COMPANION FORMS FOR UNITARY AND SYMPLECTIC GROUPS

TOBY GEE AND DAVID GERAGHTY

Abstract. We prove a companion forms theorem for ordinary n-dimensional automorphic Galois representations, by use of automorphy lifting theorems developed by the second author, and a technique for deducing companion forms theorems due to the first author. We deduce results about the possible Serre weights of mod l Galois representations corresponding to automorphic representations on unitary groups. We then use functoriality to prove similar results for automorphic representations of GSp_4 over totally real fields.

Contents

1.	Introduction.	1
2.	Notation	5
3.	Galois deformations	5
4.	Ordinary automorphic representations	14
5.	Existence of lifts	19
6.	Serre weights	24
7.	GSp_4	28
References		43

1. Introduction.

1.1. The problem of companion forms was first introduced by Serre for modular forms in his seminal paper [Ser87]. Fix a prime l, algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_l$ of $\mathbb{Q}$ and $\mathbb{Q}_l$ respectively, and an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_l$. Suppose that f is a modular newform of weight $k \geq 2$ which is ordinary at l, so that the corresponding l-adic Galois representation $\rho_{f,l}$ becomes reducible when restricted to a decomposition group $G_{\mathbb{Q}_l}$ at l. Then the companion forms problem is essentially the question of determining for which other weights k' there is an ordinary newform g of weight $k' \geq 2$ such that the Galois representations $\rho_{f,l}$ and $\rho_{g,l}$ are congruent modulo l. The problem is straightforward unless the restriction to $G_{\mathbb{Q}_l}$ of $\overline{\rho}_{f,l}$ (the reduction mod l of $\rho_{f,l}$) is split and non-scalar, in which case there are two possible Hida families whose corresponding Galois representations lift $\overline{\rho}_{f,l}$; the restrictions of the corresponding Galois representations to a decomposition group at l are either "upper-triangular" or "lower-triangular".

2000 *Mathematics Subject Classification.* 11F33.
The first author was partially supported by NSF grant DMS-0841491.

This problem was essentially resolved by Gross and Coleman-Voloch ([Gro90], [CV92]). In the paper [Gee07], the first author reproved these results, and generalised them to Hilbert modular forms, by a completely new technique. In essence, rather than working directly with modular forms, the method is to firstly obtain a Galois representation which should correspond to a modular form in the sought-after Hida family, and then to use a modularity lifting theorem to prove that this Galois representation is modular. In [Gee07] the Galois representation is obtained by using a generalisation of a lifting technique of Ramakrishna, which is proved by purely deformation theory techniques. The modularity is then obtained from the $R = T$ theorem of Kisin for Hilbert modular forms of parallel weight 2 ([Kis09]).

These techniques seem amenable to generalisation (to other reductive groups over more general number fields), subject to some important caveats. In particular, it is necessary to have modularity lifting theorems available over fields in which l is highly ramified. The current technology for modularity lifting theorems requires one to work with reductive groups which admit discrete series, and to work over totally real or CM fields; so it is impossible at present to work directly with GL_n for $n > 2$. Instead, one works with closely related groups, such as unitary or symplectic groups, which do admit discrete series.

In the present paper we make use of $R = T$ theorems for unitary groups to deduce companion forms theorems for unitary groups (in arbitrary dimension), and thus for conjugate self-dual automorphic representations of GL_n over CM fields. We then deduce similar theorems for GSp_4 by developing the relevant deformation theory and employing known instances of functoriality. The analogue for unitary groups of the $R = T$ theorems of [Kis09] seem to be out of reach at present, and we use the main theorems of [Ger09] instead. As explained below, this in fact allows us to prove stronger theorems than the natural analogue of [Kis09] would permit. We replace the use of Ramakrishna's techniques in [Gee07] with a method of Khare and Wintenberger (cf. Lemma 3.6 of [KW08]), which allows weaker hypotheses on local deformation problems. This method shows that various universal deformation rings are finite over various other universal deformation rings, and is employed in Lemmas 3.2.5 and 7.4.1 below.

To our knowledge the only results on companion forms for groups other than GL_2 are those announced for GSp_4 over $\mathbb{Q}$ in [HT08] (see also [Til09]). Our results are rather stronger than those of [HT08] in several respects. We are able to work with arbitrary totally real fields (with no restriction on ramification at l), rather than just over $\mathbb{Q}$, and we do not need any assumption that the residual Galois representation occurs at minimal level (indeed, one may deduce results on level lowering for GSp_4 from our theorem). In addition, the results of [HT08] apply only in one special case, effectively one of 8 cases (corresponding to the 8 elements of the Weyl group of GSp_4) where one could hope to prove a companion forms theorem; this is in part due to the fact that their techniques only apply to Galois representations in the Fontaine-Laffaille range. In contrast, we make no such restrictions. We hope that these results will prove useful for generalisations of the Buzzard-Taylor method to GSp_4, as part of a program of Tilouine.

In recent years there has been a good deal of interest in generalisations of Serre's conjecture (cf. [ADP02]) and in particular in the question of determining the set of weights of a given Galois representation (cf. [Her09]). One of us (T.G.) has formulated a conjecture to the effect that the set of weights should be determined completely by the existence of (local) crystalline lifts (cf. [Gee10]). In general this seems to be a very difficult conjecture to prove, but our methods give a substantial partial result; essentially we prove the conjecture (subject to mild technical hypotheses) for ordinary weights for unitary groups which are compact at infinity. See section 6 for the precise statements.

The strategy of the proofs of our main theorems is as follows. In each case, the required Galois representation is proved to exist by exhibiting it as a $\overline{\mathbb{Q}}_l$-point of an appropriate universal deformation ring. The automorphy then follows from the automorphy lifting theorems of [Ger09]. In order to see that such $\overline{\mathbb{Q}}_l$-points exist, we show that the universal deformation rings are finite over $\mathbb{Z}_l$ and have Krull dimension at least one. In both the unitary and symplectic cases, the lower bound on the Krull dimension follows from standard techniques (one computes the dimensions of the appropriate local universal lifting rings, and then applies a cohomological calculation). In the unitary case, the finiteness over $\mathbb{Z}_l$ follows from the method of Khare–Wintenberger explained above, and the automorphy lifting theorems of [Ger09], which prove the corresponding finiteness after a finite base change.

In the symplectic case, in order to prove that the symplectic deformation ring is finite over $\mathbb{Z}_l$ we proceed slightly indirectly by reducing to the unitary case. In order to do so we make a choice of a quadratic imaginary CM extension of our totally real base field, and show that the symplectic universal deformation ring is finite over the corresponding unitary universal deformation ring, using a slight variant of the method of Khare–Wintenberger. Finally, we use the results of [GT07] on the functoriality between GSp_4 and GL_4 to deduce results for automorphic representations on GSp_4.

We now outline the structure of the paper. In section 3 we develop the basic deformation theory that we need. We then recall in section 4 the necessary material on ordinary automorphic representations on unitary groups and modularity lifting theorems for the corresponding Galois representations; in particular we recall the main theorem of [Ger09].

Section 5 contains our main theorems for unitary groups; the corresponding Galois representations are conjugate self-dual representations of the absolute Galois group of an imaginary CM field. Using the results of section 3 we give a lower bound for the dimension of a universal deformation ring, and the results of section 4 then permit us to prove that this universal deformation ring is finite over $\mathbb{Z}_l$, which implies that it has $\overline{\mathbb{Q}}_l$-points, which correspond to the Galois representations we seek. The automorphy of these Galois representations follows at once from the modularity lifting theorems recalled in section 4. The particular universal deformation ring we consider is one for representations of the absolute Galois group of a totally real field, valued in a group $\mathcal{G}_n$ defined in [CHT08]. Representations valued in this group correspond to representations which are self-dual with respect to some pairing; this permits us to prove results for both the conjugate self-dual representations considered in section 5, and the symplectic representations studied in later sections.

We remark that the $\overline{\mathbb{Q}}_l$-points of universal deformation rings that we study in section 5 always correspond to ordinary crystalline representations of a certain weight. This is in contrast to the approach of [Gee07], which used potentially crystalline representations corresponding to Hilbert modular forms of parallel weight 2 and non-trivial level at l. The required automorphic representations were then obtained by specialising Hida families through these points at the sought-for weight. The difficulty with following this approach in general is that if the weight is not sufficiently regular a specialisation of a Hida family at this weight may fail to be an unramified principal series at places dividing l (for example, a specialisation of a Hida family of modular forms in weight 2 can correspond to a Steinberg representation at l). It is for this reason that we use modularity lifting theorems for crystalline lifts instead.

In section 6 we deduce results about the possible Serre weights of mod l Galois representations corresponding to automorphic representations of compact at infinity unitary groups. In particular, we deduce that the possible ordinary weights are determined by the existence of local crystalline lifts. We remark that these are the first results in anything approaching this level of generality for any groups other than GL_2.

Finally in section 7 we study the analogous questions for automorphic representations of GSp_4 over totally real fields. We use the known functoriality between globally generic cuspidal representations of GSp_4 and GL_4 to apply the methods of the earlier sections. In particular, we prove results analogous to those of section 3 for Galois representations valued in GSp_4, and obtain a lower bound for the dimension of a universal deformation ring as in section 5. We then prove that this universal deformation ring is finite over the corresponding one for unitary representations, which allows us to deduce that our symplectic universal deformation ring is also finite over $\mathbb{Z}_l$. Our main results for symplectic representations follow from this.

We remark that in all our main theorems we actually obtain somewhat more precise results; we are also able to control the ramification of our Galois representations at places not dividing l, and we are able to choose our Galois representations so as to correspond to points on any particular set of irreducible components of the local deformation rings. Thus as a direct corollary of our results one obtains strong results on level lowering and level raising for ordinary automorphic Galois representations. Similarly, our method yields modularity lifting theorems for ordinary representations of GSp_4 which are rather stronger than those of [GT05]; for example, we do not need to assume any form of level-lowering for GSp_4, we work over general totally real fields, and we are not restricted to weights in the Fontaine-Laffaille range.

The recent preprint [BLGGT10] contains some slight improvements to the results of this paper. In particular, thanks to the work of Thorne ([Tho10]), one can slightly weaken the "big image" assumptions made in this paper. One can also relax the assumption that the Galois representations we work with are ordinary to the assumption that they are potentially diagonalizable (see [BLGGT10] for this notion).

We would like to thank Wee Teck Gan, Florian Herzig, Mark Kisin and Richard Taylor for helpful conversations. We would like to thank the anonymous referees for an extremely thorough reading, and for numerous helpful corrections and suggestions.

2. Notation

If M is a field, we let G_M denote its absolute Galois group. Let ϵ denote the l-adic or mod l cyclotomic character of G_M. If M is a finite extension of $\mathbb{Q}_p$ for some p, we write I_M for the inertia subgroup of G_M. We write all matrix transposes on the left; so ${}^tA$ is the transpose of A. If R is a local ring we write $\mathfrak{m}_R$ for the maximal ideal of R. We let $\mathbb{Z}^n_+$ denote the subset of elements $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \ldots \geq \lambda_n$.

3. Galois deformations

3.1. Local deformation rings. Let l be a prime number and K a finite extension of $\mathbb{Q}_l$ with residue field k and ring of integers $\mathcal{O}$. Let M be a finite extension of $\mathbb{Q}_p$ (with p possibly equal to l). Let $\overline{\rho} : G_M \to \mathrm{GL}_n(k)$ be a continuous representation. Let $\mathcal{C}_{\mathcal{O}}$ be the category of complete local Noetherian $\mathcal{O}$-algebras with residue field k. Then the functor from $\mathcal{C}_{\mathcal{O}}$ to *Sets* which takes $A \in \mathcal{C}_{\mathcal{O}}$ to the set of liftings of $\overline{\rho}$ to a continuous homomorphism $\rho : G_M \to \mathrm{GL}_n(A)$ is represented by a complete local Noetherian $\mathcal{O}$-algebra $R^{\square}_{\overline{\rho}}$. We call this ring the *universal $\mathcal{O}$-lifting ring* of $\overline{\rho}$. We write $\rho^{\square} : G_M \to \mathrm{GL}_n(R^{\square}_{\overline{\rho}})$ for the universal lifting. We will need to consider certain quotients of $R^{\square}_{\overline{\rho}}$.

3.1.1. *The case where $p \neq l$.* Firstly, we consider the case $p \neq l$. In this case, the quotients we wish to consider will correspond to particular inertial types. Recall that τ is an *inertial type* for G_M over K if τ is a K-representation of I_M with open kernel which extends to a representation of G_M, and that we say that an l-adic representation of G_M has *type* τ if the restriction of the corresponding Weil-Deligne representation to I_M is equivalent to τ. For any such τ there is a unique reduced, l-torsion free quotient $R^{\square,\tau}_{\overline{\rho}}$ of $R^{\square}_{\overline{\rho}}$ with the property that if E/K is a finite extension, then a map of $\mathcal{O}$-algebras $R^{\square}_{\overline{\rho}} \to E$ factors through $R^{\square,\tau}_{\overline{\rho}}$ if and only if the corresponding E-representation has type τ. Furthermore, we have:

Lemma 3.1.1. *For any τ, if $R^{\square,\tau}_{\overline{\rho}} \neq 0$ then $R^{\square,\tau}_{\overline{\rho}}[1/l]$ is equidimensional of dimension n^2 and admits a dense open subscheme which is formally smooth over K.*

Proof. This is Theorem 2.1.6 of [Gee10]. □

Of course, $R^{\square,\tau}_{\overline{\rho}} \neq 0$ if and only if $\overline{\rho}$ has a lift of type τ.

3.1.2. *The case where $p = l$.* Now assume that $p = l$. In this case, we wish to consider crystalline ordinary deformations of fixed weight. We assume from now on that K is large enough that any embedding $M \hookrightarrow \overline{K}$ has image contained in K.

Notation. Recall that $\mathbb{Z}^n_+$ is the set of non-increasing n-tuples of integers. Let ϵ be the l-adic cyclotomic character and let $\mathrm{Art}_M : M^{\times} \to W^{\mathrm{ab}}_M$ be the Artin map (normalized to take uniformizers to lifts of geometric Frobenius).

Definition 3.1.2. Let λ be an element of $(\mathbb{Z}^n_+)^{\mathrm{Hom}(M,K)}$. We associate to λ an n-tuple of characters $I_M \to \mathcal{O}^{\times}$ as follows. For $j = 1, \ldots, n$ define

$$\begin{aligned} \chi^{\lambda}_j : I_M &\to \mathcal{O}^{\times} \\ \sigma &\mapsto \epsilon(\sigma)^{-(j-1)} \prod_{\tau : M \hookrightarrow K} \tau(\mathrm{Art}_M^{-1}(\sigma))^{-\lambda_{\tau,n-j+1}}. \end{aligned}$$

Note that χ_j^λ can also be thought of as the restriction to I_M of any crystalline character $G_M \to \overline{\mathbb{Q}}_l^\times$ whose Hodge-Tate weight with respect to $\tau : M \hookrightarrow \overline{\mathbb{Q}}_l$ is given by $(j-1)+\lambda_{\tau,n-j+1}$ for all τ (we use the convention that the Hodge-Tate weights of ϵ are all -1).

Let λ be an element of $(\mathbb{Z}^n_+)^{\mathrm{Hom}(M,K)}$. We associate to λ an *l-adic Hodge type* $\mathbf{v}_\lambda$ in the sense of section 2.6 of [Kis08] as follows. Let D_K denote the vector space K^n. Let $D_{K,M} = D_K \otimes_{\mathbb{Q}_l} M$. For each embedding $\tau : M \hookrightarrow K$, we let $D_{K,\tau} = D_{K,M} \otimes_{K\otimes M, 1\otimes\tau} K$ so that $D_{K,M} = \oplus_\tau D_{K,\tau}$. For each τ choose a decreasing filtration $\mathrm{Fil}^i D_{K,\tau}$ of $D_{K,\tau}$ so that $\dim_K \mathrm{gr}^i D_{K,\tau} = 0$ unless $i = (j-1)+\lambda_{\tau,n-j+1}$ for some $j = 1,\dots,n$ in which case $\dim_K \mathrm{gr}^i D_{K,\tau} = 1$. We define a decreasing filtration of $D_{K,M}$ by $K\otimes_{\mathbb{Q}_l} M$-submodules by setting

$$\mathrm{Fil}^i D_{K,M} = \oplus_\tau \mathrm{Fil}^i D_{K,\tau}.$$

Let $\mathbf{v}_\lambda = \{D_K, \mathrm{Fil}^i D_{K,M}\}$.

Let B denote a finite, local K-algebra and $\rho_B : G_M \to \mathrm{GL}_n(B)$ a crystalline representation. Then $D_B := D_{\mathrm{cris}}(\rho_B) = (\rho_B \otimes_{\mathbb{Q}_l} B_{\mathrm{cris}})^{G_M}$ is a free $B\otimes_{\mathbb{Q}_l} M_0$-module of rank n where M_0 is the maximal subfield of M which is unramified over $\mathbb{Q}_l$. Moreover, D_B is equipped with a B-linear and φ_0-semilinear morphism φ_B where φ_0 denotes the arithmetic Frobenius on M_0. For each embedding $\tau : M_0 \to K$, let $D_{B,\tau} = D_B \otimes_{B\otimes M_0, 1\otimes\tau} B$. Then $D_B = \oplus_\tau D_{B,\tau}$. Also, for each τ, φ_B defines an isomorphism of B-modules $\varphi_B : D_{B,\tau} \xrightarrow{\sim} D_{B,\tau\circ\varphi_0^{-1}}$. Let $f = [M_0 : \mathbb{Q}_l]$. Then φ_B^f is a B-linear endomorphism of D_B which preserves each $D_{B,\tau}$. For each τ, the isomorphism $\varphi_B : D_{B,\tau} \to D_{B,\tau\circ\varphi_0^{-1}}$ takes φ_B^f to φ_B^f.

Let $\mathcal{F}$ denote the flag variety over $\mathrm{Spec}\,\mathcal{O}$ whose set of A-points, for any $\mathcal{O}$-algebra A, corresponds to filtrations $0 = \mathrm{Fil}_0 \subset \mathrm{Fil}_1 \subset \ldots \subset \mathrm{Fil}_n = A^n$ of A^n by locally free submodules which, locally, are direct summands and are such that Fil_j has rank j.

Definition 3.1.3. Let E be an algebraic extension of K let B be a finite local E-algebra. Let $\rho : G_M \to \mathrm{GL}_n(B)$ be a continuous homomorphism. We say that ρ is *ordinary of weight* $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(M,K)}$ if ρ is conjugate to a representation of the form

$$\begin{pmatrix} \psi_1 & * & \dots & * & * \\ 0 & \psi_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \psi_{n-1} & * \\ 0 & 0 & \dots & 0 & \psi_n \end{pmatrix}$$

where for each $j = 1,\dots,n$ the character ψ_j agrees on an open subgroup of I_M with the character χ_j^λ introduced above.

Equivalently, ρ is ordinary of weight λ if there is a full flag $0 = \mathrm{Fil}_0 \subset \mathrm{Fil}_1 \subset \ldots \subset \mathrm{Fil}_n = B^n$ of B^n which is preserved by G_M and such that the representation of G_M on $\mathrm{gr}_j = \mathrm{Fil}_j / \mathrm{Fil}_{j-1}$ is potentially semistable and for each embedding $\tau : M \hookrightarrow K$, the Hodge-Tate weight of gr_j with respect to τ is $(j-1)+\lambda_{\tau,n-j+1}$.

Lemma 3.1.4. *Suppose that E is an algebraic extension of K and $\rho : G_M \to \mathrm{GL}_n(E)$ is ordinary of weight λ. Let $\psi_1,\dots,\psi_n : G_M \to E^\times$ be as above. Then*

(1) *ρ is potentially semistable.*

(2) *If each* ψ_j *is crystalline (which occurs if and only if* ψ_j *agrees with* χ_j^λ *on all of* I_M*), then* ρ *is semistable.*
(3) *If each* ψ_j *is crystalline and if for each* $j = 1, \ldots, n-1$ *there exists* $\tau : M \hookrightarrow K$ *with* $\lambda_{\tau,j} > \lambda_{\tau,j+1}$*, then* ρ *is crystalline.*

Proof. Part 2 follows from Proposition 1.28(2) of [Nek93] and part 1 follows from part 2. Part 3 follows from Proposition 1.26 of [Nek93] and the formulae in Proposition 1.24 of [Nek93]. □

Lemma 3.1.5. *Let* $\psi_i : G_M \to E^\times$ *be as above (with respect to some* $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(M,K)}$*), with each* ψ_i *crystalline. Suppose that* $\overline{\rho} : G_M \to \mathrm{GL}_n(k)$ *is of the form*

$$\begin{pmatrix} \overline{\mu}_1 & * & \ldots & * & * \\ 0 & \overline{\mu}_2 & \ldots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \overline{\mu}_{n-1} & * \\ 0 & 0 & \ldots & 0 & \overline{\mu}_n \end{pmatrix}$$

where $\overline{\psi}_i = \overline{\mu}_i$ *for each* $1 \leq i \leq n$*. Suppose that for each* $i < j$ *we have* $\overline{\mu}_i \overline{\mu}_j^{-1} \neq \overline{\epsilon}$*. Then* $\overline{\rho}$ *has a lift to a crystalline representation* $\rho : G_M \to \mathrm{GL}_n(E)$ *of the form*

$$\begin{pmatrix} \psi_1 & * & \ldots & * & * \\ 0 & \psi_2 & \ldots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \psi_{n-1} & * \\ 0 & 0 & \ldots & 0 & \psi_n \end{pmatrix}.$$

Proof. The fact that any upper-triangular representation of this form is crystalline follows easily as in the proof of Lemma 3.1.4, because the assumption that $\overline{\mu}_i \overline{\mu}_j^{-1} \neq \overline{\epsilon}$ implies that $\psi_i \psi_j^{-1} \neq \epsilon$. The fact that such an upper-triangular lift exists follows from the fact that $H^2(G_M, \mathfrak{u}) = 0$, where $\mathfrak{u}$ is the subspace of the Lie algebra $\mathrm{ad}\,\overline{\rho}$ consisting of strictly upper-triangular matrices. The vanishing of this cohomology group follows from Tate local duality and the existence of a filtration on $\mathfrak{u}$ whose graded pieces are one-dimensional with G_M acting via the characters $\overline{\mu}_i \overline{\mu}_j^{-1} \neq \overline{\epsilon}$, $i < j$ (cf. Lemma 3.2.3 of [Ger09]). □

We now recall some results of Kisin. Let λ be an element of $(\mathbb{Z}^n_+)^{\mathrm{Hom}(M,K)}$ and let $\mathbf{v}_\lambda$ be the associated l-adic Hodge type.

Definition 3.1.6. If B is a finite K-algebra and V_B is a free B-module of rank n with a continuous action of G_M that makes V_B into a de Rham representation, then we say that V_B *is of* l*-adic Hodge type* $\mathbf{v}_\lambda$ if for each i there is an isomorphism of $B \otimes_{\mathbb{Q}_l} M$-modules

$$\mathrm{gr}^i (V_B \otimes_{\mathbb{Q}_l} B_{dR})^{G_M} \xrightarrow{\sim} B \otimes_K (\mathrm{gr}^i D_{K,M}).$$

For example, if E is a finite extension of K and $\rho : G_M \to \mathrm{GL}_n(E)$ is ordinary of weight λ, then ρ is of l-adic Hodge type $\mathbf{v}_\lambda$.

Corollary 2.7.7 of [Kis08] implies that there is a unique l-torsion-free quotient $R^{\mathbf{v}_\lambda, cr}_{\overline{\rho}}$ of $R^\square_{\overline{\rho}}$ with the property that for any finite K-algebra B, a homomorphism of $\mathcal{O}$-algebras $\zeta : R^\square_{\overline{\rho}} \to B$ factors through $R^{\mathbf{v}_\lambda, cr}_{\overline{\rho}}$ if and only if $\zeta \circ \rho^\square$ is crystalline of l-adic Hodge type $\mathbf{v}_\lambda$. Moreover, Theorem 3.3.8 of [Kis08] implies that $\operatorname{Spec} R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}[1/l]$ is formally smooth over K and equidimensional of dimension $n^2+\frac{1}{2}n(n-1)[M:\mathbb{Q}_l]$. In particular, $R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}$ is reduced.

Let $\mathcal{F}$ be the flag variety over $\operatorname{Spec}\mathcal{O}$ as introduced above and let $\mathcal{G}^\lambda$ be the closed subscheme of $\mathcal{F}\times_{\operatorname{Spec}\mathcal{O}}\operatorname{Spec} R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}$ corresponding to filtrations Fil which (i) are preserved by the induced action of G_M and (ii) are such that I_M acts on $\operatorname{gr}_j=\operatorname{Fil}_j/\operatorname{Fil}_{j-1}$ via the character χ_j^λ for each $j=1,\dots,n$. The fact that $\mathcal{G}^\lambda$ is a closed subscheme can be proved in the same way as Lemma 3.1.2 of [Ger09]. Let $R_{\overline{\rho}}^{\triangle_\lambda,cr}$ be the image of

$$R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}\to\mathcal{O}_{\mathcal{G}^\lambda}(\mathcal{G}^\lambda[1/l]).$$

In other words, $\operatorname{Spec} R_{\overline{\rho}}^{\triangle_\lambda,cr}$ is the scheme theoretic image of the morphism $\mathcal{G}^\lambda[1/l]\to\operatorname{Spec} R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}$. The next result follows from Lemma 3.3.3 of [Ger09].

Lemma 3.1.7. *For any finite local K-algebra B, a homomorphism of $\mathcal{O}$-algebras $\zeta: R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}\to B$ factors through $R_{\overline{\rho}}^{\triangle_\lambda,cr}$ if and only if $\zeta\circ\rho^\square$ is ordinary of weight λ. Moreover, the underlying topological space of* $\operatorname{Spec} R_{\overline{\rho}}^{\triangle_\lambda,cr}$ *is a union of irreducible components of* $\operatorname{Spec} R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}$.

We note that since $R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}$ is reduced, the last statement determines $R_{\overline{\rho}}^{\triangle_\lambda,cr}$ uniquely as a quotient of $R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}$.

3.1.3. *The $p=l$ case with a slight refinement.* We continue to consider, as above, crystalline lifts of $\overline{\rho}$ which are ordinary of a given weight λ. A necessary condition for such lifts to exist is that $\overline{\rho}$ itself is conjugate to an upper triangular representation whose ordered n-tuple of diagonal characters, restricted to I_M, is given by $(\overline{\chi}_1^\lambda,\dots,\overline{\chi}_n^\lambda)$. Let us assume that $\overline{\rho}$ has this property. In fact, let us fix characters $\overline{\mu}_1,\dots,\overline{\mu}_n: G_M\to k^\times$ with $\overline{\mu}_j|_{I_M}=\overline{\chi}_j^\lambda$ and assume that $\overline{\rho}$ is conjugate to an upper triangular representation whose ordered n-tuple of diagonal characters is $\overline{\mu}:=(\overline{\mu}_1,\dots,\overline{\mu}_n)$. (We note that we may have more than one choice for the ordered n-tuple $\overline{\mu}$. For example, if each $\overline{\chi}_j^\lambda$ is trivial and $\overline{\rho}$ is a direct sum of distinct unramified characters, then choosing $\overline{\mu}$ amounts to choosing an ordering of these characters.) We now would like to study crystalline lifts of $\overline{\rho}$ which are ordinary of weight λ *and* are such that for each j, the character ψ_j of Definition 3.1.3 lifts $\overline{\mu}_j$.

Let $R_{\overline{\mu}}$ denote the object of $\mathcal{C}_\mathcal{O}$ representing the functor which sends an object A of $\mathcal{C}_\mathcal{O}$ to the set of lifts $(\psi_1,\dots,\psi_n)$ of the ordered n-tuple $(\overline{\mu}_1,\dots,\overline{\mu}_n)$ with $\psi_j|_{I_M}=\chi_j^\lambda$ for each j. The ring $R_{\overline{\mu}}$ is non-canonically isomorphic to $\mathcal{O}[[X_1,\dots,X_n]]$. Let $(\psi_1^{\mathrm{univ}},\dots,\psi_n^{\mathrm{univ}})$ be the universal lift of the tuple $(\overline{\mu}_1,\dots,\overline{\mu}_n)$ to $R_{\overline{\mu}}$. Let $\mathcal{G}_{\overline{\mu}}^\lambda$ denote the closed subscheme of the flag variety $\mathcal{F}\times_{\operatorname{Spec}\mathcal{O}}\operatorname{Spec}(R_{\overline{\rho}}^{\triangle_\lambda,cr}\widehat{\otimes}_\mathcal{O}R_{\overline{\mu}})$ corresponding to filtrations which are (i) preserved by the induced action of G_M and (ii) such that G_M acts on gr_j via the pushforward of ψ_j^{univ} for each $j=1,\dots,n$. Let $R_{\overline{\rho},\overline{\mu}}^{\triangle_\lambda,cr}$ be the quotient of $R_{\overline{\rho}}^{\triangle_\lambda,cr}\widehat{\otimes}_\mathcal{O}R_{\overline{\mu}}$ corresponding to the scheme theoretic image of $\mathcal{G}_{\overline{\mu}}^\lambda[1/l]$. Note that we have a natural morphism $\mathcal{G}_{\overline{\mu}}^\lambda[1/l]\to\mathcal{G}^\lambda[1/l]$ covering the morphism $\operatorname{Spec} R_{\overline{\rho},\overline{\mu}}^{\triangle_\lambda,cr}\to\operatorname{Spec} R_{\overline{\rho}}^{\triangle_\lambda,cr}$. Let B be a finite local K-algebra and $\zeta: R_{\overline{\rho}}^{\triangle_\lambda,cr}\widehat{\otimes}_\mathcal{O}R_{\overline{\mu}}\to B$ a homomorphism of $\mathcal{O}$-algebras. Then $\zeta\circ\rho^\square$ is ordinary of weight λ. Let $\psi_1,\dots,\psi_n: G_M\to B^\times$ be as in Definition 3.1.3. Then ζ factors through $R_{\overline{\rho},\overline{\mu}}^{\triangle_\lambda,cr}$ if and only if $\psi_j=\zeta\circ\psi_j^{\mathrm{univ}}$ for each j. If this is the case, then ψ_j lifts $\overline{\mu}_j$ for each j. (Note that $\psi_j \mod \mathfrak{m}_B$ takes values in the ring of integers of $E := B/\mathfrak{m}_B$ and hence can be reduced modulo $\mathfrak{m}_{\mathcal{O}_E}$. The resulting character is $\overline{\mu}_j$.)

Lemma 3.1.8. *Suppose $R^{\triangle_\lambda,cr}_{\overline{\rho},\overline{\mu}} \neq (0)$. Then after inverting l, the morphism $\operatorname{Spec} R^{\triangle_\lambda,cr}_{\overline{\rho},\overline{\mu}} \to \operatorname{Spec} R^{\triangle_\lambda,cr}_{\overline{\rho}}$ becomes a closed immersion and identifies $\operatorname{Spec} R^{\triangle_\lambda,cr}_{\overline{\rho},\overline{\mu}}[1/l]$ with a union of irreducible components of $\operatorname{Spec} R^{\triangle_\lambda,cr}_{\overline{\rho}}[1/l]$. Moreover, every irreducible component of $\operatorname{Spec} R^{\triangle_\lambda,cr}_{\overline{\rho}}[1/l]$ is in the image of $\operatorname{Spec} R^{\triangle_\lambda,cr}_{\overline{\rho},\overline{\mu}}[1/l] \to \operatorname{Spec} R^{\triangle_\lambda,cr}_{\overline{\rho}}[1/l]$ for a unique $\overline{\mu}$.*

We note that the irreducible components of $\operatorname{Spec} R^{\triangle_\lambda,cr}_{\overline{\rho}}[1/l]$ and $\operatorname{Spec} R^{\triangle_\lambda,cr}_{\overline{\rho}}$ are in bijection.

Proof. Let $R = R^{\triangle_\lambda,cr}_{\overline{\rho}}$ and let $\rho_R : G_M \to \mathrm{GL}_n(R)$ denote the push-forward of the universal lift of $\overline{\rho}$. Let $V = (R[1/l])^n$, endowed with the action of G_M coming from ρ_R. Choose an element $\sigma \in I_M$ such that $\chi^\lambda_i(\sigma) \neq \chi^\lambda_j(\sigma)$ for all $i \neq j$. For $1 \leq j \leq n-1$, let $P_j(X)$ denote the polynomial $\prod_{i>j}(X - \chi^\lambda_i(\sigma))$ and let $V_j = P_j(\rho_R(\sigma))V \subset V$. Note that for each maximal ideal $\wp$ of $R[1/l]$, we have $\dim_{R[1/l]/\wp}((V/V_j) \otimes_{R[1/l]} R[1/l]/\wp) = n - j$ by Lemma 3.1.7. It follows that V/V_j is locally free of rank $n - j$. We then deduce that V_j is locally free of rank j and a direct summand of V. Note that V_j is G_M-stable. (For each $\tau \in G_M$, the quotient $V_j/(V_j \cap \rho_R(\tau)(V_j))$ vanishes modulo $\wp$ for each maximal ideal $\wp$ of $R[1/l]$ and hence is zero.) Note also that $V_1 \subset V_2 \subset \cdots \subset V_{n-1}$. (We have $V_j = (\rho_R(\sigma) - \chi^\lambda_{j+1}(\sigma))V_{j+1}$ for $j = 1, \ldots, n-2$.) Set $V_n = V$ and $V_0 = (0)$. For $j = 1, \ldots, n$, let $\psi_j : G_M \to R[1/l]^\times$ denote the character giving the action of G_M on the rank 1 locally free $R[1/l]$-module V_j/V_{j-1}. We have $\psi_j|_{I_M} = \chi^\lambda_j$ by Lemma 3.1.7 and the fact that R is reduced.

Let $\widetilde{R}$ denote the normalisation of R. Note that $\widetilde{R}[1/l] = R[1/l]$ since normalisation commutes with localisation and $R[1/l]$ is normal (being formally smooth over K). It follows that $R \subset \widetilde{R} \subset R[1/l]$. Note also that $\widetilde{R}$ is a product of complete local Noetherian integral domains whose residue fields are finite extensions of k (this follows from the fact that $\widetilde{R}$ is finite over R) and that the irreducible components of $\operatorname{Spec} \widetilde{R}$ biject with those of $\operatorname{Spec} R[1/l]$. For each maximal ideal $\wp$ of $R[1/l]$, the character $(\psi_j \mod \wp) : G_M \to (R[1/l]/\wp)^\times$ takes values in the ring of integers $\mathcal{O}_\wp$ of $R[1/l]/\wp$ (since G_M is compact and $\psi_j \mod \wp$ is continuous). By Theorem 7.4.1 and Lemma 7.1.9 of [dJ95], the character ψ_j takes values in $\widetilde{R}^\times$.

For each maximal ideal $\wp$ of $\operatorname{Spec} R[1/l]$ let $\overline{\psi}_{\wp,j}$ denote the composition of the character $\psi_j \mod \wp : G_M \to \mathcal{O}_\wp^\times$ with the reduction map $\mathcal{O}_\wp^\times \to \bar{k}(\wp)^\times$ where $\bar{k}(\wp)$ denotes the residue field of $\mathcal{O}_\wp$. We now show that the ordered n-tuple $\overline{\psi}_\wp := (\overline{\psi}_{\wp,1}, \ldots, \overline{\psi}_{\wp_n})$ is constant for $\wp$ varying in a fixed irreducible component of $R[1/l]$: let $\mathfrak{q}$ be a minimal prime of $R[1/l]$ and $\widetilde{\mathfrak{q}} = \mathfrak{q} \cap \widetilde{R}$ the corresponding minimal prime of $\widetilde{R}$. Let $\mathfrak{n}$ denote the maximal ideal of the local ring $\widetilde{R}/\widetilde{\mathfrak{q}}$. We also regard $\mathfrak{n}$ as an ideal of $\widetilde{R}$. For $j = 1, \ldots, n$, let $\overline{\psi}_{\mathfrak{q},j} : G_M \to (\widetilde{R}/\mathfrak{n})^\times$ denote the character ψ_j mod $\mathfrak{n}$. Since $\overline{\rho}^{\mathrm{ss}} \cong \overline{\psi}_{\mathfrak{q},1} \oplus \cdots \oplus \overline{\psi}_{\mathfrak{q},n}$, we see that each $\overline{\psi}_{\mathfrak{q},j}$ is valued in $k^\times$. Let $\overline{\psi}_\mathfrak{q} = (\overline{\psi}_{\mathfrak{q},1}, \ldots, \overline{\psi}_{\mathfrak{q},n})$. It is then tautological that $\overline{\psi}_\wp = \overline{\psi}_\mathfrak{q}$ for each maximal ideal $\wp$ of $R[1/l]$ containing the minimal prime $\mathfrak{q}$.

The assumption that $R^{\triangle_\lambda, cr}_{\overline{\rho},\overline{\mu}} \neq (0)$ implies that the set of minimal primes $\mathfrak{q}$ of $R[1/l]$ such that $\overline{\psi}_{\mathfrak{q}} = \overline{\mu}$ is non-empty. Let I denote the intersection of these minimal primes. Then the map $\mathcal{G}^{\lambda}_{\overline{\mu}}[1/l] \to \operatorname{Spec} R[1/l]$ factors through $\operatorname{Spec} R[1/l]/I$. To prove the first statement of the lemma, it suffices to show that the natural map from $R[1/l]/I$ to the image of the map

$$\alpha : (R\widehat{\otimes}_{\mathcal{O}} R_{\overline{\mu}})[1/l] \to \mathcal{O}_{\mathcal{G}^{\lambda}_{\overline{\mu}}}(\mathcal{G}^{\lambda}_{\overline{\mu}}[1/l])$$

is an isomorphism. The injectivity of the map $R[1/l]/I \to \operatorname{Im}\alpha$ follows from the fact that the map $\mathcal{G}^{\lambda}_{\overline{\mu}} \to \operatorname{Spec} R[1/l]/I$ is dominant (as every closed point of the target is in the image).

We now establish surjectivity. Let $\widetilde{I} = \widetilde{R} \cap I$. The characters ψ_j give rise to a map $R_{\overline{\mu}} \to \widetilde{R}/\widetilde{I}$ under which ψ_j^{univ} pushes forward to $\psi_j \mod \widetilde{I}$. This gives rise to an R-algebra surjection $\beta : (R\widehat{\otimes}_{\mathcal{O}} R_{\overline{\mu}})[1/l] \twoheadrightarrow R[1/l]/I$. Moreover, for each closed point x of $\mathcal{G}^{\lambda}_{\overline{\mu}}$, the composite

$$(R\widehat{\otimes}_{\mathcal{O}} R_{\overline{\mu}})[1/l] \to \mathcal{O}_{\mathcal{G}^{\lambda}_{\overline{\mu}}}(\mathcal{G}^{\lambda}_{\overline{\mu}}[1/l]) \to k(x)$$

factors through β. (This follows from the fact that G_M acts on V_j/V_{j-1} via ψ_j.) It follows that the map α factors through β and in particular, $R[1/l]/I$ surjects onto the image of α. Thus we have established the first statement of the lemma. The second statement is immediate. $\square$

3.1.4. *The $p = l$ case in non-fixed weight.* In this section we assume that $p = l$, that $\overline{\rho} : G_M \to \mathrm{GL}_n(k)$ is the trivial homomorphism. Let $R^{\square}_{\overline{\rho}}$ denote the universal $\mathcal{O}$-lifting ring of $\overline{\rho}$ and let $\Lambda_M = \mathcal{O}[[I_{M^{\mathrm{ab}}/M}(l)^n]]$ where for a group H, $H(l)$ denotes its pro-l completion. Then Λ_M represents the functor $\mathcal{C}_{\mathcal{O}} \to Sets$ sending an algebra A to the set of ordered n-tuples $(\chi_1, \ldots, \chi_n)$ of characters $\chi_j : I_{M^{\mathrm{ab}}/M} \to A^{\times}$ lifting the trivial character modulo $\mathfrak{m}_A$. Let $\rho^{\square}$ denote the universal lift of $\overline{\rho}$ to $R^{\square}_{\overline{\rho}}$ and let $(\chi_1^{\mathrm{univ}}, \ldots, \chi_n^{\mathrm{univ}})$ denote the universal n-tuple of characters $I_{M^{\mathrm{ab}}/M} \to \Lambda_M^{\times}$.

Let $R^{\square}_{\overline{\rho},\Lambda_M} = R^{\square}_{\overline{\rho}}\widehat{\otimes}_{\mathcal{O}}\Lambda_M$. Let $\mathcal{G}$ denote the closed subscheme of the flag variety $\mathcal{F} \times_{\operatorname{Spec}\mathcal{O}} \operatorname{Spec} R^{\square}_{\overline{\rho},\Lambda_M}$ corresponding to filtrations which are (i) preserved by the induced action of G_M and (ii) such that I_M acts on gr_j via the pushforward of χ_j^{univ}. Let $R^{\triangle}_{\overline{\rho},\Lambda_M}$ be the quotient of $R^{\square}_{\overline{\rho},\Lambda_M}$ corresponding to the scheme theoretic image of the morphism

$$\mathcal{G}[1/l] \to \operatorname{Spec} R^{\square}_{\overline{\rho},\Lambda_M}.$$

If E is a finite extension of K, a homomorphism of $\mathcal{O}$-algebras $\zeta : R^{\square}_{\overline{\rho},\Lambda_M} \to E$ factors through $R^{\triangle}_{\overline{\rho},\Lambda_M}$ if and only if $\zeta\circ\rho^{\square}$ is conjugate to an upper triangular representation whose ordered n-tuple of diagonal characters, restricted to I_M, is the pushforward of $(\chi_1^{\mathrm{univ}}, \ldots, \chi_n^{\mathrm{univ}})$. A quotient $R^{\triangle,ar}_{\overline{\rho},\Lambda_M}$ of $R^{\triangle}_{\overline{\rho},\Lambda_M}$ is introduced in Definition 3.4.5 of [Ger09]. This quotient is equidimensional of dimension $1+n^2+[M:\mathbb{Q}_l]n(n+1)/2$. Moreover, if $\zeta : R^{\triangle}_{\overline{\rho},\Lambda_M} \to E$ is a homomorphism of $\mathcal{O}$-algebras such that $\zeta\circ\rho^{\square}$ is potentially semistable of l-adic Hodge type $\mathbf{v}_\lambda$ for some $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(M,K)}$, then ζ factors through $R^{\triangle,ar}_{\overline{\rho},\Lambda_M}$. (See Lemmas 3.4.6 and 3.4.7 of [Ger09] and Lemma 3.1.4 of this paper.)

3.2. Global deformation rings.

3.2.1. *The group $\mathcal{G}_n$.* Assume from now on that $l > 2$. Let n be a positive integer, and let $\mathcal{G}_n$ be the group scheme over $\mathbb{Z}$ which is the semidirect product of $\mathrm{GL}_n \times \mathrm{GL}_1$ by the group $\{1, j\}$, which acts on $\mathrm{GL}_n \times \mathrm{GL}_1$ by

$$j(g, \mu)j^{-1} = (\mu^t g^{-1}, \mu).$$

There is a homomorphism $\nu : \mathcal{G}_n \to \mathrm{GL}_1$ sending (g, μ) to μ and j to -1. Write $\mathfrak{g}_n^0$ for the trace zero subspace of the Lie algebra of GL_n, regarded as a Lie subalgebra of the Lie algebra of $\mathcal{G}_n$.

Definition 3.2.1. Let F^+ be a totally real field, and let $r : G_{F^+} \to \mathcal{G}_n(L)$ be a continuous homomorphism, where L is a topological field. Then we say that r is *odd* if for all complex conjugations $c_v \in G_{F^+}$, $\nu \circ r(c_v) = -1$.

3.2.2. *Bigness.* Recall definition 2.5.1 of [CHT08].

Definition 3.2.2. Let k be an algebraic extension of the finite field $\mathbb{F}_l$. We say that a finite subgroup $H \subset \mathrm{GL}_n(k)$ is *big* if the following conditions are satisfied.

- H has no quotient of l-power order.
- $H^0(H, \mathfrak{g}_n^0(k)) = (0)$.
- $H^1(H, \mathfrak{g}_n^0(k)) = (0)$.
- For all irreducible $k[H]$-submodules W of $\mathfrak{g}_n^0(k)$ we can find $h \in H$ and $\alpha \in k$ such that the α-generalised eigenspace $V_{h,\alpha}$ of h in k^n is one-dimensional and furthermore $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq (0)$. Here $\pi_{h,\alpha} : k^n \to V_{h,\alpha}$ is the h-equivariant projection of k^n to $V_{h,\alpha}$, and $i_{h,\alpha}$ is the h-equivariant injection of $V_{h,\alpha}$ into k^n.

We call a finite subgroup $H \subset \mathcal{G}_n(k)$ *big* if H surjects onto $\mathcal{G}_n(k)/\mathcal{G}_n^0(k)$ and $H \cap \mathcal{G}_n^0(k)$ is big.

3.2.3. *Deformation problems.* Let F/F^+ be a totally imaginary quadratic extension of a totally real field F^+. Let c denote the non-trivial element of $\mathrm{Gal}(F/F^+)$. Let k denote a finite field of characteristic l and K a finite extension of $\mathbb{Q}_l$, inside a fixed algebraic closure $\overline{\mathbb{Q}}_l$, with ring of integers $\mathcal{O}$ and residue field k. Assume that K contains the image of every embedding $F \hookrightarrow \overline{\mathbb{Q}}_l$ and that the prime l is odd. Assume that every place in F^+ dividing l splits in F. Let S denote a finite set of finite places of F^+ which split in F, and assume that S contains every place dividing l. Let S_l denote the set of places of F^+ lying over l. Let $F(S)$ denote the maximal extension of F unramified away from S. Let $G_{F^+,S} = \mathrm{Gal}(F(S)/F^+)$ and $G_{F,S} = \mathrm{Gal}(F(S)/F)$. For each $v \in S$ choose a place $\widetilde{v}$ of F lying over v and let $\widetilde{S}$ denote the set of $\widetilde{v}$ for $v \in S$. For each place $v|\infty$ of F^+ we let c_v denote a choice of a complex conjugation at v in $G_{F^+,S}$. For each place w of F we have a $G_{F,S}$-conjugacy class of homomorphisms $G_{F_w} \to G_{F,S}$. For $v \in S$ we fix a choice of homomorphism $G_{F_{\widetilde{v}}} \to G_{F,S}$.

If R is a ring and $r : G_{F^+,S} \to \mathcal{G}_n(R)$ is a homomorphism with $r^{-1}(\mathrm{GL}_n(R) \times \mathrm{GL}_1(R)) = G_{F,S}$, we will make a slight abuse of notation and write $r|_{G_{F,S}}$ (respectively $r|_{G_{F_w}}$ for w a place of F) to mean $r|_{G_{F,S}}$ (respectively $r|_{G_{F_w}}$) composed with the projection $\mathrm{GL}_n(R) \times \mathrm{GL}_1(R) \to \mathrm{GL}_n(R)$.

Fix a continuous homomorphism

$$\bar{r} : G_{F^+,S} \to \mathcal{G}_n(k)$$

such that $G_{F,S} = \bar{r}^{-1}(\mathrm{GL}_n(k)\times\mathrm{GL}_1(k))$ and fix a continuous character $\chi : G_{F^+,S} \to \mathcal{O}^\times$ such that $\nu \circ \bar{r} = \overline{\chi}$. Assume that $\bar{r}|_{G_{F,S}}$ is absolutely irreducible. As in Definition 1.2.1 of [CHT08], we define

- a *lifting* of $\bar{r}$ to an object A of $\mathcal{C}_{\mathcal{O}}$ to be a continuous homomorphism $r : G_{F^+,S} \to \mathcal{G}_n(A)$ lifting $\bar{r}$ and with $\nu \circ r = \chi$;
- two liftings r, r' of $\bar{r}$ to A to be *equivalent* if they are conjugate by an element of $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$;
- a *deformation* of $\bar{r}$ to an object A of $\mathcal{C}_{\mathcal{O}}$ to be an equivalence class of liftings.

For each place $v \in S$, let $R^{\square}_{\bar{r}|_{G_{F_{\widetilde{v}}}}}$ denote the universal $\mathcal{O}$-lifting ring of $\bar{r}|_{G_{F_{\widetilde{v}}}}$ and let $R_{\widetilde{v}}$ denote a quotient of $R^{\square}_{\bar{r}|_{G_{F_{\widetilde{v}}}}}$ which satisfies the following property:

(*) let A be an object of $\mathcal{C}_{\mathcal{O}}$ and let $\zeta, \zeta' : R^{\square}_{\bar{r}|_{G_{F_{\widetilde{v}}}}} \to A$ be homomorphisms corresponding to two lifts r and r' of $\bar{r}|_{G_{F_{\widetilde{v}}}}$ which are conjugate by an element of $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$. Then ζ factors through $R_{\widetilde{v}}$ if and only if ζ' does.

We consider the *deformation problem*

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \bar{r}, \chi, \{R_{\widetilde{v}}\}_{v\in S})$$

(see sections 2.2 and 2.3 of [CHT08] for this terminology). We say that a lifting $r : G_{F^+,S} \to \mathcal{G}_n(A)$ is *of type* $\mathcal{S}$ if for each place $v \in S$, the homomorphism $R^{\square}_{\bar{r}|_{G_{F_{\widetilde{v}}}}} \to A$ corresponding to $r|_{G_{F_{\widetilde{v}}}}$ factors through $R_{\widetilde{v}}$. We also define deformations of type $\mathcal{S}$ in the same way.

Let $\mathrm{Def}_{\mathcal{S}}$ be the functor $\mathcal{C}_{\mathcal{O}} \to \mathit{Sets}$ which sends an algebra A to the set of deformations of $\bar{r}$ to A of type $\mathcal{S}$. By Proposition 2.2.9 of [CHT08] this functor is represented by an object $R^{\mathrm{univ}}_{\mathcal{S}}$ of $\mathcal{C}_{\mathcal{O}}$. In the statement of the next lemma, we note that the rings $R^{\square,\tau}_{\overline{\rho}}$ and $R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}$ are reduced and hence any union of irreducible components corresponds to a unique quotient ring.

Lemma 3.2.3. *Let M be a finite extension of $\mathbb{Q}_p$ for some prime p and $\overline{\rho} : G_M \to \mathrm{GL}_n(k)$ a continuous homomorphism. If $p \neq l$, let τ be an inertial type for G_M over K and let R be a quotient of $R^{\square,\tau}_{\overline{\rho}}$ corresponding to a union of irreducible components. If $p = l$, assume that K contains the image of every embedding $M \hookrightarrow \overline{K}$, let $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(M,K)}$ and let R be a quotient of $R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}$ corresponding to a union of irreducible components. Then R satisfies property (*) above.*

Proof. We consider the case $p = l$; the other case is similar. Let $R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}[[\underline{X}]] = R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}[[X_{ij} : 1 \leq i,j \leq n]]$ and consider the lift of $\overline{\rho}$ to $R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}[[\underline{X}]]$ given by $(1_n + (X_{ij}))\rho^{\square}(1_n + (X_{ij}))^{-1}$. This lift gives rise to an $\mathcal{O}$-algebra homomorphism $R^{\square}_{\overline{\rho}} \to R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}[[\underline{X}]]$. We claim that this homomorphism factors through $R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}$. This follows from the fact that $R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}[[\underline{X}]]$ is reduced and l-torsion-free and every $\overline{\mathbb{Q}}_l$-point of this ring gives rise to a lift of $\overline{\rho}$ which is crystalline of l-adic Hodge type $\mathbf{v}_\lambda$. Let α denote the resulting $\mathcal{O}$-algebra homomorphism $R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}} \to R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}[[\underline{X}]]$ and let $\iota : R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}} \to R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}[[\underline{X}]]$ denote the standard $R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}$-algebra structure on $R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}[[\underline{X}]]$.

The irreducible components of $\mathrm{Spec}\, R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}[[\underline{X}]]$ and $\mathrm{Spec}\, R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}$ are in natural bijection (if $\wp$ is a minimal prime of $R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}$, then $\iota(\wp)$ generates a minimal prime of $R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}[[\underline{X}]]$). Let $\wp$ be a minimal prime of $R^{\mathbf{v}_\lambda,cr}_{\overline{\rho}}$. We claim that the kernel of the map $\beta : R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr} \to R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}[[\underline{X}]]/\iota(\wp) = (R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}/\wp)[[\underline{X}]]$ induced by α is $\wp$. To see this note that the map $\gamma : R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}[[\underline{X}]] \twoheadrightarrow R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}[[\underline{X}]]/(X_{ij}) \cong R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}$ satisfies $\gamma \circ \alpha = \mathrm{id}$. From this it follows that $\ker \beta \subset \wp$. Since $\wp$ is minimal, we must have $\ker \beta = \wp$. If $\wp_1, \ldots, \wp_k$ are minimal primes of $R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}$ and $I = \wp_1 \cap \cdots \cap \wp_k$, we deduce that the kernel of the map $R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr} \stackrel{\alpha}{\to} (R_{\overline{\rho}}^{\mathbf{v}_\lambda,cr}/I)[[\underline{X}]]$ is I. The lemma follows. □

3.2.4. *A lower bound.* Let $F, F^+, S, \widetilde{S}$ and $\bar{r}$ be as in the previous section. In this section we will give a lower bound on the Krull dimension of the ring $R_{\mathcal{S}}^{\mathrm{univ}}$ for certain deformation problems $\mathcal{S}$.

For each place $v \in S$ away from l, fix an inertial type τ_v for $I_{F_{\widetilde{v}}}$ and assume that $\bar{r}|_{G_{F_{\widetilde{v}}}}$ has a lift of type τ_v (in other words, $R_{\bar{r}|_{G_{F_{\widetilde{v}}}}}^{\square,\tau_v}$ is non-zero). Let $R_{\widetilde{v}}$ be a quotient of $R_{\bar{r}|_{G_{F_{\widetilde{v}}}}}^{\square,\tau_v}$ corresponding to a union of irreducible components.

For each place $v \in S$ lying above l, let $\lambda_{\widetilde{v}}$ be an element of $(\mathbb{Z}^n_+)^{\mathrm{Hom}(F_{\widetilde{v}},K)}$, and assume that $\bar{r}|_{G_{F_{\widetilde{v}}}}$ has a crystalline lift which is ordinary of weight $\lambda_{\widetilde{v}}$ and let $R_{\widetilde{v}}$ be a quotient of the ring $R_{\bar{r}|_{G_{F_{\widetilde{v}}}}}^{\triangle_{\lambda_{\widetilde{v}}},cr}$ corresponding to a union of irreducible components. Let

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \bar{r}, \chi, \{R_{\widetilde{v}}\}_{v \in S}).$$

Write $\mathrm{ad}\,\bar{r}$ for the adjoint action of $G_{F^+,S}$ on $M_n(k)$.

Lemma 3.2.4. *Assume that $\bar{r}$ is odd, and that $H^0(G_{F^+,S}, \mathrm{ad}\,\bar{r}(1)) = \{0\}$. For $\mathcal{S}$ as above, the Krull dimension of $R_{\mathcal{S}}^{univ}$ is at least* 1.

Proof. By Corollary 2.3.5 of [CHT08] (noting that $\chi(c_v) = -1$ for all $v|\infty$) we see that this dimension is at least

$$1 + \sum_{v \in S} (\dim R_{\widetilde{v}} - n^2 - 1) - \dim_k H^0(G_{F^+,S}, \mathrm{ad}\,\bar{r}(1)) - \sum_{v|\infty} n(n-1)/2.$$

For $v \in S$ away from l, we have $\dim R_{\widetilde{v}} = n^2 + 1$ by Lemma 3.1.1. For $v \in S$ lying over l we have $\dim R_{\widetilde{v}} = n^2 + 1 + \frac{1}{2}n(n-1)[F_v^+ : \mathbb{Q}_l]$ by Lemma 3.1.7 and the remark preceding it. We therefore have

$$\begin{aligned}\sum_{v \in S} (\dim R_{\widetilde{v}} - n^2 - 1) &= \sum_{v|l} \frac{1}{2} n(n-1)[F_v^+ : \mathbb{Q}_l] \\ &= \frac{1}{2} n(n-1)[F^+ : \mathbb{Q}] \\ &= \sum_{v|\infty} n(n-1)/2,\end{aligned}$$

giving the required bound. □

3.2.5. *A finiteness result.* Let $F, F^+, S, \widetilde{S}$ and $\bar{r}$ be as in the previous two sections. Suppose that L^+/F^+ is a finite totally real extension. Let $L = L^+F$. Let S' (resp. $\widetilde{S}'$) denote a set of places of L^+ (resp. L) all of which split in L, containing all places lying over a place in S (resp. containing exactly one place above each place in S', and containing every place lying above a place in $\widetilde{S}$). Let $G_{L^+,S'} = \mathrm{Gal}(L(S')/L^+)$, where $L(S')$ is the maximal extension of L unramified outside S'. Let $G_{L,S'} = \mathrm{Gal}(L(S')/L)$. We assume that $\bar{r}|_{G_{L,S'}}$ is absolutely irreducible.

Let

$$\mathcal{S}_0 = (F/F^+, S, \widetilde{S}, \mathcal{O}, \bar{r}, \chi, \{R^{\square}_{\bar{r}|_{G_{F_{\tilde{v}}}}}\}_{v\in S})$$

and

$$\mathcal{S}'_0 = (L/L^+, S', \widetilde{S}', \mathcal{O}, \bar{r}|_{G_{L^+,S'}}, \chi|_{G_{L^+,S'}}, \{R^{\square}_{\bar{r}|_{G_{L_{\tilde{v}'}}}}\}_{v'\in S'})$$

and let $R^{\mathrm{univ}}_{\mathcal{S}_0}$ and $R^{\mathrm{univ}}_{\mathcal{S}'_0}$ denote the rings representing the functors $\mathrm{Def}_{\mathcal{S}_0}$ and $\mathrm{Def}_{\mathcal{S}'_0}$. Restricting the universal deformation valued in $R^{\mathrm{univ}}_{\mathcal{S}_0}$ to $G_{L^+,S'}$ gives $R^{\mathrm{univ}}_{\mathcal{S}_0}$ the structure of a $R^{\mathrm{univ}}_{\mathcal{S}'_0}$-algebra.

Lemma 3.2.5. *$R^{\mathrm{univ}}_{\mathcal{S}_0}$ is a finite $R^{\mathrm{univ}}_{\mathcal{S}'_0}$-algebra.*

Proof. The argument is extremely similar to that of Lemma 3.6 of [KW08]. We will follow the proof of Lemma 1.2.2 of [BLGGT10]. Write $\mathfrak{m}_{L^+}$ for the maximal ideal of $R^{\mathrm{univ}}_{\mathcal{S}'_0}$, and let r_{F^+,L^+} denote the $R^{\mathrm{univ}}_{\mathcal{S}_0}/\mathfrak{m}_{L^+}R^{\mathrm{univ}}_{\mathcal{S}_0}$-representation of $G_{F^+,S}$ obtained from the universal representation over $R^{\mathrm{univ}}_{\mathcal{S}_0}$. By definition, $r_{F^+,L^+}|_{G_{L^+,S'}}$ is equivalent to $\bar{r}|_{G_{L^+,S'}}$. As a consequence, if M denotes the normal closure of the composite of L^+ and the fixed field of $\ker \bar{r}$, then r_{F^+,L^+} factors through $\mathrm{Gal}(M/F^+)$, and the image of r_{F^+,L^+} is necessarily finite. [Alternatively, note that the image of $G_{L^+,S'}$ in $G_{F^+,S}$ has finite index.]

Let m be the order of the image of r_{F^+,L^+}, and choose elements $g_1, \ldots, g_m \in G_{F^+,S}$ whose images exhaust the image of r_{F^+,L^+}. Let

$$f(T) = \prod_{(\zeta_1,\ldots,\zeta_n)\in\mu_m(\bar{k})^n} (T - (\zeta_1 + \cdots + \zeta_n)) \in k[T]$$

and let A denote the maximal quotient of $k[X_{i,j}]_{i,j=1,\ldots,n}$ over which the m^{th}-power of the matrix $(X_{i,j})$ is 1_n. If $\wp$ is a prime ideal of A then all the roots of the characteristic polynomial of $(X_{i,j})$ over $A_\wp/\wp$ are m^{th} roots of unity and hence $f(\mathrm{tr}(X_{i,j})) = 0$ in $A/\wp \subset A_\wp/\wp$. Thus there is a positive integer a such that $f(\mathrm{tr}(X_{i,j}))^a = 0$ in A. Then we get a map

$$\begin{array}{rcl} k[T_1,\ldots,T_m]/(f(T_1)^a,\ldots,f(T_m)^a) & \longrightarrow & R^{\mathrm{univ}}_{\mathcal{S}_0}/\mathfrak{m}_{L^+}R^{\mathrm{univ}}_{\mathcal{S}_0} \\ T_i & \longmapsto & \mathrm{tr}\, r_{F^+,L^+}(g_i). \end{array}$$

By Lemma 2.1.12 of [CHT08] we see that this map has dense image. Since the source of the map is finite, we see that $R^{\mathrm{univ}}_{\mathcal{S}_0}/\mathfrak{m}_{L^+}R^{\mathrm{univ}}_{\mathcal{S}_0}$ is finite, and the result follows from the topological form of Nakayama's lemma. □

4. Ordinary automorphic representations

4.1. Ordinary automorphic representations of GL_n. Let L be either a totally real number field or a quadratic totally imaginary extension of a totally real number field. Let $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(L,\mathbb{C})}$. Let π be an automorphic representation of $GL_n(\mathbb{A}_L)$ which is

- RAESDC (regular, algebraic, essentially-self-dual, cuspidal) of weight λ if L is totally real, or
- RACSDC (regular, algebraic, conjugate-self-dual, cuspidal) of weight λ if L is totally imaginary.

See section 5 of [Tay08] or section 4 of [CHT08] for definitions of these terms. Let l be a prime number and $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ an isomorphism of fields. Let v be a place of L dividing l and ϖ_v a uniformizer in $\mathcal{O}_{L_v}$. For each $b > 0$, let $\mathrm{Iw}(v^{b,b})$ denote the open compact subgroup of $GL_n(\mathcal{O}_{L_v})$ consisting of matrices which reduce modulo v^b to a unipotent upper triangular matrix. The space $(\iota^{-1}\pi_v)^{\mathrm{Iw}(v^{b,b})}$ carries commuting actions of the scaled Hecke operators

$$U^{(j)}_{\iota^*\lambda,\varpi_v} := \left(\prod_{i=1}^{j}\prod_{\tau:L_v\hookrightarrow\overline{\mathbb{Q}}_l} \tau(\varpi_v)^{-\lambda_{\iota\tau|_L,n-i+1}}\right)\left[\mathrm{Iw}(v^{b,b})\begin{pmatrix}\varpi_v 1_j & 0\\ 0 & 1_{n-j}\end{pmatrix}\mathrm{Iw}(v^{b,b})\right]$$

for $j = 1,\dots,n$. We define the ordinary part $(\iota^{-1}\pi_v)^{\mathrm{Iw}(v^{b,b}),\mathrm{ord}}$ of $(\iota^{-1}\pi_v)^{\mathrm{Iw}(v^{b,b})}$ to be the maximal subspace which is invariant under each $U^{(j)}_{\iota^*\lambda,\varpi_v}$ and such that every eigenvalue of each $U^{(j)}_{\iota^*\lambda,\varpi_v}$ is an l-adic unit. We define

$$(\iota^{-1}\pi_v)^{\mathrm{ord}} := \varinjlim_{b>0}(\iota^{-1}\pi_v)^{\mathrm{Iw}(v^{b,b}),\mathrm{ord}}.$$

We say that π is *ι-ordinary* at v if the space $(\iota^{-1}\pi_v)^{\mathrm{ord}}$ is non-zero.

4.2. l-adic automorphic forms on definite unitary groups. Let F^+ denote a totally real number field and n a positive integer. Let F/F^+ be a totally imaginary quadratic extension of F^+ and let c denote the non-trivial element of $\mathrm{Gal}(F/F^+)$. Suppose that the extension F/F^+ is unramified at all finite places. Assume that $n[F^+ : \mathbb{Q}]$ is divisible by 4. Under this assumption, we can find a reductive algebraic group G over F^+ with the following properties:

- G is an outer form of GL_n with $G_{/F} \cong \mathrm{GL}_{n/F}$;
- for every finite place v of F^+, G is quasi-split at v;
- for every infinite place v of F^+, $G(F^+_v) \cong U_n(\mathbb{R})$.

We can and do fix a model for G over the ring of integers $\mathcal{O}_{F^+}$ of F^+ as in section 2.1 of [Ger09]. For each place v of F^+ which splits as ww^c in F there is a natural isomorphism

$$\iota_w : G(F^+_v) \xrightarrow{\sim} \mathrm{GL}_n(F_w)$$

which restricts to an isomorphism between $G(\mathcal{O}_{F^+_v})$ and $\mathrm{GL}_n(\mathcal{O}_{F_w})$. If v is a place of F^+ which splits in F and $\widetilde{v}$ is a place of F dividing v, then we let

- $\mathrm{Iw}(\widetilde{v})$ denote the subgroup of $\mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$ consisting of matrices which reduce to an upper triangular matrix modulo $\widetilde{v}$.
- $\mathrm{Iw}(\widetilde{v}^{b,c})$, for $0 \le b \le c$, denote the subgroup of $\mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$ consisting of matrices which reduce to an upper triangular matrix modulo $\widetilde{v}^c$ and to a unipotent matrix modulo $\widetilde{v}^b$. In particular $\mathrm{Iw}(\widetilde{v}^{0,0}) = \mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$.

Let $l > n$ be prime number with the property that every place of F^+ dividing l splits in F. Fix an algebraic closure $\overline{\mathbb{Q}}_l$ of $\mathbb{Q}_l$. Let K be an algebraic extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ such that every embedding $F \hookrightarrow \overline{\mathbb{Q}}_l$ has image contained in K and such that K contains a primitive l-th root of unity. Let $\mathcal{O}$ denote the ring of integers in K and k the residue field. Let S_l denote the set of places of F^+ dividing l and for each $v \in S_l$, let $\widetilde{v}$ be a place of F over v. Let $\tilde{S}_l$ be the set of $\widetilde{v}$ for $v \in S_l$. We write $\mathcal{O}_{F^+,l} := \mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_l$, $F^+_l := F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_l$.

Let W be an $\mathcal{O}$-module with an action of $G(\mathcal{O}_{F^+,l})$. Let $V \subset G(\mathbb{A}^\infty_{F^+})$ be a compact open subgroup with $v_l \in G(\mathcal{O}_{F^+,l})$ for all $v \in V$, where v_l denotes the projection of v to $G(F_l^+)$. We let $S(V,W)$ denote the space of l-adic automorphic forms on G of weight W and level V, that is, the space of functions

$$f : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty) \to W$$

with $f(gv) = v_l^{-1}f(g)$ for all $v \in V$.

Let $\widetilde{I}_l$ denote the set of embeddings $F \hookrightarrow K$ giving rise to a place in $\tilde{S}_l$. To each $\lambda \in (\mathbb{Z}_+^n)^{\widetilde{I}_l}$ we associate a finite free $\mathcal{O}$-module M_λ with a continuous action of $G(\mathcal{O}_{F^+,l})$ as in Definition 2.2.3 of [Ger09]. The representation M_λ is the tensor product over $\tau \in \widetilde{I}_l$ of the irreducible algebraic representations of GL_n of highest weights given by the λ_τ. We write $S_\lambda(V,\mathcal{O})$ instead of $S(V,M_\lambda)$ and similarly for any $\mathcal{O}$-module A, we write $S_\lambda(V,A)$ for $S(V, M_\lambda \otimes_{\mathcal{O}} A)$.

Assume from now on that K is a finite extension of $\mathbb{Q}_l$ containing a primitive l-th root of unity. Let $\mathfrak{l}$ denote the product of all places in S_l. Let R and S_a denote finite sets of finite places of F^+ disjoint from each other and from S_l and consisting only of places which split in F. Assume that $\mathbf{N}(v) \equiv 1 \mod l$ for each $v \in R$. Assume also that each $v \in S_a$ is unramified over a rational prime p with $[F(\zeta_p):F] > n$. Let $T = S_l \coprod R \coprod S_a$. For each $v \in T$ fix a place $\widetilde{v}$ of F dividing v, extending the choice of $\widetilde{v}$ for $v \in S_l$. We henceforth identify $G(F_v^+)$ with $GL_n(F_{\widetilde{v}})$ via $\iota_{\widetilde{v}}$ for $v \in T$ without comment. Let $U = \prod_v U_v$ be a compact open subgroup of $G(\mathbb{A}_{F^+}^\infty)$ with

- $U_v = G(\mathcal{O}_{F_v^+})$ if $v \notin R \cup S_a$ splits in F;
- $U_v = \mathrm{Iw}(\widetilde{v})$ if $v \in R$;
- $U_v = \ker(\mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}}) \to \mathrm{GL}_n(k(\widetilde{v})))$ if $v \in S_a$;
- U_v is a hyperspecial maximal compact subgroup of $G(F_v^+)$ if v is inert in F.

If $0 \le b \le c$, we let $U(\mathfrak{l}^{b,c}) = U^l \times \prod_{v \in S_l} \mathrm{Iw}(\widetilde{v}^{b,c})$. We note that if S_a is non-empty then U is sufficiently small (which means that its projection to $G(F_v^+)$ for some finite place v of F^+ contains no finite order elements other than the identity).

For each $v \in S_l$ fix a uniformizer $\varpi_{\widetilde{v}}$ in $\mathcal{O}_{F_{\widetilde{v}}}$. For $0 \le b \le c$ with $c > 0$ and $j = 1, \ldots, n$, consider the scaled Hecke operator

$$U_{\lambda,\varpi_{\widetilde{v}}}^{(j)} := \left(\prod_{i=1}^{j} \prod_{\tau : F_{\widetilde{v}} \hookrightarrow \overline{\mathbb{Q}}_l} \tau(\varpi_{\widetilde{v}})^{-\lambda_{\tau|_F, n-i+1}} \right) \left[\mathrm{Iw}(\widetilde{v}^{b,c}) \begin{pmatrix} \varpi_{\widetilde{v}} 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix} \mathrm{Iw}(\widetilde{v}^{b,c}) \right]$$

acting on the space $S_\lambda(U(\mathfrak{l}^{b,c}),\mathcal{O})$ (see Definition 2.3.1 of [Ger09]). We let $S_\lambda^{\mathrm{ord}}(U(\mathfrak{l}^{b,c}),\mathcal{O})$ denote the ordinary part of $S_\lambda(U(\mathfrak{l}^{b,c}),\mathcal{O})$ as defined in section 2.4 of [Ger09] (noting that the space $S_\lambda(U(\mathfrak{l}^{b,c}),\mathcal{O})$ is denoted $S_{\lambda,\{1\}}(U(\mathfrak{l}^{b,c}),\mathcal{O})$ in [Ger09]). This is the maximal submodule on which each of the operators $U_{\lambda,\varpi_{\widetilde{v}}}^{(j)}$ acts invertibly. This space is preserved by the Hecke operators

- $$T_w^{(j)} := \iota_w^{-1} \left(\left[GL_n(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix} GL_n(\mathcal{O}_{F_w}) \right] \right)$$
  for w a place of F, split over F^+ and not lying over T, $j = 1, \ldots, n$ and ϖ_w a uniformizer in $\mathcal{O}_{F_w}$, and
- $$\langle u \rangle := \prod_{v \in S_l} \left[\mathrm{Iw}(\widetilde{v}^{b,c}) \,\mathrm{diag}(u_{\widetilde{v}})\, \mathrm{Iw}(\widetilde{v}^{b,c}) \right]$$

for $u = (u_{\tilde{v}})_{v\in S_l} \in \prod_{v\in S_l}(\mathcal{O}_{F_{\tilde{v}}}^{\times})^n$.

We let $\mathbb{T}_{\lambda}^{T,\mathrm{ord}}(U(\mathfrak{l}^{b,c}),\mathcal{O})$ denote the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(S_{\lambda}^{\mathrm{ord}}(U(\mathfrak{l}^{b,c}),\mathcal{O}))$ generated by the operators $T_w^{(j)}$, $(T_w^{(n)})^{-1}$ and $\langle u\rangle$. It is commutative. We let

$$\mathbb{T}_{\lambda}^{T,\mathrm{ord}}(U(\mathfrak{l}^{\infty}),\mathcal{O}) := \varprojlim_{c} \mathbb{T}_{\lambda}^{T,\mathrm{ord}}(U(\mathfrak{l}^{c,c}),\mathcal{O}).$$

Let T_n denote the diagonal torus in GL_n. We define $T_n(\mathfrak{l})$ as the pro-l part of $T_n(\mathcal{O}_{F^+,l}) = \prod_{v\in S_l} T_n(\mathcal{O}_{F_v^+})$. In other words, we have an exact sequence

$$0 \to T_n(\mathfrak{l}) \to T_n(\mathcal{O}_{F^+,l}) \to T_n(\mathcal{O}_{F^+}/\mathfrak{l}) \to 0.$$

Define the completed group algebras

$$\begin{aligned} \Lambda^+ &:= \mathcal{O}[[T_n(\mathcal{O}_{F^+,l})]] \\ \Lambda &:= \mathcal{O}[[T_n(\mathfrak{l})]]. \end{aligned}$$

Identifying $T_n(\mathcal{O}_{F^+,l})$ with $\prod_{v\in S_l} T_n(\mathcal{O}_{F_{\tilde{v}}})$ in the natural way gives $\mathbb{T}_{\lambda}^{T,\mathrm{ord}}(U(\mathfrak{l}^{\infty}),\mathcal{O})$ the structure of a Λ^+-algebra (via the operators $\langle u\rangle$).

It is shown in section 2.6 of [Ger09] that the algebra $\mathbb{T}_{\lambda}^{T,\mathrm{ord}}(U(\mathfrak{l}^{\infty}),\mathcal{O})$ is independent of the weight λ in the sense that for each λ there is a natural isomorphism $\mathbb{T}_{\lambda}^{T,\mathrm{ord}}(U(\mathfrak{l}^{\infty}),\mathcal{O}) \cong \mathbb{T}_{0}^{T,\mathrm{ord}}(U(\mathfrak{l}^{\infty}),\mathcal{O})$. We let $\mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^{\infty}),\mathcal{O})$ denote the universal ordinary Hecke algebra as in Definition 2.6.2 of [Ger09]. By definition, this is just $\mathbb{T}_{0}^{T,\mathrm{ord}}(U(\mathfrak{l}^{\infty}),\mathcal{O})$ with a modified Λ^+-structure which is more convenient from the point of view of Galois representations.

4.3. An $R^{\mathrm{red}} = \mathbb{T}$ Theorem. Let $\mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^{\infty}),\mathcal{O})$ be the algebra introduced above. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^{\infty}),\mathcal{O})$ with residue field k which is non-Eisenstein in the sense of section 2.7 of [Ger09]. According to propositions 2.7.3 and 2.7.4 of [Ger09] one can choose a continuous homomorphism

$$\bar{r}_{\mathfrak{m}} : G_{F^+} \to \mathcal{G}_n(\mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^{\infty}),\mathcal{O})/\mathfrak{m})$$

with $\bar{r}_{\mathfrak{m}}|_{G_F}$ absolutely irreducible and a continuous lifting

$$r_{\mathfrak{m}} : G_{F^+} \to \mathcal{G}_n(\mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^{\infty}),\mathcal{O})_{\mathfrak{m}})$$

with the following properties:

(0) $r_{\mathfrak{m}}^{-1}(\mathrm{GL}_n \times \mathrm{GL}_1)(\mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^{\infty}),\mathcal{O})_{\mathfrak{m}}) = G_F$.

(1) $r_{\mathfrak{m}}$ is unramified outside T.

(2) If $v \notin T$ is a place of F^+ which splits as ww^c in F and Frob_w is the geometric Frobenius element of G_{F_w}/I_{F_w}, then $r_{\mathfrak{m}}(\mathrm{Frob}_w)$ has characteristic polynomial

$$X^n - T_w^{(1)}X^{n-1} + \ldots + (-1)^j(\mathbf{N}w)^{j(j-1)/2}T_w^{(j)}X^{n-j} + \ldots + (-1)^n(\mathbf{N}w)^{n(n-1)/2}T_w^{(n)}.$$

(3) $\nu \circ r_{\mathfrak{m}} = \epsilon^{1-n}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}$ where δ_{F/F^+} is the non-trivial character of $\mathrm{Gal}(F/F^+)$ and $\mu_{\mathfrak{m}} \in \mathbb{Z}/2\mathbb{Z}$.

(4) If $v \in R$ and $\sigma \in I_{F_{\tilde{v}}}$, then $r_{\mathfrak{m}}(\sigma)$ has characteristic polynomial $(X-1)^n$.

We make the following *assumptions*:

(a) The subgroup $\bar{r}_{\mathfrak{m}}(G_{F^+(\zeta_l)})$ of $\mathcal{G}_n(k)$ is big;

(b) For $v \in S_l \cup R$, $\bar{r}_{\mathfrak{m}}(G_{F_{\tilde{v}}}) = \{1_n\}$;

(c) The set S_a is non-empty and for $v \in S_a$, $\bar{r}_{\mathfrak{m}}|_{G_{F_{\tilde{v}}}}$ is unramified and $H^0(G_{F_{\tilde{v}}}, \mathrm{ad}\,\bar{r}_{\mathfrak{m}}(1)) = \{0\}$.

For $v \in S_l$, let

$$\Lambda_{F_{\widetilde{v}}} \quad := \quad \mathcal{O}[[I_{F_{\widetilde{v}}^{\mathrm{ab}}/F_{\widetilde{v}}}(l)^n]]$$

where for a group H, $H(l)$ denotes the pro-l completion. The inverses of the Artin maps $\mathrm{Art}_{F_{\widetilde{v}}}$ for $v \in S_l$ give rise to an isomorphism

$$\prod_{v \in S_l} (I_{F_{\widetilde{v}}^{\mathrm{ab}}/F_{\widetilde{v}}}(l))^n \xrightarrow{\sim} \prod_{v \in S_l} (1 + \varpi_{\widetilde{v}} \mathcal{O}_{F_{\widetilde{v}}})^n \cong T_n(\mathfrak{l})$$

and hence an isomorphism

$$\widehat{\otimes}_{v \in S_l} \Lambda_{F_{\widetilde{v}}} \xrightarrow{\sim} \Lambda.$$

Corollary 3.4.8 of [Ger09] shows that $r_{\mathfrak{m}}$ satisfies the following property, in addition to (0)-(4) above:

(5) For $v \in S_l$, the homomorphism $R^{\square}_{\bar{r}|_{G_{F_{\widetilde{v}}}}} \widehat{\otimes}_{\mathcal{O}} \Lambda_{F_{\widetilde{v}}} \to \mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}$ coming from $r_{\mathfrak{m}}|_{G_{F_{\widetilde{v}}}}$ and the $\Lambda_{F_{\widetilde{v}}}$-algebra structure on $\mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}$ factors through the quotient $R^{\triangle,ar}_{\bar{r}|_{G_{F_{\widetilde{v}}}}, \Lambda_{F_{\widetilde{v}}}}$ of $R^{\square}_{\bar{r}|_{G_{F_{\widetilde{v}}}}} \widehat{\otimes}_{\mathcal{O}} \Lambda_{F_{\widetilde{v}}}$ introduced in section 3.1.4.

We now turn to deformation rings. For v in R, let $R^1_{\bar{r}_{\mathfrak{m}}|_{G_{F_{\widetilde{v}}}}}$ denote the quotient of $R^{\square}_{\bar{r}_{\mathfrak{m}}|_{G_{F_{\widetilde{v}}}}}$ corresponding to lifts r for which $r(\sigma)$ has characteristic polynomial $(X-1)^n$ for each $\sigma \in I_{F_{\widetilde{v}}}$. This ring is studied in section 3 of [Tay08]. Let

$$\mathcal{S}_\Lambda = \left(F/F^+, T, \widetilde{T}, \Lambda, \bar{r}, \epsilon^{1-n}\delta^{\mu_{\mathfrak{m}}}_{F/F^+}, \{R^{\square}_{\bar{r}_{\mathfrak{m}}|_{G_{F_{\widetilde{v}}}}}\}_{v \in S_a}, \{R^1_{\bar{r}_{\mathfrak{m}}|_{G_{F_{\widetilde{v}}}}}\}_{v \in R}, \{R^{\triangle,ar}_{\bar{r}_{\mathfrak{m}}|_{G_{F_{\widetilde{v}}}}, \Lambda_{F_{\widetilde{v}}}}\}_{v \in S_l} \right)$$

Let $\mathcal{C}_\Lambda$ denote the category of complete local Noetherian Λ-algebras with residue field k. We say that a lift r of $\bar{r}$ to an object A of $\mathcal{C}_\Lambda$ is *of type* $\mathcal{S}_\Lambda$ if $\nu \circ r = \epsilon^{1-n}\delta^{\mu_{\mathfrak{m}}}_{F/F^+}$, and for each $v \in S_l$, the homomorphism $R^{\square}_{\bar{r}|_{G_{F_{\widetilde{v}}}}} \widehat{\otimes}_{\mathcal{O}} \Lambda_{F_{\widetilde{v}}} \to A$ coming from $r|_{G_{F_{\widetilde{v}}}}$ and the Λ-structure on A factors through $R^{\triangle,ar}_{\bar{r}|_{G_{F_{\widetilde{v}}}}, \Lambda_{F_{\widetilde{v}}}}$ and if for each $v \in R$ the homomorphism $R^{\square}_{\bar{r}|_{G_{F_{\widetilde{v}}}}} \to A$ coming from $r|_{G_{F_{\widetilde{v}}}}$ factors through $R^1_{\bar{r}_{\mathfrak{m}}|_{G_{F_{\widetilde{v}}}}}$. We define deformations of type $\mathcal{S}_\Lambda$ in the same way. Let $\mathrm{Def}_{\mathcal{S}_\Lambda} : \mathcal{C}_\Lambda \to Sets$ be the functor which sends an object A to the set of deformations of $\bar{r}$ to A of type $\mathcal{S}_\Lambda$. This functor is represented by an object $R^{\mathrm{univ}}_{\mathcal{S}_\Lambda}$ of $\mathcal{C}_\Lambda$.

Properties (0)-(5) above imply that the lift $r_{\mathfrak{m}}$ of $\bar{r}_{\mathfrak{m}}$ to $\mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}$ is of type $\mathcal{S}_\Lambda$ and hence gives rise to a homomorphism of Λ-algebras

$$R^{\mathrm{univ}}_{\mathcal{S}_\Lambda} \to \mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}.$$

The following result is contained in Theorem 4.3.1 of [Ger09].

Theorem 4.3.1. *The map* $R^{\mathrm{univ}}_{\mathcal{S}_\Lambda} \to \mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}$ *induces an isomorphism*

$$(R^{\mathrm{univ}}_{\mathcal{S}_\Lambda})^{\mathrm{red}} \xrightarrow{\sim} \mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}}$$

and $\mu_{\mathfrak{m}} \equiv n \mod 2$ *so that* $\bar{r}_{\mathfrak{m}}$ *is odd.*

Let $\lambda \in (\mathbb{Z}^n_+)^{\widetilde{I}_l}$ and for each $v \in S_l$, let $\lambda_{\widetilde{v}}$ denote the element of $(\mathbb{Z}^n_+)^{\mathrm{Hom}(F_{\widetilde{v}}, K)}$ given by the $\lambda_{\tau|_F}$ for $\tau : F_{\widetilde{v}} \hookrightarrow K$. In section 3.1.2 we associated to $\lambda_{\widetilde{v}}$ an n-tuple of characters $(\chi_1^{\lambda_{\widetilde{v}}}, \ldots, \chi_n^{\lambda_{\widetilde{v}}})$ from $I_{F_{\widetilde{v}}^{ab}/F_{\widetilde{v}}}$ to $\mathcal{O}^\times$. The restrictions of these characters to $I_{F_{\widetilde{v}}^{ab}/F_{\widetilde{v}}}(l)$ induce an $\mathcal{O}$-algebra homomorphism

$$\chi^{\lambda_{\widetilde{v}}} : \Lambda_{F_{\widetilde{v}}} \to \mathcal{O}$$

and taking the tensor product over the places $v \in S_l$, we get a homomorphism

$$\chi^\lambda : \Lambda \to \mathcal{O}.$$

We denote the kernels of these homomorphisms by $\wp_{\lambda_{\widetilde{v}}}$ and $\wp_\lambda$. The next result follows from Corollary 2.5.4 and Lemma 2.6.4 of [Ger09] (noting that U is sufficiently small since S_a is non-empty).

Proposition 4.3.2. *The algebra $\mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty),\mathcal{O})$ is finite and faithful as a Λ-module and for every $\lambda \in (\mathbb{Z}^n_+)^{\widetilde{I}_l}$ there is a natural surjection*

$$\mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty),\mathcal{O}) \otimes_\Lambda \Lambda_{\wp_\lambda}/\wp_\lambda \twoheadrightarrow \mathbb{T}^{T,\mathrm{ord}}_\lambda(U(\mathfrak{l}^{1,1}),\mathcal{O}) \otimes_{\mathcal{O}} K$$

whose kernel is nilpotent.

Let λ and $\lambda_{\widetilde{v}}$ for $v \in S_l$ be as above. Consider the deformation problem

$$\mathcal{S}_\lambda = (F/F^+, T, \widetilde{T}, \mathcal{O}, \bar{r}, \epsilon^{1-n}\delta^{\mu_{\mathfrak{m}}}_{F/F^+}, \{R^{\square}_{\bar{r}_{\mathfrak{m}}|_{G_{F_{\widetilde{v}}}}}\}_{v\in S_a}, \{R^{1}_{\bar{r}_{\mathfrak{m}}|_{G_{F_{\widetilde{v}}}}}\}_{v\in R}, \{R^{\triangle_{\lambda_{\widetilde{v}}},cr}_{\bar{r}_{\mathfrak{m}}|_{G_{F_{\widetilde{v}}}}}\}_{v\in S_l})$$

and the corresponding deformation ring $R^{\mathrm{univ}}_{\mathcal{S}_\lambda}$. Consider $R^{\mathrm{univ}}_{\mathcal{S}_\lambda}$ as a Λ-algebra via $\Lambda/\wp_\lambda \xrightarrow{\sim} \mathcal{O} \to R^{\mathrm{univ}}_{\mathcal{S}_\lambda}$. The universal deformation over $R^{\mathrm{univ}}_{\mathcal{S}_\lambda}$ is of type $\mathcal{S}_\Lambda$ and hence gives rise to a map $R^{\mathrm{univ}}_{\mathcal{S}_\Lambda} \to R^{\mathrm{univ}}_{\mathcal{S}_\lambda}$ which is surjective (to see that it is surjective, note that a lift $r : G_{F^+} \to \mathcal{G}_n(A)$ of type $\mathcal{S}_\Lambda$ is of type $\mathcal{S}_\lambda$ if and only if for each $v \in S_l$, the map $R^{\triangle,ar}_{\bar{r}_{\mathfrak{m}}|_{G_{F_{\widetilde{v}}}},\Lambda_{F_{\widetilde{v}}}} \to A$ corresponding to $r|_{G_{F_{\widetilde{v}}}}$ factors through $R^{\triangle_{\lambda_{\widetilde{v}}},cr}_{\bar{r}_{\mathfrak{m}}|_{G_{F_{\widetilde{v}}}}}$.)

Corollary 4.3.3. *The ring $R^{\mathrm{univ}}_{\mathcal{S}_\lambda}$ is a finite $\mathcal{O}$-algebra.*

Proof. First of all, observe that if R is an object of $\mathcal{C}_{\mathcal{O}}$, then R is a finite $\mathcal{O}$-algebra if R^{red} is. Indeed, if R^{red} is finite over $\mathcal{O}$ then $R/\mathfrak{m}_{\mathcal{O}}$ is Noetherian and 0-dimensional and hence Artinian. It follows from the topological Nakayama lemma that R is finite over $\mathcal{O}$.

The ring $(R^{\mathrm{univ}}_{\mathcal{S}_\lambda})^{\mathrm{red}}$ is a quotient of $(R^{\mathrm{univ}}_{\mathcal{S}_\Lambda})^{\mathrm{red}}/\wp_\lambda$. By Theorem 4.3.1 and Proposition 4.3.2, $(R^{\mathrm{univ}}_{\mathcal{S}_\Lambda})^{\mathrm{red}}/\wp_\lambda$ is a finite $\mathcal{O}$-algebra. The result follows. □

5. Existence of lifts

5.1. Let F be an imaginary CM field, F^+ its maximal totally real subfield and c the non-trivial element of $\mathrm{Gal}(F/F^+)$. Let π be a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and ι an isomorphism $\overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. In [CH09] it is shown that there is a semisimple representation

$$r_{l,\iota}(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

uniquely determined by the following properties:

(1) $r_{l,\iota}(\pi)^c = r_{l,\iota}(\pi)^\vee \epsilon^{1-n}$;

(2) for w a place of F not dividing l we have

$$\iota\,\mathrm{WD}(r_{l,\iota}(\pi)|_{G_{F_w}})^{\mathrm{ss}} \cong \mathrm{rec}(\pi_w \otimes |\det|^{(1-n)/2})^{\mathrm{ss}}$$

where $\mathrm{WD}(r_{l,\iota}(\pi)|_{G_{F_w}})$ denotes the Weil-Deligne representation associated to $r_{l,\iota}(\pi)|_{G_{F_w}}$ and rec is the local Langlands correspondence of [HT01];

(3) for w a place of F not dividing l, $r_{l,\iota}(\pi)|_{G_{F_w}}$ is unramified if π_w is unramified.

If F and c are as above, we let $(\mathbb{Z}^n_+)_c^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_l)}$ denote the subset of $(\mathbb{Z}^n_+)^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_l)}$ consisting of elements λ with $\lambda_{\tau c,j} = -\lambda_{\tau,n-j+1}$ for all $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ and $j = 1,\dots,n$. If $\lambda \in (\mathbb{Z}^n_+)_c^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_l)}$ and w is a place of F dividing l, we let $\lambda_w = (\lambda_{w,\sigma})_\sigma$ be the element of $(\mathbb{Z}^n_+)^{\mathrm{Hom}(F_w,\overline{\mathbb{Q}}_l)}$ determined by $\lambda_{w,\sigma} = \lambda_{\sigma|_F}$ for all $\sigma : F_w \hookrightarrow \overline{\mathbb{Q}}_l$.

The representation $r_{l,\imath}(\pi)$ can be conjugated to take values in $\mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$. Reducing modulo the maximal ideal of $\mathcal{O}_{\overline{\mathbb{Q}}_l}$ and semisimplifying, one obtains a representation $\bar{r}_{l,\imath}(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ which is independent of any choices made. Furthermore, if $r_{l,\imath}(\pi)$ is irreducible, then by Lemma 2.1.4 of [CHT08], one can extend $r_{l,\imath}(\pi)$ to a continuous representation $r : G_{F^+} \to \mathcal{G}_n(\overline{\mathbb{Q}}_l)$ with $r|_{G_F} = (r_{l,\imath}(\pi), \epsilon^{1-n})$ and $G_F = r^{-1}(\mathrm{GL}_n(\overline{\mathbb{Q}}_l) \times \overline{\mathbb{Q}}_l^\times)$. After conjugating by an element of $\mathrm{GL}_n(\overline{\mathbb{Q}}_l) \subset \mathcal{G}_n(\overline{\mathbb{Q}}_l)$, we may assume that the extension r takes values in $\mathcal{G}_n(\mathcal{O}_K)$ for some finite extension $K/\mathbb{Q}_l$ (see Lemma 2.1.5 of [CHT08]) .

If K (resp. k) is an algebraic extension of $\mathbb{Q}_l$ (resp. $\mathbb{F}_l$) and $\rho : G_F \to \mathrm{GL}_n(K)$ (resp. $\overline{\rho} : G_F \to \mathrm{GL}_n(k)$) is a continuous representation, we say that ρ (resp. $\overline{\rho}$) is *automorphic* if there exists a π and $\imath$ as above with $r_{l,\imath}(\pi)$ (resp. $\bar{r}_{l,\imath}(\pi)$) isomorphic to $\rho \otimes_K \overline{\mathbb{Q}}_l$ (resp. $\overline{\rho} \otimes_k \overline{\mathbb{F}}_l$). We say that ρ (or $\overline{\rho}$) is *ordinarily automorphic* if in addition π and $\imath$ can be chosen so that π is $\imath$-ordinary at every place dividing l. We say that ρ is *ordinary automorphic* of weight $\lambda \in (\mathbb{Z}^n_+)_c^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_l)}$ if ρ is automorphic and $\rho|_{G_{F_w}}$ is ordinary of weight $\lambda_w \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F_w,\overline{\mathbb{Q}}_l)}$ for each place $w|l$ of F. We say that ρ is *ordinary automorphic* if it is ordinary automorphic of some weight. If ρ is ordinarily automorphic and its reduction $\overline{\rho}$ is absolutely irreducible, then ρ is ordinary automorphic by Proposition 5.3.1 of [Ger09].

We are now ready to prove our main theorem. For the convenience of the reader, we recall all our assumptions in the statement of the theorem.

Theorem 5.1.1. *Let F be an imaginary CM field with maximal totally real subfield F^+. Let $n \geq 2$ be an integer and $l > n$ a prime number. Assume that the extension F/F^+ is split at all places dividing l. Suppose that*

$$\overline{\rho} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$$

is an irreducible representation satisfying the following assumptions.

1. *The representation $\overline{\rho}$ is ordinarily automorphic (so in particular $\overline{\rho}^c \cong \overline{\rho}^\vee \bar{\epsilon}^{1-n}$); say $\overline{\rho} \cong \bar{r}_{l,\imath}(\pi)$.*
2. *Any place of F at which $\overline{\rho}$ is ramified splits over F^+.*
3. *The image $\overline{\rho}(G_{F(\zeta_l)})$ is big.*
4. *$\overline{F}^{\ker \mathrm{ad}\,\overline{\rho}}$ does not contain $F(\zeta_l)$ (so in particular, $\zeta_l \notin F$).*
5. *There is an element $\lambda \in (\mathbb{Z}^n_+)_c^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_l)}$ such that for every place $w|l$ of F, $\overline{\rho}|_{G_{F_w}}$ has a crystalline lift which is ordinary of weight λ_w.*

Then $\overline{\rho}$ has a lift ρ which is crystalline and ordinary of weight λ_w at each place w of F dividing l, and which is ordinarily automorphic of level prime to l.

In fact, suppose we are given a finite set of places S of F^+ which split in F, a choice of a place $\tilde{v}$ of F above each place v of F^+, and an inertial type $\tau_{\tilde{v}}$ for $I_{F_{\tilde{v}}}$ for each $v \in S$ not dividing l such that $\overline{\rho}|_{G_{F_{\tilde{v}}}}$ has a lift of type $\tau_{\tilde{v}}$. Then ρ can be chosen to be of type $\tau_{\tilde{v}}$ at $\tilde{v}$ for all places $v \in S$, $v \nmid l$. More precisely, choose a model $r : G_{F^+} \to \mathcal{G}_n(\mathcal{O}_K)$ of an extension to $\mathcal{G}_n(\overline{\mathbb{Q}}_l)$ of $r_{l,\imath}(\pi)$, where $K/\mathbb{Q}_l(\zeta_l)$ is a finite extension in $\overline{\mathbb{Q}}_l$ which contains the image of every embedding $F \hookrightarrow \overline{\mathbb{Q}}_l$.

Assume moreover that each $\tau_{\widetilde{v}}$ is defined over K. Then, given a choice of irreducible component of each $\mathcal{O}_K$-lifting ring $R^{\square,\tau_{\widetilde{v}}}_{\bar{r}|_{G_{F_{\widetilde{v}}}}}$ (resp. $R^{\triangle_{\lambda_{\widetilde{v}}},cr}_{\bar{r}|_{G_{F_{\widetilde{v}}}}}$) with $v \in S$ and $v \nmid l$ (resp. $v|l$), we may choose ρ so as to give a point on each of these components and, if S contains all places at which $\overline{\rho}$ is ramified, we may choose ρ to be automorphic of level dividing $S \setminus \{v|l\}$.

Proof. It suffices to prove the final statement. Let ι, π, r and K be as in the final statement and let $\mathcal{O} = \mathcal{O}_K$ and $k = \mathcal{O}/\mathfrak{m}_{\mathcal{O}}$. By a slight abuse of notation, we will let $\overline{\rho}$ denote the representation $G_F \to \mathrm{GL}_n(k)$ obtained from $\bar{r}|_{G_F}$. Extending S if necessary, we may assume that S contains all places above l and that $\overline{\rho}$ is unramified away from S. Indeed, for the places $v \nmid l$ just added to S, the lift $r|_{G_{F_{\widetilde{v}}}}$ determines an inertial type $\tau_{\widetilde{v}}$ for $I_{F_{\widetilde{v}}}$ and at least one irreducible component of $R^{\square,\tau_{\widetilde{v}}}_{\bar{r}|_{G_{F_{\widetilde{v}}}}}$. For the places $v|l$ just added to S, assumption (5) guarantees that $R^{\triangle_{\lambda_{\widetilde{v}}},cr}_{\bar{r}|_{G_{F_{\widetilde{v}}}}}$ is non-zero and hence we can choose an irreducible component of this ring.

By Lemma 2.1.4 of [CHT08], $\nu \circ \bar{r} = \bar{\epsilon}^{1-n}\delta^{\mu}_{F/F^+}$ for some $\mu \in \mathbb{Z}/2\mathbb{Z}$, where δ_{F/F^+} is the quadratic character of G_{F^+} corresponding to F. By Theorem 4.3.1, $\bar{r}$ is odd, so in fact $\mu \equiv n \mod 2$. For each $v \in S$, let $R_{\widetilde{v}}$ be the chosen irreducible component of $R^{\square,\tau_{\widetilde{v}}}_{\overline{\rho}|_{G_{F_{\widetilde{v}}}}}$ when $v \nmid l$ or $R^{\triangle_{\lambda_{\widetilde{v}}},cr}_{\overline{\rho}|_{G_{F_{\widetilde{v}}}}}$ when $v|l$. Let $\widetilde{S}$ denote the set of $\widetilde{v}$ for $v \in S$ and let

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \bar{r}, \epsilon^{1-n}\delta^{\mu}_{F/F^+}, \{R_{\widetilde{v}}\}_{v\in S}).$$

To prove the theorem it suffices to show that we can find a closed point of $R^{\mathrm{univ}}_{\mathcal{S}}[1/l]$ so that the corresponding representation restricted to G_F is automorphic of level dividing $S \setminus \{v|l\}$.

Choose a finite place v_1 of F not lying over S so that

- v_1 is unramified over a rational prime p with $[F(\zeta_p) : F] > n$;
- v_1 does not split completely in $F(\zeta_l)$;
- $\mathrm{ad}\,\bar{r}(\mathrm{Frob}_{v_1}) = 1$.

The last two conditions imply that $H^0(G_{F_{v_1}}, \mathrm{ad}\,\bar{r}(1)) = \{0\}$. Choose a CM extension L of F with the following properties:

- L/F is Galois and soluble;
- L is linearly disjoint from $\overline{F}^{\ker(\mathrm{ad}\,\bar{r})}(\zeta_l)$ over F;
- all primes of L lying above S or $\{v_1\}$ are split over L^+ where L^+ is the maximal totally real subfield of L;
- the extension L/L^+ is unramified at all finite places;
- $4|[L^+ : F^+]$;
- for each place $\widetilde{v} \in \widetilde{S}$ away from l and each place w of L lying over $\widetilde{v}$, we have $\mathbf{N}w \equiv 1 \mod l$, $\overline{\rho}(G_{L_w}) = \{1_n\}$, the type $\tau_{\widetilde{v}}$ becomes trivial upon restriction to I_{L_w} and if π_L denotes the base change of π to L, then $(\pi_L)^{\mathrm{Iw}(w)}_w \neq 0$.
- the places $\{v_1, cv_1\}$ split completely in L;
- for each place $\widetilde{v} \in \widetilde{S}$ dividing l and each place w of L lying over $\widetilde{v}$ we have $\overline{\rho}(G_{L_w}) = \{1_n\}$.
- if w is a place of L not lying over l such that $(\pi_L)_w$ is ramified, then w lies over a place of L^+ which splits in L, and $(\pi_L)^{\mathrm{Iw}(w)}_w \neq 0$.

Let T denote the set of places of L^+ comprised of those lying above $S \cup \{v_1|_{F^+}\}$, together with any places of L^+ over which there is a place w of L with $(\pi_L)_w$ ramified. Let $\widetilde{T}$ denote a set of places of L, containing all places lying above $\widetilde{S} \cup \{v_1\}$, such that $\widetilde{T}$ consists of one place $\widetilde{w}$ for each place $w \in T$. For each $\widetilde{w} \in \widetilde{T}$ lying above v_1, let $R_{\widetilde{w}} = R^{\square}_{\overline{\rho}|_{G_{L_{\widetilde{w}}}}}$. For $\widetilde{w} \in \widetilde{T}$ not dividing l or v_1, let $R_{\widetilde{w}}$ denote the quotient $R^{1}_{\overline{\rho}|_{G_{L_{\widetilde{w}}}}}$ of $R^{\square}_{\overline{\rho}|_{G_{L_{\widetilde{w}}}}}$ corresponding to lifts for which each element of inertia has characteristic polynomial $(X-1)^n$. Let λ_L be the element of $(\mathbb{Z}^n_+)_c^{\mathrm{Hom}(L,\overline{\mathbb{Q}}_l)}$ determined by $(\lambda_L)_\tau = \lambda_{\tau|_F}$ for all $\tau : L \hookrightarrow \overline{\mathbb{Q}}_l$. Extend K if necessary so that it contains the image of every embedding $L \hookrightarrow \overline{\mathbb{Q}}_l$. For $\widetilde{w} \in \widetilde{T}$ lying above l, let $R_{\widetilde{w}} = R^{\triangle_{(\lambda_L)_{\widetilde{w}}}, cr}_{\overline{\rho}|_{G_{L_{\widetilde{w}}}}}$. Let

$$\mathcal{S}' = (L/L^+, T, \widetilde{T}, \mathcal{O}, \bar{r}|_{G_{L^+,T}}, \epsilon^{1-n}\delta^{\mu}_{L/L^+}, \{R_{\widetilde{w}}\}_{w\in T}).$$

Restricting the universal deformation over $R^{\mathrm{univ}}_{\mathcal{S}}$ to $G_{L^+,T}$ gives rise to a map $R^{\mathrm{univ}}_{\mathcal{S}'} \to R^{\mathrm{univ}}_{\mathcal{S}}$ and by Lemma 3.2.5, this map is finite (this lemma shows the finiteness of the corresponding map of unrestricted deformation rings, and the finiteness of the map $R^{\mathrm{univ}}_{\mathcal{S}'} \to R^{\mathrm{univ}}_{\mathcal{S}}$ follows by taking the appropriate quotients).

Now, let G be a reductive group over $\mathcal{O}_{L^+}$ as in section 4.2 (with L^+ replacing F^+). By Théorème 5.4 and Corollaire 5.3 of [Lab09] and the assumption that L/L^+ is unramified at all finite places, π_L is the strong base change of an automorphic representation Π of $G(\mathbb{A}_{L^+})$. By Lemma 5.1.6 of [Ger09] π_L is ι-ordinary at each place of L dividing l. Let $U \subset G(\mathbb{A}^{\infty}_{L^+})$ be a compact open subgroup defined as in section 4.2 with S_a the set of places of T above $v_1|_{F^+}$ and R the set of places of T not dividing l and not in S_a. Then extending $\mathcal{O}$ if necessary, the Hecke eigenvalues on $(\iota^{-1}\Pi^{l,\infty})^{U^l} \otimes \bigotimes_{v|l}(\iota^{-1}\Pi_v)^{\mathrm{ord}}$ give rise to an $\mathcal{O}$-algebra homomorphism $\mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty), \mathcal{O}) \to \mathcal{O}$. Reducing this modulo the maximal ideal of $\mathcal{O}$ gives a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty), \mathcal{O})$ which is non-Eisenstein by the second of our conditions on L above. All of the hypotheses of section 4.3 are satisfied and we deduce from Corollary 4.3.3 that $R^{\mathrm{univ}}_{\mathcal{S}'}$ is finite over $\mathcal{O}$. Theorem 4.3.1 and Proposition 4.3.2 imply that every closed point of $R^{\mathrm{univ}}_{\mathcal{S}'}[1/l]$ gives rise to a representation of G_L which is ordinarily automorphic of level dividing T.

Since $R^{\mathrm{univ}}_{\mathcal{S}}$ is finite over $\mathcal{O}$ and has Krull dimension at least one by Lemma 3.2.4, the ring $R^{\mathrm{univ}}_{\mathcal{S}}[1/l]$ is non-zero. Any closed point on this ring gives rise to a crystalline ordinary representation ρ of G_F which is ordinarily automorphic of level dividing T upon restriction to G_L. By Lemma 1.4 of [BLGHT09] any such ρ is automorphic and hence, by Lemma 5.1.6 of [Ger09], is in fact ordinarily automorphic. The fact that $\rho|_{G_L}$ is automorphic of level dividing T implies that ρ is automorphic of level dividing $S \cup \{v_1, v_1^c\}$. (Suppose $\rho \cong r_{l,\imath}(\pi')$. Then for $v \notin S \cup \{v_1, v_1^c\}$ a prime of F and $w|v$ a prime of L, we have that $\mathrm{BC}_{L_w/F_v}(\pi'_v)$ is unramified since $w \notin T$. In particular, the monodromy operator N is zero on $\mathrm{rec}(\pi'_v)$. We deduce from property (2) of $r_{l,\imath}(\pi')$ (at the beginning of section 5) that $\iota\,\mathrm{WD}(\rho|_{G_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}(\pi'_v \otimes |\det|^{(1-n)/2})$, where F-ss denotes Frobenius semisimplification. Since ρ is unramified at v, we see that π'_v is unramified.) By varying the choice of v_1, we see that ρ must in fact be automorphic of level dividing S. Finally, Theorem 5.3.2 of [Ger09] implies that ρ is automorphic of level prime to l. $\square$

We can frequently make this rather more explicit.

Corollary 5.1.2. *Let F be an imaginary CM field with maximal totally real subfield F^+. Let $n \geq 2$ be an integer and $l > n$ a prime number. Suppose that $\zeta_l \notin F$. Assume that the extension F/F^+ is split at all places dividing l. Let $\tilde{S}_l$ be a set of places of F lying over l, containing exactly one place above each place of F^+ dividing l. Suppose that*

$$\overline{\rho} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$$

is an irreducible representation satisfying the following assumptions.

1. *The representation $\overline{\rho}$ is ordinarily automorphic (so in particular $\overline{\rho}^c = \overline{\rho}^\vee \epsilon^{1-n}$).*
2. *Any place of F at which $\overline{\rho}$ is ramified splits over F^+.*
3. *The image $\overline{\rho}(G_{F(\zeta_l)})$ is big.*
4. *$\overline{F}^{\ker \mathrm{ad}\, \overline{\rho}}$ does not contain $F(\zeta_l)$.*
5. *There is an element $\lambda \in (\mathbb{Z}^n_+)_c^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_l)}$ such that for every place $\tilde{v} \in \tilde{S}_l$, $\overline{\rho}|_{G_{F_{\tilde{v}}}}$ is isomorphic to a representation*

$$\begin{pmatrix} \overline{\mu}_{\tilde{v},1} & * & \dots & * & * \\ 0 & \overline{\mu}_{\tilde{v},2} & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \overline{\mu}_{\tilde{v},n-1} & * \\ 0 & 0 & \dots & 0 & \overline{\mu}_{\tilde{v},n} \end{pmatrix}$$

   where $\overline{\mu}_{\tilde{v},i}|_{I_{F_{\tilde{v}}}} = \overline{\chi}_i^{\lambda_{\tilde{v}}}|_{I_{F_{\tilde{v}}}}$ (where $\chi_i^{\lambda_{\tilde{v}}}$ is the crystalline character of Definition 3.1.2), and for each $i < j$ we have $\overline{\mu}_{\tilde{v},i}\overline{\mu}_{\tilde{v},j}^{-1} \neq \overline{\epsilon}$.

Then $\overline{\rho}$ has an ordinarily automorphic lift (of level prime to l) ρ which is crystalline and ordinary of weight λ_w at each place $w|l$; furthermore for each place $\tilde{v} \in \tilde{S}_l$, $\rho|_{G_{F_{\tilde{v}}}}$ is isomorphic to a representation

$$\begin{pmatrix} \psi_{\tilde{v},1} & * & \dots & * & * \\ 0 & \psi_{\tilde{v},2} & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \psi_{\tilde{v},n-1} & * \\ 0 & 0 & \dots & 0 & \psi_{\tilde{v},n} \end{pmatrix}$$

with $\psi_{\tilde{v},i}$ a lift of $\overline{\mu}_{\tilde{v},i}$ agreeing with $\chi_i^{\lambda_{\tilde{v}}}$ on $I_{F_{\tilde{v}}}$.

Proof. This is an immediate consequence of Theorem 5.1.1, Lemma 3.1.5 and Lemma 3.1.8. □

We now state a corollary to the proof of Theorem 5.1.1, which we will use in section 7.

Corollary 5.1.3. *Let $l, n, F, F^+, \overline{\rho}, \pi, \bar{r}, S, \widetilde{S}, \{\tau_{\widetilde{v}}\}_{v \in S, v \nmid l}, \{\lambda_{\widetilde{v}}\}_{v \in S_l}$ and $\{R_{\widetilde{v}}\}_{v \in S}$ be as in Theorem 5.1.1 and its proof. Let*

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \bar{r}, \epsilon^{1-n}\delta^n_{F/F^+}, \{R_{\widetilde{v}}\}_{v \in S}).$$

Then $R^{\mathrm{univ}}_{\mathcal{S}}$ is a finite $\mathcal{O}$-module of rank at least 1.

6. Serre weights

6.1. We now put ourselves in the setting of section 4.2. In particular, we let F^+ denote a totally real number field and n a positive integer. Let F/F^+ be a totally imaginary quadratic extension of F^+ and let c denote the non-trivial element of $\mathrm{Gal}(F/F^+)$. Suppose that the extension F/F^+ is unramified at all finite places. Assume that $n[F^+:\mathbb{Q}]$ is divisible by 4. We note that we could make somewhat weaker assumptions, but the necessity of considering definite unitary groups which fail to be quasi-split at some finite places would complicate the notation unnecessarily.

Let G be the reductive algebraic group over F^+ defined in section 4.2, together with a fixed model over $\mathcal{O}_{F^+}$ as before. Again, we take a prime number $l>n$ so that every place in the set S_l of places of F^+ dividing l splits in F. Fix a set $\tilde{S}_l$ of places of F consisting of exactly one place above each place in S_l. Let $\mathcal{O}$ be the ring of integers of $\overline{\mathbb{Q}}_l$, with residue field $\overline{\mathbb{F}}_l$. Let $\tilde{I}_l$ denote the set of embeddings $F\hookrightarrow\overline{\mathbb{Q}}_l$ giving rise to one of the places $\tilde{v}\in\tilde{S}_l$. Let $\tilde{I}_{\tilde{v}}$ denote the subset of $\tilde{I}_l$ giving rise to $\tilde{v}$. Let the residue field of $F_{\tilde{v}}$ be $k(\tilde{v})$. Then any element $\sigma\in\tilde{I}_{\tilde{v}}$ induces an embedding $\overline{\sigma}:k(\tilde{v})\hookrightarrow\overline{\mathbb{F}}_l$. For an embedding $\tau:k(\tilde{v})\hookrightarrow\overline{\mathbb{F}}_l$, we let $\tilde{I}_\tau$ denote the subset of $\tilde{I}_{\tilde{v}}$ consisting of the σ with $\overline{\sigma}=\tau$. We let $\overline{I}_l$ be the set of embeddings $\tau:k(\tilde{v})\hookrightarrow\overline{\mathbb{F}}_l$ (running over all v).

Define $(\mathbb{Z}^n_+)^{\tilde{I}_l}$ as in section 4.2. Let $(\mathbb{Z}^n_{+,\mathrm{res}})^{\overline{I}_l}$ be the subset of $(\mathbb{Z}^n)^{\overline{I}_l}$ consisting of λ with $l-1\geq\lambda_{\tau,i}-\lambda_{\tau,i+1}\geq 0$ for all τ and all $i=1,\dots,n-1$. Let $(\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$ denote the subset of $(\mathbb{Z}^n_+)^{\tilde{I}_l}$ consisting of weights λ with the property that for each $\tilde{v}$ and $\tau:k(\tilde{v})\hookrightarrow\overline{\mathbb{F}}_l$, it is possible to write $\tilde{I}_\tau=\{\sigma_{\tau,1},\dots,\sigma_{\tau,e}\}$ with $\lambda_{\sigma_{\tau,i},j}=0$ if $i>1$ and $l-1\geq\lambda_{\sigma_{\tau,1},j}-\lambda_{\sigma_{\tau,1},j+1}\geq 0$ for all $j=1,\dots,n-1$. This being the case, we define $\lambda_{\tau,j}:=\lambda_{\sigma_{\tau,1},j}$. In this way, we define a surjective map π from $(\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$ to $(\mathbb{Z}^n_{+,\mathrm{res}})^{\overline{I}_l}$.

Fix $\lambda\in(\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$. We now consider the finite free $\mathcal{O}$-module M_λ of Definition 2.2.3 of [Ger09]. This has a continuous action of $G(\mathcal{O}_{F^+,l})=\prod_{v\in S_l}\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$. The action on $M_\lambda\otimes\overline{\mathbb{F}}_l$ factors through $\prod_{v\in S_l}\mathrm{GL}_n(k(\tilde{v}))$.

Let S_a be a nonempty set of finite places of F^+, disjoint from S_l, such that any place v of S_a splits in F and is unramified over a rational prime p with $[F(\zeta_p):F]>n$. Choose a place $\tilde{v}$ of F lying over each $v\in S_a$. Let $U=\prod_v U_v$ be a compact open subgroup of $G(\mathbb{A}^\infty_{F^+})$ such that

- $U_v\subset G(\mathcal{O}_{F^+_v})$ for all v which split in F;
- $U_v=\iota_{\tilde{v}}^{-1}\ker(\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}})\to\mathrm{GL}_n(k(\tilde{v})))$ if $v\in S_a$;
- $U_v=G(\mathcal{O}_{F^+_v})$ if $v|l$;
- U_v is a hyperspecial maximal compact subgroup of $G(F^+_v)$ if v is inert in F.

Then (because S_a is nonempty) U is sufficiently small, and

$$S_\lambda(U,\mathcal{O})\otimes\overline{\mathbb{F}}_l=S(U,M_\lambda)\otimes\overline{\mathbb{F}}_l=S(U,M_\lambda\otimes\overline{\mathbb{F}}_l).$$

Let T be a finite set of finite places of F^+ which split in F, containing S_l and all the places v which split in F for which $U_v\neq G(\mathcal{O}_{F^+_v})$. We let $\mathbb{T}^{T,univ}_\lambda$ be the commutative $\mathcal{O}$-polynomial algebra generated by formal variables $T^{(j)}_w$ for all $1\leq j\leq n$, w a place of F lying over a place v of F^+ which splits in F and is not contained in T, together with variables $T^{(j)}_{\lambda,\tilde{v}}$ for all $v \in S_l$ and $j = 1, \ldots, n$. Let $\mathbb{T}^{T,univ}$ denote the subalgebra generated by the operators $T^{(j)}_w$. We now fix a uniformiser $\varpi_{\tilde{v}}$ of $\mathcal{O}_{F_{\tilde{v}}}$ for each $v \in S_l$. The algebra $\mathbb{T}^{T,univ}_\lambda$ acts on $S(U, M_\lambda)$ via the following Hecke operators:

- $$T^{(j)}_w := \iota_w^{-1}\left[GL_n(\mathcal{O}_{F_w})\begin{pmatrix}\varpi_w 1_j & 0\\ 0 & 1_{n-j}\end{pmatrix}GL_n(\mathcal{O}_{F_w})\right]$$
  for w not lying over a place in T and ϖ_w a uniformiser in $\mathcal{O}_{F_w}$, and
- $$T^{(j)}_{\lambda,\tilde{v}} := \left(\prod_{i=1}^{j}\prod_{\tau: F_{\tilde{v}} \hookrightarrow \overline{\mathbb{Q}}_l}\tau(\varpi_{\tilde{v}})^{-\lambda_{\tau|_F, n-i+1}}\right)\iota_{\tilde{v}}^{-1}\left[GL_n(\mathcal{O}_{F_{\tilde{v}}})\begin{pmatrix}\varpi_{\tilde{v}} 1_j & 0\\ 0 & 1_{n-j}\end{pmatrix}GL_n(\mathcal{O}_{F_{\tilde{v}}})\right].$$
  for $v \in S_l$.

Let $T_{\lambda,l} = \prod_{v|l}\prod_{j=1}^n T^{(j)}_{\lambda,\tilde{v}}$. Then there is a (unique) $\mathbb{T}^{T,univ}_\lambda$-stable decomposition $S(U, M_\lambda) = S^{\mathrm{ord}}(U, M_\lambda) \oplus S^{\text{non-ord}}(U, M_\lambda)$ with $T_{\lambda,l}$ being topologically nilpotent on $S^{\text{non-ord}}(U, M_\lambda)$ and every eigenvalue of $T_{\lambda,l}$ on $S^{\mathrm{ord}}(U, M_\lambda)$ being an l-adic unit. Suppose that $\mathfrak{m}$ is a maximal ideal of $\mathbb{T}^{T,univ}$ with residue field $\overline{\mathbb{F}}_l$ such that $S^{\mathrm{ord}}(U, M_\lambda)_\mathfrak{m} \neq 0$. Then, by Proposition 2.7.3 of [Ger09], there is a continuous semisimple representation

$$\bar{r}_\mathfrak{m} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$$

naturally associated to $\mathfrak{m}$. We have the following definition.

Definition 6.1.1. Suppose that $\overline{\rho} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is a continuous irreducible representation. Then we say that $\overline{\rho}$ is *modular and ordinary of weight* $\lambda \in (\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$ if there is a U, T as above and a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{T,univ}$ with residue field $\overline{\mathbb{F}}_l$ such that

- $S^{\mathrm{ord}}(U, M_\lambda)_\mathfrak{m} \neq 0$, and
- $\overline{\rho} \cong \bar{r}_\mathfrak{m}$.

We say that $\overline{\rho}$ is *modular and ordinary* if it is modular and ordinary of some weight $\lambda \in (\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$.

Note in particular that if $\overline{\rho}$ is modular and ordinary then it is unramified at any place of F which doesn't split over F^+. We have the following theorem.

Theorem 6.1.2. *Suppose that*

$$\overline{\rho} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$$

is an irreducible representation satisfying the following assumptions.

1. *The representation $\overline{\rho}$ is modular and ordinary (so in particular $\overline{\rho}^c = \overline{\rho}^\vee\epsilon^{1-n}$).*
2. *The image $\overline{\rho}(G_{F(\zeta_l)})$ is big.*
3. *$\overline{F}^{\ker \mathrm{ad}\,\overline{\rho}}$ does not contain $F(\zeta_l)$.*

Then $\overline{\rho}$ is modular and ordinary of weight $\lambda \in (\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$ if and only if

- *For every place $v|l$ of F^+, $\overline{\rho}|_{G_{F_{\tilde{v}}}}$ has a crystalline lift which is ordinary of weight $\lambda_{\tilde{v}}$.*

Proof. Suppose firstly that $\overline{\rho}$ is modular and ordinary of weight $\lambda \in (\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$. Then by definition we see that there is a U, T, $\mathfrak{m}$ as above such that $S^{\mathrm{ord}}(U, M_\lambda)_{\mathfrak{m}} \neq 0$ and $\bar{r}_{\mathfrak{m}} \cong \overline{\rho}$. Choose an isomorphism $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ and let $\iota_*\lambda$ be the element of $(\mathbb{Z}^n_+)^{\mathrm{Hom}(F,\mathbb{C})}$ with $(\iota_*\lambda)_{\iota\circ\tau,j}$ equal to $\lambda_{\tau,j}$ if $\tau \in \widetilde{I}_l$ and $-\lambda_{\tau,n-j+1}$ if $\tau c \in \widetilde{I}_l$. We see by Corollaire 5.3 of [Lab09] and Lemmas 2.2.5 and 2.7.6 of [Ger09] that there is a RACSDC representation π of weight $\iota_*\lambda$ of $\mathrm{GL}_n(\mathbb{A}_F)$ which is unramified at l, ι-ordinary at all $v|l$ and which satisfies $\bar{r}_{l,\iota}(\pi) \cong \overline{\rho}$. (The cuspidality of π follows from the irreducibility of $\overline{\rho}$.) Thus, for each prime $v|l$ of F^+, $r_{l,\iota}(\pi)|_{G_{F_{\tilde{v}}}}$ is crystalline and ordinary of weight $\lambda_{\tilde{v}}$ and provides the required lift of $\overline{\rho}|_{G_{F_{\tilde{v}}}}$.

For the converse, if the condition holds then by Theorem 5.1.1, $\overline{\rho}$ has a lift to a representation ρ which is crystalline and ordinary of weight λ and ordinarily automorphic of level dividing a (finite) set of places which split over F^+ and don't divide l. The result now follows from Corollaire 5.3 and Théorème 5.4 of [Lab09], the assumption that F/F^+ is unramified at all finite places and Lemmas 2.2.5 and 2.7.6 of [Ger09]. □

We now show that if $\overline{\rho}$ is modular and ordinary of weight $\lambda \in (\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$, then it is modular of weight $\pi(\lambda)$ in the sense of generalisations of Serre's conjecture (cf. [Her09]). This is a straightforward consequence of the elementary calculations underlying Hida theory, as we now explain.

Let v_λ be the rank one $\mathcal{O}$-submodule of M_λ on which the usual maximal torus of GL_n acts via the highest weight λ. Let $v_{w_0\lambda}$ be the rank one $\mathcal{O}$-submodule of M_λ on which the usual maximal torus of GL_n acts via the lowest weight $w_0\lambda$.

The irreducible $\overline{\mathbb{F}}_l$-representations of $\prod_{v\in S_l}\mathrm{GL}_n(k(\tilde{v}))$ are tensor products of irreducible representations of the $\mathrm{GL}_n(k(\tilde{v}))$. From the standard classification of the irreducible $\overline{\mathbb{F}}_l$-representations of $\mathrm{GL}_n(k(\tilde{v}))$ (see for example the appendix to [Her09]), one sees that:

1. There is an irreducible $\overline{\mathbb{F}}_l$-representation F_λ of $\prod_{v\in S_l}\mathrm{GL}_n(k(\tilde{v}))$ for each $\lambda \in (\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$, and every irreducible $\overline{\mathbb{F}}_l$-representation of $\prod_{v\in S_l}\mathrm{GL}_n(k(\tilde{v}))$ is equivalent to some F_λ.
2. Take $\lambda \in (\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$. Let P_λ be the sub-$\prod_{v\in S_l}\mathrm{GL}_n(k(\tilde{v}))$-representation of $M_\lambda \otimes \overline{\mathbb{F}}_l$ generated by $v_\lambda \otimes \overline{\mathbb{F}}_l$. Then $P_\lambda \cong F_{\pi(\lambda)}$ (since both representations are obtained by restriction from the corresponding algebraic group, it suffices to establish the analogous result for representations of GL_n, for which see II.8.8(1) of [Jan03]).
3. P_λ contains $v_{w_0\lambda} \otimes \overline{\mathbb{F}}_l$.

For $\lambda \in (\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$, we have an exact sequence

$$0 \to S(U, P_\lambda) \to S(U, M_\lambda \otimes \overline{\mathbb{F}}_l) \to S(U, (M_\lambda \otimes \overline{\mathbb{F}}_l)/P_\lambda) \to 0$$

of $\mathbb{T}^{T,univ}$-modules (as U is sufficiently small). Let $T_{\lambda,l} = \prod_{v|l}\prod_{j=1}^n T^{(j)}_{\lambda,\tilde{v}}$, regarded as an endomorphism of $S(U, M_\lambda \otimes \overline{\mathbb{F}}_l)$. We claim that $T_{\lambda,l}$ preserves $S(U, P_\lambda)$ and is zero on the quotient $S(U, (M_\lambda \otimes \overline{\mathbb{F}}_l)/P_\lambda)$. To see this, let $\alpha^{(j)}_{\varpi_{\tilde{v}}}$ denote the matrix $\begin{pmatrix} \varpi_{\tilde{v}} 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix}$, regarded both as an element of $GL_n(F_{\tilde{v}})$ and of $G(F^+_v)$ (via $\iota_{\tilde{v}}$). Let $\alpha = \prod_{v|l}\prod_{j=1}^n \alpha^{(j)}_{\varpi_{\tilde{v}}} \in G(F^+_l) \subset G(\mathbb{A}^\infty_{F^+})$. Decompose $U\alpha U = \coprod_i x_i \alpha U$. Then the action of $T_{\lambda,l}$ on $S(U,M_\lambda)$ is given by sending an element $f(g)$ to the function

$$g \mapsto \sum_i (x_i)_l((w_0\lambda)(\alpha)^{-1}\alpha)f(gx_i\alpha).$$

(Here $(x_i)_l$ denotes the l-part of x_i and $(w_0\lambda)(\alpha)$ is the element of $\overline{\mathbb{Q}}_l$ defined in Definition 2.2.3(2) of [Ger09].) The operator $(w_0\lambda)(\alpha)^{-1}\alpha$ acts trivially on $v_{w_0\lambda}$ and acts via multiplication by an element of $\mathfrak{m}_{\mathcal{O}}$ on all other weight spaces of M_λ. It follows that $(w_0\lambda)(\alpha)^{-1}\alpha$ induces a projection onto $v_{w_0\lambda}\otimes\overline{\mathbb{F}}_l \subset M_\lambda\otimes\overline{\mathbb{F}}_l$. The claimed result now follows from property (3) above and we have established the following lemma.

Lemma 6.1.3. *Let $T_{\lambda,l}$ be the Hecke operator $\prod_{v|l,1\le j\le n} T^{(j)}_{\lambda,\tilde{v}}$. Then $T_{\lambda,l}$ preserves $S(U,P_\lambda)$ and acts by 0 on $S(U,(M_\lambda\otimes\overline{\mathbb{F}}_l)/P_\lambda)$.*

We let $S^{\mathrm{ord}}(U,P_\lambda)$ denote the maximal $\overline{\mathbb{F}}_l$-subspace of $S(U,P_\lambda)$ on which every eigenvalue of $T_{\lambda,l}$ is non-zero. Then $S^{\mathrm{ord}}(U,P_\lambda)$ is a $\mathbb{T}^{T,univ}$-direct summand of $S(U,P_\lambda)$ and $S^{\mathrm{ord}}(U,P_\lambda)\cong S^{\mathrm{ord}}(U,M_\lambda)\otimes\overline{\mathbb{F}}_l$. In particular, if $\mathfrak{m}$ is a maximal ideal of $\mathbb{T}^{T,univ}$ with residue field $\overline{\mathbb{F}}_l$ such that $S^{\mathrm{ord}}(U,P_\lambda)_{\mathfrak{m}}\neq 0$, then $S^{\mathrm{ord}}(U,M_\lambda)_{\mathfrak{m}}\neq 0$, and we have a Galois representation $\bar{r}_{\mathfrak{m}}: G_F\to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ as before.

Corollary 6.1.4. *$\overline{\rho}$ is modular and ordinary of weight $\lambda\in(\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$ if and only if there is a U, T as above and a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{T,univ}$ with residue field $\overline{\mathbb{F}}_l$ such that*

- *$S^{\mathrm{ord}}(U,P_\lambda)_{\mathfrak{m}}\neq 0$, and*
- *$\overline{\rho}\cong\bar{r}_{\mathfrak{m}}$.*

Proof. This is an immediate consequence of the definitions and of Lemma 6.1.3. □

Fix now an element $\mu\in(\mathbb{Z}^n_{+,\mathrm{res}})^{\overline{I}_l}$. Fix $\lambda\in(\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$ with $\pi(\lambda)=\mu$. Then there is an equivalence $P_\lambda\cong F_\mu$, so that $\mathbb{T}^{T,univ}$ acts on $S(U,F_\mu)$ and we can define a subspace $S^{\mathrm{ord}}_\lambda(U,F_\mu)$ of $S(U,F_\mu)$ corresponding to $S^{\mathrm{ord}}(U,P_\lambda)$. Suppose that $\lambda'\in(\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$ with $\pi(\lambda')=\mu$. Then we obtain another action of $\mathbb{T}^{T,univ}$ on $S(U,F_\mu)$ and a subspace $S^{\mathrm{ord}}_{\lambda'}(U,F_\mu)$ of $S(U,F_\mu)$. It is easy to check that the two actions of $\mathbb{T}^{T,univ}$ on $S(U,F_\mu)$ coincide. Moreover, we have $S^{\mathrm{ord}}_\lambda(U,F_\mu)=S^{\mathrm{ord}}_{\lambda'}(U,F_\mu)$ and we denote this space unambiguously by $S^{\mathrm{ord}}(U,F_\mu)$. (Note that the induced action of $T_{\lambda',l}$ on $S(U,F_\mu)$ differs from the induced action of $T_{\lambda,l}$ by multiplication by the image of $(w_0\lambda)(\alpha)(w_0\lambda')(\alpha)^{-1}\in\mathcal{O}^\times$ in $\overline{\mathbb{F}}_l^\times$.) We can therefore make the following definition.

Definition 6.1.5. Suppose that $\overline{\rho}: G_F\to\mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is a continuous irreducible representation. Then we say that $\overline{\rho}$ is *modular and ordinary* of weight $\mu\in(\mathbb{Z}^n_{+,\mathrm{res}})^{\overline{I}_l}$ if there is a U, T as above, and for some (equivalently, any) $\lambda\in(\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$ with $\pi(\lambda)=\mu$ there is a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{T,univ}$ with residue field $\overline{\mathbb{F}}_l$ such that

- $S^{\mathrm{ord}}(U,F_\mu)_{\mathfrak{m}}\neq 0$, and
- $\overline{\rho}\cong\bar{r}_{\mathfrak{m}}$.

We can then reinterpret Theorem 6.1.2.

Theorem 6.1.6. *Suppose that*

$$\overline{\rho}: G_F\to\mathrm{GL}_n(\overline{\mathbb{F}}_l)$$

is an irreducible representation satisfying the following assumptions.

(1) *The representation* $\overline{\rho}$ *is modular and ordinary (so in particular* $\overline{\rho}^c = \overline{\rho}^\vee \epsilon^{1-n}$*).*
(2) *The image* $\overline{\rho}(G_{F(\zeta_l)})$ *is big.*
(3) $\overline{F}^{\ker \operatorname{ad} \overline{\rho}}$ *does not contain* $F(\zeta_l)$*.*

Then $\overline{\rho}$ *is modular and ordinary of weight* $\mu \in (\mathbb{Z}^n_{+,\mathrm{res}})^{\overline{I}_l}$ *if and only if for some (equivalently, any)* $\lambda \in (\mathbb{Z}^n_{+,\mathrm{res}})^{\tilde{I}_l}$ *with* $\pi(\lambda) = \mu$*, the following condition holds.*

- *For every place* $v|l$ *of* F^+*,* $\overline{\rho}|_{G_{F_{\tilde{v}}}}$ *has a crystalline lift which is ordinary of weight* $\lambda_{\tilde{v}}$*.*

Proof. This follows at once from Theorem 6.1.2, Lemma 6.1.3, and Definition 6.1.5. □

7. GSp_4

7.1. Definitions. We define GSp_4 to be the reductive group over $\mathbb{Z}$ defined as a subgroup of GL_4 by

$$\mathrm{GSp}_4(R) = \{g \in \mathrm{GL}_4(R) : gJ^tg = \mu(g)J\}$$

where $\mu(g)$ is the similitude factor (which is uniquely determined by g), and J is the antisymmetric matrix

$$\begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}$$

where X is the 2×2 antidiagonal matrix with all entries on the antidiagonal equal to 1. Note that the map $\mu : g \mapsto \mu(g)$ gives a homomorphism $\mathrm{GSp}_4 \to \mathbb{G}_m$.

Lemma 7.1.1. *Let* Γ *be a profinite group, and* $S \subset R$ *be complete local Noetherian rings with* $\mathfrak{m}_R \cap S = \mathfrak{m}_S$ *and common residue field* k *of characteristic* > 2*. Let* $\rho : \Gamma \to \mathrm{GSp}_4(R)$ *be a continuous representation. Suppose that* ρ *mod* $\mathfrak{m}_R$ *is absolutely irreducible (when considered as a representation to* $\mathrm{GL}_4(k)$*) and that* $\operatorname{tr}\rho(\Gamma) \subset S$*. Then there is a* $\ker(\mathrm{GSp}_4(R) \to \mathrm{GSp}_4(k))$*-conjugate of* ρ *whose image is contained in* $\mathrm{GSp}_4(S)$*.*

Proof. By Lemma 2.1.10 of [CHT08], we see that ρ is $\ker(\mathrm{GL}_4(R) \to \mathrm{GL}_4(k))$-conjugate to a representation ρ' valued in $\mathrm{GL}_4(S)$. Now, $(\mu \circ \rho)^2 = \det \rho = \det \rho'$ is valued in S, which by Hensel's lemma means that $\mu \circ \rho$ is valued in S. Thus ${}^t(\rho')^{-1}(\mu\circ\rho)$ is also valued in $\mathrm{GL}_4(S)$. Because ρ' and ${}^t(\rho')^{-1}(\mu\circ\rho)$ are conjugate in $\mathrm{GL}_4(R)$ they are also conjugate in $\mathrm{GL}_4(S)$, by Théorème 1 of [Car94]. Suppose that $\rho' = B^t(\rho')^{-1}(\mu\circ\rho)B^{-1}$. The matrix B is antisymmetric (because ρ is symplectic). By choosing a symplectic basis for the symplectic form determined by B, we see that ρ is $\mathrm{GL}_4(R)$-conjugate to a representation valued in $\mathrm{GSp}_4(S)$, and it is easy to check that one may choose the symplectic basis so that the conjugating matrix is in $\ker(\mathrm{GL}_4(R) \to \mathrm{GL}_4(k))$. It remains to check that the conjugating matrix is also in $\mathrm{GSp}_4(R)$; but this is an immediate consequence of Schur's lemma. □

7.2. Symplectic lifting rings (local case). Fix as before a finite field k of characteristic $l > 2$, and a finite totally ramified extension K of $W(k)[1/l]$ with ring of integers $\mathcal{O}$. Let the maximal ideal of $\mathcal{O}$ be $\mathfrak{m}_K = (\pi_K)$. Let M be a finite extension of $\mathbb{Q}_p$ for some prime p, possibly equal to l. In the case where $p = l$, we assume that K contains the image of every embedding of M into $\overline{K}$. Let

$$\overline{\rho} : G_M \to \mathrm{GSp}_4(k)$$

be a continuous representation. Since GSp_4 is an algebraic subgroup of GL_4, we can also view it as a representation to $\mathrm{GL}_4(k)$. Then there is a universal $\mathcal{O}$-lifting

$$\rho^{\square} : G_M \to \mathrm{GL}_4(R^{\square}_{\overline{\rho}}),$$

and it is immediate that there is a quotient $R^{\square,sympl}_{\overline{\rho}}$ of $R^{\square}_{\overline{\rho}}$ and a universal symplectic lifting

$$\rho^{\square,sympl} : G_M \to \mathrm{GSp}_4(R^{\square,sympl}_{\overline{\rho}}).$$

Fix a character $\psi : G_M \to \mathcal{O}^{\times}$ lifting $\mu \circ \overline{\rho}$, which we assume to be crystalline if $p = l$; then there is a quotient $R^{\square,sympl,\psi}_{\overline{\rho}}$, the universal lifting ring for lifts with similitude factor ψ.

We will need to study certain refined lifting problems. Suppose that $p = l$. Let λ be an element of $(\mathbb{Z}^4_+)^{\mathrm{Hom}(M,K)}$ and let $\mathbf{v}_\lambda$ be the associated l-adic Hodge type (see section 3.1.2). Corollary 2.7.7 of [Kis08] shows that there is a unique l-torsion-free quotient $R^{sympl,\mathbf{v}_\lambda,cr,\psi}_{\overline{\rho}}$ of $R^{\square,sympl,\psi}_{\overline{\rho}}$ with the property that for any finite K-algebra B, a homomorphism of $\mathcal{O}$-algebras $R^{\square,sympl,\psi}_{\overline{\rho}} \to B$ factors through $R^{sympl,\mathbf{v}_\lambda,cr,\psi}_{\overline{\rho}}$ if and only if the corresponding representation is crystalline of l-adic Hodge type $\mathbf{v}_\lambda$ (where as usual we define a homomorphism $G_M \to \mathrm{GSp}_4(B)$ to be crystalline of a particular Hodge type if and only if the same is true of the composite homomorphism to $\mathrm{GL}_4(B)$).

The following discussion will be useful below. Let E be a finite extension of K and let $\mathcal{C}_E$ be the category of finite, local E-algebras with residue field E. If B is an object of $\mathcal{C}_E$, a *symplectic B-module* is a pair (V_B, α_B) where V_B is a free B-module of rank 4 with a continuous action of G_M and α_B is a perfect symplectic pairing $V_B \times V_B \to B$ satisfying

$$\alpha_B(gx, gy) = \psi_B(g)\alpha_B(x, y)$$

for all $x, y \in V_B$ and $g \in G_M$, for some continuous character $\psi_B : G_M \to B^{\times}$. A *symplectic basis* of such a pair (V_B, α_B) is a basis $\beta_B = \{e_1, e_2, e_3, e_4\}$ of V_B where the matrix $(\alpha_B(e_i, e_j))$ equals λJ for some $\lambda \in B^{\times}$. A symplectic basis always exists. Two symplectic B-modules (V_B, α_B) and (V'_B, α'_B) are isomorphic if there is an isomorphism of $B[G_M]$-modules $V_B \cong V'_B$ under which α'_B pulls back to α_B. In this case $\psi_B = \psi'_B$.

Fix a symplectic E-module (V_E, α_E) together with a symplectic basis β_E. A *deformation* of (V_E, α_E) to an object B of $\mathcal{C}_E$ is a triple (V_B, α_B, ι_B) where (V_B, α_B) is a symplectic B-module and ι_B is an isomorphism $(V_B \otimes_B B/\mathfrak{m}_B, \alpha_B \otimes_B B/\mathfrak{m}_B) \cong (V_E, \alpha_E)$ of symplectic E-modules. A *framed deformation* of (V_E, α_E, β_E) is a deformation (V_B, α_B, ι_B) together with a symplectic basis β_B of (V_B, α_B) reducing to β_E under ι_B. We say that two framed deformations $(V_B, \alpha_B, \iota_B, \beta_B)$ and $(V'_B, \alpha'_B, \iota'_B, \beta'_B)$ are isomorphic if there is an isomorphism $f : V_B \to V'_B$ which is compatible with ι_B and ι'_B, which is compatible up to a scalar with α_B and α'_B, and which takes β_B to β'_B; in particular, multiplication by any scalar in $1+\mathfrak{m}_B$ preserves the isomorphism class of a framed deformation. Let $\rho_E : G_M \to \mathrm{GSp}_4(E)$ be the matrix of V_E with respect to β_E. For an object B of $\mathcal{C}_E$ there is a natural bijection between the set of isomorphism classes of framed deformations of (V_E, α_E, β_E) to B and the set of lifts $\rho_B : G_M \to \mathrm{GSp}_4(B)$: the class of a framed deformation (V_B, α_B, β_B) corresponds to the matrix representation of V_B with respect to the basis β_B. Similarly, there is a natural bijection between the set of isomorphism classes of deformations of (V_B, α_B) to B and the set of *deformations of* ρ_E to B, that is, $\ker(\mathrm{GSp}_4(B) \to \mathrm{GSp}_4(E))$-conjugacy classes of lifts $\rho_B : G_M \to \mathrm{GSp}_4(B)$ of ρ_E: the class of a deformation (V_B, α_B) corresponds to the conjugacy class of the matrix representation of V_B with respect to any symplectic basis β_B lifting β_E.

Suppose that (V_B, α_B) is a *crystalline* symplectic B-module and let $D_B := D_{\mathrm{cris}}(V_B) = (V_B \otimes_{\mathbb{Q}_l} B_{\mathrm{cris}})^{G_M}$ be the associated weakly admissible filtered φ-module. Let $D_{\psi_B} = D_{\mathrm{cris}}(\psi_B)$. There is an associated alternating pairing

$$D(\alpha_B) : D_B \times D_B \to D_{\psi_B}$$

which is a map of filtered φ-modules and is non-degenerate in the sense that it induces an isomorphism $D_B \to \mathrm{Hom}(D_B, D_{\psi_B})$. This pairing is defined by taking the B_{cris}-linear extension of α_B to $V_B \otimes_{\mathbb{Q}_l} B_{\mathrm{cris}}$ and then taking G_M-invariants. Suppose in addition that V_B has l-adic Hodge type $\mathbf{v}_\lambda$. Let $\tau : M \hookrightarrow K$ be an embedding and let $D_{B,\tau} = (D_B \otimes_{M_0} M) \otimes_{B \otimes M, 1\otimes\tau} B$ and $D_{\psi_B,\tau} = (D_{\psi_B} \otimes_{M_0} M) \otimes_{B \otimes M, 1\otimes\tau} B$. Then $D(\alpha_B)$ induces a symplectic pairing $D_{B,\tau} \times D_{B,\tau} \to D_{\psi_B,\tau}$. For $j = 1, \ldots, 4$, let $i_j = \lambda_{\tau,j} + (4-j)$ be the Hodge-Tate weights of V_B with respect to τ. Let i_ψ be the Hodge-Tate weight of ψ_B with respect to τ. Then $i_\psi = i_1 + i_4 = i_2 + i_3$ since $V_B \cong \mathrm{Hom}_B(V_B, \psi_B)$. Let Fil^i be the filtration on $D_{B,\tau}$. In order for $D(\alpha_B)$ to respect filtrations and to be non-degenerate we must have $D(\alpha_B)(\mathrm{Fil}^{i_1}, \mathrm{Fil}^{i_3}) = \{0\}, D(\alpha_B)(\mathrm{Fil}^{i_2}, \mathrm{Fil}^{i_3}) = D_{\psi_B,\tau}$ and $D(\alpha_B)(\mathrm{Fil}^{i_1}, \mathrm{Fil}^{i_4}) = D_{\psi_B,\tau}$. In other words, we can find a symplectic basis e_1, e_2, e_3, e_4 for $D_{B,\tau}$ such that $\mathrm{Fil}^{i_j} = Be_1 + \ldots + Be_j$ for $j = 1, \ldots, 4$.

We define a *symplectic filtered* φ*-module* over an object B in $\mathcal{C}_E$ to be a pair $(D_B, D(\alpha_B))$ consisting of a weakly admissible rank 4 filtered φ-module D_B over $B \otimes_{\mathbb{Q}_l} M_0$ and an alternating, non-degenerate morphism of filtered φ-modules

$$D(\alpha_B) : D_B \times D_B \to D_{\psi_B}$$

where D_{ψ_B} is a weakly admissible rank 1 filtered φ-module over $B \otimes_{\mathbb{Q}_l} M_0$. There is an obvious notion of isomorphism between symplectic filtered φ-modules and also an obvious notion of a deformation of a symplectic filtered φ-module over E to an object B of $\mathcal{C}_E$. The functors D_{cris} and V_{cris} are quasi-inverse equivalences of categories between the category of crystalline symplectic B-modules and the category of symplectic filtered φ-modules over B (all morphisms in these categories are isomorphisms).

Suppose now that M is a finite extension of $\mathbb{Q}_p$, $p \neq l$. Then it is easy to check (for example by considering the Weil-Deligne representation corresponding to the universal lifting) that the inertial type at a closed point of the generic fibre is an invariant of the irreducible components of $R_{\overline{\rho}}^{\square,sympl,\psi}[1/l]$. Thus for any 4-dimensional inertial type τ of I_M which is defined over K, there is a unique reduced l-torsion-free quotient $R_{\overline{\rho}}^{sympl,\tau,\psi}$ of $R_{\overline{\rho}}^{\square,sympl,\psi}$, corresponding to a union of irreducible components of $R_{\overline{\rho}}^{\square,sympl,\psi}[1/l]$, with the property that for any finite extension L of K, a homomorphism of $\mathcal{O}$-algebras $R_{\overline{\rho}}^{\square,sympl,\psi} \to L$ factors through $R_{\overline{\rho}}^{sympl,\tau,\psi}$ if and only if the corresponding lifting of $\overline{\rho}$ (considered as a representation to $\mathrm{GL}_4(L)$) has type τ.

Let $\mathrm{ad}\,\overline{\rho}$ denote the Lie algebra of GSp_4 over k, and $\mathrm{ad}^0\,\overline{\rho}$ the Lie algebra of Sp_4. These have a natural action of G_M via $\overline{\rho}$ and the adjoint action of $\mathrm{GSp}_4(k)$, and are respectively 11-dimensional and 10-dimensional k-vector spaces.

We have the following result on the dimensions of these local lifting rings.

Proposition 7.2.1. *Let M be a finite extension of $\mathbb{Q}_p$. If $p \neq l$, and τ is such that the ring $R_{\overline{\rho}}^{sympl,\tau,\psi}$ is non-zero, then any irreducible component of $R_{\overline{\rho}}^{sympl,\tau,\psi}$ has dimension at least 11. If $p = l$ and $\mathbf{v}_\lambda$ is such that $R_{\overline{\rho}}^{sympl,\mathbf{v}_\lambda,cr,\psi}[1/l]$ is non-zero, then this ring is formally smooth over K of relative dimension $10 + 4[M : \mathbb{Q}_l]$.*

Proof. Firstly, suppose $p = l$ and let $X = \operatorname{Spec} R_{\overline{\rho}}^{sympl,\mathbf{v}_\lambda,cr,\psi}$. Let x be a closed point of $X[1/l]$ with residue field E. We need to show that the completed local ring $\mathcal{O}_{X,x}^\wedge$ is formally smooth over E of dimension $10 + 4[M : \mathbb{Q}_l]$. We first establish formal smoothness. Let $\rho_E : G_M \to \mathrm{GSp}_4(E)$ be the representation associated to x. Let B denote a finite local E-algebra with residue field E and let I be an ideal of B with $\mathfrak{m}_B I = \{0\}$. Let $\zeta : R_{\overline{\rho}}^{sympl,\mathbf{v}_\lambda,cr,\psi} \to B/I$ be an $\mathcal{O}$-algebra homomorphism corresponding to a crystalline lift $\rho_{B/I} : G_M \to \mathrm{GSp}_4(B/I)$ of ρ_E. We need to show that we can lift ζ to B, or equivalently, that we can find a crystalline lift $G_M \to \mathrm{GSp}_4(B)$ of $\rho_{B/I}$ with similitude character ψ.

Let $V_{B/I} = (B/I)^4$ regarded as G_M-module via $\rho_{B/I}$ and let $\alpha_{B/I} : V_{B/I} \times V_{B/I} \to (B/I)(\psi)$ be the symplectic pairing associated to the matrix J (that is, $\alpha_{B/I}(x, y) = {}^t x J y$ where x and y are regarded as column vectors). Let $(D_{B/I}, D(\alpha_{B/I}))$ be the symplectic, filtered φ-module over B/I associated to $(V_{B/I}, \alpha_{B/I})$. To construct the required lift of $\rho_{B/I}$, it suffices (by applying V_{cris}) to construct a symplectic filtered φ-module $(D_B, D(\alpha_B))$ over B (with $D(\alpha_B)$ valued in $B \otimes_E D_{\mathrm{cris}}(E(\psi))$) lifting $(D_{B/I}, D(\alpha_{B/I}))$.

Let b be an $E \otimes_{\mathbb{Q}_l} M_0$-generator of $D_\psi := D_{\mathrm{cris}}(E(\psi))$. Choose a $(B/I) \otimes_{\mathbb{Q}_l} M_0$-basis e_1, e_2, e_3, e_4 for $D_{B/I}$ so that the matrix $(D(\alpha_{B/I})(e_i, e_j))$ is $(1 \otimes b)J \in M_{4\times 4}((B/I) \otimes_E D_\psi)$. The matrix M_φ of φ with respect to this basis is an element of $\mathrm{GSp}_4((B/I) \otimes_{\mathbb{Q}_l} M_0)$ with similitude factor '$\varphi(b)/b$'$\in (E \otimes_{\mathbb{Q}_l} M_0)^\times \subset ((B/I) \otimes_{\mathbb{Q}_l} M_0)^\times$. Let $\widetilde{M_\varphi}$ be a lifting of this matrix to an element of $\mathrm{GSp}_4(B \otimes_{\mathbb{Q}_l} M_0)$ with the same similitude factor. Let D_B be the free $B \otimes_{\mathbb{Q}_l} M_0$-module on generators $\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4$. Endow it with the symplectic form $D(\alpha_B) : D_B \times D_B \to B \otimes_E D_\psi$ defined by $(D(\alpha_B)(\widetilde{e}_i, \widetilde{e}_j)) = (1 \otimes b)J$. Let $\widetilde{\varphi}$ be the φ_0-semilinear automorphism of D_B whose matrix with respect to the basis $\widetilde{e}_i$ is $\widetilde{M_\varphi}$. Now choose a filtration on $D_B \otimes_{M_0} M$ lifting the filtration on $D_{B/I} \otimes_{M_0} M$ and such that D_B becomes a weakly admissible symplectic filtered φ-module and we have shown that $\mathcal{O}_{X,x}^\wedge$ is formally smooth over E.

We now determine the relative dimension d of $\mathcal{O}_{X,x}^\wedge$ over E. Let $\mathfrak{g}$ denote the Lie algebra of $\mathrm{GSp}_4(E)$ and $\mathfrak{g}^\circ$ the Lie algebra of $\mathrm{Sp}_4(E)$. Let $D_{\rho_E}^\square(E[\varepsilon])$ (resp. $D_{\rho_E}(E[\varepsilon])$) denote the set of crystalline lifts (resp. deformations) $G_M \to \mathrm{GSp}_4(E[\varepsilon])$ of ρ_E with similitude character ψ. These sets are naturally E-vector spaces. Since the natural map $D_{\rho_E}^\square(E[\varepsilon]) \twoheadrightarrow D_{\rho_E}(E[\varepsilon])$ is a $\mathfrak{g}/\mathfrak{g}^{G_M} = \mathfrak{g}^\circ/(\mathfrak{g}^\circ)^{G_M}$-torsor, we have

$$d = \dim_E D_{\rho_E}^\square(E[\varepsilon]) = \dim_E \left(\mathfrak{g}^\circ/(\mathfrak{g}^\circ)^{G_M}\right) + \dim_E D_{\rho_E}(E[\varepsilon]).$$

Let $D_{D_E}(E[\varepsilon])$ denote the set of equivalence classes of deformations $(D, D(\alpha))$ to $E[\varepsilon]$ of the symplectic filtered φ-module $(D_E, D(\alpha_E))$ where the pairing $D(\alpha)$ takes values in $E[\varepsilon] \otimes_E D_\psi$. By the discussion preceding the proposition, we see that there is a natural bijection between $D_{\rho_E}(E[\varepsilon])$ and $D_{D_E}(E[\varepsilon])$.

Choose any deformation $(D', D(\alpha)')$ in $D_{D_E}(E[\varepsilon])$. We can choose an isomorphism of $E[\varepsilon] \otimes_{\mathbb{Q}_l} M_0$-modules $j : D' \to D_E \otimes_E E[\epsilon]$ taking $D(\alpha_E) \otimes_E E[\epsilon]$ to $D(\alpha)'$. Let φ denote the φ-operator on $D_E \otimes_E E[\epsilon]$ and Fil the filtration on

$(D_E \otimes_{M_0} M) \otimes_E E[\epsilon]$. Let φ' denote the operator on $D_E \otimes_E E[\epsilon]$ corresponding under j to the φ-operator on D'. Similarly, let Fil' denote the filtration on $(D_E \otimes_{M_0} M) \otimes_E E[\epsilon]$ corresponding under j to the filtration on $D' \otimes_{M_0} M$. Let $\mathfrak{g}_{D_E}$ and $\mathfrak{g}^\circ_{D_E}$ denote the Lie algebras of $\mathrm{GSp}(D_E, D(\alpha_E))$ and $\mathrm{Sp}(D_E, D(\alpha_E))$ respectively. Similarly, let $\mathfrak{g}_{D_{E,M}}$ denote the Lie algebra of $\mathrm{GSp}(D_E \otimes_{M_0} M, \alpha_E \otimes 1)$. Let $\mathfrak{b}_{D_{E,M}}$ denote the Lie algebra of the Borel subgroup of $\mathrm{GSp}(D_E \otimes_{M_0} M, \alpha_E \otimes 1)$ which stabilises the filtration on $D_E \otimes_{M_0} M$. Then there exists $X \in \mathfrak{g}^\circ_{D_E}$ and $Y \in \mathfrak{g}_{D_{E,M}}$ such that $\varphi' = (1 + \varepsilon X)\varphi$ and $\mathrm{Fil}' = (1 + \varepsilon Y)\,\mathrm{Fil}$. Moreover, any such pair X, Y gives rise to a deformation of $(D_E, D(\alpha_E))$ and we get a surjective linear map

$$\mathfrak{g}^\circ_{D_E} \oplus \mathfrak{g}_{D_{E,M}}/\mathfrak{b}_{D_{E,M}} \twoheadrightarrow D_{D_E}(E[\varepsilon]).$$

The kernel of this map is the image of the map

$$\mathfrak{g}^\circ_{D_E} \to \mathfrak{g}^\circ_{D_E} \oplus \mathfrak{g}_{D_{E,M}}/\mathfrak{b}_{D_{E,M}}$$

sending Z to the pair $(Z - \varphi \circ Z \circ \varphi^{-1}, Z)$. Denote the kernel of this last map by $(\mathfrak{g}^\circ_{D_E})^{\varphi=1,\mathrm{Fil}}$. We have shown that

$$d = \dim_E \left(\mathfrak{g}^\circ/(\mathfrak{g}^\circ)^{G_M}\right) + \dim_E \mathfrak{g}_{D_{E,M}}/\mathfrak{b}_{D_{E,M}} + \dim_E (\mathfrak{g}^\circ_{D_E})^{\varphi=1,\mathrm{Fil}}.$$

The result now follows from the fact that $\dim_E \mathfrak{g}^\circ = 10$, $\dim_E \mathfrak{g}_{D_{E,M}}/\mathfrak{b}_{D_{E,M}} = 4[M : \mathbb{Q}_l]$ and $(\mathfrak{g}^\circ)^{G_M} \cong (\mathfrak{g}^\circ_{D_E})^{\varphi=1,\mathrm{Fil}}$ via D_{cris}.

Now suppose that $p \neq l$. In this case we only need to establish a lower bound on the dimension, and we do this by means of a slight variant of Mazur's lower bound for the dimension of an unrestricted deformation ring (see Proposition 2 of [Maz89]). Note that by the construction of the ring $R^{sympl,\tau,\psi}_{\overline{\rho}}$, we need only show that each irreducible component of $R^{\square,sympl,\psi}_{\overline{\rho}}$ has dimension at least 11.

Let $\mathfrak{m}^{sympl}$ denote the maximal ideal of $R^{\square,sympl,\psi}_{\overline{\rho}}$. Then $R^{\square,sympl,\psi}_{\overline{\rho}}$ is the quotient of a power series ring over $\mathcal{O}$ in $\dim_k \mathfrak{m}^{sympl}/((\mathfrak{m}^{sympl})^2, \pi_K)$ variables. The argument of the proof of Lemma 4.1.1 of [Kis07] shows that it is necessary to quotient out by at most $\dim_k H^2(G_M, \mathrm{ad}^0\,\overline{\rho})$ relations. Thus every component of $R^{\square,sympl,\psi}_{\overline{\rho}}$ has dimension at least

$$1 + \dim_k \mathfrak{m}^{sympl}/((\mathfrak{m}^{sympl})^2, \pi_K) - \dim_k H^2(G_M, \mathrm{ad}^0\,\overline{\rho}).$$

Now, $\mathfrak{m}^{sympl}/((\mathfrak{m}^{sympl})^2, \pi_K)$ is dual to the tangent space

$$D^{\square,sympl}(k[\epsilon]/(\epsilon^2)),$$

where $D^{\square,sympl}$ is the functor represented by $R^{\square,sympl,\psi}_{\overline{\rho}}$. The elements of this space are 1-cocycles in $Z^1(G_M, \mathrm{ad}^0\,\overline{\rho})$, so we see that

$$\begin{aligned}\dim_k \mathfrak{m}^{sympl}/((\mathfrak{m}^{sympl})^2, \pi_K) &= \dim_k Z^1(G_M, \mathrm{ad}^0\,\overline{\rho}) \\ &= \dim_k H^1(G_M, \mathrm{ad}^0\,\overline{\rho}) + \dim_k \mathrm{ad}^0\,\overline{\rho} - \dim_k H^0(G_M, \mathrm{ad}^0\,\overline{\rho}).\end{aligned}$$

Thus every component of $R^{\square,sympl,\psi}_{\overline{\rho}}$ has dimension at least

$$\begin{aligned}1 + \dim_k H^1(G_M, \mathrm{ad}^0\,\overline{\rho}) + \dim_k \mathrm{ad}^0\,\overline{\rho} - \dim_k H^0(G_M, \mathrm{ad}^0\,\overline{\rho}) - \dim_k H^2(G_M, \mathrm{ad}^0\,\overline{\rho}) \\ = 1 + \dim_k \mathrm{ad}^0\,\overline{\rho} \\ = 11\end{aligned}$$

by the local Euler characteristic formula, as required. □

Remark 7.2.2. In the case where $p = l$, it follows immediately that $R_{\overline{\rho}}^{sympl,\mathbf{v}_\lambda,cr,\psi}$ is reduced and equidimensional of dimension $11+4[M:\mathbb{Q}_l]$ (whenever it is non-zero).

Remark 7.2.3. It is presumably possible to use the techniques of [Kis08] to prove that if $p \neq l$, and τ is such that the ring $R^{sympl,\tau,\psi}$ is non-zero, then it is equidimensional of dimension 11. As we do not need this result we have not attempted to verify this.

The following lemma can be proved in exactly the same way as Lemma 3.3.3 of [Ger09].

Lemma 7.2.4. *Let M be a finite extension of $\mathbb{Q}_l$. There is a quotient $R_{\overline{\rho}}^{sympl,\triangle_\lambda,cr,\psi}$ of $R_{\overline{\rho}}^{sympl,\mathbf{v}_\lambda,cr,\psi}$ corresponding to a union of irreducible components such that for any finite local K-algebra B, a homomorphism of $\mathcal{O}$-algebras $\zeta : R_{\overline{\rho}}^{sympl,\mathbf{v}_\lambda,cr,\psi} \to B$ factors through $R_{\overline{\rho}}^{sympl,\triangle_\lambda,cr,\psi}$ if and only if $\zeta \circ \rho^\square$ is ordinary of weight λ (when considered as a representation valued in $\mathrm{GL}_4(B)$).*

(We remark that any $\zeta \circ \rho^\square : G_M \to \mathrm{GSp}_4(B)$ as above is ordinary of weight λ if and only if it is conjugate in $\mathrm{GSp}_4(B)$ to an upper triangular representation of the form appearing in Definition 3.1.3. We note that since $R_{\overline{\rho}}^{sympl,\mathbf{v}_\lambda,cr,\psi}$ is reduced, the last statement determines $R_{\overline{\rho}}^{sympl,\triangle_\lambda,cr,\psi}$ uniquely as a quotient of $R_{\overline{\rho}}^{sympl,\mathbf{v}_\lambda,cr,\psi}$.)

7.3. A lower bound on the dimension of a symplectic deformation ring. In this section we outline a proof of a lower bound on the dimension of a global deformation ring. This material is by now rather standard (see for example section 4 of [Kis07] or Corollary 2.3.5 of [CHT08]), and we content ourselves with a sketch of the proofs.

Suppose that F^+ is a totally real field. Let $\overline{\rho} : G_{F^+} \to \mathrm{GSp}_4(k)$ be absolutely irreducible. Let S be a finite set of finite places of F^+, including all places at which $\overline{\rho}$ is ramified, and all places dividing l. Let $F^+(S)$ be the maximal extension of F^+ unramified outside of S and infinity, and write $G_{F^+(S)/F^+}$ for the Galois group $\mathrm{Gal}(F^+(S)/F^+)$ (note that we do not use the more usual notation $G_{F^+,S}$ for this group, as to do so would be inconsistent with [CHT08] and the earlier sections of this paper). Fix a crystalline character $\psi : G_{F^+(S)/F^+} \to \mathcal{O}^\times$ lifting the character $\mu \circ \overline{\rho}$ (note that such a character need not exist - this is an assumption on $\overline{\rho}$ and S). If R is a complete local Noetherian $\mathcal{O}$-algebra with residue field k, then an *R-valued deformation* of $\overline{\rho}$ is a $\ker(\mathrm{GSp}_4(R) \to \mathrm{GSp}_4(k))$-conjugacy class of liftings of $\overline{\rho}$ to $\mathrm{GSp}_4(R)$. Since $\overline{\rho}$ is absolutely irreducible, it is an easy consequence of Schur's lemma that $\overline{\rho}$ has a universal symplectic deformation with fixed similitude factor ψ to a complete local Noetherian $\mathcal{O}$-algebra $R_{F^+,S}^{sympl,\psi}$ (see for example Theorem 3.3 of [Til96]).

Let $R_{F^+,S}^{\square,sympl,\psi}$ denote the complete local Noetherian $\mathcal{O}$-algebra representing the functor $\mathcal{D}_{F^+,S}^{\square,sympl,\psi}$ which assigns to a complete local Noetherian $\mathcal{O}$-algebra R with residue field k the set of equivalence classes of tuples $(\rho, \{\alpha_v\}_{v\in S})$ where ρ is a lifting of $\overline{\rho}$ to R with similitude character ψ and for each $v \in S$, $\alpha_v \in \ker(\mathrm{GSp}_4(R) \to \mathrm{GSp}_4(k))$. Two such tuples $(\rho, \{\alpha_v\}_{v\in S})$ and $(\rho', \{\alpha'_v\}_{v\in S})$ are said to be *equivalent* if there exists an element $\beta \in \ker(\mathrm{GSp}_4(R) \to \mathrm{GSp}_4(k))$ with $\rho' = \beta\rho\beta^{-1}$ and $\alpha'_v = \beta\alpha_v$ for all $v \in S$. Note that $R_{F^+,S}^{\square,sympl,\psi}$ is formally smooth over $R_{F^+,S}^{sympl,\psi}$ of relative dimension $11|S| - 1$. For each $v \in S$ let $R_v^{\square,sympl,\psi}$ denote the ring

$R^{\square,sympl,\psi}_{\overline{\rho}|_{G_{F_v^+}}}$ defined above. Let $R^\psi_S = \widehat{\otimes}_{v\in S} R_v^{\square,sympl,\psi}$. There is a natural map $R^\psi_S \to R^{\square,sympl,\psi}_{F^+,S}$ given on R-points by sending a tuple $(\rho,\{\alpha_v\}_{v\in S})$ to the tuple $(\alpha_v^{-1}\rho|_{G_{F_v^+}}\alpha_v)_{v\in S}$ (note that this map is well-defined by the definition of equivalence for these tuples).

We let h^2_S denote the k-dimension of the kernel of the natural map

$$\theta^2 : H^2(G_{F^+(S)/F^+}, \mathrm{ad}^0\,\overline{\rho}) \to \prod_{v\in S} H^2(G_{F_v^+}, \mathrm{ad}^0\,\overline{\rho}).$$

Let $\mathfrak{m}_{F^+,S}$ denote the maximal ideal of $R^{\square,sympl,\psi}_{F^+,S}$, and $\mathfrak{m}_S$ the maximal ideal of R^ψ_S.

Proposition 7.3.1. *Let*

$$\eta : \mathfrak{m}_S/(\mathfrak{m}_S^2,\pi_K) \to \mathfrak{m}_{F^+,S}/(\mathfrak{m}^2_{F^+,S},\pi_K)$$

be the natural map. Then $R^{\square,sympl}_{F^+,S}$ is a quotient of a power series ring over R^ψ_S in $\dim_k \operatorname{coker}\eta$ variables by at most $\dim_k\ker\eta + h^2_S$ relations.

Proof. This may be proved in exactly the same fashion as Proposition 4.1.4 of [Kis07]. □

Corollary 7.3.2. *Suppose that $H^0(G_{F^+(S)/F^+},(\mathrm{ad}^0\,\overline{\rho})^*(1)) = 0$. Let $s = \sum_{v|\infty}\dim_k H^0(G_{F_v^+},\mathrm{ad}^0\,\overline{\rho})$. Then for some non-negative integer r and some $f_1,\dots,f_{r+s}$, there is an isomorphism*

$$R^{\square,sympl,\psi}_{F^+,S} \xrightarrow{\sim} R^\psi_S[[x_1,\dots,x_{r+|S|-1}]]/(f_1,\dots,f_{r+s}).$$

Proof. This is very similar to the proof of Proposition 4.1.5 of [Kis07]. By Proposition 7.3.1 we see that the result will hold with s chosen such that

$$|S| - s - 1 = \dim_k \mathfrak{m}_{F^+,S}/(\mathfrak{m}^2_{F^+,S},\pi_K) - \dim_k \mathfrak{m}_S/(\mathfrak{m}_S^2,\pi_K) - h^2_S,$$

so it suffices to show that this agrees with the value of s in the statement of the corollary. Note firstly that $\mathrm{Hom}_k(\mathfrak{m}_{F^+,S}/(\mathfrak{m}^2_{F^+,S},\pi_K),k)$ is naturally isomorphic to $\mathcal{D}^{\square,sympl,\psi}_{F^+,S}(k[\epsilon]/(\epsilon^2))$. Consideration of the equivalence relation defining $\mathcal{D}^{\square,sympl,\psi}_{F^+,S}$ shows that this space has k-dimension

$$11|S| + \dim_k H^1(G_{F^+(S)/F^+},\mathrm{ad}^0\,\overline{\rho}) - \dim_k H^0(G_{F^+(S)/F^+},\mathrm{ad}\,\overline{\rho}).$$

Similarly,

$$\begin{aligned}\dim_k \mathfrak{m}_S/(\mathfrak{m}_S^2,\pi_K) &= \sum_{v\in S}(\dim \mathrm{ad}^0\,\overline{\rho} + \dim_k H^1(G_{F_v^+},\mathrm{ad}^0\,\overline{\rho}) - \dim_k H^0(G_{F_v^+},\mathrm{ad}^0\,\overline{\rho}))\\ &= \sum_{v\in S}(10 + \dim_k H^1(G_{F_v^+},\mathrm{ad}^0\,\overline{\rho}) - \dim_k H^0(G_{F_v^+},\mathrm{ad}^0\,\overline{\rho})).\end{aligned}$$

The condition that $H^0(G_{F^+(S)/F^+},(\mathrm{ad}^0\,\overline{\rho})^*(1)) = 0$, together with the last 3 terms of the Poitou-Tate sequence, shows that the map θ^2 is surjective, so that

$$h^2_S = \dim_k H^2(G_{F^+(S)/F^+},\mathrm{ad}^0\,\overline{\rho}) - \sum_{v\in S}\dim_k H^2(G_{F_v^+},\mathrm{ad}^0\,\overline{\rho}).$$

Thus

$$\dim_k \mathfrak{m}_{F^+,S}/(\mathfrak{m}^2_{F^+,S},\pi_K) - \dim_k \mathfrak{m}_S/(\mathfrak{m}_S^2,\pi_K) - h^2_S = |S| + \sum_{v\in S}\chi(G_{F_v^+},\mathrm{ad}^0\,\overline{\rho}) - \chi(G_{F^+(S)/F^+},\mathrm{ad}^0\,\overline{\rho}) - 1,$$

where χ denotes the Euler characteristic as a k-vector space, and it suffices to show that

$$\sum_{v\in S}\chi(G_{F_v^+},\mathrm{ad}^0\,\overline{\rho})-\chi(G_{F^+(S)/F^+},\mathrm{ad}^0\,\overline{\rho})=\sum_{v|\infty}\dim_k H^0(G_{F_v^+},\mathrm{ad}^0\,\overline{\rho}).$$

This follows at once from the local and global Euler characteristic formulae. □

For each place $v\in S$ not dividing l we fix a type τ_v such that $\overline{\rho}|_{G_{F_v^+}}$ has a symplectic lifting of type τ_v and similitude character $\psi|_{G_{F_v^+}}$, and we fix an l-torsion free quotient R_v of $R^{sympl,\tau_v,\psi}_{\overline{\rho}|_{G_{F_v^+}}}$ corresponding to a union of irreducible components. For each $v|l$ we fix a weight λ_v such that $\overline{\rho}|_{G_{F_v^+}}$ has a crystalline symplectic ordinary lift of weight λ_v and similitude character $\psi|_{G_{F_v^+}}$, and we fix an l-torsion free quotient R_v of $R^{sympl,\Delta_{\lambda_v},cr,\psi}_{\overline{\rho}|_{G_{F_v^+}}}$ corresponding to a union of irreducible components. Let $R_S^{\tau,\psi}:=\widehat{\otimes}_{v\in S}R_v$, and let $R^{\square,\tau,\psi}_{F^+,S}=R^{\square,sympl,\psi}_{F^+,S}\widehat{\otimes}_{R_S^\psi}R_S^{\tau,\psi}$. Let $R^{sympl,\tau,\psi}_{F^+,S}$ be the universal deformation $\mathcal{O}$-algebra representing the functor which assigns to R the $\ker(\mathrm{GSp}_4(R)\to\mathrm{GSp}_4(k))$-conjugacy classes of liftings of $\overline{\rho}$ with the property that for each $v\in S$ the corresponding lifting of $\overline{\rho}|_{G_{F_v^+}}$ gives an R-point of R_v (that this functor is well defined follows from the symplectic analogue of Lemma 3.2.3 which can be proved in the same way). Thus $R^{\square,\tau,\psi}_{F^+,S}$ is formally smooth over $R^{sympl,\tau,\psi}_{F^+,S}$ of relative dimension $11|S|-1$.

Definition 7.3.3. We say that $\overline{\rho}$ is *odd* if for all complex conjugations $c\in G_{F^+}$, $(\mu\circ\overline{\rho})(c)=-1$.

Proposition 7.3.4. *Assume that $\overline{\rho}$ is odd and that $H^0(G_{F^+(S)/F^+},(\mathrm{ad}^0\,\overline{\rho})^*(1))=0$. Then the Krull dimension of $R^{sympl,\tau,\psi}_{F^+,S}$ is at least one.*

Proof. It suffices to check that the dimension of $R^{\square,\tau,\psi}_{F^+,S}$ is at least $11|S|$. By Corollary 7.3.2, it would be enough to check that

$$\dim R_S^{\tau,\psi}+|S|-1-\sum_{v|\infty}\dim_k H^0(G_{F_v^+},\mathrm{ad}^0\,\overline{\rho})\geq 11|S|.$$

By Propositions 7.2.1 and 7.2.4, together with Lemma 3.3 of [BLGHT09],

$$\dim R_S^{\tau,\psi}\geq 1+10|S|+4[F^+:\mathbb{Q}].$$

An easy calculation using the fact that $\overline{\rho}$ is odd shows that for each $v|\infty$, $\dim_k H^0(G_{F_v^+},\mathrm{ad}^0\,\overline{\rho})=4$ (for example, one easily checks that if c_v is a corresponding complex conjugation then $\overline{\rho}(c_v)$ is conjugate to the diagonal matrix $diag(1,1,-1,-1)$, and one may then compute explicitly). Thus

$$\dim R_S^{\tau,\psi}+|S|-1-\sum_{v|\infty}\dim_k H^0(G_{F_v^+},\mathrm{ad}^0\,\overline{\rho})\geq 10|S|+4[F^+:\mathbb{Q}]+|S|-4[F^+:\mathbb{Q}]=11|S|,$$

as required. □

7.4. Relationship to unitary representations. Let F be a totally imaginary CM field with maximal totally real field F^+, with the property that all primes in S split in F. Let $\tilde{S}$ denote a set of places of F consisting of one place dividing each place in S. Recall that we let $G_{F^+,S} = \mathrm{Gal}(F(S)/F^+)$. Let $\rho : G_{F^+} \to \mathrm{GSp}_4(R)$ be a continuous representation, with R a complete local Noetherian ring. Assume that F is linearly disjoint from $(F^+)^{\ker \overline{\rho}}$ over F^+. Then, as in Lemma 2.1.2 of [CHT08], there is a continuous homomorphism $r : G_{F^+} \to \mathcal{G}_4(R)$ determined by

$$r(g) = (\rho(g), (\mu \circ \rho)(g))$$

if $g \in G_F$, and

$$r(g) = (\rho(g)J, -(\mu \circ \rho)(g))j$$

if $g \notin G_F$. We have

$$\nu \circ r = \mu \circ \rho.$$

Furthermore, this construction is obviously compatible with deformations, in the sense that if $B \in \ker(\mathrm{GSp}_4(R) \to \mathrm{GSp}_4(k))$ and ρ is replaced by ρ_B with

$$\rho_B(g) := B\rho(g)B^{-1},$$

then r is replaced by r_B with

$$r_B(g) := (aB, 1)r(g)(aB, 1)^{-1},$$

where $a \in 1 + \mathfrak{m}_R$ satisfies $a^2 = \mu(B)^{-1}$ (such an a exists because $\mu(B) \in 1 + \mathfrak{m}_R$ and $l > 2$). Applying this construction to the universal symplectic deformation of the previous section

$$\rho^{univ} : G_{F^+(S)/F^+} \to \mathrm{GSp}_4(R^{sympl,\tau,\psi}_{F^+,S}),$$

we obtain a deformation

$$r^{sympl} : G_{F^+(S)/F^+} \to \mathcal{G}_4(R^{sympl,\tau,\psi}_{F^+,S}).$$

We may also consider the corresponding residual representation

$$\bar{r} : G_{F^+,S} \to \mathcal{G}_4(k),$$

and (in the notation of sections 2.2 and 2.3 of [CHT08]) the deformation problem

$$\mathcal{S} = (F/F^+, S, \tilde{S}, \mathcal{O}, \bar{r}, \psi, R_v)$$

with corresponding universal deformation

$$r_{\mathcal{S}} : G_{F^+,S} \to \mathcal{G}_4(R^{\mathrm{univ}}_{\mathcal{S}}).$$

(Here R_v is the quotient defined in the previous section, regarded now as a lifting ring for $\overline{\rho}|_{G_{F_{\tilde{v}}}}$.) Since $G_{F^+(S)/F^+}$ is a quotient of $G_{F^+,S}$, there is a homomorphism $\theta : R^{\mathrm{univ}}_{\mathcal{S}} \to R^{sympl,\tau,\psi}_{F^+,S}$ such that there is an equality of deformations

$$r^{sympl} = \theta \circ r_{\mathcal{S}}.$$

Lemma 7.4.1. *$R^{sympl,\tau,\psi}_{F^+,S}$ is finite over $R^{\mathrm{univ}}_{\mathcal{S}}$.*

Proof. Let ρ_{F,F^+} denote the $\mathrm{GSp}_4(R^{sympl,\tau,\psi}_{F^+,S}/\theta(\mathfrak{m}_{R^{\mathrm{univ}}_{\mathcal{S}}}))$-valued representation obtained from ρ^{univ}, and let r_{F,F^+} denote the corresponding representation to $\mathcal{G}_4(R^{sympl,\tau,\psi}_{F^+,S}/\theta(\mathfrak{m}_{R^{\mathrm{univ}}_{\mathcal{S}}}))$. Then r_{F,F^+} is equivalent to $\bar{r}$, so it has finite image, and thus the image of ρ_{F,F^+} is also finite. An argument exactly as in the proof of Lemma 3.2.5 (using Lemma 7.1.1 in place of Lemma 2.1.12 of [CHT08] to see that the universal deformation ring is generated by traces) shows that $R^{sympl,\tau,\psi}_{F^+,S}/\theta(\mathfrak{m}_{R^{\mathrm{univ}}_{\mathcal{S}}})$ is finite, as required. $\square$

7.5. Companion forms for symplectic Galois representations and automorphic representations for GL_4**.** We now prove our first companion forms theorem for symplectic representations. This theorem applies to automorphic representations of GL_4; in the next section we will use functoriality to deduce a result for automorphic representations of GSp_4.

Suppose that π is a RAESDC representation of $\mathrm{GL}_4(\mathbb{A}_{F^+})$, with $\pi^\vee \cong \chi\pi$. Let $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Then there is a continuous semisimple representation

$$\rho_{l,\iota}(\pi) : G_{F^+} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_l)$$

associated to π (see theorem 1.1 of [BLGHT09]). We say that a representation $\rho : G_{F^+} \to \mathrm{GL}_4(\overline{\mathbb{Q}}_l)$ is *automorphic* if $\rho \cong \rho_{l,\iota,}(\pi)$ for some ι, π.

The representation $\overline{\rho}_{l,\iota}(\pi)$ may be conjugated to be valued in the ring of integers of a finite extension of $\mathbb{Q}_l$, and we may reduce it modulo the maximal ideal of this ring of integers and semisimplify to obtain a well-defined continuous representation

$$\overline{\rho}_{l,\iota}(\pi) : G_{F^+} \to \mathrm{GL}_4(\overline{\mathbb{F}}_l).$$

We say that a representation $\overline{\rho} : G_{F^+} \to \mathrm{GL}_4(\overline{\mathbb{F}}_l)$ is *automorphic* if $\overline{\rho} \cong \overline{\rho}_{l,\iota}(\pi)$ for some ι, π. We say that $\overline{\rho}$ is *symplectic ordinarily automorphic* if $\overline{\rho} \cong \overline{\rho}_{l,\iota}(\pi)$, where π is ι-ordinary and $\rho_{l,\iota}(\pi)$ is symplectic. We say that $\overline{\rho}$ is *symplectic ordinarily automorphic of level prime to* l if furthermore π may be taken to be unramified at all places dividing l.

Corollary 7.5.1. *Assume that $\overline{\rho}$ is symplectic ordinarily automorphic, that $\overline{\rho}(G_{F^+(\zeta_l)})$ is big, and that $(\overline{F^+})^{\ker \mathrm{ad}\,\overline{\rho}}$ does not contain $F^+(\zeta_l)$. Then $R^{sympl,\tau,\psi}_{F^+,S}$ is a finite $\mathcal{O}$-module of rank at least one.*

Proof. This follows from Lemma 7.4.1, Corollary 5.1.3 and Proposition 7.3.4 (note we are free to choose F linearly disjoint from $(\overline{F^+})^{\ker \mathrm{ad}\,\overline{\rho}}(\zeta_l)$ over F^+). □

Suppose that $\rho : G_{F^+} \to \mathrm{GSp}_4(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ is crystalline. Then the similitude factor ψ of ρ is a crystalline character of G_{F^+}, so there is an integer n such that for all places $v|l$, $\psi|_{I_{F^+_v}} = \epsilon^n$. Suppose now that $\rho' : G_{F^+} \to \mathrm{GSp}_4(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ is another crystalline representation with similitude factor ψ', and that $\overline{\rho} = \overline{\rho}'$. Then $\overline{\psi} = \overline{\psi}'$, and there is an integer n' such that for all places $v|l$, $\psi|_{I_{F^+_v}} = \epsilon^{n'}$. Thus $\epsilon^{n'-n}$ is a crystalline character of G_{F^+} whose reduction mod l is everywhere unramified. This motivates the choice of similitude factor in the following theorem (in particular, it shows that our choice of similitude factor does not exclude any possibilities for the Hodge-Tate weights of the Galois representations we construct).

Theorem 7.5.2. *Let F^+ be a totally real field. Let $l \geq 5$ be a prime number such that $[F^+(\zeta_l) : F^+] > 2$. Suppose that*

$$\overline{\rho} : G_{F^+} \to \mathrm{GSp}_4(\overline{\mathbb{F}}_l)$$

is an irreducible representation, and let n be an integer such that $\overline{\epsilon}^n$ is an unramified character of G_{F^+}. Suppose that $\overline{\rho}$ satisfies the following assumptions.

1. *There are finite fields $\mathbb{F}_l \subset k \subset k'$ such that $\mathrm{Sp}_4(k) \subset \overline{\rho}(G_{F^+}) \subset (k')^\times \mathrm{GSp}_4(k)$.*
2. *The representation $\overline{\rho}$ is symplectic ordinarily automorphic of level prime to l; say $\overline{\rho} \cong \overline{\rho}_{l,\iota}(\pi)$, and write ψ for the similitude factor of $\rho_{l,\iota}(\pi)$.*

(3) *Define* $\psi_n := \psi\epsilon^n\tilde{\omega}^{-n}$, *where* $\tilde{\omega}$ *is the Teichmüller lift of the mod* l *cyclotomic character (so* $\overline{\psi}_n = \overline{\psi}$, *and* ψ_n *is crystalline). There is an element* $\lambda \in (\mathbb{Z}^4_+)^{\mathrm{Hom}(F^+,\overline{\mathbb{Q}}_l)}$ *such that*
   - *for every place* $v|l$ *of* F^+, $\overline{\rho}|_{G_{F_v^+}}$ *has a crystalline symplectic lift which is ordinary of weight* $(\lambda_\tau)_\tau$ *(where the indexing set runs over the embeddings* $\tau \in \mathrm{Hom}(F^+,\overline{\mathbb{Q}}_l)$ *inducing* v*) and similitude factor* ψ_n.

Then $\overline{\rho}$ *has a lift to a representation* $\rho : G_{F^+} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_l)$ *which is automorphic of level prime to* l, *such that for each place* $v|l$, $\rho|_{G_{F_v^+}}$ *is crystalline and ordinary of weight* λ_v.

Given any finite set of places S *of* F^+, *and an inertial type* τ_v *for each* $v \in S$ *not dividing* l *such that* $\overline{\rho}|_{G_{F_v^+}}$ *has a symplectic lift of type* τ_v *and similitude factor* ψ_n, ρ *can be chosen to be of similitude factor* ψ_n *and of type* τ_v *at* v *for all places* $v \in S$, $v \nmid l$. *More precisely, choose a model* $G_{F^+} \to \mathrm{GSp}_4(\mathcal{O}_K)$ *for* $\rho_{l,\imath}(\pi)$ *where* $K/\mathbb{Q}_l(\zeta_l)$ *is a finite extension in* $\overline{\mathbb{Q}}_l$ *containing the image of each embedding* $F^+ \hookrightarrow \overline{\mathbb{Q}}_l$. *Assume moreover that each* $\tau_{\tilde{v}}$ *is defined over* K. *Then, given a choice of an irreducible component of each* $\mathcal{O}_K$*-lifting ring* $R^{sympl,\tau_v,\psi_n}_{\overline{\rho}|_{G_{F_v^+}}}$ *(resp.* $R^{sympl,\triangle_{\lambda_v},cr,\psi_n}_{\overline{\rho}|_{G_{F_v^+}}}$*) for* $v \in S$, $v \nmid l$ *(resp.* $v|l$*), we may choose* ρ *may be chosen so as to give a point on each of these components.*

We remark that condition (1) implies that the similitude factor of $\rho_{l,\imath}(\pi)$ is uniquely determined. (If we had $\rho_{l,\imath}(\pi)^\vee \cong \rho_{l,\imath}(\pi)\psi'$ for some $\psi' \neq \psi$, then ρ would become reducible upon restriction to G_L for some finite abelian extension L/F^+.)

Proof. It suffices to prove the last statement. Enlarge S if necessary so that S contains all places of F^+ dividing l and all places at which $\overline{\rho}$ is ramified. Choose a totally imaginary quadratic extension F/F^+ such that all places in S split in F, and such that F is linearly disjoint from $(\overline{F^+})^{\ker\overline{\rho}}$. Note that we can choose a type τ_v at any place not dividing l such that $\overline{\rho}|_{G_{F_v^+}}$ has a symplectic lift of type τ_v and similitude factor ψ_n; $\rho_{l,\iota}(\pi)|_{G_{F_v^+}}$ provides such a lift if $n = 0$, and we may twist this lift in the general case (note that $\psi_n\psi^{-1}$ is unramified at v, and there is no obstruction to taking a square root of an unramified character). We then consider deformation problems as in the previous section. By Lemma 2.5.5 of [CHT08], and the fact that $\mathrm{Sp}_4(k)/\pm 1$ is simple, we see that $\overline{\rho}(G_{F^+(\zeta_l)})$ is big. Again, because $\mathrm{Sp}_4(k)/\pm 1$ is simple, the abelianisation of $\mathrm{ad}\,\overline{\rho}(G_{F^+})$ is a subgroup of

$$\mathrm{PGSp}_4(k)/(\mathrm{Sp}_4(k)/\pm 1) \xrightarrow{\sim} k^\times/(k^\times)^2.$$

As this latter group has cardinality 2 and $[F^+(\zeta_l) : F^+] > 2$, we see that $\overline{F^+}^{\ker\mathrm{ad}\,\overline{\rho}}$ does not contain $F^+(\zeta_l)$. Then Corollary 7.5.1 gives the existence of a Galois representation satisfying every property except possibly automorphicity, which follows from Theorem 4.3.1 and Lemmas 1.4 and 1.5 of [BLGHT09]. □

7.6. Companion forms for GSp_4**.** We now prove results for automorphic representations for GSp_4 over totally real fields by making use of known cases of functoriality between GSp_4 and GL_4. The main result we need is the following.

Theorem 7.6.1. *Let* M *be a number field. There is an injective map* $\pi \mapsto \Pi \boxtimes \theta$ *from the set of globally generic cuspidal representations* π *of* GSp_4 *over* M *to the* *set of globally generic representations* $\Pi \boxtimes \theta$ *of* $\mathrm{GL}_4 \times \mathrm{GL}_1$ *over* M*. This map has the following properties:*

(1) $\theta = \omega_\pi$ *(the central character of* π*), and the central character of* Π *is* ω_π^2*.*
(2) $\Pi \cong \Pi^\vee \otimes \omega_\pi$*.*
(3) *For each place* v *of* M *there is an equality of Weil-Deligne representations* $\mathrm{rec}_{\mathrm{GT}}(\pi_v) = \mathrm{rec}(\Pi_v)$*, where we denote the local Langlands correspondence of* [GT07] *by* $\mathrm{rec}_{\mathrm{GT}}$*, and consider* GSp_4 *as a subgroup of* GL_4*.*
(4) *If* $\Pi \boxtimes \theta$ *is such that* Π *is cuspidal, then* $\Pi \boxtimes \theta$ *is in the image of the map if and only if the partial L-function* $L^S(s, \Pi, \bigwedge^2 \otimes \theta^{-1})$ *has a pole at* $s = 1$ *(where* S *is any finite set of places of* M*).*
(5) *If* $\Pi \boxtimes \theta$ *is in the image of the map and* Π *is not cuspidal, then* Π *is an isobaric direct sum of two cuspidal representations of* GL_2*.*

Proof. This is a special case of Theorem 12.1 of [GT07]. □

Definition 7.6.2. Let F^+ be a totally real field, and let π be a cuspidal automorphic representation of GSp_4 over F^+. Assume further that π is automorphic of weight $\eta = (\eta_{v,1}, \eta_{v,2}; \alpha_v)_{v|\infty} \in (\mathbb{Z}^3)^{\mathrm{Hom}(F^+,\mathbb{R})}$, in the sense that for each $v|\infty$, π_v is a discrete series representation with the same central and infinitesimal characters as the finite-dimensional irreducible algebraic representation of highest weight given by

$$t = \mathrm{diag}(t_1, t_2, t_3, t_4) \mapsto t_1^{\eta_{v,1}} t_2^{\eta_{v,2}} \eta(t)^{-(\eta_{v,1}+\eta_{v,2}+\alpha_v)/2}.$$

Here $\eta_{v,1} \geq \eta_{v,2} \geq 0$ and $\eta_{v,1} + \eta_{v,2}$ has the same parity as α_v. Fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Then we say that *there is a Galois representation associated to* π if there is a continuous semisimple representation

$$\rho_{\pi,\iota} : G_{F^+} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_l)$$

such that:

- for each finite place $v \nmid l$,
$$\iota\, \mathrm{WD}(\rho_{\pi,\iota}|_{W_{F_v^+}})^{\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GT}}(\pi_v \otimes |\cdot|^{-3/2})^{\mathrm{ss}},$$
where $|\cdot|$ is the composition of the similitude character and the norm character.
- If π_v is unramified at a place $v|l$ then $\rho_{\pi,\iota}$ is crystalline at v, and in any case it is de Rham.
- Define $\lambda_{\iota,\eta} \in (\mathbb{Z}^4_+)^{\mathrm{Hom}(F^+,\overline{\mathbb{Q}}_l)}$ by letting
$$\lambda_{\iota,\eta,\tau} = (\delta_{\iota\circ\tau} + \eta_{\iota\circ\tau,1} + \eta_{\iota\circ\tau,2}, \delta_{\iota\circ\tau} + \eta_{\iota\circ\tau,1}, \delta_{\iota\circ\tau} + \eta_{\iota\circ\tau,2}, \delta_{\iota\circ\tau})$$
for each embedding $\tau : F^+ \hookrightarrow \overline{\mathbb{Q}}_l$, where
$$\delta_v := -\frac{1}{2}(\eta_{v,1} + \eta_{v,2} + \alpha_v)$$
for each $v|\infty$. Then for each $\tau : F^+ \hookrightarrow \overline{\mathbb{Q}}_l$ lying over a place v of F^+, the Hodge-Tate weights of $\rho_{\pi,\iota}|_{G_{F_v^+}}$ with respect to τ are the $\lambda_{\iota,\eta,\tau,j} + 4 - j$.

We remark that if $\rho_{\pi,\iota}$ is irreducible and is not induced from the Galois group of a finite extension of F^+, then it follows from Schur's lemma that it is unique up to isomorphism as a symplectic representation.

We now define what it means for a cuspidal automorphic representation of GSp_4 to be ordinary. We could do this directly in terms of Hecke operators on GSp_4, but for the sake of brevity we use the local Langlands correspondences for GL_4 and GSp_4 and the definition of ordinarity for GL_4.

Definition 7.6.3. Let $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ be an isomorphism, and let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{F^+})$ which is of weight $\eta = (\eta_{v,1}, \eta_{v,2}; \alpha_v)_{v|\infty}$ in the above sense. Let $\lambda \in (\mathbb{Z}^4_+)^{\mathrm{Hom}(F^+,\mathbb{R})}$ be defined by $\lambda_v = (\delta_v + \eta_{v,1} + \eta_{v,2}, \delta_v + \eta_{v,1}, \delta_v + \eta_{v,2}, \delta_v)$. We say that π is *ι-ordinary* if for each $v|l$, the irreducible admissible representation Π_v of $\mathrm{GL}_4(F^+_v)$ with

$$\mathrm{rec}_{\mathrm{GT}}(\pi_v) = \mathrm{rec}(\Pi_v)$$

satisfies $(\iota^{-1}\Pi_v)^{\mathrm{ord}} \neq 0$, where the space $(\iota^{-1}\Pi_v)^{\mathrm{ord}}$ is defined as in section 4.1. (We remind the reader that this notion depends on λ, and thus on η.)

Definition 7.6.4. We say that a continuous irreducible representation

$$\rho : G_{F^+} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_l)$$

is GSp_4-*automorphic* (of weight $\lambda \in (\mathbb{Z}^4_+)^{\mathrm{Hom}(F^+,\overline{\mathbb{Q}}_l)}$) if there is a π and an $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ with $\rho \cong \rho_{\pi,\iota}$, and for each $\tau : F^+ \hookrightarrow \overline{\mathbb{Q}}_l$ lying over a place v of F^+, the Hodge-Tate weights of $\rho_{\pi,\iota}$ with respect to τ are the $\lambda_{\tau,j} + 4 - j$. By the above definitions, we see that this is equivalent to π being automorphic of weight η with $\lambda_{\iota,\eta} = \lambda$. We say that ρ is GSp_4-*automorphic and holomorphic* if π can be chosen to be a holomorphic discrete series at all infinite places, and that ρ is GSp_4-automorphic and generic if π can be chosen to be globally generic (note that it is possible for ρ to be both holomorphic and generic, corresponding to different choices of π in the same global L-packet). We say that ρ is GSp_4-*ordinarily automorphic* if π can be chosen to be ι-ordinary. We say that ρ is GSp_4-*ordinarily automorphic and holomorphic* (respectively *generic*) if π may be chosen to be simultaneously ι-ordinary and holomorphic discrete series at all infinite places (respectively globally generic). Finally, we say in addition that ρ is automorphic of level prime to l if π_l is unramified. We say that ρ is automorphic of level prime to l and holomorphic if π can be chosen to be simultaneously of level prime to l, and to be holomorphic discrete series at all infinite places.

In recent work ([Sor10]) Sorensen has used Theorem 7.6.1 and the constructions of [HT01] to associate Galois representations to certain globally generic cuspidal representations of GSp_4 over totally real fields. In particular, he obtains the following theorem, which gives a ready supply of Galois representations associated to automorphic representations of GSp_4.

Theorem 7.6.5. *Let F^+ be a totally real field, and let π be a globally generic cuspidal automorphic representation of GSp_4 over F^+ of weight η for some η. Assume that for some finite place v the local component π_v is an unramified twist of the Steinberg representation. Then there is a Galois representation associated to π.*

It is now straightforward to use the results of the previous sections to deduce a theorem about companion forms for automorphic representations of GSp_4 over F^+.

Theorem 7.6.6. *Let F^+ be a totally real field. Let $l \geq 5$ be a prime number such that $[F^+(\zeta_l) : F^+] > 2$. Fix $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Suppose that*

$$\overline{\rho} : G_{F^+} \to \mathrm{GSp}_4(\overline{\mathbb{F}}_l)$$

is an irreducible representation, and let n *be an integer such that* $\overline{\epsilon}^n$ *is an unramified character of* G_{F^+}*. Suppose that* $\overline{\rho}$ *satisfies the following assumptions.*

(1) *There are finite fields* $\mathbb{F}_l \subset k \subset k'$ *such that* $\mathrm{Sp}_4(k) \subset \overline{\rho}(G_{F^+}) \subset (k')^\times \mathrm{GSp}_4(k)$*.*
(2) *The representation* $\overline{\rho}$ *has a lift which is* GSp_4*-ordinarily automorphic and generic of level prime to* l*; say* $\overline{\rho} \cong \overline{\rho}_{\pi,\iota}$*, and write* ψ *for the similitude factor of* $\rho_{\pi,\iota}$*.*
(3) *Define* $\psi_n := \psi\epsilon^n\tilde{\omega}^{-n}$*, where* $\tilde{\omega}$ *is the Teichmüller lift of the mod* l *cyclotomic character (so* $\overline{\psi}_n = \overline{\psi}$*, and* ψ_n *is crystalline). There is a* $\lambda \in (\mathbb{Z}^4_+)^{\mathrm{Hom}(F^+,\overline{\mathbb{Q}}_l)}$ *such that*
    - *for every place* $v|l$ *of* F^+*,* $\overline{\rho}|_{G_{F^+_v}}$ *has an ordinary crystalline symplectic lift of weight* $(\lambda_\tau)_\tau$ *(where the indexing set runs over the embeddings* $\tau \in \mathrm{Hom}(F^+, \overline{\mathbb{Q}}_l)$ *inducing* v*) and similitude factor* ψ_n*.*

Then $\overline{\rho}$ *has an ordinary crystalline symplectic lift* ρ *of weight* λ *and similitude factor* ψ_n*, which is* GSp_4*-ordinarily automorphic of level prime to* l *and generic. If* $F^+ = \mathbb{Q}$ *then* ρ *is also* GSp_4*-ordinarily automorphic of level prime to* l *and holomorphic.*

Given any finite set of places S *of* F^+*, and an inertial type* τ_v *for each* $v \in S$ *not dividing* l *such that* $\overline{\rho}|_{G_{F^+_v}}$ *has a symplectic lift of type* τ_v *and similitude factor* ψ_n*,* ρ *can be chosen to have type* τ_v *at* v *for all places* $v \in S$*,* $v \nmid l$*. More precisely, choose a model* $G_{F^+} \to \mathrm{GSp}_4(\mathcal{O}_K)$ *for* $\rho_{\pi,\iota}$ *where* $K/\mathbb{Q}_l(\zeta_l)$ *is a finite extension in* $\overline{\mathbb{Q}}_l$ *containing the image of each embedding* $F^+ \hookrightarrow \overline{\mathbb{Q}}_l$*. Assume moreover that each* $\tau_{\tilde{v}}$ *is defined over* K*. Then, given a choice of an irreducible component of each* $\mathcal{O}_K$*-lifting ring* $R^{sympl,\tau_v,\psi_n}_{\overline{\rho}|_{G_{F^+_v}}}$ *(resp.* $R^{sympl,\triangle_{\lambda_v},cr,\psi_n}_{\overline{\rho}|_{G_{F^+_v}}}$*) for* $v \in S$*,* $v \nmid l$ *(resp.* $v|l$*), we may choose* ρ *may be chosen so as to give a point on each of these components.*

Proof. This follows from Theorems 7.6.1 and 7.5.2. Note that if π is a globally generic automorphic representation of GSp_4 with $\overline{\rho}_{\pi,\iota} \cong \overline{\rho}$, then the transfer of π to GL_4 is cuspidal (because $\overline{\rho}$ is irreducible). Conversely, if Π is a RAESDC automorphic representation of $\mathrm{GL}_4(\mathbb{A}_{F^+})$ with $\Pi^\vee \cong \chi\Pi$, and $\rho_{l,\iota}(\Pi)$ is symplectic, it follows that $L^S(s, \Pi, \bigwedge^2 \otimes \chi^{-1})$ has a pole at $s = 1$ (because the corresponding statement is true for $\rho_{l,\iota}(\Pi)$).

In the case $F^+ = \mathbb{Q}$, the fact that ρ is also GSp_4-automorphic and holomorphic follows from Proposition 1.5 of [Wei05] (because our assumptions on $\overline{\rho}$ obviously imply that if $\overline{\rho} \cong \overline{\rho}_{\pi,\iota}$ then π is neither CAP nor weak endoscopic). □

In many cases we can make this rather more explicit, just as in the unitary case.

Lemma 7.6.7. *Let* M *be a finite extension of* $\mathbb{Q}_l$*. Take* $\lambda \in (\mathbb{Z}^4_+)^{\mathrm{Hom}(M,\overline{\mathbb{Q}}_l)}$*. Let* E *be a finite extension of* $\mathbb{Q}_l$ *with residue field* k*. Let* ψ_i*,* $1 \le i \le 4$*, be crystalline characters* $G_M \to E^\times$*, with* $\psi_i|_{I_M} = \chi_i^\lambda|_{I_M}$ *in the notation of Definition 3.1.2. Assume that* $\psi_1\psi_4 = \psi_2\psi_3$*. Suppose that* $\overline{\rho} : G_M \to \mathrm{GSp}_4(k)$ *is of the form*

$$\begin{pmatrix} \overline{\mu}_1 & * & * & * \\ 0 & \overline{\mu}_2 & * & * \\ 0 & 0 & \overline{\mu}_3 & * \\ 0 & 0 & 0 & \overline{\mu}_4 \end{pmatrix}$$

where $\overline{\psi}_i = \overline{\mu}_i$ *for* $1 \le i \le 4$*. Suppose that none of the characters* $\overline{\mu}_i\overline{\mu}_j^{-1}$*,* $i < j$*, are equal to* $\overline{\epsilon}$*. Then* $\overline{\rho}$ *has a lift to a crystalline representation* $\rho : G_M \to \mathrm{GSp}_4(E)$ *of* *the form*

$$\begin{pmatrix} \psi_1 & * & * & * \\ 0 & \psi_2 & * & * \\ 0 & 0 & \psi_3 & * \\ 0 & 0 & 0 & \psi_4 \end{pmatrix}$$

Proof. This is proved in exactly the same way as Lemma 3.1.5. □

Just as in section 3.1.3, we can consider ordinary crystalline lifts of a particular form. Given $\overline{\rho}$, λ as in the previous lemma (but no longer requiring that the characters $\overline{\mu}_i\overline{\mu}_j^{-1} \neq \overline{\epsilon}$), we can consider ordinary lifts where we demand that $\psi_i|_{I_M} = \chi_i^{\lambda}|_{I_M}$ and $\overline{\psi}_i = \overline{\mu}_i$, $1 \leq i \leq 4$. This gives a deformation ring $R^{sympl,\triangle_\lambda,cr,\psi}_{\overline{\rho},\overline{\mu}}$, and the following lemma may be proved in exactly the same way as Lemma 3.1.8.

Lemma 7.6.8. *After inverting l, the morphism* $\mathrm{Spec}\, R^{sympl,\triangle_\lambda,cr,\psi}_{\overline{\rho},\overline{\mu}} \to \mathrm{Spec}\, R^{sympl,\triangle_\lambda,cr,\psi}_{\overline{\rho}}$ *becomes a closed immersion identifying* $\mathrm{Spec}\, R^{sympl,\triangle_\lambda,cr,\psi}_{\overline{\rho},\overline{\mu}}[1/l]$ *with a union of irreducible components of* $\mathrm{Spec}\, R^{sympl,\triangle_\lambda,cr,\psi}_{\overline{\rho}}[1/l]$.

Theorem 7.6.9. *Let F^+ be a totally real field. Let $l \geq 5$ be a prime number such that $[F^+(\zeta_l) : F^+] > 2$. Fix $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Suppose that*

$$\overline{\rho} : G_{F^+} \to \mathrm{GSp}_4(\overline{\mathbb{F}}_l)$$

is an irreducible representation. Suppose that the following conditions hold.

1. *There are finite fields $\mathbb{F}_l \subset k \subset k'$ such that $\mathrm{Sp}_4(k) \subset \overline{\rho}(G_{F^+}) \subset (k')^\times \mathrm{GSp}_4(k)$.*
2. *The representation $\overline{\rho}$ has a lift which is GSp_4-ordinarily automorphic and generic of level prime to l.*
3. *There is a $\lambda \in (\mathbb{Z}^4_+)^{\mathrm{Hom}(F^+,\overline{\mathbb{Q}}_l)}$ such that*
    - *$m := \lambda_{\tau,1} + \lambda_{\tau,4} = \lambda_{\tau,2} + \lambda_{\tau,3}$ is independent of τ, and*
    - *for every place $v|l$, $\overline{\rho}|_{G_{F_v^+}}$ is isomorphic to a representation*

    $$\begin{pmatrix} \overline{\mu}_{v,1} & * & * & * \\ 0 & \overline{\mu}_{v,2} & * & * \\ 0 & 0 & \overline{\mu}_{v,3} & * \\ 0 & 0 & 0 & \overline{\mu}_{v,4} \end{pmatrix}$$

    where none of $\overline{\mu}_{v,i}\overline{\mu}_{v,j}^{-1}$, $i < j$, are equal to $\overline{\epsilon}$. Furthermore, $\overline{\mu}_{v,i}|_{I_{F_v^+}} = \overline{\chi}_i^{\lambda_v}|_{I_{F_v^+}}$ for each i (in the notation of Definition 3.1.2).

Then $\overline{\rho}$ has an ordinary crystalline symplectic lift ρ of weight λ, which is GSp_4-ordinarily automorphic of level prime to l and generic, with similitude factor ψ, say. Furthermore $\psi\epsilon^{m+3}$ is a finite order character, and for every place $v|l$, $\rho|_{G_{F_v^+}}$ is isomorphic to a representation of the form

$$\begin{pmatrix} \psi_{v,1} & * & * & * \\ 0 & \psi_{v,2} & * & * \\ 0 & 0 & \psi_{v,3} & * \\ 0 & 0 & 0 & \psi_{v,4} \end{pmatrix}$$

where the $\psi_{v,i}$ are crystalline characters such that $\overline{\psi}_{v,i} = \overline{\mu}_{v,i}$ and $\psi_{v,i}|_{I_{F_v^+}} = \chi_i^{\lambda_v}|_{I_{F_v^+}}$. Finally, if $F^+ = \mathbb{Q}$ then we may in addition assume that ρ is GSp_4-ordinarily automorphic and holomorphic (of level prime to l).

Proof. This follows from Theorem 7.6.6, together with Lemma 7.6.7 and Lemma 7.6.8. □

Remark 7.6.10. It is expected that whenever π is a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_M)$, M a number field, and π is neither CAP nor weak endoscopic, then π is stable. In the special case that π is a discrete series representation at each infinite place, this means that if $\pi = \pi_f \otimes \pi_\infty$ (with π_f, π_∞ respectively denoting the finite and infinite factors of π) then $\pi_f \otimes \pi'_\infty$ is also automorphic for any π'_∞ in the same L-packet as π_∞, i.e. we are free to change between holomorphic and generic discrete series at any infinite place. Assuming this result, which is expected to follow from Arthur's work on the trace formula (cf. [Art04]), one could conclude that the representation ρ in the above theorems is also GSp_4-automorphic and holomorphic, even if $F^+ \neq \mathbb{Q}$.

References

[ADP02] Avner Ash, Darrin Doud, and David Pollack, *Galois representations with conjectural connections to arithmetic cohomology*, Duke Math. J. **112** (2002), no. 3, 521–579. MR MR1896473 (2003g:11055)

[Art04] James Arthur, *Automorphic representations of* GSp(4), Contributions to automorphic forms, geometry, and number theory, Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 65–81. MR MR2058604 (2005d:11074)

[BLGGT10] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Potential automorphy and change of weight*, preprint arXiv:1010.2561, 2010.

[BLGHT09] Tom Barnet-Lamb, David Geraghty, Michael Harris, and Richard Taylor, *A family of Calabi-Yau varieties and potential automorphy II*, Preprint, 2009.

[Car94] Henri Carayol, *Formes modulaires et représentations galoisiennes à valeurs dans un anneau local complet*, p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991), Contemp. Math., vol. 165, Amer. Math. Soc., Providence, RI, 1994, pp. 213–237. MR MR1279611 (95i:11059)

[CH09] Gaëtan Chenevier and Michael Harris, *Construction of automorphic Galois representations, II*, preprint, 2009.

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Pub. Math. IHES **108** (2008), 1–181.

[CV92] Robert F. Coleman and José Felipe Voloch, *Companion forms and Kodaira-Spencer theory*, Invent. Math. **110** (1992), no. 2, 263–281. MR MR1185584 (93i:11063)

[dJ95] A. J. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Inst. Hautes Études Sci. Publ. Math. (1995), no. 82, 5–96 (1996). MR 1383213 (97f:14047)

[Gee07] Toby Gee, *Companion forms over totally real fields. II*, Duke Math. J. **136** (2007), no. 2, 275–284.

[Gee10] ———, *Automorphic lifts of prescribed types*, Math. Annalen (to appear) (2010).

[Ger09] David Geraghty, *Modularity lifting theorems for ordinary Galois representations*, `http://www.math.harvard.edu/~geraghty/`, 2009, preprint.

[Gro90] Benedict H. Gross, *A tameness criterion for Galois representations associated to modular forms (mod p)*, Duke Math. J. **61** (1990), no. 2, 445–517. MR MR1074305 (91i:11060)

[GT05] Alain Genestier and Jacques Tilouine, *Systèmes de Taylor-Wiles pour* GSp_4, Astérisque (2005), no. 302, 177–290, Formes automorphes. II. Le cas du groupe $GSp(4)$. MR MR2234862 (2007k:11086)

[GT07] Wee Teck Gan and Shuichiro Takeda, *The local Langlands conjecture for* GSp(4), to appear in Annals of Mathematics, 2007.

[Her09] Florian Herzig, *The weight in a Serre-type conjecture for tame n-dimensional Galois representations*, Duke Math. J. **149** (2009), no. 1, 37–116.

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR MR1876802 (2002m:11050)

[HT08] Florian Herzig and Jacques Tilouine, *Conjectures de type Serre et formes compagnons pour* GSp(4), preprint, 2008.

[Jan03] Jens Carsten Jantzen, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003. MR MR2015057 (2004h:20061)

[Kis07] Mark Kisin, *Modularity of 2-dimensional Galois representations*, Current developments in mathematics, 2005, Int. Press, Somerville, MA, 2007, pp. 191–230. MR MR2459302

[Kis08] ———, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21** (2008), no. 2, 513–546. MR MR2373358 (2009c:11194)

[Kis09] ———, *Moduli of finite flat group schemes, and modularity*, Annals of Math. **170** (2009), no. 3, 1085–1180.

[KW08] Chandrashekhar Khare and Jean-Pierre Wintenberger, *On Serre's conjecture for 2-dimensional mod p representations of the absolute Galois group of the rationals*, to appear in Annals of Mathematics (2008).

[Lab09] Jean-Pierre Labesse, *Changement de base CM et séries discrètes*, preprint, 2009.

[Maz89] B. Mazur, *Deforming Galois representations*, Galois groups over **Q** (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., vol. 16, Springer, New York, 1989, pp. 385–437. MR MR1012172 (90k:11057)

[Nek93] Jan Nekovář, *On p-adic height pairings*, Séminaire de Théorie des Nombres, Paris, 1990–91, Progr. Math., vol. 108, Birkhäuser Boston, Boston, MA, 1993, pp. 127–202. MR MR1263527 (95j:11050)

[Ser87] Jean-Pierre Serre, *Sur les représentations modulaires de degré* 2 *de* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Duke Math. J. **54** (1987), no. 1, 179–230. MR MR885783 (88g:11022)

[Sor10] Claus Sorensen, *Galois representations and Hilbert-Siegel modular forms*, Documenta Mathematica **15** (2010), 623–670.

[Tay08] Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations. II*, Pub. Math. IHES **108** (2008), 183–239.

[Tho10] Jack Thorne, *On the automorphy of l-adic Galois representations with small residual image*, preprint available at `http://www.math.harvard.edu/~thorne`, 2010.

[Til96] Jacques Tilouine, *Deformations of Galois representations and Hecke algebras*, Published for The Mehta Research Institute of Mathematics and Mathematical Physics, Allahabad, 1996. MR MR1643682 (99i:11038)

[Til09] ———, *Formes compagnons et complexe BGG dual pour* GSp(4), preprint, 2009.

[Wei05] Rainer Weissauer, *Four dimensional Galois representations*, Astérisque (2005), no. 302, 67–150, Formes automorphes. II. Le cas du groupe $GSp(4)$. MR MR2234860 (2007f:11057)

E-mail address: `tgee@math.harvard.edu`

Department of Mathematics, Harvard University

E-mail address: `geraghty@math.harvard.edu`

Department of Mathematics, Harvard University