SERRE WEIGHTS FOR $U(n)$.

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Abstract. We study the weight part of (a generalisation of) Serre's conjecture for mod l Galois representations associated to automorphic representations on unitary groups of rank n for odd primes l. Given a modular Galois representation, we use automorphy lifting theorems to prove that it is modular in many other weights. We make no assumptions on the ramification or inertial degrees of l. We give an explicit strengthened result when $n = 3$ and l splits completely in the underlying CM field.

Contents

1. Introduction

In recent years there has been considerable progress in formulating generalisations of Serre's conjecture, and in particular of the weight part of Serre's conjecture, for higher-dimensional groups; cf. [ADP02], [Her09], [Gee11], [EGHS14]. There has been rather less progress in proving cases of these conjectures; indeed, the only results that we are aware of are the essentially complete treatment of the ordinary case for definite unitary groups in [GG12], and the results of [EGH13] for definite unitary groups of rank 3.

In the present paper, we use the automorphy lifting theorems developed in [BLGG11], [BLGG12] and [BLGGT14b] to prove that a modular Galois representation, coming from an automorphic form on $U(n)$, is necessarily modular in a number of additional weights predicted by the conjectures of [Her09] and [EGHS14]. Rather complete results are available in the case $n = 2$, which are worked out in detail in the papers [BLGG13, GLS14, GLS13], so we concentrate in this paper on the case that $n > 2$. The additional complications are twofold. Firstly, we no longer know that any modular Galois representation admits a potentially diagonalizable lift (in the case $n = 2$, this is proved in [BLGG13] as a consequence of the

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results of [Kis09] and [Gee06]). Secondly, the relationship between being modular of some weight and having an automorphic lift of some weight is substantially more complicated for $n > 2$ than it is for $n = 2$; in particular, it is no longer the case that given an irreducible mod l representation F of $\mathrm{GL}_n(\mathbb{F}_l)$, there is necessarily an irreducible characteristic zero algebraic representation W of GL_n whose reduction modulo l is F. Instead, one finds that F is the socle of the reduction modulo l of some W, and this gives strictly weaker information.

As a result of these two difficulties, our main theorems have two restrictions. Let F be a CM field, and let $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ be our given modular Galois representation. Firstly, we must assume that $\bar{r}$ has a potentially diagonalizable automorphic lift. This assumption is perhaps not as serious as it initially sounds, as it is conjecturally always satisfied, and in particular is known to hold provided that l is unramified in F and $\bar{r}$ has an automorphic lift of sufficiently small weight. Secondly, rather than prove that $\bar{r}$ is modular of some particular weight, we typically only provide a list of weights, and guarantee that $\bar{r}$ is modular of some weight in this list. In fact, it is often the case that only one weight on this list is predicted by the conjectures of [Her09] and [EGHS14], and it should presumably be possible to prove modularity in this weight in many cases using integral p-adic Hodge theory. We carry out such an analysis in detail in the case $n = 3$, defining a list of conjectural weights $W^{\mathrm{obv}}(\bar{r})$, and obtaining the following result (Theorem 5.1.4).

Theorem A. *Let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, and that l splits completely in F. Suppose that $l > 2$, and that $\bar{r} : G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_l)$ is an irreducible representation with split ramification. Assume that there is a RACSDC automorphic representation Π of $\mathrm{GL}_3(\mathbb{A}_F)$ of weight $\mu \in (\mathbb{Z}^3_+)_0^{\mathrm{Hom}\,(F,\mathbb{C})}$ and level prime to l such that*

- $\bar{r} \cong \bar{r}_{l,\imath}(\Pi)$ *(so in particular, $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{-2}$).*
- *For each $\tau \in \mathrm{Hom}\,(F, \mathbb{C})$, $\mu_{\tau,1} - \mu_{\tau,3} \leq l - 3$.*
- $\bar{r}(G_{F(\zeta_l)})$ *is adequate.*

Let $a \in (\mathbb{Z}^3_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ be a generic Serre weight. Assume that $a \in W^{\mathrm{obv}}(\bar{r})$. Then $\bar{r}$ is modular of weight a.

(See sections 2 and 4 for any unfamiliar terminology, and section 5 for the definition of "generic" that we are using, which is extremely mild.) We should point out that we do *not* expect that $W^{\mathrm{obv}}(\bar{r})$ contains all the weights in which $\bar{r}$ is modular; rather, it consists of those weights which are "obvious" in the terminology of [EGHS14]. (It is perhaps worth remarking that despite the name, it is not obvious that $\bar{r}$ is modular in any of these weights!) In order to prove this theorem we make use of Fontaine-Laffaille theory; it seems likely that if one could compute the possible reductions of crystalline Galois representations outside of the Fontaine-Laffaille range then one could prove an analogous theorem for $n > 3$.

We now outline the structure of this paper. In Section 2 we define the spaces of automorphic forms that we work with, and define what it means for $\bar{r}$ to be modular of some weight. In Section 3 we establish the main lifting theorem that we need, a corollary of the results of [BLGGT14b]. In Section 4 we define the set of weights $W^{\mathrm{obv}}(\bar{r})$, recall some results from Fontaine-Laffaille theory, and prove our main results for arbitrary n. Finally, in Section 5 we prove Theorem A.

1.1. **Notation.** If M is a field, we let G_M denote its absolute Galois group. We write all matrix transposes on the left; so ${}^tA$ is the transpose of A. Let ϵ_l denote the l-adic cyclotomic character, and $\bar{\epsilon}_l$ or ω_l the mod l cyclotomic character. If M is a finite extension of $\mathbb{Q}_p$ for some p, we write I_M for the inertia subgroup of G_M. If R is a local ring we write $\mathfrak{m}_R$ for the maximal ideal of R.

We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. For each prime p we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

If W is a de Rham representation of G_K over $\overline{\mathbb{Q}}_l$ and if $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$ then by definition the multiset $\mathrm{HT}_\tau(W)$ of Hodge-Tate weights of W with respect to τ contains i with multiplicity $\dim_{\overline{\mathbb{Q}}_l}(W \otimes_{\tau,K} \widehat{\overline{K}}(i))^{G_K}$. Thus for example $\mathrm{HT}_\tau(\epsilon_l) = \{-1\}$.

If K is a finite extension of $\mathbb{Q}_p$ for some p, we will let rec_K be the local Langlands correspondence of [HT01], so that if π is an irreducible complex admissible representation of $\mathrm{GL}_n(K)$, then $\mathrm{rec}_K(\pi)$ is a Weil-Deligne representation of the Weil group W_K. We will write rec for rec_K when the choice of K is clear. We write $\mathrm{Art}_K : K^\times \to W_K$ for the isomorphism of local class field theory, normalised so that uniformisers correspond to geometric Frobenius elements.

Let K be a finite extension of $\mathbb{Q}_l$ with residue field k. For each $\sigma \in \mathrm{Hom}\,(k, \overline{\mathbb{F}}_l)$ we define the fundamental character ω_σ corresponding to σ to be the composite

$$I_{K^{\mathrm{ab}}/K} \xrightarrow{\mathrm{Art}_K^{-1}} \mathcal{O}_K^\times \longrightarrow k^\times \xrightarrow{\sigma^{-1}} \overline{\mathbb{F}}_l^{\,\times}.$$

Note that if $k = \mathbb{F}_l$ then $\omega_\sigma = \omega_l$. For any algebraic extension L of $\mathbb{Q}_l$, we often denote by $\mathrm{Hom}\,(K, L)$ the set of field homomorphisms from K to L which are continuous for the l-adic topologies on K and L (or equivalently, which are $\mathbb{Q}_l$-linear).

2. Definitions

2.1. Let l be a prime, and let F be an imaginary CM field with maximal totally real field subfield F^+. We assume throughout this paper that:

- F/F^+ is unramified at all finite places.
- Every place $v|l$ of F^+ splits in F.
- If n is even, then $n[F^+ : \mathbb{Q}]/2$ is also even.

Under these hypotheses, there is a reductive algebraic group G/F^+ with the following properties:

- G is an outer form of GL_n, with $G_{/F} \cong \mathrm{GL}_{n/F}$.
- If v is a finite place of F^+, G is quasi-split at v.
- If v is an infinite place of F^+, then $G(F_v^+) \cong U_n(\mathbb{R})$.

To see that such a group exists, one may argue as follows. Let B denote the matrix algebra $M_n(F)$. An involution $\ddagger$ of the second kind on B gives a reductive group $G_\ddagger$ over F^+ by setting

$$G_\ddagger(R) = \{g \in B \otimes_{F^+} R : g^\ddagger g = 1\}$$

for any F^+-algebra R. Any such $G_\ddagger$is an outer form of GL_n, with $G_{\ddagger/F} \cong \mathrm{GL}_{n/F}$. One can choose $\ddagger$ such that

- If v is a finite place of F^+, $G_\ddagger$ is quasi-split at v.
- If v is an infinite place of F^+, then $G_\ddagger(F_v^+) \cong U_n(\mathbb{R})$.

To see this, one uses the argument of Lemma I.7.1 of [HT01]. We then fix some choice of $\ddagger$ as above, and take $G = G_\ddagger$.

As in section 3.3 of [CHT08] we define a model for G over $\mathcal{O}_{F^+}$ in the following way. We choose an order $\mathcal{O}_B$ in B such that $\mathcal{O}_B^\ddagger = \mathcal{O}_B$, and $\mathcal{O}_{B,w}$ is a maximal order in B_w for all places w of F which are split over F^+ (see section 3.3 of [CHT08] for a proof that such an order exists). Then we can define G over $\mathcal{O}_{F^+}$ by setting

$$G(R) = \{g \in \mathcal{O}_B \otimes_{\mathcal{O}_{F^+}} R : g^\ddagger g = 1\}$$

for any $\mathcal{O}_{F^+}$-algebra R.

If v is a place of F^+ which splits as ww^c over F, then we choose an isomorphism

$$\iota_v : \mathcal{O}_{B,v} \xrightarrow{\sim} M_n(\mathcal{O}_{F,v}) = M_n(\mathcal{O}_{F_w}) \oplus M_n(\mathcal{O}_{F_{w^c}})$$

such that $\iota_v(x^\ddagger) = {}^t\iota_v(x)^c$. This gives rise to an isomorphism

$$\iota_w : G(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}_{F_w})$$

sending $\iota_v^{-1}(x, {}^tx^{-c})$ to x.

Let K be an algebraic extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ which contains the image of every embedding $F \hookrightarrow \overline{\mathbb{Q}}_l$, let $\mathcal{O}$ denote the ring of integers of K, and let k denote the residue field of K. Let S_l denote the set of places of F^+ lying over l, and for each $v \in S_l$ fix a place $\widetilde{v}$ of F lying over v. Let $\tilde{S}_l$ denote the set of places $\tilde{v}$ for $v \in S_l$.

Let W be an $\mathcal{O}$-module with an action of $G(\mathcal{O}_{F^+,l})$, and let $U \subset G(\mathbb{A}_{F^+}^\infty)$ be a compact open subgroup with the property that for each $u \in U$, if u_l denotes the projection of u to $G(F_l^+)$, then $u_l \in G(\mathcal{O}_{F_l^+})$. Let $S(U, W)$ denote the space of algebraic modular forms on G of level U and weight W, i.e. the space of functions

$$f : G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty) \to W$$

with $f(gu) = u_l^{-1} f(g)$ for all $u \in U$.

Let $\widetilde{I}_l$ denote the set of embeddings $F \hookrightarrow K$ giving rise to a place in $\widetilde{S}_l$. For any $\widetilde{v} \in \widetilde{S}_l$, let $\widetilde{I}_{\widetilde{v}}$ denote the set of elements of $\widetilde{I}_l$ lying over $\widetilde{v}$. Let $\mathbb{Z}_+^n$ denote the set of tuples $(\lambda_1, \ldots, \lambda_n)$ of integers with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. For any $\lambda \in \mathbb{Z}_+^n$, view λ as a dominant character of the algebraic group $\mathrm{GL}_{n/\mathcal{O}}$ in the usual way, and let M'_λ be the algebraic $\mathcal{O}$-representation of GL_n given by

$$M'_\lambda := \mathrm{Ind}_{B_n}^{\mathrm{GL}_n} (w_0\lambda)_{/\mathcal{O}}$$

where B_n is the standard upper-triangular Borel subgroup of GL_n, and w_0 is the longest element of the Weyl group (see [Jan03] for more details of these notions). Write M_λ for the $\mathcal{O}$-representation of $\mathrm{GL}_n(\mathcal{O})$ obtained by evaluating M'_λ on $\mathcal{O}$. For any $\lambda \in (\mathbb{Z}_+^n)^{\widetilde{I}_{\widetilde{v}}}$, let W_λ be the free $\mathcal{O}$-module with an action of $\mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$ given by

$$W_\lambda := \otimes_{\tau \in \widetilde{I}_{\widetilde{v}}} M_{\lambda_\tau} \otimes_{\mathcal{O}_{F_{\tilde{v}}},\tau} \mathcal{O}.$$

We give this an action of $G(\mathcal{O}_{F^+,v})$ via $\iota_{\widetilde{v}}$. For any $\lambda \in (\mathbb{Z}_+^n)^{\widetilde{I}_l}$, let W_λ be the free $\mathcal{O}$-module with an action of $G(\mathcal{O}_{F^+,l})$ given by

$$W_\lambda := \otimes_{\widetilde{v} \in \widetilde{S}_l} W_{\lambda_{\widetilde{v}}}.$$

If A is an $\mathcal{O}$-module we let

$$S_\lambda(U, A) := S(U, W_\lambda \otimes_{\mathcal{O}} A).$$

For any compact open subgroup U as above of $G(\mathbb{A}^\infty_{F^+})$ we may write $G(\mathbb{A}^\infty_{F^+}) = \coprod_i G(F^+)t_iU$ for some finite set $\{t_i\}$. Then there is an isomorphism

$$S(U,W) \to \oplus_i W^{U\cap t_i^{-1}G(F^+)t_i}$$

given by $f \mapsto (f(t_i))_i$. We say that U is *sufficiently small* if for some finite place v of F^+ the projection of U to $G(F^+_v)$ contains no element of finite order other than the identity. Suppose that U is sufficiently small. Then for each i as above we have $U \cap t_i^{-1}G(F^+)t_i = \{1\}$, so taking $W = W_\lambda \otimes_{\mathcal{O}} A$ we see that for any $\mathcal{O}$-module A, we have

$$S_\lambda(U,A) \cong S_\lambda(U,\mathcal{O}) \otimes_{\mathcal{O}} A.$$

We note when U is not sufficiently small, we still have $S_\lambda(U,A) \cong S_\lambda(U,\mathcal{O}) \otimes_{\mathcal{O}} A$ whenever A is $\mathcal{O}$-flat.

We now recall the relationship between our spaces of algebraic automorphic forms and the space of automorphic forms on G. Write $S_\lambda(\overline{\mathbb{Q}}_l)$ for the direct limit of the spaces $S_\lambda(U,\overline{\mathbb{Q}}_l)$ over compact open subgroups U as above (with the transition maps being the obvious inclusions $S_\lambda(U,\overline{\mathbb{Q}}_l) \subset S_\lambda(V,\overline{\mathbb{Q}}_l)$ whenever $V \subset U$). Concretely, $S_\lambda(\overline{\mathbb{Q}}_l)$ is the set of functions

$$f : G(F^+)\backslash G(\mathbb{A}_{F^+}) \to W_\lambda \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$$

such that there is a compact open subgroup U of $G(\mathbb{A}^{\infty,l}_{F^+}) \times G(\mathcal{O}_{F^+,l})$ with

$$f(gu) = u_l^{-1}f(g)$$

for all $u \in U$, $g \in G(\mathbb{A}_{F^+})$. This space has a natural left action of $G(\mathbb{A}^\infty_{F^+})$ via

$$(g\cdot f)(h) := g_l f(hg).$$

Fix an isomorphism $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. For each embedding $\tau : F^+ \hookrightarrow \mathbb{R}$, there is a unique embedding $\tilde{\tau} : F \hookrightarrow \mathbb{C}$ extending τ such that $\imath^{-1}\tilde{\tau} \in \widetilde{I}_l$. Let σ_λ denote the representation of $G(F^+_\infty)$ given by $W_\lambda \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l \otimes_{\overline{\mathbb{Q}}_l,\imath} \mathbb{C}$, with an element $g \in G(F^+_\infty)$ acting via $\otimes_\tau \tilde{\tau}(\iota_{\tilde{\tau}}(g))$. Let $\mathcal{A}$ denote the space of automorphic forms on $G(F^+)\backslash G(\mathbb{A}_{F^+})$. From the proof of Proposition 3.3.2 of [CHT08], one easily obtains the following.

Lemma 2.1.1. *There is an isomorphism of $G(\mathbb{A}^\infty_{F^+})$-modules*

$$S_\lambda(\overline{\mathbb{Q}}_l) \xrightarrow{\sim} \mathrm{Hom}_{G(F^+_\infty)}(\sigma_\lambda^\vee, \mathcal{A}).$$

In particular, we note that $S_\lambda(\overline{\mathbb{Q}}_l)$ is a semi-simple admissible $G(\mathbb{A}^\infty_{F^+})$-module.

Following [CHT08], we say that a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ is RACSDC (regular, algebraic, conjugate self dual, and cuspidal) if

- π_∞ has the same infinitesimal character as some irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$, and
- $\pi^c \cong \pi^\vee$.

We say that π has level prime to l if π_v is unramified for all $v|l$. If Ω is an algebraically closed field of characteristic 0 we write $(\mathbb{Z}^n_+)_0^{\mathrm{Hom}\,(F,\Omega)}$ for the subset of elements $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}\,(F,\Omega)}$ such that

$$\lambda_{\tau,i} + \lambda_{\tau\circ c,n+1-i} = 0$$

for all τ, i.

If $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}\,(F,\mathbb{C})}$ we write Σ_λ for the irreducible algebraic representation of $\mathrm{GL}_n^{\mathrm{Hom}\,(F,\mathbb{C})}$ given by the tensor product over τ of the irreducible representations with highest weights λ_τ. We say that a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ has weight $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}\,(F,\mathbb{C})}$ if π_∞ has the same infinitesimal character as $\Sigma_\lambda^\vee$. If this is the case then necessarily $\lambda \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}\,(F,\mathbb{C})}$.

Theorem 2.1.2. *If π is a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight λ, then there is a continuous semisimple representation*

$$r_{l,\imath}(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

such that

1. $r_{l,\imath}(\pi)^c \cong r_{l,\imath}(\pi)^\vee \otimes \epsilon_l^{1-n}$.
2. *The representation $r_{l,\imath}(\pi)$ is de Rham, and is crystalline if π has level prime to l. If $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ then*
$$\mathrm{HT}_\tau(r_{l,\imath}(\pi)) = \{\lambda_{\imath\tau,1} + n - 1, \dots, \lambda_{\imath\tau,n}\}.$$
3. *For each finite place v of l, we have*
$$\imath \mathrm{WD}(r_{l,\imath}(\pi)|_{G_{F_v}})^{\mathrm{F-ss}} \cong \mathrm{rec}(\pi_v^\vee \otimes |\det|^{(1-n)/2}).$$
Here $\mathrm{WD}(r_{l,\imath}(\pi)|_{G_{F_v}})^{\mathrm{F-ss}}$ denotes the Frobenius semisimplification of the Weil-Deligne representation associated to $r_{l,\imath}(\pi)|_{G_{F_v}}$, as in section 1 of [TY07].

Proof. This follows at once from the main results of [Shi11], [CH13], [Car12a], [BLGGT14a] and [Car12b]. □

We say that a continuous irreducible representation $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ (respectively $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$) is automorphic if $r \cong r_{l,\imath}(\pi)$ (respectively $\bar{r} \cong \bar{r}_{l,\imath}(\pi)$) for some RACSDC representation π of $\mathrm{GL}_n(\mathbb{A}_F)$. We say that a continuous irreducible representation $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is automorphic of weight $\lambda \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}\,(F,\overline{\mathbb{Q}}_l)}$ if $r \cong r_{l,\imath}(\pi)$ for some RACSDC representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\imath\lambda$.

The theory of base change gives a close relationship between automorphic representations of $G(\mathbb{A}_{F^+})$ and automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$. For example, one has the following consequences of Corollaire 5.3 and Théorème 5.4 of [Lab09].

Theorem 2.1.3. *Suppose that Π is a RACSDC representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\lambda \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}\,(F,\mathbb{C})}$. Then there is an automorphic representation π of $G(\mathbb{A}_{F^+})$ such that*

1. *For each embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ and each $\tilde{\tau} : F \hookrightarrow \mathbb{C}$ extending τ, we have $\pi_\tau \cong \Sigma^\vee_{\lambda_{\tilde{\tau}}} \circ \iota_{\tilde{\tau}}$.*
2. *If v is a finite place of F^+ which splits as ww^c in F, then $\pi_v \cong \Pi_w \circ \iota_w$.*
3. *If v is a finite place of F^+ which is inert in F, and Π_v is unramified, then π_v has a fixed vector for some hyperspecial maximal compact subgroup of $G(F_v^+)$.*

Theorem 2.1.4. *Suppose that π is an automorphic representation of $G(\mathbb{A}_{F^+})$. Then either:*

1. *There is an RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ of some weight $\lambda \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}\,(F,\mathbb{C})}$, or:*

(2) *There is a nontrivial partition $n = n_1 + \cdots + n_r$ and cuspidal automorphic representations Π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$ such that if $\Pi := \Pi_1 \boxplus \cdots \boxplus \Pi_r$ is the isobaric direct sum of the Π_i, then Π is regular, algebraic, and conjugate self-dual of some weight $\lambda \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}\,(F,\mathbb{C})}$*

such that in either case

(1) *For each embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ and each $\tilde{\tau} \hookrightarrow \mathbb{C}$ extending τ, we have $\pi_\tau \cong \Sigma^\vee_{\lambda_{\tilde{\tau}}} \circ \iota_{\tilde{\tau}}$.*
(2) *If v is a finite place of F^+ which splits as ww^c in F, then $\pi_v \cong \Pi_w \circ \iota_w$.*
(3) *If v is a finite place of F^+ which is inert in F, and π_v has a fixed vector for some hyperspecial maximal compact subgroup of $G(F_v^+)$, then Π_v is unramified.*

We now wish to define what it means for an irreducible representation $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ to be modular of some weight. In order to do so, we return to the spaces of algebraic modular forms considered before. For each place $w|l$ of F, let k_w denote the residue field of F_w. If w lies over a place v of F^+, write $v = ww^c$. Let $(\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ denote the subset of $(\mathbb{Z}^n_+)^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ consisting of elements a such that for each $w|l$, if $\sigma \in \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)$ and $1 \le i \le n$ then

$$a_{\sigma,i} + a_{\sigma c, n+1-i} = 0.$$

We say that an element $a \in (\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ is a *Serre weight* if for each $w|l$ and each $\sigma \in \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)$ we have

$$l - 1 \ge a_{\sigma,i} - a_{\sigma,i+1}$$

for all $1 \le i \le n-1$. Similarly, if $\mathbb{F}$ is a finite extension of $\mathbb{F}_l$, we say that an element $a \in (\mathbb{Z}^n_+)^{\mathrm{Hom}\,(\mathbb{F},\overline{\mathbb{F}}_l)}$ is a *Serre weight* if for each $\sigma \in \mathrm{Hom}\,(\mathbb{F}, \overline{\mathbb{F}}_l)$ and each $1 \le i \le n-1$ we have

$$l - 1 \ge a_{\sigma,i} - a_{\sigma,i+1}.$$

Given any $a \in \mathbb{Z}^n_+$ with $l - 1 \ge a_i - a_{i+1}$ for all $1 \le i \le n-1$, we define the $\mathbb{F}$-representation P_a of $\mathrm{GL}_n(\mathbb{F})$ to be the representation obtained by evaluating $\mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0 a)_{\mathbb{F}}$ on $\mathbb{F}$, and let N_a be the irreducible sub-$\mathbb{F}$-representation of P_a generated by the highest weight vector (that this is indeed irreducible follows for example from II.2.8(1) of [Jan03] and the appendix to [Her09]).

If $a \in (\mathbb{Z}^n_+)^{\mathrm{Hom}\,(\mathbb{F},\overline{\mathbb{F}}_l)}$ is a Serre weight then we define an irreducible $\overline{\mathbb{F}}_l$-representation F_a of $\mathrm{GL}_n(\mathbb{F})$ by

$$F_a := \otimes_{\tau \in \mathrm{Hom}\,(\mathbb{F},\overline{\mathbb{F}}_l)} N_{a_\tau} \otimes_{\mathbb{F},\tau} \overline{\mathbb{F}}_l.$$

We will also consider the $\overline{\mathbb{F}}_l$-representation P_a of $\mathrm{GL}_n(\mathbb{F})$ given by

$$P_a := \otimes_{\tau \in \mathrm{Hom}\,(\mathbb{F},\overline{\mathbb{F}}_l)} P_{a_\tau} \otimes_{\mathbb{F},\tau} \overline{\mathbb{F}}_l.$$

We say that two Serre weights a and b are *equivalent* if and only if $F_a \cong F_b$ as representations of $\mathrm{GL}_n(\mathbb{F})$. This is equivalent to demanding that for each $\sigma \in \mathrm{Hom}\,(\mathbb{F}, \overline{\mathbb{F}}_l)$, we have

$$a_{\sigma,i} - a_{\sigma,i+1} = b_{\sigma,i} - b_{\sigma,i+1},$$

for each $1 \le i \le n-1$, and the character $\mathbb{F}^\times \to \overline{\mathbb{F}}_l^{\,\times}$ given by

$$x \mapsto \prod_{\sigma \in \mathrm{Hom}\,(\mathbb{F},\overline{\mathbb{F}}_l)} \sigma(x)^{a_{\sigma,n} - b_{\sigma,n}}$$

is trivial. Every irreducible $\overline{\mathbb{F}}_l$-representation of $\mathrm{GL}_n(\mathbb{F})$ is of the form F_a for some a (see for example the appendix to [Her09]).

If $a \in (\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ is a Serre weight, we define an irreducible $\overline{\mathbb{F}}_l$-representation F_a of $G(\mathcal{O}_{F^+,l})$ as follows: we define

$$F_a = \otimes_{\overline{\mathbb{F}}_l} F_{a_{\widetilde{v}}},$$

an irreducible representation of $\prod_{\widetilde{v}\in\tilde{S}_l} \mathrm{GL}_n(k_{\widetilde{v}})$, and we let $G(\mathcal{O}_{F^+,l})$ act on $F_{a_{\widetilde{v}}}$ by the composition of $\iota_{\widetilde{v}}$ and reduction modulo l. Again, we say that two Serre weights a and b are equivalent if and only if $F_a \cong F_b$ as representations of $G(\mathcal{O}_{F^+,l})$. This is equivalent to demanding that for each place $w|l$ and each $\sigma \in \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)$ and each $1 \le i \le n-1$ we have

$$a_{\sigma,i} - a_{\sigma,i+1} = b_{\sigma,i} - b_{\sigma,i+1},$$

and the character $k_w^\times \to \overline{\mathbb{F}}_l^{\times}$ given by

$$x \mapsto \prod_{\sigma\in\mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)} \sigma(x)^{a_{\sigma,n}-b_{\sigma,n}}$$

is trivial.

Note that the representation F_a is independent of the choice of $\widetilde{S}_l$ (this follows easily from the condition that $a_{\sigma c,n+1-i} = -a_{\sigma,i}$ and the relation $\iota_{w^c}(x) = {}^t(\iota_w(x))^{-1}$).

For future use, if $a \in (\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ is a Serre weight, we also define an $\overline{\mathbb{F}}_l$-representation P_a of $G(\mathcal{O}_{F^+,l})$ as follows: we define

$$P_a = \otimes_{\overline{\mathbb{F}}_l} P_{a_{\widetilde{v}}},$$

a representation of $\prod_{\widetilde{v}\in\tilde{S}_l} \mathrm{GL}_n(k_{\widetilde{v}})$, and we let $G(\mathcal{O}_{F^+,l})$ act on $P_{a_{\widetilde{v}}}$ by the composition of $\iota_{\widetilde{v}}$ and reduction modulo l. Note that F_a is a subrepresentation of P_a.

We say that a weight $\lambda \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}\,(F,\overline{\mathbb{Q}}_l)}$ is a *lift* of a Serre weight a if for each $w|l$ and each $\sigma \in \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)$ there is an element $\tau \in \mathrm{Hom}\,(F, \overline{\mathbb{Q}}_l)$ lying over w and lifting σ such that $\lambda_\tau = a_\sigma$, and for all other $\tau' \in \mathrm{Hom}\,(F, \overline{\mathbb{Q}}_l)$ lying over w and lifting σ we have $\lambda_{\tau'} = 0$. If $\lambda \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}\,(F,\overline{\mathbb{Q}}_l)}$ and $w|l$ is a place of F, we let $\lambda_w \in (\mathbb{Z}^n_+)^{\mathrm{Hom}\,(F_w,\overline{\mathbb{Q}}_l)}$ be defined in the obvious way. If L is a finite extension of $\mathbb{Q}_l$ with residue field k_L, we say that an element $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}\,(L,\overline{\mathbb{Q}}_l)}$ is a *lift* of an element $a \in (\mathbb{Z}^n_+)^{\mathrm{Hom}\,(k_L,\overline{\mathbb{F}}_l)}$ if for each $\sigma \in \mathrm{Hom}\,(k_L, \overline{\mathbb{F}}_l)$ there is an element $\tau \in \mathrm{Hom}\,(L, \overline{\mathbb{Q}}_l)$ lifting σ such that $\lambda_\tau = a_\sigma$, and for all other $\tau' \in \mathrm{Hom}\,(L, \overline{\mathbb{Q}}_l)$ lifting σ we have $\lambda_{\tau'} = 0$.

For the rest of this section, fix $K = \overline{\mathbb{Q}}_l$.

Definition 2.1.5. We say that a compact open subgroup of $G(\mathbb{A}^\infty_{F^+})$ is *good* if $U = \prod_v U_v$ with U_v a compact open subgroup of $G(F_v^+)$ such that:

- $U_v \subset G(\mathcal{O}_{F_v^+})$ for all v which split in F;
- $U_v = G(\mathcal{O}_{F_v^+})$ if $v|l$;
- U_v is a hyperspecial maximal compact subgroup of $G(F_v^+)$ if v is inert in F.

Let U be a good compact open subgroup of $G(\mathbb{A}^\infty_{F^+})$. Let T be a finite set of finite places of F^+ which split in F, containing S_l and all the places v which split in F

for which $U_v \neq G(\mathcal{O}_{F_v^+})$. We let $\mathbb{T}^{T,\mathrm{univ}}$ be the commutative $\mathcal{O}$-polynomial algebra generated by formal variables $T_w^{(j)}$ for all $1 \leq j \leq n$, w a place of F lying over a place v of F^+ which splits in F and is not contained in T. For any $\lambda \in (\mathbb{Z}^n_+)^{\widetilde{I}_l}$, the algebra $\mathbb{T}^{T,\mathrm{univ}}$ acts on $S_\lambda(U, \mathcal{O})$ via the Hecke operators

$$T_w^{(j)} := \iota_w^{-1} \left[\mathrm{GL}_n(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix} \mathrm{GL}_n(\mathcal{O}_{F_w}) \right]$$

for $w \notin T$ and ϖ_w a uniformiser in $\mathcal{O}_{F_w}$. Similarly, for any Serre weight $a \in (\mathbb{Z}^n_+)_0^{\coprod_{v|l} \mathrm{Hom}\,(k_v, \overline{\mathbb{F}}_l)}$, $\mathbb{T}^{T,\mathrm{univ}}$ acts on $S(U, F_a)$.

Suppose that $\mathfrak{m}$ is a maximal ideal of $\mathbb{T}^{T,\mathrm{univ}}$ with residue field $\overline{\mathbb{F}}_l$ such that $S_\lambda(U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} \neq 0$. Then (cf. Proposition 3.4.2 of [CHT08]) by Lemma 2.1.1, Theorem 2.1.4, and Theorem 2.1.2, there is a continuous semisimple representation

$$\bar{r}_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$$

associated to $\mathfrak{m}$, which is uniquely determined by the properties that:

- $\bar{r}_{\mathfrak{m}}^c \cong \bar{r}_{\mathfrak{m}}^\vee \bar{\epsilon}_l^{1-n}$,
- for all finite places w of F not lying over T, $\bar{r}_{\mathfrak{m}}|_{G_{F_w}}$ is unramified, and
- if w is a finite place of F which doesn't lie over T and which splits over F^+, then the characteristic polynomial of $\bar{r}_{\mathfrak{m}}(\mathrm{Frob}_w)$ is

$$X^n - T_w^{(1)} X^{n-1} + \cdots + (-1)^j (\mathbf{N}w)^{j(j-1)/2} T_w^{(j)} X^{n-j} + \cdots + (-1)^n (\mathbf{N}w)^{n(n-1)/2} T_w^{(n)}.$$

Lemma 2.1.6. *Suppose that U is sufficiently small, and let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_\lambda^{T,\mathrm{univ}}$ with residue field $\overline{\mathbb{F}}_l$. Suppose that $a \in (\mathbb{Z}^n_+)_0^{\coprod_{v|l} \mathrm{Hom}\,(k_v, \overline{\mathbb{F}}_l)}$ is a Serre weight, and that $\lambda \in (\mathbb{Z}^n_+)^{\widetilde{I}_l}$ is a lift of a. Then*

$$S_\lambda(U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} \neq 0$$

if and only if for some Jordan-Hölder factor F of the $G(\mathcal{O}_{F^+,l})$-representation P_a,

$$S(U, F)_{\mathfrak{m}} \neq 0.$$

In particular if $S(U, F_a)_{\mathfrak{m}} \neq 0$ then $S_\lambda(U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} \neq 0$.

Proof. We have $S_\lambda(U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} = S_\lambda(U, \mathcal{O}_{\overline{\mathbb{Q}}_l})_{\mathfrak{m}} \otimes \overline{\mathbb{Q}}_l$. Since U is sufficiently small, it follows that $S_\lambda(U, \mathcal{O}_{\overline{\mathbb{Q}}_l})_{\mathfrak{m}}$ is l-torsion free. Thus $S_\lambda(U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} \neq 0$ if and only if $S_\lambda(U, \mathcal{O}_{\overline{\mathbb{Q}}_l})_{\mathfrak{m}} \neq 0$. However, using the fact that U is sufficiently small again, we have $S_\lambda(U, \overline{\mathbb{F}}_l)_{\mathfrak{m}} \neq 0$ if and only if $S_\lambda(U, \mathcal{O}_{\overline{\mathbb{Q}}_l})_{\mathfrak{m}} \neq 0$. Thus, $S_\lambda(U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} \neq 0$ if and only if $S_\lambda(U, \overline{\mathbb{F}}_l)_{\mathfrak{m}} \neq 0$.

But $S_\lambda(U, \overline{\mathbb{F}}_l)_{\mathfrak{m}} = S(U, W_\lambda \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ is nonzero if and only if $S(U, F)_{\mathfrak{m}}$ is nonzero for some Jordan-Hölder factor F of $W_\lambda \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l$. (This follows from the exactness of the functor $F \mapsto S(U, F)_{\mathfrak{m}}$ which in turn follows from the fact that U is sufficiently small.) It then suffices to note that as an immediate consequence of the definitions, we have $P_a \cong W_\lambda \otimes_{\mathcal{O}} \overline{\mathbb{F}}_l$ and F_a is a Jordan-Hölder factor of $W_\lambda \otimes_{\mathcal{O}} \mathbb{F}$. $\square$

We have the following definitions.

Definition 2.1.7. If R is a commutative ring and $r : G_F \to \mathrm{GL}_n(R)$ is a representation, we say that r has *split ramification* if $r|_{G_{F_w}}$ is unramified for any finite place $w \in F$ which does not split over F^+.

Definition 2.1.8. If π is a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, we say that π has *split ramification* if π_w is unramified for any finite place $w \in F$ which does not split over F^+.

Definition 2.1.9. Suppose that $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is a continuous irreducible representation. Then we say that $\bar{r}$ *is modular of weight* $a \in (\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ if there is a good, sufficiently small level U, a set of places T as above, and a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{T,\mathrm{univ}}$ with residue field $\overline{\mathbb{F}}_l$ such that

- $S(U, F_a)_{\mathfrak{m}} \neq 0$, and
- $\bar{r} \cong \bar{r}_{\mathfrak{m}}$.

(Note that $\bar{r}_{\mathfrak{m}}$ exists by Lemma 2.1.6 and the remarks preceding it.) We say that $\bar{r}$ is modular if it is modular of some weight.

Remark 2.1.10. Note that if $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is modular then $\bar{r}$ must have split ramification, and $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{1-n}$. Note also that this definition is independent of the choice of $\tilde{S}_l$ (because F_a is independent of this choice). We need to restrict to split ramification and good level because a development of deformation theory for local Galois representations valued in the group $\mathcal{G}_n$ of [CHT08] is currently missing from the literature; in particular, this applies to the results that we use from [BLGGT14b].

Lemma 2.1.11. *Suppose that $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is a continuous irreducible representation with split ramification. Let $a \in (\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ be a Serre weight, and let $\lambda \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}\,(F, \overline{\mathbb{Q}}_l)}$ be a lift of a. Then if $\bar{r}$ is modular of weight $a \in (\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$, there is a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\imath\lambda$ and level prime to l which has split ramification, and which satisfies $\bar{r}_{l,\imath}(\pi) \cong \bar{r}$. Conversely, if there is a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\imath\lambda$ and level prime to l which has split ramification, and which satisfies $\bar{r}_{l,\imath}(\pi) \cong \bar{r}$, then $\bar{r}$ is modular of weight $b \in (\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w, \overline{\mathbb{F}}_l)}$ for some b such that the $G(\mathcal{O}_{F^+,l})$-representation P_a has a Jordan-Hölder factor isomorphic to F_b.*

Proof. Suppose firstly that $\bar{r}$ is modular of weight a. Then by definition there is a good U and a T as above with U sufficiently small, and a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{T,\mathrm{univ}}$ with residue field $\overline{\mathbb{F}}_l$ such that

- $S(U, F_a)_{\mathfrak{m}} \neq 0$, and
- $\bar{r} \cong \bar{r}_{\mathfrak{m}}$.

By Lemma 2.1.6, the first property implies that $S_\lambda(U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} \neq 0$. Define a compact open subgroup $U' = \prod_w U'_w$ of $\mathrm{GL}_n(\mathbb{A}_F^\infty)$ by

- $U'_w = \mathrm{GL}_n(\mathcal{O}_{F_w})$ if w is not split over F^+.
- $U'_w = \iota_w(U_w)$ if w splits over F^+.

By Lemma 2.1.1, Theorem 2.1.4, and Theorem 2.1.2, there is a RACSDC automorphic representation π of weight λ which satisfies $\bar{r}_{l,\imath}(\pi) \cong \bar{r}$, and $\pi_w^{U'_w} \neq 0$ for all finite places w of F. Since U is good, we see that π has level prime to l, and it has split ramification, as required.

Conversely, suppose that there is a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight λ which has split ramification and level prime to l with $\bar{r}_{l,\imath}(\pi) \cong \bar{r}$. Then there is a compact open subgroup $U' = \prod_w U'_w$ of $\mathrm{GL}_n(\mathbb{A}_F^\infty)$ such that

- for each finite place w of F, $\pi_w^{U'_w} \neq 0$,
- $U'_w \subset \mathrm{GL}_n(\mathcal{O}_{F_w})$ for all w,
- $U'_w = \mathrm{GL}_n(\mathcal{O}_{F_w})$ for all $w|l$ and all w which are not split over F^+,
- if $v = ww^c$ is a place of F^+ which splits in F, then $U'_{w^c} = c({}^tU'^{-1}_w)$,
- there is a finite place w of F which is split over F^+ such that
  - w lies above a rational prime p with $[F(\zeta_p) : F] > n$, and
  - $U'_w = \ker(\mathrm{GL}_n(\mathcal{O}_w) \to \mathrm{GL}_n(\mathcal{O}_w/\varpi_w))$.

Define a compact open subgroup $U = \prod_v U_v$ of $G(\mathbb{A}^\infty_{F^+})$ by

- if v is inert in F, then U_v is hyperspecial, and
- if $v = ww^c$ splits in F, then $U_v = \iota_w^{-1}(U'_w)$ (which is well-defined by the fourth bullet point above).

By the final bullet point in the list of properties of U' above, U is sufficiently small. Then by Lemma 2.1.1 and Theorem 2.1.3 we have $S_\lambda(U, \overline{\mathbb{Q}}_l)_\mathfrak{m} \neq 0$. The result now follows from Lemma 2.1.6. $\square$

3. A lifting theorem

3.1. We recall some terminology from [BLGGT14b], specialized to the crystalline (as opposed to potentially crystalline) case. Fix a prime l. Let K be a finite extension of $\mathbb{Q}_l$, and $\mathcal{O}$ the ring of integers in a finite extension of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$, with residue field k. Assume that for each continuous embedding $K \hookrightarrow \overline{\mathbb{Q}}_l$, the image is contained in the field of fractions of $\mathcal{O}$.

Let $\overline{\rho} : G_K \to \mathrm{GL}_n(k)$ be a continuous representation, and let $R^\square_{\mathcal{O},\overline{\rho}}$ be the universal $\mathcal{O}$-lifting ring. Let $\{H_\tau\}$ be a collection of n element multisets of integers parametrized by $\tau \in \mathrm{Hom}_{\mathbb{Q}_l}(K, \overline{\mathbb{Q}}_l)$. Then $R^\square_{\mathcal{O},\overline{\rho}}$ has a unique quotient $R^\square_{\mathcal{O},\overline{\rho},\{H_\tau\},\mathrm{cris}}$ which is reduced and without l-torsion and such that a $\overline{\mathbb{Q}}_l$-point of $R^\square_{\mathcal{O},\overline{\rho}}$ factors through $R^\square_{\mathcal{O},\overline{\rho},\{H_\tau\},\mathrm{cris}}$ if and only if it corresponds to a representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ which is crystalline and has $\mathrm{HT}_\tau(\rho) = H_\tau$ for all $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$. We will write $R^\square_{\overline{\rho},\{H_\tau\},\mathrm{cris}} \otimes \overline{\mathbb{Q}}_l$ for $R^\square_{\mathcal{O},\overline{\rho},\{H_\tau\},\mathrm{cris}} \otimes_\mathcal{O} \overline{\mathbb{Q}}_l$. This definition is independent of the choice of $\mathcal{O}$. The scheme $\mathrm{Spec}\,(R^\square_{\overline{\rho},\{H_\tau\},\mathrm{cris}} \otimes \overline{\mathbb{Q}}_l)$ is formally smooth over $\mathrm{Spec}\,\overline{\mathbb{Q}}_l$. (See [Kis08].)

Let $\rho_1, \rho_2 : G_K \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ be continuous representations. We say that ρ_1 *connects to* ρ_2, which we denote $\rho_1 \sim \rho_2$, if and only if

- the reduction $\overline{\rho}_1 = \rho_1 \bmod \mathfrak{m}_{\mathcal{O}_{\overline{\mathbb{Q}}_l}}$ is equivalent to the reduction $\overline{\rho}_2 = \rho_2 \bmod \mathfrak{m}_{\overline{\mathbb{Q}}_l}$;
- ρ_1 and ρ_2 are both crystalline;
- for each $\tau : K \hookrightarrow \overline{\mathbb{Q}}_l$ we have $\mathrm{HT}_\tau(\rho_1) = \mathrm{HT}_\tau(\rho_2)$;
- and ρ_1 and ρ_2 define points on the same irreducible component of the scheme $\mathrm{Spec}\,(R^\square_{\overline{\rho}_1,\{\mathrm{HT}_\tau(\rho_1)\},\mathrm{cris}} \otimes \overline{\mathbb{Q}}_l)$.

We note that $\rho_1 \sim \rho_2$ in our sense if and only if both ρ_1 and ρ_2 are crystalline and $\rho_1 \sim \rho_2$ in the sense of [BLGGT14b]. As in section 2.3 of [BLGGT14b], we have the following:

(1) The relation $\rho_1 \sim \rho_2$ does not depend on the equivalence chosen between the reductions $\overline{\rho}_1$ and $\overline{\rho}_2$, nor on the $\mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$-conjugacy class of ρ_1 or ρ_2.
(2) $\sim$ is an equivalence relation.
(3) If K'/K is a finite extension and $\rho_1 \sim \rho_2$ then $\rho_1|_{G_{K'}} \sim \rho_2|_{G_{K'}}$.

(4) If $\rho_1 \sim \rho_2$ and $\rho_1' \sim \rho_2'$ then $\rho_1 \oplus \rho_1' \sim \rho_2 \oplus \rho_2'$ and $\rho_1 \otimes \rho_1' \sim \rho_2 \otimes \rho_2'$ and $\rho_1^\vee \sim \rho_2^\vee$.
(5) If $\mu : G_K \to \overline{\mathbb{Q}}_l^\times$ is a continuous unramified character with $\overline{\mu} = 1$ then $\rho_1 \sim \rho_1 \otimes \mu$.
(6) Suppose ρ_1 is crystalline and $\overline{\rho}_1$ is semisimple. Let Fil^i be a G_K-invariant filtration on ρ_1 by $\mathcal{O}_{\overline{\mathbb{Q}}_l}$-direct summands. Then $\rho_1 \sim \oplus_i \mathrm{gr}^i(\mathrm{Fil})$.

We will call a crystalline representation $\rho : G_K \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ *diagonal* if it is of the form $\chi_1 \oplus \cdots \oplus \chi_n$ with $\chi_i : G_K \to \mathcal{O}_{\overline{\mathbb{Q}}_l}^\times$. We will call a crystalline representation $\rho : G_K \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ *diagonalizable* if it connects to some diagonal representation. We will call a representation $\rho_1 : G_K \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ *potentially diagonalizable* if there is a finite extension K'/K such that $\rho_1|_{G_{K'}}$ is diagonalizable. Note that if K''/K is a finite extension and ρ_1 is diagonalizable (resp. potentially diagonalizable) then $\rho_1|_{G_{K''}}$ is diagonalizable (resp. potentially diagonalizable).

Suppose now that K is a finite extension of $\mathbb{Q}_p$ for some prime $p \neq l$ and

$$\rho_1, \rho_2 : G_K \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$$

are two continuous representations. We define the notion that ρ_1 connects to ρ_2 exactly as in [BLGGT14b]. Again, this will be denoted by $\rho_1 \sim \rho_2$.

Recall the following definition from [Tho12] (for a discussion of the equivalence of this definition to that formulated in [Tho12], see the appendix to [BLGG13]).

Definition 3.1.1. We call a finite subgroup $H \subset \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ *adequate* if the following conditions are satisfied.

(1) H has no non-trivial quotient of l-power order (i.e. $H^1(H, \overline{\mathbb{F}}_l) = (0)$).
(2) $l \nmid n$.
(3) The elements of H with order coprime to l span $M_{n\times n}(\overline{\mathbb{F}}_l)$ over $\overline{\mathbb{F}}_l$. (This implies that $\overline{\mathbb{F}}_l^n$ is an irreducible representation of H.)
(4) $H^1(H, \mathfrak{gl}_n(\overline{\mathbb{F}}_l)) = (0)$.

In particular, we have the following useful result, an immediate consequence of Theorem 9 of [GHTT10].

Theorem 3.1.2. *Suppose that $l \geq 2(n+1)$, and that H is a finite subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_l)$ which acts irreducibly. Then H is adequate.*

Fix an isomorphism $\imath : \overline{\mathbb{Q}}_l \to \mathbb{C}$. Let F be an imaginary CM field with maximal totally real subfield F^+.

Theorem 3.1.3. *Let $l > 2$ be prime, and let F be a CM field with maximal totally real subfield F^+, with $\zeta_l \notin F$. Assume that the extension F/F^+ is split at all places dividing l. Suppose that*

$$\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$$

is an irreducible representation which satisfies the following properties.

(1) *There is a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that*
   - *$\bar{r} \cong \bar{r}_{l,\imath}(\Pi)$ (so in particular, $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{1-n}$).*
   - *For each place $w|l$ of F, $r_{l,\imath}(\Pi)|_{G_{F_w}}$ is potentially diagonalizable.*
(2) *The image $\bar{r}(G_{F(\zeta_l)})$ is adequate.*

Let S be a finite set of finite places of F^+ which split in F. Assume that S contains all the places of F^+ dividing l, and all places lying under a place of F at which

$\bar{r}$ is ramified. For each $v \in S$ choose a place $\widetilde{v}$ of F above v, and a lift $\rho_{\widetilde{v}} : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ of $\bar{r}|_{G_{F_{\widetilde{v}}}}$. Assume that if $v|l$, then $\rho_{\widetilde{v}}$ is crystalline and potentially diagonalizable, and if $\tau : F_{\widetilde{v}} \hookrightarrow \overline{\mathbb{Q}}_l$ is any embedding, then $\mathrm{HT}_\tau(\rho_{\widetilde{v}})$ consists of distinct integers.

Then there is a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of level prime to l such that

- $\bar{r} \cong \bar{r}_{l,\imath}(\pi)$.
- *π_w is unramified for all w not lying over a place of S, so that $r_{l,\imath}(\pi_w)$ is unramified at all such w.*
- *$r_{l,\imath}(\pi)|_{G_{F_{\widetilde{v}}}} \sim \rho_{\widetilde{v}}$ for all $v \in S$. In particular, for each place $v|l$, $r_{l,\imath}(\pi)|_{G_{F_{\widetilde{v}}}}$ is crystalline and for each embedding $\tau : F_{\widetilde{v}} \hookrightarrow \overline{\mathbb{Q}}_l$, $\mathrm{HT}_\tau(r_{l,\imath}(\pi)|_{G_{F_{\widetilde{v}}}}) = \mathrm{HT}_\tau(\rho_{\widetilde{v}})$.*

Proof. Let $\mathcal{G}_n$ be the group scheme over $\mathbb{Z}$ defined in section 2.1 of [CHT08]. Then by the main result of [BC11], $\bar{r}$ extends to a representation $\overline{\rho} : G_{F^+} \to \mathcal{G}_n(\overline{\mathbb{F}}_l)$ with multiplier $\bar{\epsilon}_l^{1-n}$.

We will now apply Theorem A.4.1 of [BLGG13]. In fact, we need a slight strengthening of that theorem, where we remove the assumption that (π', χ') is unramified outside of the set of primes above S. (After replacing (π', χ') with its base change to a finite solvable extension of F', we may then assume that (π', χ') is unramified outside of a set of primes S' of F', which contains all primes above S, and all of whose elements are split over $(F')^+$.) The proof of Theorem A.4.1 of [BLGG13] goes over essentially unchanged to prove this stronger result: the first (and longest) step in the proof is to show that after replacing F' by a finite solvable extension F_1/F', the representation π' can be replaced by a representation π_1 with the property that π_1 is ordinary and $r_{l,\imath}(\pi)|_{G_{F_{1,v}}} \sim r_{l,\imath}(\pi_1)|_{G_{F_{1,v}}}$ for all v not above l. It is clear that the construction of π_1 can be carried out without reference to the subfield F of F'. Once π_1 has been obtained, then Proposition 1.5.1(ii) and Theorems 2.3.1 and 2.3.2 of [BLGGT14b] are applied to produce an ordinary π_1' such that $r_{l,\imath}(\pi_1')|_{G_{F_{1,v}}} \sim \rho|_{G_{F_{1,v}}}$ for all v not above l. The proof then continues unchanged.

We now apply this strengthened version of Theorem A.4.1 of [BLGG13], with

- F, n and S as in the present setting.
- $\bar{r}$ our present $\overline{\rho}$.
- ρ_v our $\rho_{\widetilde{v}}$.
- $\mu = \epsilon_l^{1-n}$.
- $F' = F$.

We conclude that $\bar{r}$ has a lift $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ (the restriction to G_F of the representation r of Theorem A.4.1 of [BLGG13]) such that

- $r^c \cong r^\vee \epsilon_l^{1-n}$.
- if $v \in S$ then $r|_{G_{F_{\widetilde{v}}}} \sim \rho_{\widetilde{v}}$.
- r is unramified outside S.
- r is automorphic of level potentially prime to l, say $r \cong r_{l,\imath}(\pi)$.

By Theorem 2.1.2, we see that (since $r|_{G_{F_w}}$ is crystalline for all $w|l$, and unramified at all places w not lying over a place in S) π_w is unramified for all $w|l$ and all w not lying over a place in S, as required. $\square$

4. Serre weight conjectures

4.1. We now briefly discuss Serre weight conjectures for GL_n. We refer the reader to the forthcoming [EGHS14] for a far more detailed discussion. In particular, in much of this section we restrict ourselves to the case that l splits completely in F, both for simplicity of notation and because in this case we can prove theorems with cleaner conditions, as representations satisfying the Fontaine-Laffaille condition are always potentially diagonalizable.

Let K be a finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}_K$ and residue field k. Let $\overline{\rho}: G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ be a continuous representation. Then it is a folklore conjecture that for each such $\overline{\rho}$, there is a set $W(\overline{\rho})$ of Serre weights of $\mathrm{GL}_n(k)$ for each K and each $\overline{\rho}$ with the following property: if F is a CM field, $\bar{r}: G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is an irreducible modular representation (so in particular it is conjugate self-dual), $w|l$ is a place of F and σ_w is an irreducible $\overline{\mathbb{F}}_l$-representation of $\mathrm{GL}_n(k_w)$, then $\bar{r}$ is modular of Serre weight $\sigma_w \otimes_{\overline{\mathbb{F}}_l} \sigma^w$ for some σ^w if and only if $\sigma_w \in W(\bar{r}|_{G_{F_w}})$.

It is natural to believe that there is a description of $W(\overline{\rho})$ in terms of the existence of crystalline lifts with particular Hodge-Tate weights, as we now explain. This is one of the motivations for the general Serre weight conjectures explained in [EGHS14].

Definition 4.1.1. Let $K/\mathbb{Q}_l$ be a finite extension, let $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}\,(K,\overline{\mathbb{Q}}_l)}$, and let $\rho: G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ be a de Rham representation. Then we say that ρ has *Hodge type* λ if for each $\tau \in \mathrm{Hom}\,(K, \overline{\mathbb{Q}}_l)$, we have $\mathrm{HT}_\tau(\rho) = \{\lambda_{\tau,1} + (n-1), \lambda_{\tau,2} + (n-2), \dots, \lambda_{\tau,n}\}$.

Remark 4.1.2. As an immediate consequence of this definition and of Theorem 2.1.2, we see that if π is a RACSDC automorphic representation of weight $\lambda \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}\,(F,\mathbb{C})}$, then for each place $w|l$, $r_{l,\imath}(\pi)|_{G_{F_w}}$ has Hodge type $(\imath^{-1}\lambda)_w$.

Lemma 4.1.3. *Let n be a positive integer, and let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F, and that if n is even then $n[F^+ : \mathbb{Q}]/2$ is even. Suppose that $\bar{r}: G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification. Let $a \in (\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ be a Serre weight, and let $\lambda \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}\,(F,\overline{\mathbb{Q}}_l)}$ be a lift of a. If $\bar{r}$ is modular of weight a, then for each place $w|l$ there is a continuous lift $r_w : G_{F_w} \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ of $\bar{r}|_{G_{F_w}}$ such that r_w is crystalline of Hodge type λ_w.*

Proof. By Lemma 2.1.11 there is a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$, which has level prime to l and weight $\imath\lambda$, such that $\bar{r}_{l,\imath}(\pi) \cong \bar{r}$. Then we may take $r_w := r_{l,\imath}(\pi)|_{G_{F_w}}$, which has the required properties by Remark 4.1.2. $\square$

This suggests the following definition.

Definition 4.1.4. Let K be a finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}_K$ and residue field k. Let $\overline{\rho}: G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ be a continuous representation. Then we let $W^{\mathrm{cris}}(\overline{\rho})$ be the set of Serre weights $a \in (\mathbb{Z}^n_+)^{\mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)}$ with the property that there is a crystalline representation $\rho: G_K \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ lifting $\overline{\rho}$, such that ρ has Hodge type λ for some lift $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}\,(K,\overline{\mathbb{Q}}_l)}$ of a.

The results of section 3 suggest the following definition.

Definition 4.1.5. Let K be a finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}_K$ and residue field k. Let $\overline{\rho} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ be a continuous representation. Then we let $W^{\mathrm{diag}}(\overline{\rho})$ be the set of Serre weights $a \in (\mathbb{Z}^n_+)^{\mathrm{Hom}\,(k,\overline{\mathbb{F}}_l)}$ with the property that there is a potentially diagonalizable crystalline representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ lifting $\overline{\rho}$, such that ρ has Hodge type λ for some lift $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}\,(K,\overline{\mathbb{Q}}_l)}$ of a.

Remark 4.1.6. If a and b are equivalent Serre weights, then $a \in W^{\mathrm{cris}}(\overline{\rho})$ (respectively $W^{\mathrm{diag}}(\overline{\rho})$) if and only if $b \in W^{\mathrm{cris}}(\overline{\rho})$ (respectively $W^{\mathrm{diag}}(\overline{\rho})$). This is an easy consequence of Lemma 4.1.15 of [BLGG13], which provides a crystalline character with trivial reduction with which one can twist the crystalline Galois representations of Hodge type some lift of a to obtain crystalline representations of Hodge type some lift of b. The same remarks apply to the set $W^{\mathrm{obv}}(\overline{\rho})$ defined below.

By definition we have $W^{\mathrm{diag}}(\overline{\rho}) \subset W^{\mathrm{cris}}(\overline{\rho})$. We "globalise" these definitions in the obvious way:

Definition 4.1.7. Let $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ be a continuous representation with $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{1-n}$. Then we let $W^{\mathrm{cris}}(\bar{r})$ (respectively $W^{\mathrm{diag}}(\bar{r})$) be the set of Serre weights $a \in (\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ such that for each place $w|l$, the corresponding Serre weight $a_w \in (\mathbb{Z}^n_+)^{\mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ is an element of $W^{\mathrm{cris}}(\bar{r}|_{G_{F_w}})$ (respectively $W^{\mathrm{diag}}(\bar{r}|_{G_{F_w}})$).

The point of these definitions is the following Corollary and Theorem.

Corollary 4.1.8. *Let n be a positive integer, let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F, and that if n is even then $n[F^+ : \mathbb{Q}]/2$ is even. Suppose that $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification. Let $a \in (\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_v,\overline{\mathbb{F}}_l)}$ be a Serre weight. If $\bar{r}$ is modular of weight a, then $a \in W^{\mathrm{cris}}(\bar{r})$.*

Proof. This is an immediate consequence of Lemma 4.1.3 and Definition 4.1.7. $\square$

Theorem 4.1.9. *Let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F, and that if n is even then $n[F^+ : \mathbb{Q}]/2$ is even. Assume that $\zeta_l \notin F$. Suppose that $l > 2$, and that $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is an irreducible representation with split ramification. Assume that*

- *There is a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that*
  - *$\bar{r} \cong \bar{r}_{l,\imath}(\Pi)$ (so in particular, $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{1-n}$).*
  - *For each place $w|l$ of F, $r_{l,\imath}(\Pi)|_{G_{F_w}}$ is potentially diagonalizable.*
  - *$\bar{r}(G_{F(\zeta_l)})$ is adequate.*

Let $a \in (\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ be a Serre weight. Assume that $a \in W^{\mathrm{diag}}(\bar{r})$. Then there is a Serre weight $b \in (\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ such that

- *$\bar{r}$ is modular of weight b.*
- *There is a Jordan-Hölder factor of the $G(\mathcal{O}_{F^+,l})$ representation P_a which is isomorphic to F_b.*

Proof. By the assumption that $a \in W^{\mathrm{diag}}(\bar{r})$, there is a lift λ of a such that for each $w|l$ there is a potentially diagonalizable crystalline lift $\rho_w : G_{F_w} \to \mathrm{GL}_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$ of $\bar{r}|_{G_{F_w}}$ of Hodge type λ_w.

By Theorem 3.1.3, there is a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\imath\lambda$, of level prime to l and with split ramification, such that $\bar{r}_{l,\imath}(\pi) \cong \bar{r}$. The result follows from Lemma 2.1.11. $\square$

Since Fontaine–Laffaille representations are potentially diagonalizable, we obtain the following Corollary.

Corollary 4.1.10. *Let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, that every place of F^+ dividing l splits completely in F, and that if n is even then $n[F^+ : \mathbb{Q}]/2$ is even. Suppose that $l > 2$, and that $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is an irreducible representation with split ramification. Assume that*

1. *l is unramified in F.*
2. *There is a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\mu \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}\,(F,\mathbb{C})}$ and level prime to l such that*
   - *$\bar{r} \cong \bar{r}_{l,\imath}(\Pi)$ (so in particular, $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{1-n}$).*
   - *For each $\tau \in \mathrm{Hom}\,(F, \mathbb{C})$, $\mu_{\tau,1} - \mu_{\tau,n} \le l - n$.*
   - *$\bar{r}(G_{F(\zeta_l)})$ is adequate.*

Let $a \in (\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ be a Serre weight. Assume that $a \in W^{\mathrm{diag}}(\bar{r})$. Then there is a Serre weight $b \in (\mathbb{Z}^n_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ such that

- *$\bar{r}$ is modular of weight b.*
- *There is a Jordan-Hölder factor of the $G(\mathcal{O}_{F^+,l})$ representation P_a which is isomorphic to F_b.*

Proof. By Theorem 4.1.9, it is enough to check that for each place $w|l$ of F, $r_{l,\imath}(\Pi)|_{G_{F_w}}$ is potentially diagonalizable. This follows from the main result of [GL12]. $\square$

As explained above, we now specialise to the case that l splits completely in F. We further assume that $\bar{r}|_{G_{F_w}}$ is semisimple for all $w|l$, and specify a set $W^{\mathrm{obv}}(\bar{r})$ of Serre weights. These weights will have the property that if $a \in W^{\mathrm{obv}}(\bar{r})$, and λ is the unique lift of a to $(\mathbb{Z}^n_+)_0^{\mathrm{Hom}\,(F,\overline{\mathbb{Q}}_l)}$, then for each place $w|l$, $\bar{r}|_{G_{F_w}}$ has a potentially diagonalizable (indeed potentially diagonal) crystalline lift of Hodge type λ_w.

Since the situation is purely local, we change notation and work with $G_{\mathbb{Q}_l}$. Let $\mathbb{Q}_{l^m}$ denote the unramified extension of $\mathbb{Q}_l$ of degree m inside $\overline{\mathbb{Q}}_l$, and let $\omega_m : G_{\mathbb{Q}_{l^m}} \to \overline{\mathbb{F}}_l^\times$ denote a choice of fundamental character of niveau m (this is given by the action of $G_{\mathbb{Q}_{l^m}}$ on the (l^m-1)-st roots of l). Given $\lambda \in \overline{\mathbb{F}}_l^\times$ and an m-tuple of integers $\underline{c} = (c_0, \dots, c_{m-1})$, we consider the representation

$$\overline{\rho}_{\lambda,\underline{c}} := \mathrm{nr}_\lambda \otimes \mathrm{Ind}\,_{G_{\mathbb{Q}_{l^m}}}^{G_{\mathbb{Q}_l}} \omega_m^{-(c_0 + lc_1 + \dots + l^{m-1}c_{m-1})},$$

where nr_λ is the unramified character taking a geometric Frobenius to λ. Given a partition $\underline{n} = n_1 + \dots + n_r$, elements $\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$ of $\overline{\mathbb{F}}_l^\times$, and a tuple $\underline{c} = (\underline{c_1}, \dots, \underline{c_r})$ of tuples $\underline{c}_i = (c_{i,0}, \dots, c_{i,n_i-1})$ of integers, we define the representation

$$\overline{\rho}_{\underline{n},\underline{\lambda},\underline{c}} := \oplus_{i=1}^r \overline{\rho}_{\lambda_i,\underline{c}_i}.$$

Note that we can we can think of $\underline{c}$ as the element $(c_{1,0}, c_{1,2}, \dots, c_{r,n_r-1})$ of $\mathbb{Z}^n$, where $n = n_1 + \dots + n_r$.

Definition 4.1.11. Let $\overline{\rho} : G_{\mathbb{Q}_l} \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ be a semisimple representation. Let $W^{\mathrm{obv}}(\overline{\rho})$ be the set of Serre weights $a \in \mathbb{Z}_+^n$ for which there exists a permutation $\sigma \in S_n$, a partition $\underline{n}$ of n and $\underline{\lambda}$ as above such that

$$\overline{\rho} \cong \overline{\rho}_{\underline{n},\underline{\lambda},\sigma(a_1+n-1,a_2+n-2,\dots,a_n)}.$$

Lemma 4.1.12. *If $\overline{\rho} : G_{\mathbb{Q}_l} \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is a semisimple representation and $a \in W^{\mathrm{obv}}(\overline{\rho})$, then $\overline{\rho}$ has a potentially diagonalizable crystalline lift of Hodge type a.*

Proof. By the definition of "Hodge type a", it is enough to show that each representation $\overline{\rho}_{\lambda,\underline{c}} : G_{\mathbb{Q}_l} \to \mathrm{GL}_m(\overline{\mathbb{F}}_l)$ defined above has a potentially diagonalizable crystalline lift with Hodge–Tate weights $c_0, \dots, c_{m-1}$ (note that the direct sum of potentially diagonalizable representations is again potentially diagonalizable). It thus suffices to show that the character $\omega_m^{-(c_0+lc_1+\cdots+l^{m-1}c_{m-1})}$ of $G_{\mathbb{Q}_{l^m}}$ has a crystalline lift with Hodge–Tate weights $c_0, \dots, c_{m-1}$ (because the induction to $G_{\mathbb{Q}_l}$ of such a lift is certainly potentially diagonalizable). This follows at once from Lemma 6.2 of [GS11] (noting that the conventions on the sign of Hodge–Tate weights in [GS11] are the opposite of those of this paper). $\square$

Again we may globalise this definition in the obvious way.

Definition 4.1.13. Continue to assume that l splits completely in F, and let $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ be a continuous representation with $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{1-n}$ and such that $\bar{r}|_{G_{F_w}}$ is semisimple for each $w|l$. Then we let $W^{\mathrm{obv}}(\bar{r})$ be the set of Serre weights $a \in (\mathbb{Z}_+^n)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ such that for each place $w|l$, the corresponding Serre weight $a_w \in (\mathbb{Z}_+^n)^{\mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ is an element of $W^{\mathrm{obv}}(\bar{r}|_{G_{F_w}})$.

Corollary 4.1.14. *Let $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ be a continuous representation satisfying the assumptions of Definition 4.1.13. Then $W^{\mathrm{obv}}(\bar{r}) \subset W^{\mathrm{diag}}(\bar{r})$.*

Proof. This follows immediately from Lemma 4.1.12. $\square$

In the case $n = 2$, which we explored more thoroughly in [BLGG13], $W^{\mathrm{obv}}(\bar{r})$ is precisely the set of weights for which $\bar{r}$ is modular. We do not conjecture this for $n > 2$; even for $n = 3$ one sees that the set of weights predicted in [Her09] is larger than $W^{\mathrm{obv}}(\bar{r})$. In fact, we expect (see [EGHS14] for a much more detailed discussion) that the set of weights for which $\bar{r}$ is modular is $W^{\mathrm{cris}}(\bar{r})$, and it is easy to see that this set is typically larger than $W^{\mathrm{obv}}(\bar{r})$. Indeed, by Lemma 2.1.11 and Theorem 2.1.2, if $\bar{r}$ is modular of some Serre weight b, and F_b is a Jordan-Hölder factor of P_a for some Serre weight a, then $a \in W^{\mathrm{cris}}(\bar{r})$. It is easy to find examples of a, b for which $b \in W^{\mathrm{obv}}(\bar{r})$ but $a \notin W^{\mathrm{obv}}(\bar{r})$. On the other hand, as explained in [EGHS14] we believe that $W^{\mathrm{cris}}(\bar{r})$ is determined by $W^{\mathrm{obv}}(\bar{r})$ and a simple combinatorial recipe, so that the weights in $W^{\mathrm{obv}}(\bar{r})$ are in some sense fundamental.

4.2. Fontaine-Laffaille theory. In applications of our results it is often useful to have information in the opposite direction; namely one wishes to have information about $\bar{r}|_{G_{F_w}}$ at places $w|p$, given that $\bar{r}$ is modular of some particular weight. In the case that l is unramified in F and the weight is sufficiently far inside the lowest alcove, this can be done by Fontaine–Laffaille theory. Again, we specialise to the case that l splits completely in F.

Lemma 4.2.1. *Let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, and that l splits completely in F. If n is even, assume that $[F^+:\mathbb{Q}]n/2$ is even. Suppose that $l>2$, and that $\bar{r}:G_F\to\mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification. Let $a\in(\mathbb{Z}^n_+)_0^{\coprod_{w|l}\mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ be a Serre weight. If $\bar{r}$ is modular of weight a, and $w|l$ is such that $a_{w,1}-a_{w,n}\le l-n$, then $a_w\in W^{\mathrm{obv}}(\bar{r}|^{\mathrm{ss}}_{G_{F_w}})$.*

Proof. This is a standard application of Fontaine–Laffaille theory. By Corollary 4.1.8, $\bar{r}|_{G_{F_w}}$ has a crystalline lift with Hodge–Tate weights $a_{w,1}+n-1,\dots,a_{w,n}$. Since by assumption we have $a_{w,1}+n-1-a_{w,n}\le l-1$, the result follows immediately from, for example, Proposition 3 of [Wor02] (note that while this reference assumes that the crystalline representation has $\mathbb{Q}_l$-coefficients, the proof goes through unchanged with $\overline{\mathbb{Q}}_l$-coefficients). □

5. Explicit results for GL_3

5.1. We now show how one can obtain cleaner results in the case $n=3$, making use of the fact that the representation theory of GL_3, while more complicated than that of GL_2, is rather simpler than that of GL_n for $n\ge 4$. The following Lemmas are key to our approach.

Lemma 5.1.1. *Let $a\in\mathbb{Z}^3_+$ be a Serre weight for $\mathrm{GL}_3(\mathbb{F}_l)$. Then*

(1) *if $l-1\le a_1-a_3$ and a_1-a_2, $a_2-a_3\le l-2$, then there is a short exact sequence*
$$0\to F_a\to P_a\to F_b\to 0$$
where $b=(a_3+l-2,a_2,a_1-l+2)$.

(2) *In all other cases, $P_a=F_a$.*

Proof. This is Proposition 3.18 of [Her09]. □

Lemma 5.1.2. *Suppose that $n=3$, and that $a\in\mathbb{Z}^3_+$ is a Serre weight for $\mathrm{GL}_3(\mathbb{F}_l)$. If $a\in W^{\mathrm{obv}}(\bar{r})$ for some representation $\bar{r}:G_{\mathbb{Q}_l}\to\mathrm{GL}_3(\mathbb{F}_l)$, then either $a_1-a_3=l-1$ and*
$$\bar{r}|_{I_{\mathbb{Q}_l}}\cong\omega^{-(a_1+1)}\oplus\omega^{-(a_2+1)}\oplus\omega^{-(a_3+1)},$$
or there is a permutation x, y, z of $-(a_1+2)$, $-(a_2+1)$, $-a_3$ such that $\bar{r}|_{I_{\mathbb{Q}_l}}$ is isomorphic to one of
$$\omega^x\oplus\omega^y\oplus\omega^z,$$
$$\omega^x\oplus\omega_2^{y+lz}\oplus\omega_2^{ly+z},$$
$$\omega_3^{x+ly+l^2z}\oplus\omega_3^{y+lz+l^2x}\oplus\omega_3^{z+lx+l^2y},$$
where in the second case we have $(l+1)\nmid ly+z$, and in the third case we have $(l^2+l+1)\nmid x+ly+l^2z$.

Proof. This is a simple calculation (it is immediate from the definition that $\bar{r}|_{I_{\mathbb{Q}_l}}$ is of the given form if one ignores the divisibility condition, so the only thing to check is when it can be the case that $ly+z$ is divisible by $l+1$ or $x+ly+l^2z$ is divisible by l^2+l+1). □

Definition 5.1.3. Let $a \in \mathbb{Z}^3_+$ be a Serre weight for $\mathrm{GL}_3(\mathbb{F}_l)$. Then we say that a is *non-generic* if it is in the upper alcove, and it is at distance exactly 1 from the boundary. More precisely, it is non-generic if one of the following three conditions hold: $a_1 - a_3 = l-1$ and $a_1 - a_2$, $a_2 - a_3 \leq l-2$; or $a_2 - a_3 = l-2$ and $a_1 - a_2 \geq 2$; or $a_1 - a_2 = l-2$ and $a_2 - a_3 \geq 2$. Otherwise we say that a is *generic*.

If l splits completely in F and $a \in (\mathbb{Z}^3_+)_0^{\mathrm{Hom}\,(F,\overline{\mathbb{Q}}_l)}$ is a Serre weight, we say that a is *generic* if for each $\tau \in \mathrm{Hom}\,(F, \overline{\mathbb{Q}}_l)$ the corresponding Serre weight $a_\tau \in \mathbb{Z}^3_+$ is generic.

We remark that this definition of generic is very mild; in particular, it is much less restrictive than the notion of generic used in [EGH13]. (See also Remark 5.1.5 below.)

Theorem 5.1.4. *Let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, and that l splits completely in F. Suppose that $l > 2$, and that $\bar{r} : G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_l)$ is an irreducible representation with split ramification. Assume that*

(1) *There is a RACSDC automorphic representation Π of $\mathrm{GL}_3(\mathbb{A}_F)$ of weight $\mu \in (\mathbb{Z}^3_+)_0^{\mathrm{Hom}\,(F,\mathbb{C})}$ and level prime to l such that*
   - *$\bar{r} \cong \bar{r}_{l,\imath}(\Pi)$ (so in particular, $\bar{r}^c \cong \bar{r}^\vee \bar{\epsilon}_l^{-2}$).*
   - *For each $\tau \in \mathrm{Hom}\,(F, \mathbb{C})$, $\mu_{\tau,1} - \mu_{\tau,3} \leq l-3$.*
   - *$\bar{r}(G_{F(\zeta_l)})$ is adequate.*

Let $a \in (\mathbb{Z}^3_+)_0^{\coprod_{w|l} \mathrm{Hom}\,(k_w,\overline{\mathbb{F}}_l)}$ be a generic Serre weight. Assume that $a \in W^{\mathrm{obv}}(\bar{r})$ (so in particular, $\bar{r}|_{G_{F_w}}$ is semisimple for all $w|l$). Then $\bar{r}$ is modular of weight a.

Remark 5.1.5. In fact, the proof below shows that it suffices to assume that a_w is generic for all places $w|l$ for which $\bar{r}|_{G_{F_w}}$ has niveau 2, and that if $\bar{r}|_{G_{F_w}}$ has niveau 1, then we do not have both $a_1 - a_3 = l-1$ and $a_1 - a_2$, $a_2 - a_3 \leq l-2$. In particular, if $\bar{r}|_{G_{F_w}}$ is irreducible for all places $w|l$ (which is the situation considered in [EGH13]), then we do not need to assume that a is generic.

Proof of Theorem 5.1.4. By Corollaries 4.1.10 and 4.1.14, $\bar{r}$ is modular of weight b for some Serre weight b with the property that F_b is a Jordan-Hölder factor of P_a. We wish to show that $F_b \cong F_a$. Assume for the sake of contradiction that $F_b \not\cong F_a$, so that there is a place $w|l$ with $F_{b_w} \not\cong F_{a_w}$. By Lemma 5.1.1, we must have $l-1 \leq a_{w,1} - a_{w,3}$ and $a_{w,1} - a_{w,2}$, $a_{w,2} - a_{w,3} \leq l-2$, and $b_w = (a_{w,3} + l - 2, a_{w,2}, a_{w,1} - l + 2)$.

Since $l - 1 \leq a_{w,1} - a_{w,3}$, we have $b_{w,1} - b_{w,3} = 2l - 4 - (a_{w,1} - a_{w,3}) \leq l - 3$. Thus the assumption that $\bar{r}$ is modular of weight b, together with Lemma 4.2.1 gives an explicit description of the possibilities for $\bar{r}|_{G_{F_w}}$ (which is assumed to be semisimple) in terms of b_w, and hence in terms of a_w. We also have another such description from the assumption that $a \in W^{\mathrm{obv}}(\bar{r})$. We will now compare these descriptions to obtain a contradiction.

It will be useful to note that since we are assuming that $a_{w,1} - a_{w,2}$, $a_{w,2} - a_{w,3} \leq l-2$, and $a_{w,1} - a_{w,3} \geq l - 1$ we have

$$1 \leq a_{w,1} - a_{w,2}, a_{w,2} - a_{w,3} \leq l - 2, \tag{5.1.1}$$

$$l - 1 \leq a_{w,1} - a_{w,3} \leq 2l - 4, \tag{5.1.2}$$

so that

$$a_{w,1} \not\equiv a_{w,2} \pmod{l-1}, \tag{5.1.3}$$

$$a_{w,2} \not\equiv a_{w,3} \pmod{l-1}, \tag{5.1.4}$$

$$a_{w,3} \not\equiv a_{w,1}+1 \pmod{l-1}. \tag{5.1.5}$$

$$a_{w,1}-a_{w,3} \not\equiv l-2 \pmod{l+1}. \tag{5.1.6}$$

If $a_{w,1}-a_{w,2}=1$ then the condition that $a_{w,1}-a_{w,3}\geq l-1$ forces $a_{w,2}-a_{w,3}=l-2$, so that a_w is not generic. Similarly if $a_{w,2}-a_{w,3}=1$ then a_w is not generic. Therefore if we assume that a_w is generic, we also have

$$a_{w,1} \not\equiv a_{w,2}+1 \pmod{l-1}, \tag{5.1.7}$$

$$a_{w,2} \not\equiv a_{w,3}+1 \pmod{l-1}. \tag{5.1.8}$$

By the second and third conditions in the definition of genericity, we also have

$$a_{w,3} \not\equiv a_{w,2}+1 \pmod{l-1}, \tag{5.1.9}$$

$$a_{w,2} \not\equiv a_{w,1}+1 \pmod{l-1}. \tag{5.1.10}$$

Niveau 1 Suppose firstly that $\bar{r}|_{G_{F_w}}$ has niveau 1, i.e. that $\bar{r}|_{I_{F_w}}$ is a direct sum of powers of the mod l cyclotomic character ω. Then since $a\in W^{\mathrm{obv}}(\bar{r})$ and a is generic, we see from Lemma 5.1.2 that

$$\bar{r}|_{I_{F_w}} \cong \omega^{-(a_{w,1}+2)}\oplus\omega^{-(a_{w,2}+1)}\oplus\omega^{-a_{w,3}}.$$

By Lemma 4.2.1 (applied to F_b), we see that we also have

$$\bar{r}|_{I_{F_w}} \cong \omega^{-(a_{w,3}+1)}\oplus\omega^{-(a_{w,2}+1)}\oplus\omega^{-(a_{w,1}+1)}.$$

Thus $a_{w,3}\equiv a_{w,1}+1 \pmod{l-1}$, contradicting (5.1.5).

Niveau 2 Suppose next that $\bar{r}|_{G_{F_w}}$ has niveau 2, i.e. that $\bar{r}|_{I_{F_w}}$ is a direct sum of a power of the mod l cyclotomic character ω and characters ω_2^n, ω_2^{ln} for some n with $(l+1)\nmid n$, where ω_2 is a choice of fundamental character of niveau 2. Then since $a\in W^{\mathrm{obv}}(\bar{r})$, we see from Lemma 5.1.2 that $\bar{r}|_{I_{F_w}}$ is isomorphic to one of the following:

$$\omega^{-(a_{w,1}+2)}\oplus\omega_2^{-(a_{w,2}+1+la_{w,3})}\oplus\omega_2^{-(l(a_{w,2}+1)+a_{w,3})}$$
$$\omega^{-(a_{w,2}+1)}\oplus\omega_2^{-(a_{w,1}+2+la_{w,3})}\oplus\omega_2^{-(l(a_{w,1}+2)+a_{w,3})}$$
$$\omega^{-a_{w,3}}\oplus\omega_2^{-(a_{w,1}+2+l(a_{w,2}+1))}\oplus\omega_2^{-(l(a_{w,1}+2)+a_{w,2}+1)}$$

By Lemma 4.2.1 (applied to F_b), we see that we also have that $\bar{r}|_{I_{F_w}}$ is isomorphic to one of the following:

$$\omega^{-(a_{w,1}+1)}\oplus\omega_2^{-(a_{w,2}+1+l(a_{w,3}+l))}\oplus\omega_2^{-(l(a_{w,2}+1)+a_{w,3}+l)}$$
$$\omega^{-(a_{w,2}+1)}\oplus\omega_2^{-(a_{w,1}-l+2+l(a_{w,3}+l))}\oplus\omega_2^{-(l(a_{w,1}-l+2)+(a_{w,3}+l))}$$
$$\omega^{-(a_{w,3}+1)}\oplus\omega_2^{-(a_{w,1}-l+2+l(a_{w,2}+1))}\oplus\omega_2^{-(l(a_{w,1}-l+2)+a_{w,2}+1)}$$

Comparing the powers of ω and using (5.1.3)–(5.1.10), the only possibility is that we simultaneously have

$$\bar{r}|_{I_{F_w}} \cong \omega^{-(a_{w,2}+1)}\oplus\omega_2^{-(a_{w,1}+2+la_{w,3})}\oplus\omega_2^{-(l(a_{w,1}+2)+a_{w,3})},$$
$$\bar{r}|_{I_{F_w}} \cong \omega^{-(a_{w,2}+1)}\oplus\omega_2^{-(a_{w,1}-l+2+l(a_{w,3}+l))}\oplus\omega_2^{-(l(a_{w,1}-l+2)+(a_{w,3}+l))}.$$

There are now two possibilities to examine. Firstly it could be the case that

$$a_{w,1} + 2 + la_{w,3} \equiv a_{w,1} - l + 2 + l(a_{w,3} + l) \pmod{l^2 - 1};$$

but this implies that $l^2 - l \equiv 0 \pmod{l^2-1}$, a contradiction. So we must have

$$a_{w,1} + 2 + la_{w,3} \equiv l(a_{w,1} - l + 2) + (a_{w,3} + l) \pmod{l^2 - 1}.$$

This simplifies to $a_{w,1} - a_{w,3} \equiv l - 2 \pmod{l+1}$, contradicting (5.1.6).

Niveau 3 Suppose finally that $\bar{r}|_{G_{F_w}}$ has niveau 3, i.e. that $\bar{r}|_{I_{F_w}}$ is of the form $\omega_3^n \oplus \omega_3^{ln} \oplus \omega_3^{l^2 n}$ for some n with $(l^2+l+1) \nmid n$, where ω_3 is a choice of fundamental character of niveau 3. Then since $a \in W^{\mathrm{obv}}(\bar{r})$, we see that $\bar{r}|_{I_{F_w}}$ is isomorphic to one of the following:

$$\omega_3^{-(a_{w,1}+2+l(a_{w,2}+1)+l^2 a_{w,3})} \oplus \omega_3^{-(a_{w,2}+1+la_{w,3}+l^2(a_{w,1}+2))} \oplus \omega_3^{-(a_{w,3}+l(a_{w,1}+2)+l^2(a_{w,2}+1))}$$

$$\omega_3^{-(a_{w,1}+2+la_{w,3}+l^2(a_{w,2}+1))} \oplus \omega_3^{-(a_{w,3}+l(a_{w,2}+1)+l^2(a_{w,1}+2))} \oplus \omega_3^{-(a_{w,2}+1+l(a_{w,1}+2)+l^2 a_{w,3})}$$

On the other hand, by Lemma 4.2.1 (applied to F_b) we also have that $\bar{r}|_{I_{F_w}}$ is isomorphic to one of the following:

$$\omega_3^{-(a_{w,1}-l+2+l(a_{w,2}+1)+l^2(a_{w,3}+l))} \oplus \omega_3^{-(a_{w,2}+1+l(a_{w,3}+l)+l^2(a_{w,1}-l+2))} \oplus \omega_3^{-(a_{w,3}+l+l(a_{w,1}-l+2)+l^2(a_{w,2}+1))}$$

$$\omega_3^{-(a_{w,1}-l+2+l(a_{w,3}+l)+l^2(a_{w,2}+1))} \oplus \omega_3^{-(a_{w,3}+l+l(a_{w,2}+1)+l^2(a_{w,1}-l+2))} \oplus \omega_3^{-(a_{w,2}+1+l(a_{w,1}-l+2)+l^2(a_{w,3}+l))}$$

Examining the exponents in these expressions, we obtain 12 possible congruences $\pmod{l^3-1}$, each of which we will now show yields a contradiction. In each case below we derive a congruence modulo l^2+l+1 or l^3-1, and it is easy to see in each case that the inequalities (5.1.1) and (5.1.2) imply that the congruence has no solutions.

(1) $a_{w,1} + 2 + l(a_{w,2}+1) + l^2 a_{w,3} \equiv a_{w,1} - l + 2 + l(a_{w,2}+1) + l^2(a_{w,3}+l) \pmod{l^3-1}$. This simplifies to $l^2 - 1 \equiv 0 \pmod{l^3-1}$, a contradiction.

(2) $a_{w,1} + 2 + l(a_{w,2}+1) + l^2 a_{w,3} \equiv a_{w,1} - l + 2 + l(a_{w,3}+l) + l^2(a_{w,2}+1) \pmod{l^3-1}$. This simplifies to $a_{w,2} - a_{w,3} + 2 \equiv 0 \pmod{l^2+l+1}$, a contradiction.

(3) $a_{w,1}+2+l(a_{w,2}+1)+l^2 a_{w,3} \equiv a_{w,2}+1+l(a_{w,1}-l+2)+l^2(a_{w,3}+l) \pmod{l^3-1}$. This simplifies to $a_{w,1} - a_{w,2} \equiv l \pmod{l^2+l+1}$, a contradiction.

(4) $a_{w,1} + 2 + l(a_{w,2}+1) + l^2 a_{w,3} \equiv a_{w,2} + 1 + l(a_{w,3}+l) + l^2(a_{w,1} - l + 2) \pmod{l^3-1}$. This simplifies to $l(a_{w,1} - a_{w,3} + 3) + (a_{w,1} - a_{w,2} + 2) \equiv 0 \pmod{l^2+l+1}$, which is easily seen to be impossible.

(5) $a_{w,1} + 2 + l(a_{w,2}+1) + l^2 a_{w,3} \equiv a_{w,3} + l + l(a_{w,1} - l + 2) + l^2(a_{w,2}+1) \pmod{l^3-1}$. This simplifies to $(a_{w,1} - a_{w,3}) + l(a_{w,2} - a_{w,3}) + 2 \equiv 0 \pmod{l^2+l+1}$, which is also impossible.

(6) $a_{w,1} + 2 + l(a_{w,2}+1) + l^2 a_{w,3} \equiv a_{w,3} + l + l(a_{w,2}+1) + l^2(a_{w,1} - l + 2) \pmod{l^3-1}$. This simplifies to $(l+1)(a_{w,1} - a_{w,3} + 2) + 1 \equiv 0 \pmod{l^2+l+1}$, which is impossible.

(7) $a_{w,1} + 2 + la_{w,3} + l^2(a_{w,2}+1) \equiv a_{w,1} - l + 2 + l(a_{w,2}+1) + l^2(a_{w,3}+l) \pmod{l^3-1}$. This simplifies to $l(a_{w,2} - a_{w,3} + 1) + 1 \equiv 0 \pmod{l^2+l+1}$, a contradiction.

(8) $a_{w,1} + 2 + la_{w,3} + l^2(a_{w,2}+1) \equiv a_{w,1} - l + 2 + l(a_{w,3}+l) + l^2(a_{w,2}+1) \pmod{l^3-1}$. This simplifies to $l^2 - l \equiv 0 \pmod{l^3-1}$, a contradiction.

(9) $a_{w,1} + 2 + la_{w,3} + l^2(a_{w,2}+1) \equiv a_{w,2} + 1 + l(a_{w,1} - l + 2) + l^2(a_{w,3}+l) \pmod{l^3-1}$. This simplifies to $l(a_{w,2} - a_{w,3} + 2) \equiv a_{w,1} - a_{w,2} \pmod{l^2+l+1}$, which is easily seen to be impossible.

(10) $a_{w,1} + 2 + la_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,2} + 1 + l(a_{w,3} + l) + l^2(a_{w,1} - l + 2) \pmod{l^3 - 1}$. This simplifies to $a_{w,1} - a_{w,2} + 2 \equiv 0 \pmod{l^2 + l + 1}$, a contradiction.
(11) $a_{w,1} + 2 + la_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,3} + l + l(a_{w,1} - l + 2) + l^2(a_{w,2} + 1) \pmod{l^3 - 1}$. This simplifies to $a_{w,1} - a_{w,3} \equiv l - 2 \pmod{l^2 + l + 1}$, a contradiction.
(12) $a_{w,1} + 2 + la_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,3} + l + l(a_{w,2} + 1) + l^2(a_{w,1} - l + 2) \pmod{l^3 - 1}$. This simplifies to $l(a_{w,1} - a_{w,2} + 1) + a_{w,1} - a_{w,3} + 3 \equiv 0 \pmod{l^2 + l + 1}$, which is impossible.

As we have obtained a contradiction in every case, we see that $F_b \cong F_a$, as required. □

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