if G is. This holds for any affine group G over k (since it is automatically flat); there is no need for reductivity or connectedness hypotheses. Furthermore, $\text{Lie } P_{\mathcal{F}} = \mathcal{F}^0(\text{Lie } \underline{\text{Aut}}^{\otimes}(\eta))$ and $\text{Lie } U_{\mathcal{F}} = \mathcal{F}^1(\text{Lie } \underline{\text{Aut}}^{\otimes}(\eta))$, by the same result.

We also have a notion of a \otimes -grading on η : a \otimes -grading of η is the specification of a grading $\eta(V) = \bigoplus_{n \in \mathbf{Z}} \eta(V)_n$ of vector bundles on each $\eta(V)$ such that

- (1) the specified gradings are functorial in V .
- (2) the specified gradings are tensor-compatible, in the sense that

$$\eta(V \otimes_k V')_n = \bigoplus_{p+q=n} (\eta(V)_p \otimes_A \eta(V')_q).$$

- (3) $\eta(\mathbf{1})_0 = \eta(\mathbf{1})$.

Equivalently, a \otimes -grading of η is a factorization of η through the category of graded vector bundles on $\text{Spec } A$. A \otimes -grading induces a homomorphism of A -group schemes $\mathbf{G}_m \rightarrow \underline{\text{Aut}}^{\otimes}(\eta)$.

Given a \otimes -grading of η , we may construct a \otimes -filtration of η , by setting

$$\mathcal{F}^n \eta(V) = \bigoplus_{n' \geq n} \eta(V)_{n'}.$$

We say that a \otimes -filtration \mathcal{F}^{\bullet} is *splittable* if it arises in this way, and we say that \mathcal{F}^{\bullet} is *locally splittable* if fpqc-locally on $\text{Spec } A$ it arises in this way. A *splitting* of \mathcal{F}^{\bullet} is a \otimes -grading on η giving rise to \mathcal{F}^{\bullet} .

Given an exact \otimes -filtration \mathcal{F}^{\bullet} on η , we may define a fiber functor $\text{gr}(\eta)$ equipped with a \otimes -grading, by setting

$$\text{gr}(\eta)(V)_n := \mathcal{F}^n(V) / \mathcal{F}^{n+1}(V)$$

Thus, a splitting of \mathcal{F}^{\bullet} is equivalent to an isomorphism of filtered fiber functors $\text{gr}(\eta) \cong \eta$.

In fact, by a theorem of Deligne (proved in [SR72, Chapter IV, 2.4]), every \otimes -filtration is locally splittable (in fact, splittable Zariski-locally on $\text{Spec } A$), because G is smooth and A has characteristic 0 (this result also holds under various other sets of hypotheses on G and A). Again, this does not require G to be reductive or connected. If $\lambda : \mathbf{G}_m \rightarrow \underline{\text{Aut}}^{\otimes}(\eta)$ is a cocharacter splitting the filtration, then $P_{\mathcal{F}} = U_{\mathcal{F}} \rtimes Z_G(\lambda)$, by [SR72, Chapter IV, 2.1.5.1]. In particular, λ factors through $P_{\mathcal{F}}$.

If \mathcal{F}^{\bullet} is a splittable filtration on η , we may consider the functor $\underline{\text{Scin}}(\eta, \mathcal{F}^{\bullet})$ of splittings. Then $\underline{\text{Scin}}(\eta, \mathcal{F}^{\bullet})$ is the same as the functor $\underline{\text{Isom}}_{\mathcal{F}}^{\otimes!}(\text{gr}_{\mathcal{F}}(\eta), \eta)$, which is the subset of $\underline{\text{Isom}}_{\mathcal{F}}^{\otimes}(\text{gr}_{\mathcal{F}}(\eta), \eta)$ inducing the identity $\text{gr}_{\mathcal{F}}(\eta) \rightarrow \text{gr}_{\mathcal{F}}(\eta)$. Thus, $\underline{\text{Scin}}(\eta, \mathcal{F}^{\bullet})$ is a left torsor under $U_{\mathcal{F}}$. It follows that the composition $\lambda : \mathbf{G}_m \rightarrow P_{\mathcal{F}} \rightarrow P_{\mathcal{F}}/U_{\mathcal{F}}$ is independent of the choice of splitting.

In other words, $P_{\mathcal{F}}$ and $U_{\mathcal{F}}$ depend only on the filtration, and if it is locally splittable, there is a homomorphism $\bar{\lambda} : \mathbf{G}_m \rightarrow P_{\mathcal{F}}/U_{\mathcal{F}}$ which also only depends on the filtration. If the filtration is actually splittable, a choice of splitting lets us lift $\bar{\lambda}$ to a cocharacter $\lambda : \mathbf{G}_m \rightarrow P_{\mathcal{F}}$. In that case, since both $\underline{\text{Scin}}(\eta, \mathcal{F})$ and the set of lifts of cocharacters from $P_{\mathcal{F}}/U_{\mathcal{F}}$ to $P_{\mathcal{F}}$ are torsors under $U_{\mathcal{F}}$, they are isomorphic.

Proposition 2.7.4. *Suppose that G is reductive (but possibly disconnected). Let $\eta : \text{Rep}_k(G) \rightarrow \text{Proj}_A$ be a fiber functor equipped with a splittable exact \otimes -filtration*

\mathcal{F}^\bullet , and let $\lambda : \mathbf{G}_m \rightarrow \underline{\text{Aut}}^\otimes(\eta)$ be a splitting. Let \mathcal{G} denote the group scheme representing $\underline{\text{Aut}}^\otimes(\eta)$. Then $P_{\mathcal{F}} = P_{\mathcal{G}}(\lambda)$ and $U_{\mathcal{F}} = U_{\mathcal{G}}(\lambda)$ is connected.

Proof. We consider the map $\mu : \mathbf{G}_m \times P_{\mathcal{F}} \rightarrow \underline{\text{Aut}}^\otimes(\eta)$ defined by $\mu(t, g) := \lambda(t)g\lambda(t^{-1})$, and for $g \in P_{\mathcal{F}}(A')$, we let $\mu_g : (\mathbf{G}_m)_{A'} \rightarrow (\underline{\text{Aut}}^\otimes(\eta))_{A'}$ be the restriction $\mu|_{\mathbf{G}_m \times \{g\}}$. Let $\sigma : G \rightarrow \text{GL}(V)$ be a representation of G . Then the pushout $\eta(V)$ is a filtered vector bundle, and if $g \in P_{\mathcal{F}}(A')$, the action of g preserves the filtration on $\eta(V)$. The choice of a splitting in particular specifies an isomorphism $\text{gr}^\bullet(\eta(V)) \xrightarrow{\sim} \eta(V)$, and $t \in \mathbf{G}_m(A')$ acts via

$$\bigoplus_{n \in \mathbf{Z}} t^n : (\eta(V))_n \rightarrow (\eta(V))_n.$$

Let $\sigma_*(\lambda)$ denote the corresponding cocharacter $\sigma_*(\lambda) : \mathbf{G}_m \rightarrow \text{Aut}_{\text{GL}(V)}(\eta(V))$. Since this cocharacter induces the filtration on $\eta(V)$, we see that the morphism

$$\sigma_*(\mu_g) := \sigma_*(\lambda)(t)g\sigma_*(\lambda)(t^{-1}) : \mathbf{G}_m \rightarrow P_{\text{Aut}_{\text{GL}(V)}(\eta(V))}(\sigma_*(\lambda))$$

extends uniquely to a morphism

$$\widetilde{\sigma_*(\mu_g)} : \mathbf{A}^1 \rightarrow P_{\text{Aut}_{\text{GL}(V)}(\eta(V))}(\sigma_*(\lambda)).$$

We claim that the collection $\{\widetilde{\sigma_*(\mu_g)}\}_\sigma$ is functorial in σ and tensor-compatible. Indeed, since the collection $\{\sigma_*(\mu_g)|_{\mathbf{G}_m}\}_\sigma$ is functorial in σ and tensor-compatible, and the extensions to \mathbf{A}^1 are unique, it follows that $\{\widetilde{\sigma_*(\mu_g)}\}_\sigma$ is functorial in σ and tensor-compatible. Thus, there is a morphism $\widetilde{\mu}_g : \mathbf{A}^1 \rightarrow \underline{\text{Aut}}_{\mathcal{F}}^\otimes(\eta)$ whose restriction to \mathbf{G}_m is μ_g . It follows that $g \in P_{\mathcal{G}}(\lambda)(A')$.

Suppose in addition that $g \in U_{\mathcal{F}}(A')$. Then for every representation $\sigma : G \rightarrow \text{GL}(V)$, g induces the identity map from $\text{gr}^\bullet(\sigma(\mathcal{F}^\bullet))$ to itself. It follows that $\widetilde{\sigma_*(\mu_g)}(0) = \mathbf{1}$ for all σ , and therefore $\widetilde{\mu}_g(0) = \mathbf{1}$.

On the other hand, if $g \in P_{\mathcal{G}}(\lambda)(A')$, then the morphism $\mu_g : (\mathbf{G}_m)_{A'} \rightarrow \underline{\text{Aut}}^\otimes(\eta)_{A'}$ defined by $t \mapsto \lambda(t)g\lambda(t^{-1})$ extends to a morphism $\widetilde{\mu}_g : (\mathbf{A}^1)_{A'} \rightarrow \underline{\text{Aut}}^\otimes(\eta)_{\mathbf{A}^1}$. It therefore induces a family of morphisms

$$\sigma_*(\widetilde{\mu}_g) : (\mathbf{A}^1)_{A'} \rightarrow \text{GL}(V)_{A'}$$

and so $\sigma_*(g) \in P_{\text{Aut}_{\text{GL}(V)}(\eta(V))}(\sigma_*(\lambda))$. But then $\sigma_*(g)$ preserves the filtration on $\eta(V)$ induced by $\sigma_*(\lambda)$; since this holds for all $V \in \text{Rep}_k(G)$, $g \in P_{\mathcal{F}}(A')$. A similar argument shows that if $g \in U_{\mathcal{G}}(\lambda)(A')$, then $g \in U_{\mathcal{F}}(A')$.

Finally, since $\widetilde{\mu}_g : \mathbf{A}^1 \rightarrow \underline{\text{Aut}}^\otimes(\eta)$ is a morphism from a connected scheme such that $\widetilde{\mu}_g(0) = \mathbf{1}$ and $\widetilde{\mu}_g(1) = g$, we see that g is in the connected component of the identity for all $g \in U_{\mathcal{F}}(A')$. \square

Thus, locally on $\text{Spec } A$ we obtain cocharacters $\lambda : \mathbf{G}_m \rightarrow G_A$. These cocharacters are not unique, but for any geometric point $x \in \text{Spec } A$, the $G^\circ(\kappa(x))$ -conjugacy class of \mathcal{F}_x^\bullet induces a unique $G^\circ(\kappa(x))$ -conjugacy class of cocharacters, and this conjugacy class is Zariski-locally constant on $\text{Spec } A$.

The geometric fibers of $\mathcal{G} \cong \underline{\text{Aut}}^\otimes(\eta)$ are isomorphic to $G_{\bar{k}}$, since η corresponds to a G -torsor.

Lemma 2.7.5. *Let \mathcal{F}^\bullet be a locally splittable exact \otimes -filtration on η . Then the geometric fibers of $P_{\mathcal{F}} \cong P_{\mathcal{G}}(\lambda)$ are parabolic subgroups of $G_{\bar{k}}$.*

Proof. Since the formation of $P_G(\lambda)$ commutes with base change on A , we may assume that $A = k = \bar{k}$ and $\mathcal{G} = G = G_{\bar{k}}$. Then $P_{G^\circ}(\lambda) \subset G^\circ$ is a parabolic subgroup, so $G^\circ/P_{G^\circ}(\lambda)$ is proper. There is a sequence of maps

$$G^\circ/P_{G^\circ}(\lambda) \rightarrow G/P_{G^\circ}(\lambda) \twoheadrightarrow G/P_G(\lambda)$$

Since $G^\circ \subset G$ has finite index, the properness of $G^\circ/P_{G^\circ}(\lambda)$ implies the properness of $G/P_G(\lambda)$. This implies that $G/P_G(\lambda)$, so $P_G(\lambda) \subset G$ is a parabolic subgroup. \square

We will also need the following result:

Theorem 2.7.6 ([SGA70, IX.3.6]). *Let S be an affine scheme, S_0 a subscheme defined by a nilpotent ideal J , H a group of multiplicative type over S , G a smooth group scheme over S , $\mu_0 : H \times_S S_0 \rightarrow G \times_S S_0$ a homomorphism of S_0 -groups.*

Then there exists a homomorphism $\mu : H \rightarrow G$ of S -groups which lift μ_0 , and any two such lifts are conjugate by an element of $G(S)$ which reduces to the identity modulo J .

Corollary 2.7.7. *Let A be an artin local k -algebra with maximal ideal \mathfrak{m}_A , and let $I \subset A$ be an ideal such that $\text{Im}_A = (0)$. Then if D_A is a G -torsor over A such that the reduction $D_{A/I} := D_A \otimes_A A/I$ is equipped with an exact \otimes -filtration $\mathcal{F}_{A/I}^\bullet$, then the set of lifts of $\mathcal{F}_{A/I}^\bullet$ to an exact \otimes -filtration on D_A is non-empty, and is a torsor under $(\text{ad}D_{A/I}/\mathcal{F}_{A/I}^0(\text{ad}D_{A/I})) \otimes_{A/\mathfrak{m}_A} I$.*

Proof. Suppose that $D_{A/I}$ is a G -torsor over $\text{Spec } A/I$, equipped with an exact \otimes -filtration $\mathcal{F}_{A/I}^\bullet$. Since A/I is local, $\mathcal{F}_{A/I}^\bullet$ is split, so it is induced by a cocharacter $\lambda_{A/I} : \mathbf{G}_m \rightarrow \text{Aut}_G(D_{A/I})$. By Theorem 2.7.6, $\lambda_{A/I}$ lifts to a cocharacter $\lambda_A : \mathbf{G}_m \rightarrow \text{Aut}_G(D_A)$. Then λ_A induces an exact \otimes -filtration \mathcal{F}_A^\bullet on D_A which lifts that on $D_{A/I}$.

Suppose there are two exact \otimes -filtrations, \mathcal{F}_A^\bullet and \mathcal{F}'_A^\bullet on D_A lifting $\mathcal{F}_{A/I}^\bullet$, induced by cocharacters λ_A and λ'_A , respectively, which lift $\lambda_{A/I}$. Then λ_A and λ'_A are conjugate by an element of $\text{Aut}_G(D_A)$ which is the identity modulo I . In other words, there is some $j \in \text{ad}D_{A/\mathfrak{m}_A} \otimes_{A/\mathfrak{m}_A} I$ such that $\lambda'_A = (1+j)\lambda_A(1-j)$. This implies that \mathcal{F}_A^\bullet and \mathcal{F}'_A^\bullet are conjugate.

On the other hand, conjugation by $1+j$ preserves \mathcal{F}_A^\bullet if and only if $1+j \in P_{\mathcal{F}_A}(\text{Aut}_G(D_A))$. This holds if and only if $j \in \mathcal{F}_{A/\mathfrak{m}_A}^0 \text{Lie } \text{Aut}_G(D_{A/\mathfrak{m}_A}) \otimes_{A/\mathfrak{m}_A} I = \mathcal{F}_{A/\mathfrak{m}_A}^0 \text{ad}D_{A/\mathfrak{m}_A} \otimes_{A/\mathfrak{m}_A} I$. \square

2.8. p -adic Hodge theory. Our goal is to study deformations of potentially semi-stable Galois representations. That is, we wish to consider deformations of representations $\rho : \text{Gal}_K \rightarrow G(E)$ such that $\rho|_{\text{Gal}_L}$ is semi-stable. Such representations can be described by linear algebra. Briefly, for every representation $\sigma : G \rightarrow \text{GL}_d$, $\sigma \circ \rho$ is a potentially semi-stable representation, and $D_{\text{st}}^L(\sigma \circ \rho)$ is a weakly admissible filtered $(\varphi, N, \text{Gal}_{L/K})$ -module. The formation of $D_{\text{st}}^L(\sigma \circ \rho)$ is exact and tensor-compatible in σ , and if $\mathbf{1}$ denotes the trivial representation of G , then $D_{\text{st}}^L(\mathbf{1} \circ \rho)$ is the trivial filtered $(\varphi, N, \text{Gal}_{L/K})$ -module.

Therefore, as in [Bel16, §A.2.8-9], $\sigma \mapsto D_{\text{st}}^L(\sigma \circ \rho)$ is a fiber functor $\eta : \text{Rep}_E(G) \rightarrow \text{Mod}_{E \otimes_{\mathbb{Q}_p} L_0}$, and we obtain from ρ a G -torsor $D = D_{\text{st}}^L(\rho)$ over L_0 equipped with

- an isomorphism $\Phi : \varphi^* D \xrightarrow{\sim} D$,
- a nilpotent element $N \in \text{Lie } \text{Aut}_G D$,

- for each $g \in \text{Gal}_{L/K}$, an isomorphism $\tau(g) : g^* D \xrightarrow{\sim} D$,
- a $\text{Gal}_{L/K}$ -stable exact \otimes -filtration on D_L , or equivalently, an exact \otimes -filtration on the $\text{Res}_{E \otimes K/E} G$ -torsor $D_L^{\text{Gal}_{L/K}}$ over K .

In particular, forgetting the filtration on $D_{\text{st}}^L(\rho)$ gives us an element of $G\text{-Mod}_{L/K, \varphi, N}$.

Definition 2.8.1. The category of G -valued filtered $(\varphi, N, \text{Gal}_{L/K})$ -modules, which we denote $G\text{-Mod}_{L/K, \varphi, N, \text{Fil}}$, is the groupoid whose fiber over an E -algebra A consists of a $\text{Res}_{E \otimes L_0/E} G$ -torsor D over A , equipped with:

- an isomorphism $\Phi : \varphi^* D \xrightarrow{\sim} D$,
- a nilpotent element $N \in \text{Lie Aut}_G D$,
- for each $g \in \text{Gal}_{L/K}$, an isomorphism $\tau(g) : g^* D \xrightarrow{\sim} D$,
- a $\text{Gal}_{L/K}$ -stable \otimes -filtration on D_L , or equivalently, a \otimes -filtration on the $\text{Res}_{E \otimes K/E} G$ -torsor $D_L^{\text{Gal}_{L/K}}$ over K .

The $\text{Res}_{E \otimes L_0/E} G$ -torsor D , together with Φ , N , and $\{\tau(g)\}_{g \in \text{Gal}_{L/K}}$, is required to be an object of $G\text{-Mod}_{L/K, \varphi, N}$.

Definition 2.8.2. Suppose that $\rho : \text{Gal}_K \rightarrow G(E)$ is a potentially semi-stable Galois representation which becomes semi-stable when restricted to Gal_L . The p -adic Hodge type \mathbf{v} of ρ is the $(\text{Res}_{E \otimes K/E} G)^\circ(\overline{E})$ -conjugacy class of cocharacters $\lambda : \mathbf{G}_m \rightarrow (\text{Res}_{E \otimes K/E} G)_{\overline{E}}$ which split the \otimes -filtration on $D_{\text{st}}^L(\rho)_L^{\text{Gal}_{L/K}}$. We let $P_{\mathbf{v}}$ denote the $(\text{Res}_{E \otimes K/E} G)^\circ(\overline{E})$ -conjugacy class of $P_{\text{Res}_{E \otimes K/E} G}(\lambda)$ for $\lambda \in \mathbf{v}$.

While we do not need it, for completeness we record the following definition and result, which control the deformation theory of filtered $(\varphi, N, \text{Gal}_{L/K})$ -modules. Given an object $D_A \in G\text{-Mod}_{L/K, \varphi, N, \text{Fil}}$, we consider the diagram

$$\begin{array}{ccc} (\text{ad} D_A)^{\text{Gal}_{L/K}} & \longrightarrow & (\text{ad} D_A)^{\text{Gal}_{L/K}} \oplus (\text{ad} D_A)^{\text{Gal}_{L/K}} \longrightarrow (\text{ad} D_A)^{\text{Gal}_{L/K}} \\ \downarrow & & \\ (\text{ad} D_{A,L}/\text{Fil}^0 \text{ad} D_{A,L})^{\text{Gal}_{L/K}} & & \end{array}$$

and its total complex, which we denote C_{Fil}^\bullet . Then C_{Fil}^\bullet controls the deformation theory of D_A :

Proposition 2.8.3. *Let A be an artin local E algebra with maximal ideal \mathfrak{m}_A and let $I \subset A$ be an ideal such that $\text{Im}_{\mathfrak{m}_A} = (0)$. Let $D_{A/I}$ be an object of $G\text{-Mod}_{L/K, \varphi, N, \text{Fil}}(A/I)$ and set $D_{A/\mathfrak{m}_A} := D_{A/I} \otimes_{A/I} A/\mathfrak{m}_A$.*

- (1) *If $H_{\text{Fil}}^2(D_{A/I}) = 0$, then there exists an object $D_A \in G\text{-Mod}_{L/K, \varphi, N, \text{Fil}}(A)$ lifting $D_{A/I}$.*
- (2) *The set of isomorphism classes of lifts of $D_{A/I}$ to $D_A \in G\text{-Mod}_{L/K, \varphi, N, \text{Fil}}(A)$ is either empty or a torsor under $H_{\text{Fil}}^1(D_{A/\mathfrak{m}_A}) \otimes_{A/\mathfrak{m}_A} I$.*

Proof. This follows by combining [Bel16, Proposition 3.2] and Corollary 2.7.7. \square

3. LOCAL DEFORMATION RINGS

As in Section 1.4.2, we let K/\mathbf{Q}_p be a finite extension for some prime p , possibly equal to l , and let $\bar{\rho} : G_K \rightarrow G(\mathbf{F})$ be a continuous representation. We have a universal framed deformation \mathcal{O} -algebra $R_{\bar{\rho}}^\square$, and if we fix a homomorphism $\psi : \Gamma \rightarrow G^{\text{ab}}(\mathcal{O})$ such that $\text{ab} \circ \bar{\rho} = \bar{\psi}$, we also have the quotient $R_{\bar{\rho}}^{\square, \psi}$ corresponding

to framed deformations ρ with $\text{ab} \circ \rho = \psi$. When we define quotients of R_ρ^\square , there are corresponding quotients of $R_\rho^{\square, \psi}$, which we will not explicitly define, but will denote by a superscript ψ .

An inertial type is by definition a $G^\circ(E)$ -conjugacy class of representations $\tau : I_K \rightarrow G(E)$ with open kernel which admit extensions to Gal_K . Any such extension factors through some finite Galois extension L/K (depending on τ). If E'/E is a finite extension, and $\rho : G_K \rightarrow G(E')$ is a representation, which we assume to be potentially semi-stable if $l = p$, then we say that ρ has type τ if the restriction to I_K (forgetting N) of the corresponding Weil–Deligne representation $\text{WD}(\rho)$ is equivalent to τ .

3.1. The case $l \neq p$. Suppose firstly that $l \neq p$. The proof of [Bal12, Prop. 3.0.12] shows that for each τ we may define a \mathbf{Z}_ℓ -flat quotient $R_\rho^{\square, \tau}$ of R_ρ^\square whose characteristic 0 points correspond to representations of type τ . The usual construction of the Weil–Deligne representation associated to a Galois representation makes sense over $R_\rho^\square[1/l]$, so we have a natural morphism

$$\text{Spec } R_\rho^{\square, \tau}[1/l] \rightarrow G - \text{WD}_E(L/K).$$

3.2. The case $l = p$. Now suppose that $l = p$. If we fix a p -adic Hodge type \mathbf{v} in the sense of Definition 2.8.2 (that is, a $(\text{Res}_{E \otimes K/E} G)^\circ(\overline{E})$ -conjugacy class of cocharacters $\lambda : \mathbf{G}_m \rightarrow (\text{Res}_{E \otimes K/E} G)_{\overline{E}}$), and an inertial type τ , then by [Bal12, Prop. 3.0.12] there is unique \mathbf{Z}_ℓ -flat quotient $R_\rho^{\square, \tau, \mathbf{v}}$ of R_ρ^\square with the property that if B is a finite local E -algebra, then a morphism $R_\rho^\square \rightarrow B$ factors through $R_\rho^{\square, \tau, \mathbf{v}}$ if and only if the corresponding representation $\rho : \text{Gal}_K \rightarrow G(B)$ is potentially semi-stable with Hodge type \mathbf{v} and inertial type τ . For each finite-dimensional representation V of G , we may push-out the representation $\text{Gal}_K \rightarrow G(R_\rho^{\square, \tau, \mathbf{v}}[1/p])$ to obtain a family of potentially semi-stable representations $\text{Gal}_K \rightarrow \text{GL}(V)(R_\rho^{\square, \tau, \mathbf{v}}[1/p])$. Then we may argue as in [Kis08, §2.4] to obtain a family of filtered $(\varphi, N, \text{Gal}_{L/K})$ -modules over $R_\rho^{\square, L/K, \mathbf{v}}[1/p]$ (as we have been working with covariant functors in this paper, we need to dualize the construction in [Kis08, §2.4]). As these filtered $(\varphi, N, \text{Gal}_{L/K})$ -modules are exact and tensor-compatible, we obtain a family of G -valued $(\varphi, N, \text{Gal}_{L/K})$ -modules over $R_\rho^{\square, L/K, \mathbf{v}}[1/p]$. By Lemma 2.6.6, we again have a natural morphism

$$\text{Spec } R_\rho^{\square, \tau, \mathbf{v}}[1/l] \rightarrow G - \text{WD}_E(L/K).$$

3.3. Denseness of very smooth points. We continue to fix an inertial type τ and (if $p = l$) a p -adic Hodge type \mathbf{v} . For convenience, if $l \neq p$ then for the rest of this section we write $R_\rho^{\square, \tau, \mathbf{v}}$ for $R_\rho^{\square, \tau}$; this notational convention allows us to treat the cases $l \neq p$ and $l = p$ simultaneously. We study the generic fibre $R_\rho^{\square, \tau, \mathbf{v}}[1/l]$ via the morphism

$$(3.3.1) \quad \text{Spec } R_\rho^{\square, \tau, \mathbf{v}}[1/l] \rightarrow G - \text{WD}_E(L/K).$$

As in [Kis09, Proposition 2.3.5], if $x \in \text{Spec } R_\rho^{\square, \tau, \mathbf{v}}[1/l]$ is a closed point corresponding to a representation ρ_x , then the completed local ring A_x at x pro-represents framed deformations of ρ_x which are potentially semi-stable of p -adic Hodge type \mathbf{v} (if $l = p$), and have inertial type τ .

Proposition 3.3.2. (1) *If x is a closed point of the Jacobson scheme $\mathrm{Spec} R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/l]$, then the completion at x of the morphism (3.3.1) is formally smooth.*
 (2) *The morphism (3.3.1) is flat.*

Proof. The formal smoothness follows from the proof of [Kis08, Proposition 3.3.1], which carries over verbatim to our setting (since the morphism of groupoids from framed deformations to unframed deformations is formally smooth). Part (2) then follows from the fact that formally smooth morphisms between locally noetherian schemes are flat, which in turn follows from [Gro64, §0 Thm. 19.7.1]. \square

In the statement of the following theorem, $\delta_{l=p}$ is as usual defined to be 1 if $l = p$, and 0 otherwise.

Theorem 3.3.3. *There is a dense open subscheme of $\mathrm{Spec} R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/l]$ consisting of smooth points; in fact, there is a Zariski dense subset of $\mathrm{Spec} R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/l]$ consisting of very smooth points. Furthermore, $\mathrm{Spec} R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/l]$ is equidimensional of dimension $\dim G + \delta_{l=p} \dim \mathrm{Res}_{E \otimes K/E} G/P_{\mathbf{v}}$.*

Similarly, $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}, \psi}$ contains a dense open subscheme consisting of smooth points and a Zariski dense subset of very smooth points, and is equidimensional of dimension $1 + \dim_E G^{\mathrm{der}} + \delta_{l=p} \dim_E (\mathrm{Res}_{E \otimes K/E} G)/P_{\mathbf{v}}$.

Proof. Since the formation of scheme-theoretic images is compatible with flat base change, the existence of such a dense open subscheme U follows from Corollary 2.3.8 and Proposition 3.3.2. The existence of a Zariski dense subset of very smooth points follows from Corollary 2.4.5

We claim that if $x \in \mathrm{Spec} R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/l]$ is a closed point in U , then the completion A_x of $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/l]$ at x is formally smooth over \mathbf{Q}_p . Indeed, if \mathfrak{m}_x is the maximal ideal of A_x , then $\mathrm{Spec} A/\mathfrak{m}_x^n \subset U$ for all $n \geq 1$ (since U is open). Let B be a local \mathbf{Q}_p -algebra with maximal ideal \mathfrak{m}_B and let $I \subset B$ be an ideal such that $I\mathfrak{m}_B = (0)$. If there is a local homomorphism $A_x \rightarrow B/I$, let $D_{B/I}$ be the induced object of $G - \mathrm{WD}_E(L/K)(B/I)$. Then $H^2(D_{B/I}) = 0$, since the homomorphism $A_x \rightarrow B/I$ factors through A/\mathfrak{m}_x^n for some n . It follows that $D_{B/I}$ lifts to $D_B \in G - \mathrm{WD}_E(L/K)(B)$. Since $\mathrm{Spf} A_x \rightarrow G - \mathrm{WD}_E(L/K)$ is formally smooth, D_B is induced from a map $A_x \rightarrow B$ lifting $A \rightarrow B/I$.

Thus, to compute the dimension of $\mathrm{Spec} R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/l]$, it is therefore enough to compute the dimension of the tangent spaces at closed points in U . Let x be such a closed point, let E' be its residue field, and write A_x for the completion of $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/l]$ at x . By Proposition 3.3.2 the morphism $\mathrm{Spf} A_x \rightarrow G - \mathrm{WD}_E(L/K)$ is versal, and in the case $l \neq p$ we see (by the equivalence between Galois representations and Weil–Deligne representations recalled in Section 2.5) that it is a \widehat{G} -torsor, where \widehat{G} is the completion of G_E along the identity section. In particular, we have $\dim A_x \times_{G - \mathrm{WD}_E(L/K)} A_x = \dim A_x + \dim \widehat{G}$, and the claim about the dimension then follows from [EG17, Lem. 2.40].

If $l = p$, let $D_x := D_{\mathrm{st}}^L(\rho_x)$; it is equipped with a filtration \mathcal{F}_x^\bullet . We consider the set $(\mathrm{Spf} A_x)(E'[\varepsilon])$. Forgetting the framing on liftings is a formally smooth morphism of groupoids and makes the tangent space at x into a Lie G -torsor over the groupoid of unframed deformations. But since $E'[\varepsilon]$ is an artin local E -algebra,

by [Bel16, Proposition 2.4] the category of (unframed) potentially semi-stable representations of Gal_K over $E'[\varepsilon]$ deforming ρ_x is equivalent to the subcategory of $G - \mathrm{Mod}_{L/K, \varphi, N, \mathrm{Fil}}(E'[\varepsilon])$ deforming $D_{\mathrm{st}}^L(\rho_x)$.

There is a natural morphism of groupoids

$$G - \mathrm{Mod}_{L/K, \varphi, N, \mathrm{Fil}} \longrightarrow G - \mathrm{Mod}_{L/K, \varphi, N}$$

and therefore a commutative diagram

$$\begin{array}{ccc} G - \mathrm{Mod}_{L/K, \varphi, N, \mathrm{Fil}}(E'[\varepsilon]) & \longrightarrow & G - \mathrm{Mod}_{L/K, \varphi, N}(E'[\varepsilon]) \\ \downarrow & & \downarrow \\ G - \mathrm{Mod}_{L/K, \varphi, N, \mathrm{Fil}}(E') & \longrightarrow & G - \mathrm{Mod}_{L/K, \varphi, N}(E') \end{array}$$

By Corollary 2.7.7, the fibers of

$$G - \mathrm{Mod}_{L/K, \varphi, N, \mathrm{Fil}}(E'[\varepsilon]) \longrightarrow G - \mathrm{Mod}_{L/K, \varphi, N}(E'[\varepsilon])$$

over the filtered G -torsor D_x are torsors under $(\mathrm{ad}D_x/\mathcal{F}^0(\mathrm{ad}D_x))^{\mathrm{Gal}_{L/K}}$. Since $G - \mathrm{Mod}_{L/K, \varphi, N} \cong G - \mathrm{WD}_E$ is equidimensional of dimension 0 and $x \in \mathrm{Spec} R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}$ is a smooth point, we conclude that

$$\begin{aligned} \dim A_x &= \dim \mathrm{Lie} G + \dim (\mathrm{ad}D_x/\mathcal{F}^0(\mathrm{ad}D_x))^{\mathrm{Gal}_{L/K}} \\ &= \dim G + \dim \mathrm{Res}_{E \otimes K/E} G/P_{\mathbf{v}} \end{aligned}$$

as desired.

The corresponding statements for $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}, \psi}$ can be proved in the same way; we leave the details to the reader. \square

The following is a generalisation of [All14, Thm. D] (which treats the case that $l = p$ and $G = \mathrm{GL}_n$). We let x be a closed point of $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/l]$ with residue field E_x (a finite extension of E), and write $\rho_x : G_K \rightarrow G(E_x)$ for the corresponding representation.

Corollary 3.3.4. *The point x is a smooth point of $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/l]$ if and only if $H^0((\mathrm{ad} \mathrm{WD}(\rho_x))^*(1)) = 0$.*

Proof. This follows immediately from Corollary 2.4.2, Proposition 3.3.2, and (the proof of) Theorem 3.3.3. \square

Remark 3.3.5. If G is the L -group of a quasisplit reductive group over K , then it seems plausible that the condition of Corollary 3.3.4 could be equivalent to the condition that the (conjectural) L -packet of representations associated to the Frobenius semisimplification of $\mathrm{WD}(\rho_x)$ contains a generic element. In the case that $G = \mathrm{GL}_n$ (where the L -packets are singletons) and $\mathrm{WD}(\rho_x)$ is Frobenius semisimple, this is proved in [All14, §1], and in the general case it is closely related to [GP92, Conj. 2.6] (which relates genericity to poles at $s = 1$ of the adjoint L -function).

Remark 3.3.6. In the case that $l \neq p$, the equivalence between Galois representations and Weil–Deligne representations means that we can rewrite the condition in Corollary 3.3.4 as $H^0(G_K, \mathrm{ad}\rho_x^*(1)) = 0$.

We can also consider the quotient $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}, N=0}$, corresponding to the union of the irreducible components of $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/l]$ for which the monodromy operator N vanishes identically (if $l = p$, this is the locus of potentially crystalline representations, and if $l \neq p$, it is the locus of potentially unramified representations).

Theorem 3.3.7. *Fix an inertial type τ , and if $l = p$ then fix a p -adic Hodge type \mathbf{v} . Then $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}, N=0}[1/l]$ is smooth, and is equidimensional of dimension $1 + \dim_E G + \delta_{l=p} \dim_E(\mathrm{Res}_{E \otimes K/E} G)/P_{\mathbf{v}}$. Similarly $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}, N=0, \psi}$ is smooth and equidimensional of dimension $1 + \dim_E G^{\mathrm{der}} + \delta_{l=p} \dim_E(\mathrm{Res}_{E \otimes K/E} G)/P_{\mathbf{v}}$.*

Proof. This can be proved in exactly the same way as Theorem 3.3.3, replacing the use of the three term complex $\mathcal{C}^{\bullet}(D)$ considered in Section 2.2.1 with the two term complex

$$(\mathrm{ad}D_A)^{I_{L/K}} \xrightarrow{1 - \mathrm{Ad}(\Phi)} (\mathrm{ad}D_A)^{I_{L/K}}$$

concentrated in degrees 0 and 1; see [Kis08, Thm. 3.3.8] for more details in the case that $l = p$ and $G = \mathrm{GL}_n$. \square

3.4. Components of deformation rings. We now prove the following reassuring lemma, which shows that the components of universal deformation rings are invariant under $G(\mathcal{O})$ -conjugacy. It is a generalization of [BLGGT14, Lem. 1.2.2], which treats the case $G = \mathrm{GL}_n$; the proof there is by an explicit homotopy, while we use the theory of reductive group schemes over \mathcal{O} to construct less explicit homotopies.

Lemma 3.4.1. *Let $h \in G(\mathcal{O}')$ be an element which reduces to the identity modulo the maximal ideal, where \mathcal{O}' is the ring of integers in a finite extension of E . Then conjugation by h induces a map $\mathrm{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}} \otimes_{\mathcal{O}} \mathcal{O}')[1/l] \rightarrow \mathrm{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}} \otimes_{\mathcal{O}} \mathcal{O}')[1/l]$, and it fixes each irreducible component.*

Before we prove it, we record a preliminary lemma on irreducible components of the generic fiber of $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}$:

Lemma 3.4.2. *Let $A := \mathcal{O}[[X_1, \dots, X_n]]/I$ be the quotient of a power series ring. If $x, x' \in (\mathrm{Spf} A)^{\mathrm{rig}}$ lie on the same irreducible component, then they lie on the same irreducible component of $\mathrm{Spec} A[1/l]$.*

Proof. Let $A \rightarrow \tilde{A}$ denote the normalization of A . Then by [Con99, Theorem 2.1.3], $(\mathrm{Spf} \tilde{A})^{\mathrm{rig}} \rightarrow (\mathrm{Spf} A)^{\mathrm{rig}}$ is a normalization of the rigid space $(\mathrm{Spf} A)^{\mathrm{rig}}$, and x, x' lift to points $\tilde{x}, \tilde{x}' \in (\mathrm{Spf} \tilde{A})^{\mathrm{rig}}$ on the same connected component. By [dJ95, Lemma 7.1.9], \tilde{x} and \tilde{x}' correspond to distinct closed points of $\mathrm{Spec} \tilde{A}[1/l]$.

If \tilde{x} and \tilde{x}' lie on distinct connected components of $\mathrm{Spec} \tilde{A}[1/l]$, there are idempotents $e_x, e_{x'} \in \tilde{A}[1/l]$ such that e_x is 1 at \tilde{x} and 0 at \tilde{x}' and $e_{x'}$ is 1 at \tilde{x}' and 0 at \tilde{x} . Again by [dJ95, Lemma 7.1.9], the natural map $(\mathrm{Spf} \tilde{A})^{\mathrm{rig}} \rightarrow \mathrm{Spec} \tilde{A}[1/l]$ induces isomorphisms on residue fields of closed points. It follows that the pullbacks of e_x and $e_{x'}$ to $(\mathrm{Spf} \tilde{A})^{\mathrm{rig}}$ have the same property. But this would contradict the fact that \tilde{x} and \tilde{x}' lie on the same connected component of $(\mathrm{Spf} \tilde{A})^{\mathrm{rig}}$, so they must actually lie on the same connected component of $\mathrm{Spec} \tilde{A}[1/l]$. This in turn implies that they lie on the same irreducible component of $\mathrm{Spec} A[1/l]$. \square

Proof of Lemma 3.4.1. Let $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}} \otimes_{\mathcal{O}} \mathcal{O}' \rightarrow \mathcal{O}''$ be a homomorphism corresponding to a lift $\rho : \mathrm{Gal}_K \rightarrow G(\mathcal{O}'')$, where \mathcal{O}'' is the ring of integers in a finite extension

of E and contains \mathcal{O}' . We continue to write h for the image of h in $G(\mathcal{O}'')$. There is a finite morphism $\mathrm{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}} \otimes_{\mathcal{O}} \mathcal{O}'')[1/p] \rightarrow \mathrm{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}} \otimes_{\mathcal{O}} \mathcal{O}')[1/p]$, so to show that conjugation by h preserves irreducible components of $\mathrm{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}} \otimes_{\mathcal{O}} \mathcal{O}')[1/l]$, it suffices to show that conjugation by h preserves irreducible components of $\mathrm{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}} \otimes_{\mathcal{O}} \mathcal{O}'')[1/p]$. Moreover, by Lemma 3.4.2, it suffices to work with the rigid analytic generic fiber $\mathrm{Spf}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}} \otimes_{\mathcal{O}} \mathcal{O}'')^{\mathrm{rig}}$ of $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}} \otimes_{\mathcal{O}} \mathcal{O}''$.

After possibly extending \mathcal{O}'' , we may assume that G splits over \mathcal{O}'' . Since h is residually the identity element of G , it is a point of G° . After possibly further increasing \mathcal{O}'' , there is some Borel subgroup $B_{\mathcal{O}''[1/p]} \subset G_{\mathcal{O}''[1/p]}^{\circ}$ containing the image of h ; it extends to a Borel subgroup $B \subset G_{\mathcal{O}''}^{\circ}$ which contains h . Since \mathcal{O}'' is local, by [Con14, Proposition 5.2.3] there is a cocharacter $\lambda : (\mathbf{G}_m)_{\mathcal{O}''} \rightarrow G_{\mathcal{O}''}^{\circ}$ such that $B = P_{G^{\circ}}(\lambda) = U_{G^{\circ}}(\lambda) \rtimes Z_{G^{\circ}}(\lambda)$. Write h_z for the projection of h to $Z_{G^{\circ}}(\lambda)$ and h_u for the projection to $U_{G^{\circ}}(\lambda)$. Since this decomposition is unique, both h_z and h_u reduce to the identity modulo ϖ (where ϖ is a uniformizer of \mathcal{O}'').

Since $Z_{G^{\circ}}(\lambda)$ is a split torus, there is a map $z_t : (\mathbf{G}_m)_{\mathcal{O}''} \rightarrow G_{\mathcal{O}''}^{\circ}$ which specializes to both h_z and the identity. After analytifying this map, h_z and the identity lie in the same residue disk. Choosing coordinates on this residue disk, and rescaling them if necessary, we obtain a Galois representation $\tilde{\rho} : \mathrm{Gal}_K \rightarrow G(\mathcal{O}''[[T]])$ by considering the conjugation map $z_t \rho z_t^{-1} : \mathrm{Gal}_K \rightarrow G(\mathcal{O}''[[T]])$. This induces a homomorphism $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}} \otimes_{\mathcal{O}} \mathcal{O}'' \rightarrow \mathcal{O}''[[T]]$, which in turn induces a morphism of rigid spaces $\mathrm{Spf}(\mathcal{O}''[[T]])^{\mathrm{rig}} \rightarrow \mathrm{Spf}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}} \otimes_{\mathcal{O}} \mathcal{O}'')^{\mathrm{rig}}$. Since the source is irreducible and its image contains points corresponding to both ρ and $h_z \rho h_z^{-1}$, they lie on the same irreducible component of $\mathrm{Spf}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}} \otimes_{\mathcal{O}} \mathcal{O}'')^{\mathrm{rig}}$.

Thus, we may assume that $h \in U_{G^{\circ}}(\lambda)$. By definition, if A is an \mathcal{O}' -algebra, $U_{G^{\circ}}(\lambda)(A) = \{g \in B(A) \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} = 1\}$, so conjugating h by λ induces a map $u_t : \mathbf{A}_{\mathcal{O}''}^1 \rightarrow G_{\mathcal{O}''}$ with $u_1 = h$ and $u_0 = 1$. We therefore obtain a Galois representation $\tilde{\rho}' : \mathrm{Gal}_K \rightarrow G(\mathcal{O}''\langle T \rangle)$ by l -adically completing the map $u_t \rho u_t^{-1} : \mathrm{Gal}_K \rightarrow G(\mathcal{O}''[[T]])$. Since u_t is the identity modulo ϖ , $\tilde{\rho}'$ in fact lands in $G(\mathcal{O}''\langle lT \rangle)$, and therefore in $G(\mathcal{O}''[[lT]])$. This induces a map $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}} \rightarrow \mathcal{O}''[[lT]]$, and therefore a morphism of rigid spaces $\mathrm{Spf}(\mathcal{O}''[[lT]])^{\mathrm{rig}} \rightarrow \mathrm{Spf}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}})^{\mathrm{rig}}$. Since the source is irreducible and its image contains points corresponding to both ρ and $h_u \rho h_u^{-1}$, they lie on the same irreducible component of $\mathrm{Spf}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}})^{\mathrm{rig}}$, as required. \square

3.5. Tensor products of components, and base change. By a ‘‘component for $\bar{\rho}$ ’’ we mean a choice of τ and \mathbf{v} (in the case $l = p$) such that $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/l] \neq 0$, and a choice of an irreducible component of $\mathrm{Spec} R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/l]$.

Let $\bar{r} : G_K \rightarrow \mathrm{GL}_n(\mathbf{F})$, $\bar{s} : G_K \rightarrow \mathrm{GL}_m(\mathbf{F})$ be representations, let C be a component for \bar{r} and let D be a component for \bar{s} . Let K'/K be a finite extension. The following lemma will be useful in section 5.

Lemma 3.5.1. *There is a unique component $C \otimes D$ for $\bar{r} \otimes \bar{s}$ with the property that, if $r : G_K \rightarrow \mathrm{GL}_n(\mathbf{Q}_{\ell})$ and $s : G_K \rightarrow \mathrm{GL}_m(\mathbf{Q}_{\ell})$ correspond to closed points of C and D respectively, then $r \otimes s$ corresponds to a closed point of $C \otimes D$. Similarly, there is a unique component $C|_{K'}$ for $\bar{r}|_{G_{K'}}$ such that for all r , $r|_{G_{K'}}$ corresponds to a closed point of $C|_{K'}$.*

Proof. If a point of $\mathrm{Spec} R_{\bar{r}}^{\square, \tau, \nu}[1/l]$ or a point of $\mathrm{Spec} R_{\bar{r} \otimes \bar{s}}^{\square, \tau, \nu}[1/l]$ is smooth, then it lies on a unique irreducible component. Then the first part follows as in the proof of Theorem 3.3.3, replacing the appeal to Corollary 2.4.5 with one to Theorem 2.3.9, applied to the tensor product map

$$\mathrm{GL}_n \times \mathrm{GL}_m \rightarrow \mathrm{GL}_{nm}.$$

The second part follows from Theorem 3.3.3. \square

In the setting of the previous lemma, we will sometimes say that the component $C \otimes D$ is the tensor product of the components C and D , and that $C|_{K'}$ is the base change to K' of the component C .

4. GLOBAL DEFORMATION RINGS

4.1. A result of Balaji. In this section we recall one of the main results of [Bal12], which we will then combine with the results of section 3 to prove Proposition 4.2.6, which gives a lower bound for the dimension of certain global deformation rings. In [Bal12, §4.2] the group G is assumed to be connected, but this is unnecessary. Indeed, the assumption is only made in order to use the results of [Til96, §5], where it is also assumed that G is connected; however, this assumption is never used in any of the arguments of [Til96, §5], which apply unchanged to general G . Accordingly, we will freely use the results of [Bal12, §4.2] without assuming that G is connected. We assume in this section that E is taken large enough that G_E is quasisplit.

Let F be a number field, and let S be a finite set of places of F containing all of the places dividing $l\infty$. We work in the fixed determinant setting, and accordingly we fix homomorphisms $\bar{\rho} : \mathrm{Gal}_{F,S} \rightarrow G(\mathbf{F})$ and $\psi : \mathrm{Gal}_{F,S} \rightarrow G^{\mathrm{ab}}(\mathcal{O})$ such that $\mathrm{ab} \circ \bar{\rho} = \bar{\psi}$.

Write $R_{F,S}^{\square, \psi} \in \mathrm{CNL}_{\mathcal{O}}$ for the universal fixed determinant framed deformation \mathcal{O} -algebra of $\bar{\rho}$. Let $\Sigma \subset S$ be a subset containing all of the places lying over l . For each $v \in \Sigma$, we let $R_v^{\square, \psi}$ denote the universal fixed determinant framed deformation \mathcal{O} -algebra of $\bar{\rho}|_{\mathrm{Gal}_{F_v}}$, and we set $R_{\Sigma}^{\square, \psi} := \widehat{\otimes}_{v \in \Sigma, \mathcal{O}} R_v^{\square, \psi}$.

The following result is a special case of [Bal12, Prop. 4.2.5].

Proposition 4.1.1. *Suppose that $H^0(\mathrm{Gal}_{F,S}, (\mathfrak{g}_{\mathbf{F}}^0)^*(1)) = 0$, and let*

$$s := (|\Sigma| - 1) \dim_{\mathbf{F}} \mathfrak{g}_{\mathbf{F}}^0 + \sum_{v|l\infty, v \notin \Sigma} \dim_{\mathbf{F}} H^0(\mathrm{Gal}_{F_v}, \mathfrak{g}_{\mathbf{F}}^0).$$

Then for some $r \geq 0$ there is a presentation

$$R_{F,S}^{\square, \psi} \xrightarrow{\sim} R_{\Sigma}^{\square, \psi} \llbracket x_1, \dots, x_r \rrbracket / (f_1, \dots, f_{r+s}).$$

4.2. Global deformation rings of fixed type. In this section we combine our local results with Proposition 4.1.1 to prove a lower bound for the Krull dimension of a global deformation ring, following Balaji. This lower bound will only be non-trivial in the following setting.

Definition 4.2.1. If $p > 2$ then we say that $\bar{\rho}$ is *discrete series and odd* if F is totally real, and if for all places $v|l\infty$ of F we have $\dim_{\mathbf{F}} H^0(\mathrm{Gal}_{F_v}, \mathfrak{g}_{\mathbf{F}}^0) = \dim_E G - \dim_E B$, where B is a Borel subgroup of G .

Remark 4.2.2. Recall that we chose E to be large enough that G_E is quasisplit, so this definition makes sense. This condition is needed to make the usual Taylor–Wiles method work; see the introduction to [CHT08]. If G is the L -group of a

simply connected group then one can check that this condition is equivalent to F being totally real and $\bar{\rho}$ being odd in the sense of [Gro07] (cf. [Bal12, Lem. 4.3.1]). We use the term “discrete series” because the (conjectural) Galois representations associated to tempered automorphic representations which are discrete series at infinite places are expected to satisfy this property; see section 5 for an example of this, and [Gro07] for a more general discussion.

Definition 4.2.3. We say that a p -adic Hodge type \mathbf{v} is *regular* if the conjugacy class $P_{\mathbf{v}}$ consists of parabolic subgroups of $\mathrm{Res}_{E \otimes K/E} G$ whose connected components are Borel subgroups of $(\mathrm{Res}_{E \otimes K/E} G)^\circ$.

Remark 4.2.4. If $G = \mathrm{GL}_n$ then Definition 4.2.3 is equivalent to the usual definition, that for each embedding $K \hookrightarrow E$ the Hodge–Tate weights are pairwise distinct.

Remark 4.2.5. If E'/E is a field extension, then $(\mathrm{Res}_{E \otimes K/E} G)_{E'} \cong \mathrm{Res}_{E' \otimes K/E'} G$. Furthermore, the formation of $P_{\mathrm{Res}_{E \otimes K/E} G}(\lambda)$ is compatible with extension of scalars from E to E' . Thus, if \mathbf{v} is regular after extending scalars, it was regular over E (and $\mathrm{Res}_{E \otimes K/E} G$ is automatically quasisplit).

Write S^∞ for the set of finite places in S . For each place $v \in S^\infty$, we fix an inertial type τ_v , and if $v|l$ then we fix a Hodge type \mathbf{v}_v . If $v \nmid l$ (resp. if $v|l$), we let \bar{R}_v be a quotient of the corresponding fixed determinant framed deformation ring $R_{\bar{\rho}|\mathrm{Gal}_{F_v}}^{\square, \tau_v, \psi}$ (resp. $R_{\bar{\rho}|\mathrm{Gal}_{F_v}}^{\square, \tau_v, \mathbf{v}_v, \psi}$) corresponding to a union of irreducible components of the generic fiber. Set $R^{\square, \mathrm{univ}} := R_{F,S}^{\square, \psi} \otimes_{R_{\Sigma, \mathcal{O}}}^{\square, \psi} \widehat{\otimes}_{v \in S^\infty} \bar{R}_v$.

Assume that $H^0(\mathrm{Gal}_{F,S}, \mathfrak{g}_{\mathbf{F}}) = \mathfrak{z}_{\mathbf{F}}$, so that $\bar{\rho}$ admits a universal fixed determinant deformation \mathcal{O} -algebra $R_{F,S}^\psi \in \mathrm{CNL}_{\mathcal{O}}$, and write R^{univ} for the quotient of $R_{F,S}$ corresponding to $R^{\square, \mathrm{univ}}$ (as in the discussion preceding [BLGGT14, Lemma 1.3.3], this quotient exists by Lemma 3.4.1). In the case that we fix potentially crystalline types at the places $v|l$, and do not fix types at places away from l , the following result is [Bal12, Thm. 4.3.2]; the general case follows from the same arguments as those of Balaji, given the input of our local results.

Proposition 4.2.6. *Assume that $l > 2$, that $\bar{\rho}$ is discrete series and odd (so that in particular F is totally real), and that $H^0(\mathrm{Gal}_{F,S}, (\mathfrak{g}_{\mathbf{F}}^0)^*(1)) = 0$.*

Suppose that for each place $v|l$ the Hodge type \mathbf{v}_v is regular. Then R^{univ} has Krull dimension at least one.

Proof. By Proposition 4.1.1 (taking $\Sigma = S^\infty$, and noting that $R^{\mathrm{univ}, \square}$ is formally smooth over R^{univ} of relative dimension $\dim_{\mathbf{F}} \mathfrak{g}_{\mathbf{F}}^0$) we see that for some $r \geq \dim_{\mathbf{F}} \mathfrak{g}_{\mathbf{F}}^0$ we have a presentation

$$R^{\mathrm{univ}} \xrightarrow{\sim} \left(\widehat{\otimes}_{v \in S^\infty} \bar{R}_v \right) \llbracket x_1, \dots, x_{r - \dim_{\mathbf{F}} \mathfrak{g}_{\mathbf{F}}^0} \rrbracket / (f_1, \dots, f_{r+s})$$

where

$$s = (|S^\infty| - 1) \dim_{\mathbf{F}} \mathfrak{g}_{\mathbf{F}}^0 + \sum_{v|l} \dim_{\mathbf{F}} H^0(\mathrm{Gal}_{F_v}, \mathfrak{g}_{\mathbf{F}}^0).$$

It follows that the Krull dimension of R^{univ} is at least

$$\dim \widehat{\otimes}_{v \in S^\infty, \mathcal{O}} \bar{R}_v - |S^\infty| \dim_{\mathbf{F}} \mathfrak{g}_{\mathbf{F}}^0 - \sum_{v|l} \dim_{\mathbf{F}} H^0(\mathrm{Gal}_{F_v}, \mathfrak{g}_{\mathbf{F}}^0),$$

which by Theorem 3.3.3, and our assumption that each Hodge type \mathbf{v}_v is regular, is equal to

$$1 + \sum_{v|p} [F_v : \mathbf{Q}_p] \dim_E G/B - \sum_{v|\infty} \dim_{\mathbf{F}} H^0(\mathrm{Gal}_{F_v}, \mathfrak{g}_{\mathbf{F}}^0),$$

which in turn (by the assumption that $\bar{\rho}$ is discrete series and odd) equals 1, as required. \square

5. UNITARY GROUPS

5.1. The group \mathcal{G}_n . Let F be a CM field with maximal totally real subfield F^+ . In this section we generalise some results of [BLGGT14] on the deformation theory of Galois representations associated to polarised representations of Gal_F , by allowing ramification at primes of F^+ which are inert or ramified in F . This allows us to make cleaner statements, and is also useful in applications; for example, in Theorem 5.2.2 we remove a “split ramification” condition in the proof of the weight part of Serre’s conjecture for rank two unitary groups. Our results are also needed in [CEG], where they are used to construct lifts with specified ramification at certain places of F^+ which are inert in F .

Recall from [CHT08] the reductive group \mathcal{G}_n over \mathbf{Z} given by the semi-direct product of $\mathcal{G}_n^0 = \mathrm{GL}_n \times \mathrm{GL}_1$ by the group $\{1, j\}$ where

$$j(g, a)j^{-1} = (a(g^t)^{-1}, a).$$

We let $\nu : \mathcal{G}_n \rightarrow \mathrm{GL}_1$ be the character which sends (g, a) to a and sends j to -1 . Our results in this section are for the most part a straightforward application of the results of the earlier sections to the particular case $G = \mathcal{G}_n$, but we need to begin by comparing our definitions to those of [CHT08]; we will follow the notation of [CHT08] where possible.

Fix a place $v|\infty$. By [CHT08, Lem. 2.1.1], for any ring R there is a natural bijection between the set of homomorphisms $\rho : \mathrm{Gal}_{F^+} \rightarrow \mathcal{G}_n(R)$ inducing an isomorphism $\mathrm{Gal}_{F^+} / \mathrm{Gal}_F \xrightarrow{\sim} \mathcal{G}_n / \mathcal{G}_n^0$, and the set of triples $(r, \mu, \langle, \rangle)$ where $r : \mathrm{Gal}_F \rightarrow \mathrm{GL}_n(R)$, $\mu : \mathrm{Gal}_{F^+} \rightarrow R^\times$, and $\langle, \rangle : R^n \times R^n \rightarrow R$ is a perfect R -linear pairing such that $\langle x, y \rangle = -\mu(c_v)\langle y, x \rangle$, and $\langle r(\delta)x, r^{c_v}(\delta)y \rangle = \mu(\delta)\langle x, y \rangle$ for all $\delta \in \mathrm{Gal}_F$. We refer to such a triple as a μ -polarised representation of Gal_F , and we will sometimes denote it as a pair (r, μ) , the pairing being implicit.

This bijection is given by setting $r := \rho|_{\mathrm{Gal}_F}$ (more precisely, the projection of $\rho|_{\mathrm{Gal}_F}$ to $\mathrm{GL}_n(R)$), $\mu := \nu \circ \rho$, and $\langle x, y \rangle = x^t A^{-1} y$, where $\rho(c_v) = (A, -\mu(c_v))j$. If v is a finite place of F^+ which is inert or ramified in F , then we have an induced bijection between representations $\mathrm{Gal}_{F_v^+} \rightarrow \mathcal{G}_n(R)$ and μ -polarised representations $\mathrm{Gal}_{F_v} \rightarrow \mathrm{GL}_n(R)$.

There is an isomorphism $\mathrm{GL}_1 \rightarrow Z_{\mathcal{G}_n}$ given by $g \mapsto (g, g^2) \in \mathrm{GL}_1 \rightarrow \mathrm{GL}_1 \subset \mathrm{GL}_n \times \mathrm{GL}_1$, and we have $\mathcal{G}_n^{\mathrm{der}} = \mathrm{GL}_n \times 1$, and $\mathcal{G}_n^{\mathrm{ab}} = \mathrm{GL}_1 \times \{1, j\}$. (It is easy to check by direct calculation that $\mathcal{G}_n^{\mathrm{der}} \subset \mathcal{G}_n^{\circ}$, and indeed $\mathcal{G}_n^{\mathrm{der}} \subset \mathrm{GL}_n \times 1$. Since $\mathrm{GL}_n^{\mathrm{der}} = \mathrm{SL}_n$, we have $\mathrm{SL}_n \times 1 \subset \mathcal{G}_n^{\mathrm{der}}$, and since $j(1, a)j^{-1}(1, a^{-1}) = (a, 1)$, we also have $\mathrm{GL}_1 \times 1 \subset \mathcal{G}_n^{\mathrm{der}}$, whence $\mathrm{GL}_n \times 1 \subset \mathcal{G}_n^{\mathrm{der}}$. Similarly, one checks easily that $Z_{\mathcal{G}_n} \subset \mathcal{G}_n^{\circ}$, so that $Z_{\mathcal{G}_n} \subset \mathrm{GL}_1 \times \mathrm{GL}_1$. If $(g, a) \in \mathrm{GL}_1 \times \mathrm{GL}_1$ then $j(g, a)j^{-1} = (ag^{-1}, a)$, so we see that $(g, a) \in Z_{\mathcal{G}_n}$ if and only if $a = g^2$, as required.)

We fix a prime $l > 2$ and a representation $\bar{\rho} : \mathrm{Gal}_{F^+} \rightarrow \mathcal{G}_n(\mathbf{F})$ with $\bar{\rho}^{-1}(\mathcal{G}_n^0(\mathbf{F})) = \mathrm{Gal}_F$. We fix a character $\mu : \mathrm{Gal}_{F^+} \rightarrow \mathcal{O}^\times$ with $\nu \circ \bar{\rho} = \bar{\mu}$. Write $\psi : \mathrm{Gal}_{F^+} \rightarrow$

$\mathcal{G}_n^{\text{ab}}(\mathcal{O})$ for the character taking $g \in \text{Gal}_F$ to $(\mu(g), 1)$ and $g \in \text{Gal}_{F^+} \setminus \text{Gal}_F$ to $(-\mu(g), j)$.

Note that if $R \in \text{CNL}_{\mathcal{O}}$ then a deformation $\rho : \text{Gal}_{F^+} \rightarrow \mathcal{G}_n(R)$ of $\bar{\rho}$ has $\text{ab} \circ \rho = \psi$ if and only if $\nu \circ \rho = \mu$. By [All16, Prop. 2.2.3], restriction to Gal_F gives an equivalence between the μ -polarised (framed) deformations of $\bar{\rho}$ and the μ -polarised (framed) deformations r of $\bar{r} := \bar{\rho}|_{\text{Gal}_F} : \text{Gal}_F \rightarrow \text{GL}_n(\mathbf{F})$, the latter by definition being those r which satisfy $r^c \cong r^\vee \mu$ (where we are writing c for c_v , as r^c is independent of the choice of $v|\infty$).

The same equivalence pertains to deformations of $\bar{\rho}|_{\text{Gal}_{F_v^+}}$, where v is inert or ramified in F . On the other hand, if v splits as $\widetilde{v}v^c$ in F , then restriction to Gal_{F_v} gives an equivalence between μ -polarised (framed) deformations of $\bar{\rho}|_{\text{Gal}_{F_v^+}}$ and (framed) deformations of $\bar{r}|_{\text{Gal}_{F_v}}$; thus at such places the deformation theory of representations valued in \mathcal{G}_n is reduced to the case of GL_n . It is for this reason that [CHT08] and its sequels only permit ramification at places which split in F .

By [CHT08, Lem. 2.1.3], $\bar{\rho}$ is discrete series and odd in the sense of Definition 4.2.1 if and only if for each place $v|\infty$ of F^+ with corresponding complex conjugation $c_v \in \text{Gal}_{F^+}$ we have $\bar{\mu}(c_v) = -1$. This is by definition equivalent to the corresponding polarised representation $(\bar{\rho}|_{\text{Gal}_F}, \bar{\mu})$ being totally odd in the sense of [BLGGT14, §2.1].

Let S be a finite set of places of F^+ , including all the places where \bar{r} or μ are ramified, all the infinite places, and all the places dividing l . The following is a generalisation of [BLGGT14, Prop. 1.5.1] (which is the case that every finite place in S splits in F , and is actually proved in [CHT08]); note that the assumption that $\bar{\rho}|_{\text{Gal}_{F(\zeta_l)}}$ is absolutely irreducible is missing from the statement of [BLGGT14, Prop. 1.5.1], but should have been included there. Note also that this assumption implies that $\bar{\rho}$ admits a universal deformation ring; indeed, we have $H^0(\text{Gal}_{F^+}, \mathfrak{gl}_{n, \mathbf{F}}) = H^0(\text{Gal}_{F^+}, \mathfrak{gl}_{n, \mathbf{F}} \times \mathfrak{gl}_{1, \mathbf{F}}) = \mathfrak{gl}_{1, \mathbf{F}}$ by Schur's lemma (note that $\text{Gal}(F/F^+)$ acts by -1 on the scalar matrices in $\mathfrak{gl}_{n, \mathbf{F}}$).

Corollary 5.1.1. *Let $l > 2$ be prime, and let $\bar{\rho} : \text{Gal}_{F^+} \rightarrow \mathcal{G}_n(\mathbf{F})$ be such that $\bar{\rho}|_{\text{Gal}_{F(\zeta_l)}}$ is absolutely irreducible. Assume that $\bar{\rho}$ is discrete series and odd.*

Let μ be a de Rham lift of $\bar{\mu}$, and let S be a finite set of places of F^+ including all the places at that which either \bar{r} or μ is ramified, and all the places dividing $l\infty$. For each finite place $v \in S$, fix an inertial type τ_v , and if $v|l$, fix a regular Hodge type \mathbf{v}_v . Fix quotients of the corresponding local μ -polarised framed deformation rings which correspond to a union of irreducible components of the generic fiber.

Let R^{univ} be the universal deformation ring for μ -polarised deformations of \bar{r} which are unramified outside S , and lie on the given union of irreducible components for each finite place $v \in S$. Then R^{univ} has Krull dimension at least one.

Proof. By Proposition 4.2.6, we need only check that $H^0(\text{Gal}_{F^+, S}, (\mathfrak{gl}_{n, \mathbf{F}})^*(1))$ vanishes, where $\mathfrak{gl}_{n, \mathbf{F}}$ is the Lie algebra of $\mathcal{G}_n^{\text{der}}$. By inflation-restriction this group injects into $H^0(\text{Gal}_{F(\zeta_l)}, (\mathfrak{gl}_{n, \mathbf{F}})^*(1))^{\text{Gal}(F(\zeta_l)/F^+)} = H^0(\text{Gal}_{F(\zeta_l)}, (\mathfrak{gl}_{n, \mathbf{F}}))^{\text{Gal}(F(\zeta_l)/F^+)}$. Since $\bar{\rho}|_{\text{Gal}_{F(\zeta_l)}}$ is absolutely irreducible by assumption, this group vanishes by Schur's lemma (noting again that $\text{Gal}(F/F^+)$ acts by -1 on the scalar matrices in $\mathfrak{gl}_{n, \mathbf{F}}$). \square

5.2. Existence of lifts and the weight part of Serre's conjecture. We now prove a strengthening of [BLGG13, Thm. A.4.1], removing the condition that the

places at which our Galois representations are ramified are split in F . We refer the reader to [BLGGT14] for any unfamiliar terminology; in particular, potential diagonalizability is defined in [BLGGT14, §1.4], while adequacy and the notion of a polarised Galois representation being potentially diagonalizably automorphic are defined in [BLGGT14, §2.1].

Theorem 5.2.1. *Let l be an odd prime not dividing n , and suppose that $\zeta_l \notin F$. Let $\bar{\rho} : \text{Gal}_{F^+} \rightarrow \mathcal{G}_n(\mathbf{F})$ be such that $\bar{\rho}|_{\text{Gal}_F(\zeta_p)}$ is absolutely irreducible. Assume that $\bar{\rho}$ is discrete series and odd. Let S be a finite set of places of F^+ , including all places dividing $l\infty$.*

Let μ be a de Rham lift of $\bar{\mu}$, and let S be a finite set of places of F^+ including all the places at which either $\bar{\rho}$ or μ is ramified, and all the places dividing $l\infty$. For each finite place $v \in S$, fix an inertial type τ_v , and if $v|l$, fix a regular Hodge type \mathbf{v}_v . Fix quotients of the corresponding local μ -polarised framed deformation rings which correspond to an irreducible component of the generic fiber; if $v|l$, assume also that this component is potentially diagonalizable.

Assume further that there is a finite extension of CM fields F'/F such that F' does not contain ζ_l , all finite places of $(F')^+$ above S split in F , and $\bar{\rho}|_{\text{Gal}_{F',S}}$ is adequate; and that there exists a lift $r' : \text{Gal}_{F^+,S} \rightarrow \mathcal{G}_n(\mathcal{O})$ of $\bar{\rho}|_{\text{Gal}_{(F')^+,S}}$ with $\nu \circ r' = \mu|_{\text{Gal}_{F^+,S}}$, with the further property that r' is potentially diagonalizably automorphic.

Then there is a lift

$$r : \text{Gal}_{F^+,S} \longrightarrow \mathcal{G}_n(\mathcal{O})$$

of $\bar{\rho}$ such that

- (1) $\nu \circ r = \mu$;
- (2) if $v \in S$ is a finite place, then $r|_{G_{F_v^+}}$ corresponds to a point on our chosen component of the local deformation ring.
- (3) $r|_{\text{Gal}_{(F')^+,S}}$ is potentially diagonalizably automorphic.

Proof. Let R^{univ} be the universal deformation ring for μ -polarised deformations of $\bar{\rho}$ which are unramified outside S , and lie on the given irreducible component for each finite place $v \in S$. Then R^{univ} has Krull dimension at least one by Corollary 5.1.1. We claim that R^{univ} is a finite \mathcal{O} -algebra. Admitting this claim, we can choose a homomorphism $R^{\text{univ}} \rightarrow E$, and let r be the corresponding representation. This satisfies properties (1) and (2) by construction, and (3) by [BLGGT14, Thm. 2.3.2].

It remains to prove the claim. Let $R_{F'}^{\text{univ}}$ be the universal deformation ring for $\mu|_{G_{(F')^+,S}}$ -polarised deformations of $\bar{\rho}|_{G_{F',S}}$ which lie on the base changes of our chosen components. By [BLGGT14, Lem. 1.2.3 (1)], R^{univ} is a finite $R_{F'}^{\text{univ}}$ -algebra, so it is enough to show that $R_{F'}^{\text{univ}}$ is a finite \mathcal{O} -algebra.

By [BLGG13, Thm. A.4.1] (with F there taken to equal F'), there is a representation $r'' : G_{(F')^+,S} \rightarrow \mathcal{G}_n(\mathcal{O})$ corresponding to an \mathcal{O} -point of $R_{F'}^{\text{univ}}$, which is furthermore potentially diagonalizably automorphic. Then $R_{F'}^{\text{univ}}$ is a finite \mathcal{O} -algebra by [BLGGT14, Thm. 2.3.2], as required. \square

We now apply this result to the weight part of Serre's conjecture for unitary groups. We restrict ourselves to the case $n = 2$, where the existing results in the literature are strongest; our results should also allow the removal of the hypothesis of "split ramification" from results in the literature for higher rank unitary groups, such as the results of [BLGG]. We recall that if K/\mathbf{Q}_ℓ is a finite extension, there is

associated to any representation $\bar{\rho} : \text{Gal}_K \rightarrow \text{GL}_2(\mathbf{F})$ a set $W(\bar{\rho})$ of Serre weights. A definition of $W(\bar{\rho})$ was first given in [BDJ10] in the case that K/\mathbf{Q}_ℓ is unramified, and various generalisations and alternative definitions have subsequently been proposed. As a result of the main theorems of [GLS15, CEGM], all of these definitions are equivalent; we refer the reader to the introductions to those papers for a discussion of the various definitions.

Suppose that F is an imaginary CM field with maximal totally real subfield F^+ , such that F/F^+ is unramified at all finite places, that each place of F^+ above l splits in F , and that $[F^+ : \mathbf{Q}]$ is even. Then as in [BLGG13] we have a unitary group G/F^+ which is quasisplit at all finite places and compact at all infinite places. If $\bar{r} : \text{Gal}_{F^+} \rightarrow \mathcal{G}_2(\overline{\mathbf{F}}_\ell)$ is irreducible, the notion of \bar{r} being modular of a Serre weight is defined in [BLGG13, Defn. 2.1.9]. This definition (implicitly) insists that \bar{r} is only ramified at places which split in F , and we relax it as follows: we change the definition of a good compact open subgroup $U \subset G(\mathbf{A}_{F^+}^\infty)$ in [BLGG13, Defn. 2.1.5] to require only that at all places $v|l$ we have $U_v = G(\mathcal{O}_{F_v^+})$, and at all places $v \nmid l$ we have $U_v \subset G(\mathcal{O}_{F_v^+})$. (Consequently, we are now considering automorphic forms of arbitrary level away from l , whereas in [BLGG13] the level is hyperspecial at all places which do not split in F .)

Having made this change, everything in [BLGG13, §2] goes through unchanged, except that all mentions of “split ramification” can be deleted. The following theorem strengthens [GLS14, Thm. A], removing a hypothesis on the ramification away from l (and also a hypothesis on the ramification at l , although that could already have been removed thanks to the results of [GLS15]).

Theorem 5.2.2. *Let F be an imaginary CM field with maximal totally real subfield F^+ , and suppose that F/F^+ is unramified at all finite places, that each place of F^+ above l splits in F , and that $[F^+ : \mathbf{Q}]$ is even. Suppose that l is odd, that $\bar{r} : G_{F^+} \rightarrow \mathcal{G}_2(\overline{\mathbf{F}}_\ell)$ is irreducible and modular, and that $\bar{r}(G_{F(\zeta_l)})$ is adequate.*

Then the set of Serre weights for which \bar{r} is modular is exactly the set of weights given by the sets $W(\bar{r}|_{G_{F_v}}), v|l$.

Proof. We begin by observing that the proof of [BLGG13, Thm. 5.1.3] goes through in our more general context (that is, without assuming “split ramification”). Indeed, we have already observed that the results of [BLGG13, §2] are valid in our context, and chasing back through the references, this follows by replacing the citation of [BLGG13, Thm. A.4.1] in the proof of [BLGG13, Thm. 3.1.2] with a reference to Theorem 5.2.1 above.

This shows that \bar{r} is modular of every weight given by the $W(\bar{r}|_{G_{F_v}}), v|l$. For the converse, observe that [BLGG13, Cor. 4.1.8] also holds in our context (again, since the results of [BLGG13, §2] go through); the result then follows immediately from [GLS15, Thm. 6.1.8]. \square

Remark 5.2.3. It is presumably possible to prove in the same way a further strengthening of Theorem 5.2.2 where we allow our unitary group to be ramified at some finite places (and thus allow $[F^+ : \mathbf{Q}]$ to be odd, and F/F^+ to be ramified at some finite places), but to do so would involve a lengthier discussion of automorphic representations on unitary groups, which would take us too far afield.

Remark 5.2.4. We have assumed that the places of F^+ above l split in F , because the weight part of Serre’s conjecture has not been considered in the literature for unitary groups which do not split above l . However, it seems likely that it is

possible to formulate and prove a generalisation of Theorem 5.2.2 which removes this assumption, following the ideas of [GK14] and [GG15] (that is, using the Breuil–Mézard conjecture for potentially Barsotti–Tate representations). Again, this would take us too far afield from the main concerns of this paper, so we do not pursue this.

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