DIMENSION THEORY AND COMPONENTS OF ALGEBRAIC STACKS

MATTHEW EMERTON AND TOBY GEE

Abstract. We prove some basic results on the dimension theory of algebraic stacks, and on the multiplicities of their irreducible components, for which we do not know a reference.

Contents

In this short note we develop some basic results related to the notions of irreducible components and dimensions of locally Noetherian algebraic stacks. We work in the basic framework of the Stacks Project [Stacks]; we also note that most of the results proved here have now been incorporated into [Stacks, Tag 0DQR] (sometimes with weaker hypotheses than in this note).

The main results on the dimension theory of algebraic stacks in the literature that we are aware of are those of [Oss15], which makes a study of the notions of codimension and relative dimension. We make a more detailed examination of the notion of the dimension of an algebraic stack at a point, and prove various results relating the dimension of the fibres of a morphism at a point in the source to the dimension of its source and target. We also prove a result (Lemma 2.40 below) which allow us (under suitable hypotheses) to compute the dimension of an algebraic stack at a point in terms of a versal ring.

While we haven't always tried to optimise our results, we have largely tried to avoid making unnecessary hypotheses. However, in some of our results, in which we compare certain properties of an algebraic stack to the properties of a versal ring to this stack at a point, we have restricted our attention to the case of algebraic stacks that are locally finitely presented over a locally Noetherian scheme base, all of whose local rings are G-rings. This gives us the convenience of having Artin approximation available to compare the geometry of the versal ring to the geometry of the stack itself. However, this restrictive hypothesis may not be necessary for the truth of all of the various statements that we prove. Since it is satisfied in the applications that we have in mind, though, we have been content to make it when it helps.

M.E. was supported in part by NSF grant DMS-1303450. T.G. was supported in part by a Leverhulme Prize, EPSRC grant EP/L025485/1, ERC Starting Grant 306326, and a Royal Society Wolfson Research Merit Award.

Acknowledgements. We would like to thank our coauthors Ana Caraiani and David Savitt for their interest in this note, as well as Brian Conrad and Johan de Jong for their valuable comments on various parts of the manuscript.

1. Multiplicities of components of algebraic stacks

If X is a locally Noetherian scheme, then we may write X (thought of simply as a topological space) as a union of irreducible components, say $X = \cup T_i$. Each irreducible component is the closure of a unique generic point ξ_i, and the local ring $\mathcal{O}_{X,\xi_i}$ is a local Artin ring. We may define the *multiplicity* $\mu_{T_i}(X)$ of X along T_i to be $\ell(\mathcal{O}_{X,\xi_i})$.

Our goal here is to generalise this definition to locally Noetherian algebraic stacks. If $\mathcal{X}$ is such a stack, then it has an underlying topological space $|\mathcal{X}|$ (see [Stacks, Definition 04Y8]), which is locally Noetherian (by [Stacks, Definition 04Z8]), and hence which may be written as a union of irreducible components; we refer to these as the irreducible components of $\mathcal{X}$. If $\mathcal{X}$ is quasi-separated, then $|\mathcal{X}|$ is sober (by [LMB00, Cor. 5.7.2]), but it need not be in the non-quasi-separated case.[1] (Consider for example the non-quasi-separated algebraic space $X := \mathbb{A}^1_{\mathbb{C}}/\mathbb{Z}$.) Furthermore, there is no structure sheaf on $|\mathcal{X}|$ whose stalks can be used to define multiplicities.

In order to define the multiplicity of a component of $|\mathcal{X}|$, we use the fact that if $U \to \mathcal{X}$ is a smooth surjection from a scheme U to $\mathcal{X}$ (such a surjection exists, since $\mathcal{X}$ is an algebraic stack), it induces a surjection $|U| \to |\mathcal{X}|$ by [Stacks, Tag 04XI] (here $|U|$ simply denotes the topological space underlying U), and for each irreducible component T of $|\mathcal{X}|$ there is an irreducible component T' of $|U|$ such that T' maps into T with dense image. (See Lemma 1.2 below for a proof.)

Definition 1.1. We define $\mu_T(\mathcal{X}) := \mu_{T'}(U)$.

Of course, we must check that this is independent of the choice of chart U, and of the choice of irreducible component T' mapping to T. We begin by making this verification, as well as proving Lemma 1.2.

Lemma 1.2. *If $U \to \mathcal{X}$ is a smooth morphism from a scheme onto a locally Noetherian algebraic stack $\mathcal{X}$, then the closure of the image of any irreducible component of $|U|$ is an irreducible component of $|\mathcal{X}|$. If this morphism is furthermore surjective, then all irreducible components of $|\mathcal{X}|$ are obtained in this way.*

Proof. This is easily verified, using the fact that $|U| \to |\mathcal{X}|$ is continuous and open by [Stacks, Lem. 04XL], and furthermore surjective if $U \to \mathcal{X}$ is, once one recalls that the irreducible components of a locally Noetherian topological space can be characterised as being the closures of irreducible open subsets of the space. □

The preceding lemma applies in particular in the case of smooth morphisms between locally Noetherian schemes. This particular case is implicitly invoked in the statement of the following lemma.

[1] We follow the Stacks Project in allowing our algebraic stacks to be non-quasi-separated. However, in the applications that we have in mind, the algebraic stacks involved will in fact be quasi-separated, and so the reader who prefers to restrict their attention to the quasi-separated case will lose nothing by doing so.

Lemma 1.3. *If $U \to X$ is a smooth morphism of locally Noetherian schemes, and if T' is an irreducible component of U, with T denoting the irreducible component of X obtained as the closure of the image of T', then $\mu_{T'}(U) = \mu_T(X)$.*

Proof. Write ξ' for the generic point of T', and ξ for the generic point of T, so that we need to show that $\ell(\mathcal{O}_{X,\xi}) = \ell(\mathcal{O}_{U,\xi'})$.

Let $n = \ell(\mathcal{O}_{X,\xi})$, and choose a sequence $\mathcal{O}_{X,\xi} = I_0 \supset I_1 \supset \cdots \supset I_n = 0$ with $I_i/I_{i+1} \cong \mathcal{O}_{X,\xi}/\mathfrak{m}_{X,\xi}$. The map $\mathcal{O}_{X,\xi} \to \mathcal{O}_{U,\xi'}$ is flat, so that we have

$$I_i\mathcal{O}_{U,\xi'}/I_{i+1}\mathcal{O}_{U,\xi'} \cong (I_i/I_{i+1}) \otimes_{\mathcal{O}_{X,\xi}} \mathcal{O}_{U,\xi'} \cong \mathcal{O}_{U,\xi'}/\mathfrak{m}_{X,\xi}\mathcal{O}_{U,\xi'},$$

so it suffices to show that $\mathfrak{m}_{X,\xi}\mathcal{O}_{U,\xi'} = \mathfrak{m}_{U,\xi'}$, or in other words that $\mathcal{O}_{U,\xi'}/\mathfrak{m}_{X,\xi}\mathcal{O}_{U,\xi'}$ is reduced.

Since the map $U \to X$ is smooth, so is its base-change $U_\xi \to \operatorname{Spec} \kappa(\xi)$. As U_ξ is a smooth scheme over a field, it is reduced, and thus so its local ring at any point. In particular, $\mathcal{O}_{U,\xi'}/\mathfrak{m}_{X,\xi}\mathcal{O}_{U,\xi'}$, which is naturally identified with the local ring of U_ξ at ξ', is reduced, as required. $\square$

Using this result, we may show that notion of multiplicity given in Definition 1.1 is in fact well-defined.

Lemma 1.4. *If $U_1 \to \mathcal{X}$ and $U_2 \to \mathcal{X}$ are two smooth surjections from schemes to the locally Noetherian algebraic stack $\mathcal{X}$, and T_1' and T_2' are irreducible components of $|U_1|$ and $|U_2|$ respectively, the closures of whose images are both equal to the same irreducible component T of $|\mathcal{X}|$, then $\mu_{T_1'}(U_1) = \mu_{T_2'}(U_2)$.*

Proof. Let V_1 and V_2 be dense subsets of T_1' and T_2', respectively, that are open in U_1 and U_2 respectively. The images of $|V_1|$ and $|V_2|$ in $|\mathcal{X}|$ are non-empty open subsets of the irreducible subset T, and therefore have non-empty intersection. By [Stacks, Tag 04XH], the map $|V_1 \times_{\mathcal{X}} V_2| \to |V_1| \times_{|\mathcal{X}|} |V_2|$ is surjective, and consequently $V_1 \times_{\mathcal{X}} V_2$ is a non-empty algebraic space; we may therefore choose an étale surjection $V \to V_1 \times_{\mathcal{X}} V_2$ whose source is a (non-empty) scheme. If we let T' be any irreducible component of V, then Lemma 1.2 shows that the closure of the image of T' in U_1 (respectively U_2) is equal to T_1' (respectively T_2').

Applying Lemma 1.3 twice we find that

$$\mu_{T_1'}(U_1) = \mu_{T'}(V) = \mu_{T_2'}(U_2),$$

as required. $\square$

It will be convenient to have a comparison between the notion of multiplicity of an irreducible component given by Definition 1.1 and the related notion of multiplicities of irreducible components of (the spectra of) versal rings of $\mathcal{X}$ at finite type points. In order to have a robust theory of versal rings at finite type points, we assume for the remainder of this note that $\mathcal{X}$ is locally of finite presentation over a locally Noetherian scheme S, all of whose local rings are G-rings. (This hypothesis on the local rings may not be necessary for all the assertions that follow, but it makes the arguments straightforward, and in any case seems to be necessary for the actual comparison of multiplicities. We also note that condition this is equivalent to the apparently weaker condition that the local rings of S at finite type points are G-rings; indeed, the finite type points are dense in S [Stacks, Lem. 02J4], and essentially by definition, any localization of a G-ring is again a G-ring.)

We begin by recalling the following standard consequence of Artin approximation.

Lemma 1.5. *Let $\mathcal{X}$ be an algebraic stack locally of finite presentation over a locally Noetherian scheme S, all of whose local rings are G-rings, and let $x : \operatorname{Spec} k \to \mathcal{X}$ be a morphism whose source is the spectrum of a field of finite type over $\mathcal{O}_S$.*

If A_x is a versal ring to $\mathcal{X}$ at x, then we may find a smooth morphism $U \to \mathcal{X}$ whose source is a scheme, containing a point $u \in U$ of residue field k, such that the induced morphism $u = \operatorname{Spec} k \to U \to \mathcal{X}$ coincides with the given morphism x, and such that there is an isomorphism $\widehat{\mathcal{O}}_{U,u} \cong A_x$ compatible with the versal morphism $\operatorname{Spf} A_x \to \mathcal{X}$ and the induced morphism $\operatorname{Spf} \widehat{\mathcal{O}}_{U,u} \to U \to \mathcal{X}$.

Proof. Since $\mathcal{X}$ is an algebraic stack, the versal morphism $\operatorname{Spf} A_x \to \mathcal{X}$ is effective, i.e. can be promoted to a morphism $\operatorname{Spec} A_x \to \mathcal{X}$ [Stacks, Lem. 07X8]. By assumption $\mathcal{X}$ is locally of finite presentation over S, and hence limit preserving [EG14, Lem. 2.1.9], and so Artin approximation (see [Stacks, Lem. 07XH] and its proof) shows that we may find a morphism $U \to \mathcal{X}$ with source a finite type S-scheme, containing a point $u \in U$ of residue field k, satisfying all of the required properties except possibly the smoothness of $U \to \mathcal{X}$.

Since $\mathcal{X}$ is an algebraic stack, we see that if we replace U by a sufficient small neighbourhood of u, we may in addition assume that $U \to \mathcal{X}$ is smooth (see e.g. [EG14, Lem. 2.4.7 (4)]), as required. □

Lemma 1.6. *Let $\mathcal{X}$ be an algebraic stack locally of finite presentation over a locally Noetherian scheme S, all of whose local rings are G-rings, and let $x : \operatorname{Spec} k \to \mathcal{X}$ be a morphism whose source is the spectrum of a field of finite type over $\mathcal{O}_S$. If A_x and A'_x are two versal rings to $\mathcal{X}$ at x, then the multi-sets of irreducible components of $\operatorname{Spec} A_x$ and of $\operatorname{Spec} A'_x$ (in which each component is counted with its multiplicity), are in canonical bijection.*

Furthermore, there is a natural surjection from the set of irreducible components of each of $\operatorname{Spec} A_x$ and $\operatorname{Spec} A'_x$ to the set of irreducible components of $|\mathcal{X}|$ containing the class of x in $|\mathcal{X}|$; this surjection sends components that correspond by the above bijection to the same component of $|\mathcal{X}|$; and this surjection preserves multiplicities.

Proof. By Lemma 1.5 we can find smooth morphisms $U, U' \to \mathcal{X}$ whose sources are schemes, and points u, u' of U, U' respectively, both with residue field k, such that the induced morphisms $\widehat{\mathcal{O}}_{U,u} \to U \to \mathcal{X}$ and $\widehat{\mathcal{O}}_{U',u'} \to U' \to \mathcal{X}$ can be identified respectively with the versal morphisms $\operatorname{Spf} A_x \to \mathcal{X}$ and $\operatorname{Spf} A'_x \to \mathcal{X}$. We then form the fibre product $U'' := U \times_{\mathcal{X}} U'$; this is an algebraic space over S, and the two monomorphisms $u = \operatorname{Spec} k \to U$ and $u' = \operatorname{Spec} k \to U'$ induce a monomorphism $u'' = \operatorname{Spec} k \to U''$. Following [EG14, Prop. 2.2.14, Def. 2.2.16], we consider the complete local ring $\widehat{\mathcal{O}}_{U'',u}$ of U'' at u''.

Since $U'' \to U$ is smooth, we see that the induced morphism $A_x = \widehat{\mathcal{O}}_{U,u} \to \widehat{\mathcal{O}}_{U'',u''}$ induces a smooth morphism of representable functors, in the sense of [Stacks, Def. 06HG], and hence, by [Stacks, Lem. 06HL], we see that $\widehat{\mathcal{O}}_{U'',u''}$ is a formal power series ring over A_x. Similarly, it is a formal power series ring over A'_x. Recall that if A is a complete local ring and B is a formal power series ring in finitely many variables over A, then the irreducible components of $\operatorname{Spec} B$ are in a natural multiplicity preserving bijection with the irreducible components of $\operatorname{Spec} A$. Thus, we obtain multiplicity preserving bijections between the multi-sets of irreducible components of each of $\operatorname{Spec} A_x$ and $\operatorname{Spec} A'_x$ with the multi-set of irreducible components of $\operatorname{Spec} \widehat{\mathcal{O}}_{U'',u''}$, and hence between these two multi-sets themselves.

The morphism $\operatorname{Spec} A_x \to \mathcal{X}$ factors through U, and the scheme-theoretic image of each irreducible component of $\operatorname{Spec} A_x$ is an irreducible component of U (as follows from the facts that $\operatorname{Spec} A_x \to U$ is flat, and that flat morphisms satisfy the *going-down theorem*). Composing with the natural map from the set of irreducible components of U to the set of irreducible components of $\mathcal{X}$, we obtain a morphism from the set of irreducible components of $\operatorname{Spec} A_x$ to the set of irreducible components of $|\mathcal{X}|$. A consideration of the commutative diagram

$$\begin{array}{ccc} |\operatorname{Spec} \mathcal{O}_{U'',u''}| & \longrightarrow & |U''| \\ \downarrow & & \downarrow \\ |\operatorname{Spec} A_x| & \longrightarrow & |\mathcal{X}| \end{array}$$

and of the analogous diagram with A'_x in place of A_x, shows that this map, and the corresponding map for A'_x, are compatible with the bijection constructed above between the irreducible components of $\operatorname{Spec} A_x$ and the irreducible components of $\operatorname{Spec} A'_x$.

It remains to show that this map, from the irreducible components of $\operatorname{Spec} A_x$ to those of $\mathcal{X}$, is multiplicity preserving. A consideration of the definition of the multiplicity of an irreducible component of $\mathcal{X}$, and of the preceding constructions, shows that it suffices to show that the map from the set of irreducible components of $\operatorname{Spec} \widehat{\mathcal{O}}_{U'',u''}$ to the set of irreducible components of U'', given by taking Zariski closures, is multiplicity preserving. As we will see, this follows from the assumption that the local rings of S are G-rings.

More precisely, noting that it suffices to compare these multiplicities after making an étale base-change, we may replace U'' by a scheme which covers it via an étale map, and hence assume that U'' itself is a scheme, so that the local ring $\mathcal{O}_{U'',u''}$ is defined. (Alternatively, we could apply Artin approximation to the versal morphism $\operatorname{Spf} \widehat{\mathcal{O}}_{U'',u''} \to U''$, so as to replace U'' by a scheme.) The scheme U'' is of finite type over S, and hence the local ring $\mathcal{O}_{U'',u''}$ is a G-ring. Let $\mathfrak{p}$ be a minimal prime ideal of $\mathcal{O}_{U'',u''}$, corresponding to an irreducible component of U'' passing through u'', and let $\mathfrak{q}$ be a minimal prime of $\widehat{\mathcal{O}}_{U'',u''}$ lying over $\mathfrak{p}$ (corresponding to an irreducible component of $\operatorname{Spec} \widehat{\mathcal{O}}_{U'',u''}$ whose closure in U'' is the irreducible component corresponding to $\mathfrak{p}$); we have to show that the length of $(\widehat{\mathcal{O}}_{U'',u''})_{\mathfrak{q}}$ is equal to the length of $(\mathcal{O}_{U'',u''})_{\mathfrak{p}}$. Since $\mathfrak{q}$ lies over $\mathfrak{p}$, there is a natural isomorphism

$$(\widehat{\mathcal{O}}_{U'',u''})_{\mathfrak{q}} \cong \big(\widehat{\mathcal{O}}_{U'',u''} \otimes (\mathcal{O}_{U'',u''})_{\mathfrak{p}}\big)_{\mathfrak{q}}.$$

Now if ℓ is the length of $(\mathcal{O}_{U'',u''})_{\mathfrak{p}}$, then we may find a filtration of length ℓ on $(\mathcal{O}_{U'',u''})_{\mathfrak{p}}$, each of whose graded pieces is isomorphic to $\kappa(\mathfrak{p})$. This induces a corresponding filtration on $\big(\widehat{\mathcal{O}}_{U'',u''} \otimes (\mathcal{O}_{U'',u''})_{\mathfrak{p}}\big)_{\mathfrak{q}}$, each of whose graded pieces is isomorphic to $\big(\widehat{\mathcal{O}}_{U'',u''} \otimes \kappa(\mathfrak{p})\big)_{\mathfrak{q}}$. Since $\mathcal{O}_{U'',u''}$ is a G-ring, the formal fibre $\widehat{\mathcal{O}}_{U'',u''} \otimes \kappa(\mathfrak{p})$ is regular. Since $\mathfrak{q}$ is a minimal prime in this ring, the localization $\big(\widehat{\mathcal{O}}_{U'',u''} \otimes \kappa(\mathfrak{p})\big)_{\mathfrak{q}}$ is thus a field, and hence equal to $\kappa(\mathfrak{q})$. We conclude that $(\widehat{\mathcal{O}}_{U'',u''})_{\mathfrak{q}}$ has length ℓ, as required.

$\square$

Definition 1.7. If $\mathcal{X}$ is an algebraic stack locally of finite presentation over a locally Noetherian scheme S all of whose local rings are G-rings, if $x : \operatorname{Spec} k \to \mathcal{X}$ is a morphism whose source is the spectrum of a field of finite type over $\mathcal{O}_S$, and if A_x is a versal ring to $\mathcal{X}$ at x, then we define the set of *formal branches of* $\mathcal{X}$ *through* x to be the set of irreducible components of $\operatorname{Spec} A_x$, and we define the multiplicity of a branch to be the multiplicity of the corresponding component in $\operatorname{Spec} A_x$.

Lemma 1.6 shows, in the context of the preceding definition, that the set of formal branches of $\mathcal{X}$ through x, and their multiplicities, are well-defined independently of the choice of versal ring used to compute them. It also shows that there is a natural map from the set of formal branches of $\mathcal{X}$ through x to the set of irreducible components of $|\mathcal{X}|$ containing the class of x, and that this map preserves multiplicities.

As a closing remark, we note that it is sometimes convenient to think of an irreducible component of $\mathcal{X}$ as a closed substack. To this end, if $\mathcal{T}$ is an irreducible component of $\mathcal{X}$, i.e. an irreducible component of $|\mathcal{X}|$, then we endow $\mathcal{T}$ with its induced reduced substack structure (see [Stacks, Def. 050C]).

2. Dimension theory of algebraic stacks

In this section we discuss some concepts related to the dimension theory of locally Noetherian algebraic stacks. Since we intend to make arguments with them, it will be helpful to recall the basic definitions related to dimensions, beginning with the case of schemes, and then the case of algebraic spaces.

Definition 2.1. If X is a scheme, then we define the dimension $\dim(X)$ of X to be the Krull dimension of the topological space underlying X, while if x is a point of X, then we define the dimension $\dim_x(X)$ of X at x to be the minimum of the dimensions of the open subsets U of X containing x [Stacks, Def. 04MT]. One has the relation $\dim(X) = \sup_{x \in X} \dim_x(X)$ [Stacks, Lem. 04MU].

If X is locally Noetherian, then $\dim_x(X)$ coincides with the supremum of the dimensions at x of the irreducible components of X passing through x.

Definition 2.2. If X is an algebraic space and $x \in |X|$, then we define $\dim_x X = \dim_u U$, where U is any scheme admitting an étale surjection $U \to X$, and $u \in U$ is any point lying over x [Stacks, Def. 04N5]. We set $\dim(X) = \sup_{x \in |X|} \dim_x(X)$.

Remark 2.3. In general, the dimension of the algebraic space X at a point x may not coincide with the dimension of the underlying topological space $|X|$ at x. E.g. if k is a field of characteristic zero and $X = \mathbb{A}^1_k/\mathbb{Z}$, then X has dimension 1 (the dimension of $\mathbb{A}^1_k$) at each of its points, while $|X|$ has the indiscrete topology, and hence is of Krull dimension zero. On the other hand, in [Stacks, Ex. 02Z8] there is given an example of an algebraic space which is of dimension 0 at each of its points, while $|X|$ is irreducible of Krull dimension 1, and admits a generic point (so that the dimension of $|X|$ at any of its points is 1); see also the discussion of this example in [Stacks, Tag 04N3]

On the other hand, if X is a *decent* algebraic space, in the sense of [Stacks, Def. 03I8] (in particular, if X is quasi-separated; see [Stacks, Def. 03I7]), then in fact the dimension of X at x does coincide with the dimension of $|X|$ at x; see [Stacks, Lem. 0A4J].

In order to define the dimension of an algebraic stack, it will be useful to first have the notion of the relative dimension, at a point in the source, of a morphism whose source is an algebraic space, and whose target is an algebraic stack. The definition is slightly involved, just because (unlike in the case of schemes) the points of an algebraic stack, or an algebraic space, are not describable as morphisms from the spectrum of a field, but only as equivalence classes of such.

Definition 2.4. If $f : T \to \mathcal{X}$ is a locally of finite type morphism from an algebraic space to an algebraic stack, and if $t \in |T|$ is a point with image $x \in |\mathcal{X}|$, then we define *the relative dimension* of f at t, denoted $\dim_t(T_x)$, as follows: choose a morphism $\operatorname{Spec} k \to \mathcal{X}$, with source the spectrum of a field, which represents x, and choose a point $t' \in |T \times_{\mathcal{X}} \operatorname{Spec} k|$ mapping to t under the projection to $|T|$ (such a point t' exists, by [Stacks, Lem. 04XH]); then

$$\dim_t(T_x) := \dim_{t'}(T \times_{\mathcal{X}} \operatorname{Spec} k).$$

(Note that since T is an algebraic space and $\mathcal{X}$ is an algebraic stack, the fibre product $T \times_{\mathcal{X}} \operatorname{Spec} k$ is an algebraic space, and so the quantity on the right hand side of this proposed definition is in fact defined, by Definition 2.2.)

Remark 2.5. (1) One easily verifies (for example, by using the invariance of the relative dimension of locally of finite type morphisms of schemes under base-change; see e.g. [Stacks, Lem. 02FY]) that $\dim_t(T_x)$ is well-defined, independently of the choices used to compute it.

(2) In the case that $\mathcal{X}$ is also an algebraic space, it is straightforward to confirm that this definition agrees with the definition of relative dimension given in [Stacks, Def. 04NM (3)].

We next recall the following lemma, on which the definition of the dimension of a locally Noetherian algebraic stack is founded.

Lemma 2.6. *If $f : U \to X$ is a smooth morphism of locally Noetherian algebraic spaces, and if $u \in |U|$ with image $x \in |X|$, then*

$$\dim_u(U) = \dim_x(X) + \dim_u(U_x)$$

(*where of course* $\dim_u(U_x)$ *is defined via Definition* 2.4).

Proof. See [Stacks, Lem. 0AFI], noting that the definition of $\dim_u(U_x)$ used here coincides with the definition used there, by Remark 2.5 (2). ☐

Definition 2.7. If $\mathcal{X}$ is a locally Noetherian algebraic stack, and $x \in |\mathcal{X}|$, then we define the dimension $\dim_x(\mathcal{X})$ of $\mathcal{X}$ at x as follows: let $U \to \mathcal{X}$ be a smooth morphism from a scheme (or, more generally, from an algebraic space) to $\mathcal{X}$ containing x in its image, let u be any point of $|U|$ mapping to x, and define

$$\dim_x(\mathcal{X}) := \dim_u(U) - \dim_u(U_x)$$

(where the relative dimension $\dim_u(U_x)$ is defined by Definition 2.4).

Remark 2.8. The preceding definition is justified by the formula of Lemma 2.6, and one can use that lemma to verify that $\dim_x(\mathcal{X})$ is well-defined, independently of the choices used to compute it. Alternatively (employing the notation of the definition, and choosing U to be a scheme), one can compute $\dim_u(U_x)$ by choosing the representative of x to be the composite $\operatorname{Spec} \kappa(u) \to U \to \mathcal{X}$, where the first morphism is the canonical one with image $u \in U$. Then, if we write $R := U \times_{\mathcal{X}} U$, and let

$e : U \to R$ denote the diagonal morphism, the invariance of relative dimension under base-change shows that $\dim_u(U_x) = \dim_{e(u)}(R_u)$, and thus the preceding definition of $\dim_x(\mathcal{X})$ coincides with the definition as $\dim_u(U) - \dim_{e(u)}(R_u)$ given in [Stacks, Def. 0AFN], which is shown to be independent of choices in [Stacks, Lem. 0AFM].

Remark 2.9. For Deligne–Mumford stacks which are suitably decent (e.g. quasi-separated), it will again be the case that $\dim_x(\mathcal{X})$ coincides with the topologically defined quantity $\dim_x |\mathcal{X}|$. However, for more general Artin stacks, this will typically not be the case. For example, if $\mathcal{X} := [\mathbb{A}^1/\mathbb{G}_m]$ (over some field, with the quotient being taken with respect to the usual multiplication action of $\mathbb{G}_m$ on $\mathbb{A}^1$), then $|\mathcal{X}|$ has two points, one the specialisation of the other (corresponding to the two orbits of $\mathbb{G}_m$ on $\mathbb{A}^1$), and hence is of dimension 1 as a topological space; but $\dim_x(\mathcal{X}) = 0$ for both points $x \in |\mathcal{X}|$. (An even more extreme example is given by the classifying space $[\operatorname{Spec} k/\mathbb{G}_m]$, whose dimension at its unique point is equal to -1.)

We can now extend Definition 2.4 to the context of (locally finite type) morphisms between (locally Noetherian) algebraic stacks.

Definition 2.10. If $f : \mathcal{T} \to \mathcal{X}$ is a locally of finite type morphism between locally Noetherian algebraic stacks, and if $t \in |\mathcal{T}|$ is a point with image $x \in |\mathcal{X}|$, then we define the *relative dimension* of f at t, denoted $\dim_t(\mathcal{X}_x)$, as follows: choose a morphism $\operatorname{Spec} k \to \mathcal{X}$, with source the spectrum of a field, which represents x, and choose a point $t' \in |\mathcal{T} \times_{\mathcal{X}} \operatorname{Spec} k|$ mapping to t under the projection to $|\mathcal{T}|$ (such a point t' exists, by [Stacks, Lem. 04XH]); then

$$\dim_t(\mathcal{T}_x) := \dim_{t'}(\mathcal{T} \times_{\mathcal{X}} \operatorname{Spec} k).$$

(Note that since $\mathcal{T}$ is an algebraic stack and $\mathcal{X}$ is an algebraic stack, the fibre product $\mathcal{T} \times_{\mathcal{X}} \operatorname{Spec} k$ is an algebraic stack, which is locally Noetherian by [Stacks, Lem. 06R6]. Thus the quantity on the right side of this proposed definition is defined by Definition 2.7.)

Remark 2.11. Standard manipulations show that $\dim_t(\mathcal{T}_x)$ is well-defined, independently of the choices made to compute it.

We now establish some basic properties of relative dimension, which are obvious generalisations of the corresponding statements in the case of morphisms of schemes.

Lemma 2.12. *Suppose given a Cartesian square of morphisms of locally Noetherian stacks*

$$\begin{tikzcd} \mathcal{T}' \arrow[r] \arrow[d] & \mathcal{T} \arrow[d] \\ \mathcal{X}' \arrow[r] & \mathcal{X} \end{tikzcd}$$

in which the vertical morphisms are locally of finite type. If $t' \in |\mathcal{T}'|$, with images t, x', and x in $|\mathcal{T}|$, $|\mathcal{X}'|$, and $|\mathcal{X}|$ respectively, then $\dim_{t'}(\mathcal{T}'_{x'}) = \dim_t(\mathcal{T}_x)$.

Proof. Both sides can (by definition) be computed as the dimension of the same fibre product. $\square$

Lemma 2.13. *If $f : \mathcal{U} \to \mathcal{X}$ is a smooth morphism of locally Noetherian algebraic stacks, and if $u \in |\mathcal{U}|$ with image $x \in |\mathcal{X}|$, then*

$$\dim_u(\mathcal{U}) = \dim_x(\mathcal{X}) + \dim_u(\mathcal{U}_x).$$

Proof. Choose a smooth surjective morphism $V \to \mathcal{U}$ whose source is a scheme, and let $v \in |V|$ be a point mapping to u. Then the composite $V \to \mathcal{U} \to \mathcal{X}$ is also smooth, and by definition we have $\dim_x(\mathcal{X}) = \dim_v(V) - \dim_v(V_x)$, while $\dim_u(\mathcal{U}) = \dim_v(V) - \dim_v(V_u)$. Thus

$$\dim_u(\mathcal{U}) - \dim_x(\mathcal{X}) = \dim_v(V_x) - \dim_v(V_u).$$

Choose a representative $\operatorname{Spec} k \to \mathcal{X}$ of x and choose a point $v' \in |V \times_{\mathcal{X}} \operatorname{Spec} k|$ lying over v, with image u' in $|\mathcal{U} \times_{\mathcal{X}} \operatorname{Spec} k|$; then by definition $\dim_u(\mathcal{U}_x) = \dim_{u'}(\mathcal{U} \times_{\mathcal{X}} \operatorname{Spec} k)$, and $\dim_v(V_x) = \dim_{v'}(V \times_{\mathcal{X}} \operatorname{Spec} k)$.

Now $V \times_{\mathcal{X}} \operatorname{Spec} k \to \mathcal{U} \times_{\mathcal{X}} \operatorname{Spec} k$ is a smooth surjective morphism (being the base-change of such a morphism) whose source is an algebraic space (since V and $\operatorname{Spec} k$ are schemes, and $\mathcal{X}$ is an algebraic stack). Thus, again by definition, we have

$$\begin{aligned}\dim_{u'}(\mathcal{U} \times_{\mathcal{X}} \operatorname{Spec} k) &= \dim_{v'}(V \times_{\mathcal{X}} \operatorname{Spec} k) - \dim_{v'}\big((V \times_{\mathcal{X}} \operatorname{Spec} k)_{u'}\big) \\ &= \dim_v(V_x) - \dim_{v'}\big((V \times_{\mathcal{X}} \operatorname{Spec} k)_{u'}\big).\end{aligned}$$

Now $V \times_{\mathcal{X}} \operatorname{Spec} k \xrightarrow{\sim} V \times_{\mathcal{U}} (\mathcal{U} \times_{\mathcal{X}} \operatorname{Spec} k)$, and so Lemma 2.12 shows that $\dim_{v'}\big((V \times_{\mathcal{X}} \operatorname{Spec} k)_{u'}\big) = \dim_v(V_u)$. Putting everything together, we find that

$$\dim_u(\mathcal{U}) - \dim_x(\mathcal{X}) = \dim_u(\mathcal{U}_x),$$

as required. ☐

Lemma 2.14. *Let $f : \mathcal{T} \to \mathcal{X}$ be a locally of finite type morphism of algebraic stacks.*

(1) The function $t \mapsto \dim_t(\mathcal{T}_{f(t)})$ is upper semi-continuous on $|\mathcal{T}|$.
(2) If f is smooth, then the function $t \mapsto \dim_t(\mathcal{T}_{f(t)})$ is locally constant on $|\mathcal{T}|$.

Proof. Suppose to begin with that $\mathcal{T}$ is a scheme T, let $U \to \mathcal{X}$ be a smooth surjective morphism whose source is a scheme, and let $T' := T \times_{\mathcal{X}} U$. Let $f' : T' \to U$ be the pull-back of f over U, and let $g : T' \to T$ be the projection.

Lemma 2.12 shows that $\dim_{t'}(T'_{f'(t')}) = \dim_{g(t')}(T_{f(g(t'))})$, for $t' \in T'$, while, since g is smooth and surjective (being the base-change of a smooth surjective morphism) the map induced by g on underlying topological spaces is continuous and open (by [Stacks, Lem. 04XL]), and surjective. Thus it suffices to note that part (1) for the morphism f' follows from [Stacks, Tag 04NT], and part (2) from either of [Stacks, Lem. 02NM] or [Stacks, Lem. 02G1] (each of which gives the result for schemes, from which the analogous results for algebraic spaces can be deduced exactly as in [Stacks, Tag 04NT]).

Now return to the general case, and choose a smooth surjective morphism $h : V \to \mathcal{T}$ whose source is a scheme. If $v \in V$, then, essentially by definition, we have

$$\dim_{h(v)}(\mathcal{T}_{f(h(v))}) = \dim_v(V_{f(h(v))}) - \dim_v(V_{h(v)}).$$

Since V is a scheme, we have proved that the first of the terms on the right hand side of this equality is upper semi-continuous (and even locally constant if f is smooth), while the second term is in fact locally constant. Thus their difference is upper semi-continuous (and locally constant if f is smooth), and hence the function $\dim_{h(v)}(\mathcal{T}_{f(h(v))})$ is upper semi-continuous on $|V|$ (and locally constant if f is smooth). Since the morphism $|V| \to |\mathcal{T}|$ is open and surjective, the lemma follows. $\square$

Before continuing with our development, we prove two lemmas related to the dimension theory of schemes.

To put the first lemma in context, we note that if X is a finite-dimensional scheme, then since $\dim X$ is defined to equal the supremum of the dimensions $\dim_x X$, there exists a point $x \in X$ such that $\dim_x X = \dim X$. The following lemma shows that we may furthermore take the point x to be of finite type.

Lemma 2.15. *If X is a finite-dimensional scheme, then there exists a closed (and hence finite type) point $x \in X$ such that $\dim_x X = \dim X$.*

Proof. Let $d = \dim X$, and choose a maximal strictly decreasing chain of irreducible closed subsets of X, say

$$Z_0 \supset Z_1 \supset \cdots \supset Z_d. \tag{2.16}$$

The subset Z_d is a minimal irreducible closed subset of X, and thus any point of Z_d is a generic point of Z_d. Since the underlying topological space of the scheme X is sober, we conclude that Z_d is a singleton, consisting of a single closed point $x \in X$. If U is any neighbourhood of x, then the chain

$$U \cap Z_0 \supset U \cap Z_1 \supset \cdots \supset U \cap Z_d = Z_d = \{x\}$$

is then a strictly descending chain of irreducible closed subsets of U, showing that $\dim U \geq d$. Thus we find that $\dim_x X \geq d$. The other inequality being obvious, the lemma is proved. $\square$

The next lemma shows that $\dim_x X$ is a *constant* function on an irreducible scheme satisfying some mild additional hypotheses. (See Lemma 2.35 below for a related result.)

Lemma 2.17. *If X is an irreducible, Jacobson, catenary, and locally Noetherian scheme of finite dimension, then $\dim U = \dim X$ for every non-empty open subset U of X. Equivalently, $\dim_x X$ is a constant function on X.*

Proof. The equivalence of the two claims follows directly from the definitions. Suppose, then, that $U \subset X$ is a non-empty open subset. Certainly $\dim U \leq \dim X$, and we have to show that $\dim U \geq \dim X$. Write $d := \dim X$, and choose a maximal strictly decreasing chain of irreducible closed subsets of X, say

$$X = Z_0 \supset Z_1 \supset \cdots \supset Z_d.$$

Since X is Jacobson, the minimal irreducible closed subset Z_d is equal to $\{x\}$ for some closed point x.

If $x \in U$, then

$$U = U \cap Z_0 \supset U \cap Z_1 \supset \cdots \supset U \cap Z_d = \{x\}$$

is a strictly decreasing chain of irreducible closed subsets of U, and so we conclude that $\dim U \geq d$, as required. Thus we may suppose that $x \notin U$.

Consider the flat morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to X$. The non-empty (and hence dense) open subset U of X pulls back to an open subset $V \subset \operatorname{Spec} \mathcal{O}_{X,x}$. Replacing U by a non-empty quasi-compact, and hence Noetherian, open subset, we may assume that the inclusion $U \to X$ is a quasi-compact morphism. Since the formation of scheme-theoretic images of quasi-compact morphisms commutes with flat base-change [Stacks, Tag 081I]), we see that V is dense in $\operatorname{Spec} \mathcal{O}_{X,x}$, and so in particular non-empty, and of course $x \notin V$. (Here we use x also to denote the closed point of $\operatorname{Spec} \mathcal{O}_{X,x}$, since its image is equal to the given point $x \in X$.) Now $\operatorname{Spec} \mathcal{O}_{X,x} \setminus \{x\}$ is Jacobson [Stacks, Tag 02IM], and hence V contains a closed point z of $\operatorname{Spec} \mathcal{O}_{X,x} \setminus \{x\}$. The closure in X of the image of z is then an irreducible closed subset Z of X containing x, whose intersection with U is non-empty, and for which there is no irreducible closed subset properly contained in Z and properly containing $\{x\}$ (because pull-back to $\operatorname{Spec} \mathcal{O}_{X,x}$ induces a bijection between irreducible closed subsets of X containing x and irreducible closed subsets of $\operatorname{Spec} \mathcal{O}_{X,x}$). Since $U \cap Z$ is a non-empty closed subset of U, it contains a point u that is closed in X (since X is Jacobson), and since $U \cap Z$ is a non-empty (and hence dense) open subset of the irreducible set Z (which contains a point not lying in U, namely x), the inclusion $\{u\} \subset U \cap Z$ is proper.

As X is catenary, the chain

$$X = Z_0 \supseteq Z \supset \{x\} = Z_d$$

can be refined to a chain of length $d + 1$, which must then be of the form

$$X = Z_0 \supset W_1 \supset \cdots \supset W_{d-1} = Z \supset \{x\} = Z_d.$$

Since $U \cap Z$ is non-empty, we then find that

$$U = U \cap Z_0 \supset U \cap W_1 \supset \cdots \supset U \cap W_{d-1} = U \cap Z \supset \{u\}$$

is a strictly decreasing chain of irreducible closed subsets of U of length $d + 1$, showing that $\dim U \geq d$, as required. □

We will prove a stack-theoretic analogue of Lemma 2.17 in Lemma 2.21 below, but before doing so, we have to introduce an additional definition, necessitated by the fact that the notion of a scheme being catenary is not an étale local one (see the example of [Stacks, Tag 0355]), which makes it difficult to define what it means for an algebraic space or algebraic stack to be catenary (see the discussion of [Oss15, p. 3]). For certain aspects of dimension theory, the following definition seems to provide a good substitute for the missing notion of a catenary algebraic stack.

Definition 2.18. We say that a locally Noetherian algebraic stack $\mathcal{X}$ is *pseudo-catenary* if there exists a smooth and surjective morphism $U \to \mathcal{X}$ whose source is a universally catenary scheme.

Example 2.19. If $\mathcal{X}$ is locally of finite type over a universally catenary locally Noetherian scheme S, and $U \to \mathcal{X}$ is a smooth surjective morphism whose source is a scheme, then the composite $U \to \mathcal{X} \to S$ is locally of finite type, and so U is universally catenary [Stacks, Tag 02J9]. Thus $\mathcal{X}$ is pseudo-catenary.

The following lemma shows that the property of being pseudo-catenary passes through finite type morphisms.

Lemma 2.20. *If $\mathcal{X}$ is a pseudo-catenary locally Noetherian algebraic stack, and if $\mathcal{Y} \to \mathcal{X}$ is a locally of finite type morphism, then there exists a smooth surjective morphism $V \to \mathcal{Y}$ whose source is a universally catenary scheme; thus $\mathcal{Y}$ is again pseudo-catenary.*

Proof. By assumption we may find a smooth surjective morphism $U \to \mathcal{X}$ whose source is a universally catenary scheme. The base-change $U \times_{\mathcal{X}} \mathcal{Y}$ is then an algebraic stack; let $V \to U \times_{\mathcal{X}} \mathcal{Y}$ be a smooth surjective morphism whose source is a scheme. The composite $V \to U \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ is then smooth and surjective (being a composite of smooth and surjective morphisms), while the morphism $V \to U \times_{\mathcal{X}} \mathcal{Y} \to U$ is locally of finite type (being a composite of morphisms that are locally finite type). Since U is universally catenary, we see that V is universally catenary (by [Stacks, Tag 02J9]), as claimed. $\square$

We now study the behaviour of the function $\dim_x(\mathcal{X})$ on $|\mathcal{X}|$ (for some locally Noetherian stack $\mathcal{X}$) with respect to the irreducible components of $|\mathcal{X}|$, as well as various related topics.

Lemma 2.21. *If $\mathcal{X}$ is a Jacobson, pseudo-catenary, and locally Noetherian algebraic stack for which $|\mathcal{X}|$ is irreducible, then $\dim_x(\mathcal{X})$ is a constant function on $|\mathcal{X}|$.*

Proof. It suffices to show that $\dim_x(\mathcal{X})$ is locally constant on $|\mathcal{X}|$, since it will then necessarily be constant (as $|\mathcal{X}|$ is connected, being irreducible). Since $\mathcal{X}$ is pseudo-catenary, we may find a smooth surjective morphism $U \to \mathcal{X}$ with U being a univesally catenary scheme. If $\{U_i\}$ is an cover of U by quasi-compact open subschemes, we may replace U by $\coprod U_i$,, and it suffices to show that the function $u \mapsto \dim_{f(u)}(\mathcal{X})$ is locally constant on U_i. Since we check this for one U_i at a time, we now drop the subscript, and write simply U rather than U_i. Since U is quasi-compact, it is the union of a finite number of irreducible components, say $T_1 \cup \cdots \cup T_n$. Note that each T_i is Jacobson, catenary, and locally Noetherian, being a closed subscheme of the Jacobson, catenary, and locally Noetherian scheme U.

By definition, we have $\dim_{f(u)}(\mathcal{X}) = \dim_u(U) - \dim_u(U_{f(u)})$. Lemma 2.14 (2) shows that the second term in the right hand expression is locally constant on U, as f is smooth, and hence we must show that $\dim_u(U)$ is locally constant on U. Since $\dim_u(U)$ is the maximum of the dimensions $\dim_u T_i$, as T_i ranges over the components of U containing u, it suffices to show that if a point u lies on two distinct components, say T_i and T_j (with $i \neq j$), then $\dim_u T_i = \dim_u T_j$, and then to note that $t \mapsto \dim_t T$ is a constant function on an irreducible Jacobson, catenary, and locally Noetherian scheme T (as follows from Lemma 2.17).

Let $V = T_i \setminus \left(\bigcup_{i' \neq i} T_{i'}\right)$ and $W = T_j \setminus \left(\bigcup_{i' \neq j} T_{i'}\right)$. Then each of V and W is a non-empty open subset of U, and so each has non-empty open image in $|\mathcal{X}|$. As $|\mathcal{X}|$ is irreducible, these two non-empty open subsets of $|\mathcal{X}|$ have a non-empty intersection. Let x be a point lying in this intersection, and let $v \in V$ and $w \in W$ be points mapping to x. We then find that

$$\dim T_i = \dim V = \dim_v(U) = \dim_x(\mathcal{X}) + \dim_v(U_x)$$

and similarly that

$$\dim T_j = \dim W = \dim_w(U) = \dim_x(\mathcal{X}) + \dim_w(U_x).$$

Since $u \mapsto \dim_u(U_{f(u)})$ is locally constant on U, and since $T_i \cup T_j$ is connected (being the union of two irreducible, hence connected, sets that have non-empty intersection), we see that $\dim_v(U_x) = \dim_w(U_x)$, and hence, comparing the preceding two equations, that $\dim T_i = \dim T_j$, as required. $\square$

Lemma 2.22. *If $\mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed immersion of locally Noetherian schemes, and if $z \in |\mathcal{Z}|$ has image $x \in |\mathcal{X}|$, then $\dim_z(\mathcal{Z}) \leq \dim_x(\mathcal{X})$.*

Proof. Choose a smooth surjective morphism $U \to \mathcal{X}$ whose source is a scheme; the base-changed morphism $V := U \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Z}$ is then also smooth and surjective, and the projection $V \to U$ is a closed immersion. If $v \in |V|$ maps to $z \in |\mathcal{Z}|$, and if we let u denote the image of v in $|U|$, then clearly $\dim_v(V) \leq \dim_u(U)$, while $\dim_v(V_z) = \dim_u(U_x)$, by Lemma 2.12. Thus

$$\dim_z(\mathcal{Z}) = \dim_v(V) - \dim_v(V_z) \leq \dim_u(U) - \dim_u(U_x) = \dim_x(\mathcal{X}),$$

as claimed. □

Lemma 2.23. *If $\mathcal{X}$ is a locally Noetherian algebraic stack, and if $x \in |\mathcal{X}|$, then $\dim_x(\mathcal{X}) = \sup_{\mathcal{T}}\{\dim_x(\mathcal{T})\}$, where $\mathcal{T}$ runs over all the irreducible components of $|\mathcal{X}|$ passing through x (endowed with their induced reduced structure).*

Proof. Lemma 2.22 shows that $\dim_x(\mathcal{T}) \leq \dim_x(\mathcal{X})$ for each irreducible component $\mathcal{T}$ passing through the point x. Thus to prove the lemma, it suffices to show that

$$\dim_x(\mathcal{X}) \leq \sup_{\mathcal{T}}\{\dim_x(\mathcal{T})\}. \tag{2.24}$$

Let $U \to \mathcal{X}$ be a smooth cover by a scheme. If T is an irreducible component of U then we let $\mathcal{T}$ denote the closure of its image in $\mathcal{X}$, which is an irreducible component of $\mathcal{X}$. Let $u \in U$ be a point mapping to x. Then we have $\dim_x(\mathcal{X}) = \dim_u U - \dim_u U_x = \sup_T \dim_u T - \dim_u U_x$, where the supremum is over the irreducible components of U passing through u. Choose a component T for which the supremum is achieved, and note that $\dim_x(\mathcal{T}) = \dim_u T - \dim_u T_x$. The desired inequality (2.24) now follows from the evident inequality $\dim_u T_x \leq \dim_u U_x$. (Note that if $\operatorname{Spec} k \to \mathcal{X}$ is a representative of x, then $T \times_{\mathcal{X}} \operatorname{Spec} k$ is a closed subspace of $U \times_{\mathcal{X}} \operatorname{Spec} k$.) □

Lemma 2.25. *If $\mathcal{X}$ is a locally Noetherian algebraic stack, and if $x \in |\mathcal{X}|$, then for any open substack $\mathcal{V}$ of $\mathcal{X}$ containing x, there is a finite type point $x_0 \in |\mathcal{V}|$ such that $\dim_{x_0}(\mathcal{X}) = \dim_x(\mathcal{V})$.*

Proof. Choose a smooth surjective morphism $f : U \to \mathcal{X}$ whose source is a scheme, and consider the function $u \mapsto \dim_{f(u)}(\mathcal{X})$; since the morphism $|U| \to |\mathcal{X}|$ induced by f is open (as f is smooth) as well as surjective (by assumption), and takes finite type points to finite type points (by the very definition of the finite type points of $|\mathcal{X}|$), it suffices to show that for any $u \in U$, and any open neighbourhood of u, there is a finite type point u_0 in this neighbourhood such that $\dim_{f(u_0)}(\mathcal{X}) = \dim_{f(u)}(\mathcal{X})$. Since, with this reformulation of the problem, the surjectivity of f is no longer required, we may replace U by the open neighbourhood of the point u in question, and thus reduce to the problem of showing that for each $u \in U$, there is a finite type point $u_0 \in U$ such that $\dim_{f(u_0)}(\mathcal{X}) = \dim_{f(u)}(\mathcal{X})$. By definition $\dim_{f(u)}(\mathcal{X}) = \dim_u(U) - \dim_u(U_{f(u)})$, while $\dim_{f(u_0)}(\mathcal{X}) = \dim_{u_0}(U) - \dim_{u_0}(U_{f(u_0)})$. Since f is smooth, the expression $\dim_{u_0}(U_{f(u_0)})$ is locally constant as u_0 varies over U (by Lemma 2.14 (2)), and so shrinking U further around u if necessary, we may assume it is constant. Thus the problem becomes to show that we may find a finite type point $u_0 \in U$ for which $\dim_{u_0}(U) = \dim_u(U)$. Since by definition $\dim_u U$ is the minimum of the dimensions $\dim V$, as V ranges over the open neighbourhoods V of u in U, we may shrink U down further around u so that $\dim_u U = \dim U$. The existence of desired point u_0 then follows from Lemma 2.15. □

Lemma 2.26. *Let $\mathcal{T} \hookrightarrow \mathcal{X}$ be a locally of finite type monomorphism of algebraic stacks, with $\mathcal{X}$ (and thus also $\mathcal{T}$) being Jacobson, pseudo-catenary, and locally Noetherian. Suppose further that $\mathcal{T}$ is irreducible of some (finite) dimension d, and that $\mathcal{X}$ is reduced and of dimension less than or equal to d. Then there is a non-empty open substack $\mathcal{V}$ of $\mathcal{T}$ such that the induced monomorphism $\mathcal{V} \hookrightarrow \mathcal{X}$ is an open immersion which identifies $\mathcal{V}$ with an open subset of an irreducible component of $\mathcal{X}$.*

Proof. Choose a smooth surjective morphism $f : U \to \mathcal{X}$ with source a scheme, necessarily reduced since $\mathcal{X}$ is, and write $U' := \mathcal{T} \times_{\mathcal{X}} U$. The base-changed morphism $U' \to U$ is a monomorphism of algebraic spaces, locally of finite type, and thus representable [Stacks, Tag 0418, Tag 0463]; since U is a scheme, so is U'. The projection $f' : U' \to \mathcal{T}$ is again a smooth surjection. Let $u' \in U'$, with image $u \in U$. Lemma 2.12 shows that $\dim_{u'}(U'_{f(u')}) = \dim_u(U_{f(u)})$, while $\dim_{f'(u')}(\mathcal{T}) = d \geq \dim_{f(u)}(\mathcal{X})$ by Lemma 2.21 and our assumptions on $\mathcal{T}$ and $\mathcal{X}$. Thus we see that

$$
\begin{aligned}
\dim_{u'}(U') &= \dim_{u'}(U'_{f(u')}) + \dim_{f'(u')}(\mathcal{T}) \\
&\geq \dim_u(U_{f(u)}) + \dim_{f(u)}(\mathcal{X}) = \dim_u(U).
\end{aligned} \tag{2.27}
$$

Since $U' \to U$ is a monomorphism, locally of finite type, it is in particular unramified, and so by the étale local structure of unramified morphisms [Stacks, Tag 04HJ], we may find a commutative diagram

$$
\begin{array}{ccc}
V' & \longrightarrow & V \\
\downarrow & & \downarrow \\
U' & \longrightarrow & U
\end{array}
$$

in which the scheme V' is non-empty, the vertical arrows are étale, and the upper horizontal arrow is a closed immersion. Replacing V by a quasi-compact open subset whose image has non-empty intersection with the image of U', and replacing V' by the preimage of V, we may further assume that V (and thus V') is quasi-compact. Since V is also locally Noetherian, it is thus Noetherian, and so is the union of finitely many irreducible components.

Since étale morphisms preserve pointwise dimension [Stacks, Tag 04N4], we deduce from (2.27) that for any point $v' \in V'$, with image $v \in V$, we have $\dim_{v'}(V') \geq \dim_v(V)$. In particular, the image of V' can't be contained in the intersection of two distinct irreducible components of V, and so we may find at least one irreducible open subset of V which has non-empty intersection with V'; replacing V by this subset, we may assume that V is integral (being both reduced and irreducible). From the preceding inequality on dimensions, we conclude that the closed immersion $V' \hookrightarrow V$ is in fact an isomorphism. If we let W denote the image of V' in U', then W is a non-empty open subset of U' (as étale morphisms are open), and the induced monomorphism $W \to U$ is étale (since it is so étale locally on the source, i.e. after pulling back to V'), and hence is an open immersion (being an étale monomorphism). Thus, if we let $\mathcal{V}$ denote the image of W in $\mathcal{T}$, then $\mathcal{V}$ is a dense (equivalently, non-empty) open substack of $\mathcal{T}$, whose image is dense in an irreducible component of $\mathcal{X}$. Finally, we note that the morphism is $\mathcal{V} \to \mathcal{X}$ is smooth (since its composite with the smooth morphism $W \to \mathcal{V}$ is smooth), and also a monomorphism, and thus is an open immersion. $\square$

Lemma 2.28. *Let $f : \mathcal{T} \to \mathcal{X}$ be a locally of finite type morphism of Jacobson, pseudo-catenary, and locally Noetherian algebraic stacks, whose source is irreducible and whose target is quasi-separated, and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ denote the scheme-theoretic image of $\mathcal{T}$. Then for every finite type point $t \in |T|$, we have that $\dim_t(\mathcal{T}_{f(t)}) \geq \dim \mathcal{T} - \dim \mathcal{Z}$, and there is a non-empty (equivalently, dense) open subset of $|\mathcal{T}|$ over which equality holds.*

Proof. Replacing $\mathcal{X}$ by $\mathcal{Z}$, we may and do assume that f is scheme theoretically dominant, and also that $\mathcal{X}$ is irreducible. By the upper semi-continuity of fibre dimensions (Lemma 2.14 (1)), it suffices to prove that the equality $\dim_t(\mathcal{T}_{f(t)}) = \dim \mathcal{T} - \dim \mathcal{Z}$ holds for t lying in some non-empty open substack of $\mathcal{T}$. For this reason, in the argument we are always free to replace $\mathcal{T}$ by a non-empty open substack.

Let $T' \to \mathcal{T}$ be a smooth surjective morphism whose source is a scheme, and let T be a non-empty quasi-compact open subset of T'. Since $\mathcal{Y}$ is quasi-separated, we find that $T \to \mathcal{Y}$ is quasi-compact (by [Stacks, Tag 050Y], applied to the morphisms $T \to \mathcal{Y} \to \operatorname{Spec} \mathbb{Z}$). Thus, if we replace $\mathcal{T}$ by the image of T in $\mathcal{T}$, then we may assume (appealing to [Stacks, Tag 050X]) that the morphism $f : \mathcal{T} \to \mathcal{X}$ is quasi-compact.

If we choose a smooth surjection $U \to \mathcal{X}$ with U a scheme, then Lemma 1.2 ensures that we may find an irreducible open subset V of U such that $V \to \mathcal{X}$ is smooth and scheme-theoretically dominant. Since scheme-theoretic dominance for quasi-compact morphisms is preserved by flat base-change, the base-change $\mathcal{T} \times_{\mathcal{X}} V \to V$ of the scheme-theoretically dominant morphism f is again scheme-theoretically dominant. We let Z denote a scheme admitting a smooth surjection onto this fibre product; then $Z \to \mathcal{T} \times_{\mathcal{X}} V \to V$ is again scheme-theoretically dominant. Thus we may find an irreducible component C of Z which scheme-theoretically dominates V. Since the composite $Z \to \mathcal{T} \times_{\mathcal{X}} V \to \mathcal{T}$ is smooth, and since $\mathcal{T}$ is irreducible, Lemma 1.2 shows that any irreducible component of the source has dense image in $|\mathcal{T}|$. We now replace C by a non-empty open subset W which is disjoint from every other irreducible component of Z, and then replace $\mathcal{T}$ and $\mathcal{X}$ by the images of W and V (and apply Lemma 2.21 to see that this doesn't change the dimension of either $\mathcal{T}$ or $\mathcal{X}$). If we let $\mathcal{W}$ denote the image of the morphism $W \to \mathcal{T} \times_{\mathcal{X}} V$, then $\mathcal{W}$ is open in $\mathcal{T} \times_{\mathcal{X}} V$ (since the morphism $W \to \mathcal{T} \times_{\mathcal{X}} V$ is smooth), and is irreducible (being the image of an irreducible scheme). Thus we end up with a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[dr] & \mathcal{W} \arrow[r] \arrow[d] & V \arrow[d] \\ & \mathcal{T} \arrow[r] & \mathcal{X} \end{tikzcd}$$

in which W and V are schemes, the vertical arrows are smooth and surjective, the diagonal arrows and the left-hand upper horizontal arrow are smooth, and the induced morphism $\mathcal{W} \to \mathcal{T} \times_{\mathcal{X}} V$ is an open immersion. Using this diagram, together with the definitions of the various dimensions involved in the statement of the lemma, we will reduce our verification of the lemma to the case of schemes, where it is known.

Fix $w \in |W|$ with image $w' \in |\mathcal{W}|$, image $t \in |\mathcal{T}|$, image v in $|V|$, and image x in $|\mathcal{X}|$. Essentially by definition (using the fact that $\mathcal{W}$ is open in $\mathcal{T} \times_{\mathcal{X}} V$, and that the fibre of a base-change is the base-change of the fibre), we obtain the equalities

$$\dim_v V_x = \dim_{w'} \mathcal{W}_t$$

and

$$\dim_t \mathcal{T}_x = \dim_{w'} \mathcal{W}_v.$$

Again by definition (the diagonal arrow and right-hand vertical arrow in our diagram realise W and V as smooth covers by schemes of the stacks $\mathcal{T}$ and $\mathcal{X}$), we find that

$$\dim_t \mathcal{T} = \dim_w W - \dim_w W_t$$

and

$$\dim_x \mathcal{X} = \dim_v V - \dim_v V_x.$$

Combining the equalities, we find that

$$\begin{aligned}\dim_t \mathcal{T}_x - \dim_t \mathcal{T} + \dim_x \mathcal{X}\\ = \dim_{w'} \mathcal{W}_v - \dim_w W + \dim_w W_t + \dim_v V - \dim_{w'} \mathcal{W}_t.\end{aligned}$$

Since $W \to \mathcal{W}$ is a smooth surjection, the same is true if we base-change over the morphism $\operatorname{Spec} \kappa(v) \to V$ (thinking of $W \to \mathcal{W}$ as a morphism over V), and from this smooth morphism we obtain the first of the following two equalities

$$\dim_w W_v - \dim_{w'} \mathcal{W}_v = \dim_w (W_v)_{w'} = \dim_w W_{w'};$$

the second equality follows via a direct comparison of the two fibres involved. Similarly, if we think of $W \to \mathcal{W}$ as a morphism of schemes over $\mathcal{T}$, and base-change over some representative of the point $t \in |\mathcal{T}|$, we obtain the equalities

$$\dim_w W_t - \dim_{w'} \mathcal{W}_t = \dim_w (W_t)_{w'} = \dim_w W_{w'}.$$

Putting everything together, we find that

$$\dim_t \mathcal{T}_x - \dim_t \mathcal{T} + \dim_x \mathcal{X} = \dim_w W_v - \dim_w W + \dim_v V.$$

Our goal is to show that the left-hand side of this equality vanishes for a non-empty open subset of t. As w varies over a non-empty open subset of W, its image $t \in |\mathcal{T}|$ varies over a non-empty open subset of $|\mathcal{T}|$ (as $W \to \mathcal{T}$ is smooth).

We are therefore reduced to showing that if $W \to V$ is a scheme-theoretically dominant morphism of irreducible locally Noetherian schemes that is locally of finite type, then there is a non-empty open subset of points $w \in W$ such that $\dim_w W_v = \dim_w W - \dim_v V$ (where v denotes the image of w in V). This is a standard fact, whose proof we recall for the convenience of the reader.

We may replace W and V by their underlying reduced subschemes without altering the validity (or not) of this equation, and thus we may assume that they are in fact integral schemes. Since $\dim_w W_v$ is locally constant on W, replacing W by a non-empty open subset if necessary, we may assume that $\dim_w W_v$ is constant, say equal to d. Choosing this open subset to be affine, we may also assume that the morphism $W \to V$ is in fact of finite type. Replacing V by a non-empty open subset if necessary (and then pulling back W over this open subset; the resulting pull-back is non-empty, since the flat base-change of a quasi-compact and scheme-theoretically dominant morphism remains scheme-theoretically dominant), we may furthermore assume that W is flat over V. The morphism $W \to V$ is thus of relative dimension d in the sense of [Stacks, Tag 02NJ], and it follows from [Stacks, Tag 0AFE] that $\dim_w(W) = \dim_v(V) + d$, as required. $\square$

Remark 2.29. We note that in the context of the preceding lemma, it need not be that $\dim \mathcal{T} \geq \dim \mathcal{Z}$; this does not contradict the inequality in the statement of the lemma, because the fibres of the morphism f are again algebraic stacks, and so may have negative dimension. This is illustrated by taking k to be a field, and applying the lemma to the morphism $[\operatorname{Spec} k/\mathbb{G}_m] \to \operatorname{Spec} k$.

If the morphism f in the statement of the lemma is assumed to be quasi-DM (in the sense of [Stacks, Tag 04YW]; e.g. morphisms that are representable by algebraic spaces are quasi-DM), then the fibres of the morphism over points of the target are quasi-DM algebraic stacks, and hence are of non-negative dimension. In this case, the lemma implies that indeed $\dim \mathcal{T} \geq \dim \mathcal{Z}$. In fact, we obtain the following more general result.

Corollary 2.30. *Let $f : \mathcal{T} \to \mathcal{X}$ be a locally of finite type morphism of Jacobson, pseudo-catenary, and locally Noetherian algebraic stacks which is quasi-DM, whose source is irreducible and whose target is quasi-separated, and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ denote the scheme-theoretic image of $\mathcal{T}$. Then $\dim \mathcal{Z} \leq \dim \mathcal{T}$, and furthermore, exactly one of the following two conditions holds:*

(1) for every finite type point $t \in |T|$, we have $\dim_t(\mathcal{T}_{f(t)}) > 0$, in which case $\dim \mathcal{Z} < \dim \mathcal{T}$; or

(2) $\mathcal{T}$ and $\mathcal{Z}$ are of the same dimension.

Proof. As was observed in the preceding remark, the dimension of a quasi-DM stack is always non-negative, from which we conclude that $\dim_t \mathcal{T}_{f(t)} \geq 0$ for all $t \in |\mathcal{T}|$, with the equality

$$\dim_t \mathcal{T}_{f(t)} = \dim_t \mathcal{T} - \dim_{f(t)} \mathcal{Z}$$

holding for a dense open subset of points $t \in |\mathcal{T}|$. $\square$

We close this note by establishing a formula allowing us to compute $\dim_x(\mathcal{X})$ in terms of properties of the versal ring to $\mathcal{X}$ at x. In order to state a clean result, we will make certain hypotheses on the base-scheme S (which has remained implicit up to this point). As with the discussion at the end of Section 1, these hypotheses may not be needed for the result to hold, but they allow for a simple argument.

Before stating our hypotheses, we recall some topological results. These results are essentially contained in [EGAIV$_1$, §0.14.3]. However, as is pointed out in [Hei17], the key proposition of that discussion, namely [EGAIV$_1$, §0 Prop. 14.3.3], is in error. As is made implicit in the examples of [Hei17], and was pointed out explicitly to us by Brian Conrad, the error occurs because a topological space can be equicodimensional without its irreducible components themselves being equicodimensional. We now state (and recall the proof of) a corrected version of that proposition (and of Cor. 14.3.5, which is deduced from Prop. 14.3.3).

We first recall the definition of equicodimensionality.

Definition 2.31. A finite-dimensional topological space X is called *equicodimensional* if $\operatorname{codim}(Y, X)$ is constant as Y ranges over all the minimal irreducible closed subsets of X. (By considering a maximal chain of irreducible closed subsets of X, we then see that this constant is equal to $\dim X$.)

Lemma 2.32. *If X is an irreducible, equicodimensional, finite-dimensional topological space, then the following are equivalent:*

(1) X is catenary.

(2) For any irreducible closed subsets $Y \subseteq Z$ of X, we have

$$\dim Y + \operatorname{codim}(Y, Z) = \dim Z.$$

(3) All maximal chains of irreducible closed subsets of X have the same length.

Furthermore, if these equivalent conditions hold, then any irreducible closed subset of X is also equicodimensional.

Proof. Let Y be an irreducible closed subset of X, and consider maximal chains of irreducible closed sets

$$Y_0 \subset Y_1 \subset \cdots \subset Y_a = Y \tag{2.33}$$

and

$$Y = X_0 \subset X_1 \subset \cdots \subset X_b = X \tag{2.34}$$

Since (2.33) is maximal, we see that Y_0 is a minimal irreducible closed subset of Y (or, equivalently, of X). Concatenating these two chains yields a maximal chain of irreducible closed subsets in X.

Suppose now that (1) holds, i.e. that X is catenary. Then all maximal chains joining Y_0 to X have the same length, which is then $\operatorname{codim}(Y_0, X)$, which also equals $\dim X$ (since X is equicodimensional, by assumption). Thus we find that $a + b = \dim X$, and in particular is independent of the choice of either chain. Varying (2.33), while leaving (2.34) fixed, we find that the value of a is independent of the choice of the maximal chain (2.33). Thus we see that $a = \dim Y$, and also that $\operatorname{codim}(Y_0, Y) = \dim Y$ for any minimal irreducible closed subset of Y (so that Y is again equicodimensional, as claimed).

Now fixing the maximal chain (2.33), and varying the maximal chain (2.34), we find that the value of b is independent of the choice of chain (2.34), and in particular that $b = \operatorname{codim}(Y, X)$. Thus we may rewrite the equation $a + b = \dim X$ as $\dim Y + \operatorname{codim}(Y, X) = \dim X$, showing that (2) holds in the case when $Z = X$. If we consider the general case of (2), then since Z is irreducible (by assumption), finite-dimensional and catenary (being a closed subset of a finite-dimensional and catenary space), and equicodimensional (as we proved above), we may replace X by Z, and hence deduce the general case of (2) from the special case already proved.

Suppose next that (2) holds, and consider a maximal chain of irreducible closed subsets of X, say

$$X_0 \subset X_1 \subset \cdots \subset X_d = X.$$

Noting that $\dim X_0 = 0$ (as X_0 is minimal), and also that $\operatorname{codim}(X_i, X_{i+1}) = 1$ (since by assumption there is no irreducible closed subset lying strictly between X_i and X_{i+1}), we find, by repeated application of (2), that $\dim X_i = i$. In particular, $d = \dim X$ is independent of the chain chosen, so that (3) holds.

Finally, suppose that (3) holds; we wish to show that (1) also holds. If we consider chains of the form (2.33) and (2.34), and their concatenation, then (3) implies that $a + b = \dim X$ is indepedent of the choice of either chain, and thus, by varying these chains independently, that each of a and b is independent of the choice of chain.

Now let $Y \subseteq Z$ be an inclusion of irreducible closed subsets of X. We wish to show that all maximal chains of irreducible closed subsets joining Y and Z are of the same length. By applying what we have just proved to Z, we find that Z also satisfies (3). Thus we may replace X by Z, and hence assume that $Z = X$. But we have already shown that all maximal chains of the form (2.34) are of the same length. Thus X is indeed catenary. $\square$

Although we don't need it, we also note the following result, which among other things provides a purely topological variant of Lemma 2.17. (Note, though, that [EGAIV$_3$, (10.7.3)] gives an example of a Jacobson, universally catenary, integral, Noetherian scheme S which is not equicodimensional; this gives an example of a situation to which Lemma 2.17 applies, although Lemma 2.35 does not.)

Lemma 2.35. *Let X be an irreducible, equicodimensional, finite-dimensional, Jacobson topological space. If U is a non-empty open subset of X, then U is also irreducible, equicodimensional, finite-dimensional, and Jacobson. Furthermore, we have that $\dim U = \dim X$.*

Proof. It is standard that U is again irreducible and Jacobson. The function $T \mapsto \overline{T}$ (closure in X) induces an order-preserving bijection between irreducible closed subsets of U and irreducible closed subsets of X that have non-empty intersection with U; thus U is certainly also finite-dimensional.

Since X and U are Jacobson, the minimal irreducible closed subsets of either X or U are just the closed points, and the closed points of U are precisely the closed points of X that lie in U. Thus, under the bijection $T \mapsto \overline{T}$ described above, the collection of maximal chains of irreducible closed subsets of U containing some given closed point $u \in U$ maps bijectively to the collection of maximal chains of irreducible closed subsets of X containing the same closed point u. In particular, we find that

$$\operatorname{codim}(u, U) = \operatorname{codim}(u, X) = \dim X$$

(the last equality hoding since X is equidimensional, by assumption.) We thus see that $\operatorname{codim}(u, U)$ is independent of the particular closed point $u \in U$, so U is equicodimensional. Furthermore, it is then necessarily equal to $\dim U$, and so we also find that $\dim U = \dim X$, as claimed. $\square$

We next note the following scheme-theoretic result.

Lemma 2.36. *If S is a Jacobson, catenary, locally Noetherian scheme, all of whose irreducible components are of finite dimension and equicodimensional, and if $s \in S$ is a finite type point (or equivalently, a closed point, by Jacobsonness), then $\dim_s S = \dim \mathcal{O}_{S,s}$.*

Proof. We have the equality $\dim \mathcal{O}_{S,s} = \operatorname{codim}(s, S)$ [Stacks, Tag 02IZ]. If we let $T_1, \ldots, T_n$ denote the irreducible components of S passing through s, then $\operatorname{codim}(s, S) = \max_{i=1,\ldots,n} \operatorname{codim}(s, T_i)$, and similarly, $\dim_s S = \max_{i=1,\ldots,n} \dim_s T_i$. Thus it suffices to show that $\operatorname{codim}(s, T_i) = \dim_s T_i$ for each T_i. This follows from Lemma 2.17, which shows that $\dim_s T_i = \dim T_i$, together with the assumption that T_i is equicodimensional. $\square$

We now state the hypothesis that we will make on our base scheme S.

Hypothesis 2.37. *We assume that S is a Jacobson, universally catenary, locally Noetherian scheme, all of whose local rings are G-rings, and with the further property that each irreducible component of S is of finite dimension and equicodimensional.*

Remark 2.38. Since S is catenary by assumption, we see that the equivalent conditions of Lemma 2.32 hold for S. The conditions of Lemma 2.35 also hold. Combining these lemmas, we find in particular that each irreducible locally closed subset of T is equicodimensional. Since S is Jacobson, so is its locally closed subset T. The finite type points in T are then the same as the closed points [Stacks, Tag 01TB], and these are also the minimal irreducible closed subsets of T. Thus to say that T is equicodimensional is to say that $\operatorname{codim}(t, T)$ is constant (equal to $\dim T$) as t ranges over all closed points of T.

Lemma 2.39. *If $X \to S$ is a locally finite type morphism of schemes, and if S satisfies Hypothesis* 2.37*, then so does X.*

Proof. The properties of being Jacobson, of being universally catenary, and of the local rings being G-rings, all pass through a finite type morphism. (In the case of being universally catenary, this is immediate from the definition; for Jacobson see [Stacks, Tag 02J5]; and for local rings being G-rings, see [Stacks, Tag 07PV]). Suppose then that T is an irreducible component of X; we must show that T is finite-dimensional and equicodimensional.

We regard T as an integral scheme, by endowing it with its induced reduced structure. The composite $T \to X \to S$ is again locally of finite type, and so replacing X by T, and S by the closure of the image of T, also endowed with its reduced induced structure (note that by Remark 2.38, this closure is again equicodimensional, and since closed immersions are finite type, the discussion of the preceding paragraph shows that it also satisfies the other conditions of Hypothesis 2.37), we may assume that each of X and S are integral. We now have to show that X is finite-dimensional, and that $\operatorname{codim}(x, X)$ is independent of the closed point $x \in X$.

If $x \in X$, we may find an affine neighbourhood U of x, as well as an affine open subset $V \subset S$ containing the image of U in S. By assumption V is finite-dimensional, and U is of finite type over V; thus U is also finite-dimensional. Since U is furthermore irreducible, Jacobson, catenary, and locally Noetherian (being open in the irreducible, Jacobson, catenary, locally Noetherian scheme X), we see from Lemma 2.17 that the function $u \mapsto \dim_u U$ is constant on U. Since U is open in X, we have an equality $\dim_u U = \dim_u X$ for each point $u \in U$, and hence the function $u \mapsto \dim_u X$ is constant on U. Thus each point of X has a neighbourhood over which $\dim_x X$ is constant (and finite valued). Thus $\dim_x X$ is a locally constant finite valued function on X. Since X is irreducible (and so in particular connected) we find that $\dim_x X$ is constant (and finite valued), and consquently that X is finite-dimensional.

We turn to proving that X equicodimensional. To this end, let $x \in X$ be a closed point. Since S is universally catenary, the dimension formula [Stacks, Tag 02JU], together with the formula of [Stacks, Tag 02IZ], shows that

$$\operatorname{codim}(x, X) = \operatorname{codim}(s, S) + \operatorname{trdeg}_{R(S)} R(X) + \operatorname{trdeg}_{\kappa(s)} \kappa(x);$$

here s denotes the image of x in S, which, being a finite type point of the Jacobson scheme S, is closed in S, and $R(X)$ (resp. $R(S)$) is the function field of X (resp. S). Since $\operatorname{Spec} \kappa(x) \to \operatorname{Spec} \kappa(s)$ is of finite type, we have that $\kappa(x)$ is a finite extension of $\kappa(s)$, so that the final term on the right-hand side of the formula vanishes. Thus $\operatorname{codim}(x, X) - \operatorname{codim}(s, S)$ is constant (i.e. independent of the closed point $x \in X$). Since S is equicodimensional, the term $\operatorname{codim}(s, S)$ is also independent of the closed point $s \in S$; thus $\operatorname{codim}(x, X)$ is indeed independent of the closed point $x \in X$. $\square$

We are now able to state and prove the following result, which relates the dimension of an algebraic stack at a point to the dimension of a corresponding versal ring.

Lemma 2.40. *Suppose that $\mathcal{X}$ is an algebraic stack, locally of finite presentation over a scheme S which satisfies Hypothesis* 2.37*. Suppose further that $x : \operatorname{Spec} k \to \mathcal{X}$ is a a morphism whose source is the spectrum of a field of finite type over $\mathcal{O}_S$, and that $[U/R] \xrightarrow{\sim} \widehat{\mathcal{X}}_x$ is a presentation of $\widehat{\mathcal{X}}_x$ by a smooth groupoid in functors, with U and R both Noetherianly pro-representable*[2]*, by* $\operatorname{Spf} A_x$ *and* $\operatorname{Spf} B_x$ *respectively. Then we have the following formula:*

$$2 \dim A_x - \dim B_x = \dim_x(\mathcal{X}).$$

Proof. By Lemma 1.5, we may find a smooth morphism $V \to \mathcal{X}$, whose source is a scheme, containing a point $v \in V$ of residue field k, such that induced morphism $v = \operatorname{Spec} k \to V \to \mathcal{X}$ coincides with x, and such that $\widehat{\mathcal{O}}_{V,x}$ may be identified with A_x. If we write $W := V \times_{\mathcal{X}} V$, and we write $w := (v, v) \in W$, then we may furthermore identify $\widehat{\mathcal{O}}_{W,w}$ with B_x. Now Remark 2.8 shows that

$$\dim_x \mathcal{X} = \dim_v V - \dim_w(W_v) = \dim_v V - (\dim_w W - \dim_v V) = 2 \dim_v V - \dim_w W.$$

Since v is a finite type point of V, we have that $\dim_v V = \dim \mathcal{O}_{V,v} = \dim \widehat{\mathcal{O}_{V,v}} = \dim A_x$ (where we apply Lemmas 2.39 and 2.36 to obtain the first equality), and similarly $\dim_w W = \dim B_x$. Thus the formula of the lemma is proved. □

References

[EG14] Matthew Emerton and Toby Gee. "“Scheme-theoretic images” of certain morphisms of stacks". 2014.

[EGAIV$_1$] A. Grothendieck. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I". In: *Inst. Hautes Études Sci. Publ. Math.* 20 (1964), p. 259.

[EGAIV$_3$] A. Grothendieck. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III". In: *Inst. Hautes Études Sci. Publ. Math.* 28 (1966), p. 255.

[Hei17] Katharina Heinrich. "Some remarks on biequidimensionality of topological spaces and Noetherian schemes". In: *J. Commut. Algebra* 9.1 (2017), pp. 49–63.

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques.* Vol. 39. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Berlin: Springer-Verlag, 2000, pp. xii+208.

[Oss15] Brian Osserman. "Relative dimension of morphisms and dimension for algebraic stacks". In: *J. Algebra* 437 (2015), pp. 52–78.

[Stacks] The Stacks Project Authors. *Stacks Project.* `http://stacks.math.columbia.edu`. 2013.

[2]We say that a functor on the category of S-schemes is Noetherianally pro-representable if is isomorphic to a functor of the form $\operatorname{Spf} A$, where A is a complete Noetherian local $\mathcal{O}_S$-algebra equipped with its $\mathfrak{m}$-adic topology; here $\mathfrak{m}$ is the maximal ideal of A, of course.

E-mail address: `emerton@math.uchicago.edu`

Department of Mathematics, University of Chicago, 5734 S. University Ave., Chicago, IL 60637, USA

E-mail address: `toby.gee@imperial.ac.uk`

Department of Mathematics, Imperial College London, London SW7 2AZ, UK