SERRE WEIGHTS FOR $U(n)$.

THOMAS BARNET-LAMB, TOBY GEE, AND DAVID GERAGHTY

Abstract. We study the weight part of (a generalisation of) Serre’s conjecture for mod $l$ Galois representations associated to automorphic representations on unitary groups of rank $n$ for odd primes $l$. Given a modular Galois representation, we use automorphy lifting theorems to prove that it is modular in many other weights. We make no assumptions on the ramification or inertial degrees of $l$. We give an explicit strengthened result when $n = 3$ and $l$ splits completely in the underlying CM field.

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1. Introduction

In recent years there has been considerable progress in formulating generalisations of Serre’s conjecture, and in particular of the weight part of Serre’s conjecture, for higher-dimensional groups; cf. [ADP02], [Her09], [Gee11], [EGHS14]. There has been rather less progress in proving cases of these conjectures; indeed, the only results that we are aware of are the essentially complete treatment of the ordinary case for definite unitary groups in [GG12], and the results of [EGH13] for definite unitary groups of rank 3.

In the present paper, we use the automorphy lifting theorems developed in [BLGG11], [BLGG12] and [BLGGT14b] to prove that a modular Galois representation, coming from an automorphic form on $U(n)$, is necessarily modular in a number of additional weights predicted by the conjectures of [Her09] and [EGHS14]. Rather complete results are available in the case $n = 2$, which are worked out in detail in the papers [BLGG13, GLS14, GLS13], so we concentrate in this paper on the case that $n > 2$. The additional complications are twofold. Firstly, we no longer know that any modular Galois representation admits a potentially diagonalizable lift (in the case $n = 2$, this is proved in [BLGG13] as a consequence of the
Theorem A. Let $F$ be an imaginary CM field with maximal totally real subfield $F^+$, and suppose that $F/F^+$ is unramified at all finite places, and that $l$ splits completely in $F$. Suppose that $l > 2$, and that $\bar{\rho} : G_F \to \GL_n(F_l)$ is an irreducible representation with split ramification. Assume that there is a RACSDC automorphic representation $\Pi$ of $\GL_n(k_F)$ of weight $\mu \in (\mathbb{Z}^+)_{l,1} \Hom(k,F^+) \otimes \varepsilon_{C_l}$ and level prime to $l$ such that

- $\bar{\rho} \cong \bar{\rho_{l,1}}(\Pi)$ (so in particular, $\bar{\rho}^c \cong \bar{\rho}^\vee \tau_l^{-2}$).
- For each $\tau \in \Hom(F,C)$, $\mu_{\tau,1} + \mu_{\tau,3} \leq l - 3$.
- $\bar{\rho}(G_{F(\zeta_l)})$ is adequate.

Let $a \in (\mathbb{Z}^+)_{l,1} \bigcup \Hom(k,F_l)$ be a generic Serre weight. Assume that $a \in W^{obv}(\bar{\rho})$. Then $\bar{\rho}$ is modular of weight $a$.

(See sections 2 and 4 for any unfamiliar terminology, and section 6 for the definition of “generic” that we are using, which is extremely mild.) We should point out that we do not expect that $W^{obv}(\bar{\rho})$ contains all the weights in which $\bar{\rho}$ is modular; rather, it consists of those weights which are “obvious” in the terminology of [EGHS14]. (It is perhaps worth remarking that despite the name, it is not obvious that $\bar{\rho}$ is modular in any of these weights!) In order to prove this theorem we make use of Fontaine-Laffaille theory; it seems likely that if one could compute the possible reductions of crystalline Galois representations outside of the Fontaine-Laffaille range then one could prove an analogous theorem for $n > 3$.

We now outline the structure of this paper. In Section 2 we define the spaces of automorphic forms that we work with, and define what it means for $\bar{\rho}$ to be modular of some weight. In Section 3 we establish the main lifting theorem that we need, a corollary of the results of [BLGGT14b]. In Section 4 we define the set of weights $W^{obv}(\bar{\rho})$, recall some results from Fontaine-Laffaille theory, and prove our main results for arbitrary $n$. Finally, in Section 5 we prove Theorem A.
1.1. Notation. If $M$ is a field, we let $G_M$ denote its absolute Galois group. We write all matrix transposes on the left; so $^tA$ is the transpose of $A$. Let $\epsilon_l$ denote the $l$-adic cyclotomic character, and $\bar{\epsilon}_l$ or $\omega_l$ the mod $l$ cyclotomic character. If $M$ is a finite extension of $\mathbb{Q}_p$ for some $p$, we write $I_M$ for the inertia subgroup of $G_M$. If $R$ is a local ring we write $m_R$ for the maximal ideal of $R$.

We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. For each prime $p$ we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and we fix an embedding $\overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$.

If $W$ is a de Rham representation of $G_K$ over $\overline{\mathbb{Q}}_l$ and if $\tau : K \twoheadrightarrow \mathbb{Q}_l$ then by definition the multiset $\text{HT}_\tau(W)$ of Hodge-Tate weights of $W$ with respect to $\tau$ contains $i$ with multiplicity $\dim_{\mathbb{Q}_l}(W \otimes_{\tau, K} \mathbb{Q}_l(i))^G_K$. Thus for example $\text{HT}_\tau(\epsilon_l) = \{-1\}$.

If $K$ is a finite extension of $\mathbb{Q}_p$ for some $p$, we will let $\text{rec}_K$ be the local Langlands correspondence of [HT01], so that if $\pi$ is an irreducible complex admissible representation of $\text{GL}_n(K)$, then $\text{rec}_K(\pi)$ is a Weil-Deligne representation of the Weil group $W_K$. We will write $\text{rec}$ for $\text{rec}_K$ when the choice of $K$ is clear. We write $\text{Art}_K : K^\times \rightarrow W_K$ for the isomorphism of local class field theory, normalised so that uniformisers correspond to geometric Frobenius elements.

Let $L$ be a finite extension of $\mathbb{Q}_l$ with residue field $k$. For each $\sigma \in \text{Hom}(k, \mathbb{F}_l)$ we define the fundamental character $\omega_\sigma$ corresponding to $\sigma$ to be the composite

$$I_{K^\text{ab}/K} \xrightarrow{\text{Art}_K^{-1}} \mathcal{O}_K^\times \xrightarrow{k^\times \sigma^{-1}} \mathbb{F}_l^\times.$$ 

Note that if $k = \mathbb{F}_l$ then $\omega_\sigma = \omega_l$. For any algebraic extension $L$ of $\mathbb{Q}_l$, we often denote by $\text{Hom}(K, L)$ the set of field homomorphisms from $K$ to $L$ which are continuous for the $l$-adic topologies on $K$ and $L$ (or equivalently, which are $\mathbb{Q}_l$-linear).

2. Definitions

2.1. Let $l$ be a prime, and let $F$ be an imaginary CM field with maximal totally real field subfield $F^+$. We assume throughout this paper that:

- $F/F^+$ is unramified at all finite places.
- Every place $v|l$ of $F^+$ splits in $F$.
- If $n$ is even, then $n[F^+/Q]/2$ is also even.

Under these hypotheses, there is a reductive algebraic group $G/F^+$ with the following properties:

- $G$ is an outer form of $\text{GL}_n$, with $G/F \cong \text{GL}_n/F$.
- If $v$ is a finite place of $F^+$, $G$ is quasi-split at $v$.
- If $v$ is an infinite place of $F^+$, then $G(F_v^+) \cong U_n(\mathbb{R})$.

To see that such a group exists, one may argue as follows. Let $B$ denote the matrix algebra $M_n(F)$. An involution $^\dagger$ of the second kind on $B$ gives a reductive group $G_1$ over $F^+$ by setting

$$G_1(R) = \{ g \in B \otimes_{F^+} R : g^\dagger g = 1 \}$$

for any $F^+$-algebra $R$. Any such $G_1$ is an outer form of $\text{GL}_n$, with $G_1/F \cong \text{GL}_n/F$.

One can choose $^\dagger$ such that

- If $v$ is a finite place of $F^+$, $G_1$ is quasi-split at $v$.
- If $v$ is an infinite place of $F^+$, then $G_1(F_v^+) \cong U_n(\mathbb{R})$. 

To see this, one uses the argument of Lemma I.7.1 of [HT01]. We then fix some choice of $\mathfrak{f}$ as above, and take $G = G_1$.

As in section 3.3 of [CHT08] we define a model for $G$ over $\mathcal{O}_{F^+}$ in the following way. We choose an order $\mathcal{O}_B$ in $B$ such that $\mathcal{O}_B^{\mathfrak{f}} = \mathcal{O}_B$, and $\mathcal{O}_{B,w}$ is a maximal order in $B_w$ for all places $w$ of $F$ which are split over $F^+$ (see section 3.3 of [CHT08] for a proof that such an order exists). Then we can define $G$ over $\mathcal{O}_{F^+}$ by setting

$$G(R) = \{ g \in \mathcal{O}_B \otimes \mathcal{O}_{F^+} : g^\mathfrak{f} g = 1 \}$$

for any $\mathcal{O}_{F^+}$-algebra $R$.

If $v$ is a place of $F^+$ which splits as $ww^c$ over $F$, then we choose an isomorphism

$$\iota_v : \mathcal{O}_{B,v} \xrightarrow{\sim} \mathcal{M}_n(\mathcal{O}_{F,v}) = \mathcal{M}_n(\mathcal{O}_{F,w}) \oplus \mathcal{M}_n(\mathcal{O}_{F,w^c})$$

such that $\iota_w(x^\mathfrak{f}) = \iota_v(x)^c$. This gives rise to an isomorphism

$$\iota_w : G(\mathcal{O}_{F^+}) \xrightarrow{\sim} \text{GL}_n(\mathcal{O}_{F,w})$$

sending $\iota_v^{-1}(x, t x^{-c})$ to $x$.

Let $K$ be an algebraic extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}_l}$ which contains the image of every embedding $F \rightarrow \overline{\mathbb{Q}_l}$, let $\mathcal{O}$ denote the ring of integers of $K$, and let $k$ denote the residue field of $K$. Let $\mathcal{S}_l$ denote the set of places of $F^+$ lying over $l$, and for each $v \in \mathcal{S}_l$ fix a place $\mathfrak{v}$ of $F$ lying over $v$. Let $\mathcal{S}_l^c$ denote the set of places $\mathfrak{v}$ for $v \in \mathcal{S}_l$.

Let $W$ be an $\mathcal{O}$-module with an action of $G(\mathcal{O}_{F^+, l})$, and let $U \subset G(\mathbb{A}_F^\infty)$ be a compact open subgroup with the property that for each $u \in U$, if $u_l$ denotes the projection of $u$ to $G(F_l^+)$, then $u_l \in G(\mathcal{O}_{F^+, l})$. Let $S(U, W)$ denote the space of algebraic modular forms on $G$ of level $U$ and weight $W$, i.e. the space of functions

$$f : G(F^+) \backslash G(\mathbb{A}_{F^+, l}) \rightarrow W$$

with $f(gu) = u_l^{-1} f(g)$ for all $u \in U$.

Let $\tilde{\mathcal{S}}_l$ denote the set of embeddings $F \rightarrow K$ giving rise to a place in $\overline{\mathcal{S}}_l$. For any $\tilde{\mathfrak{v}} \in \tilde{\mathcal{S}}_l$, let $\tilde{\mathcal{S}}_l$ denote the set of elements of $\tilde{\mathcal{S}}_l$ lying over $\tilde{\mathfrak{v}}$. Let $\mathbb{Z}_l^n$ denote the set of tuples $(\lambda_1, \ldots, \lambda_n)$ of integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For any $\lambda \in \mathbb{Z}_l^n$, view $\lambda$ as a dominant character of the algebraic group $\text{GL}_n$ in the usual way, and let $M'_\lambda$ be the algebraic $\mathcal{O}$-representation of $\text{GL}_n$ given by

$$M'_\lambda := \text{Ind}_{\mathcal{B}_n}^{\text{GL}_n}(w_0 \lambda)/\mathcal{O}$$

where $\mathcal{B}_n$ is the standard upper-triangular Borel subgroup of $\text{GL}_n$, and $w_0$ is the longest element of the Weyl group (see [Jan03] for more details of these notions). Write $M_\lambda$ for the $\mathcal{O}$-representation of $\text{GL}_n(\mathcal{O})$ obtained by evaluating $M'_\lambda$ on $\mathcal{O}$. For any $\lambda \in (\mathbb{Z}_l^n)^\mathfrak{f}_l$, let $W_\lambda$ be the free $\mathcal{O}$-module with an action of $\text{GL}_n(\mathcal{O}_{F^+})$ given by

$$W_\lambda := \otimes_{\mathfrak{v} \in \mathfrak{f}_l} M_{\lambda_{\mathfrak{v}}} \otimes \mathcal{O}_{F^+, \mathfrak{v}} \otimes \mathcal{O}.$$

We give this an action of $G(\mathcal{O}_{F^+, w})$ via $\iota_w$. For any $\lambda \in (\mathbb{Z}_l^n)^{\mathfrak{f}_l}$, let $W_\lambda$ be the free $\mathcal{O}$-module with an action of $G(\mathcal{O}_{F^+, l})$ given by

$$W_\lambda := \otimes_{\mathfrak{v} \in \mathfrak{f}_l} W_{\lambda_{\mathfrak{v}}}.$$

If $A$ is an $\mathcal{O}$-module we let

$$S_\lambda(U, A) := S(U, W_\lambda \otimes \mathcal{O} A).$$
For any compact open subgroup $U$ as above of $G(\mathbb{A}_{F}^{\infty})$ we may write $G(\mathbb{A}_{F}^{\infty}) = \coprod G(F^+)^{t_i} U$ for some finite set $\{t_i\}$. Then there is an isomorphism

$$S(U,W) \rightarrow \oplus_i W^{U \cap t_i^{-1}G(F^+)t_i}$$

given by $f \mapsto (f(t_i))_i$. We say that $U$ is sufficiently small if for some finite place $v$ of $F^+$ the projection of $U$ to $G(F_v^+)$ contains no element of finite order other than the identity. Suppose that $U$ is sufficiently small. Then for each $i$ as above we have $U \cap t_i^{-1}G(F^+)t_i = \{1\}$, so taking $W = W_{\lambda} \otimes_{O} A$ we see that for any $O$-module $A$, we have

$$S_{\lambda}(U,A) \cong S_{\lambda}(U,O) \otimes_{O} A.$$  

We note when $U$ is not sufficiently small, we still have $S_{\lambda}(U,A) \cong S_{\lambda}(U,O) \otimes_{O} A$ whenever $A$ is $O$-flat.

We now recall the relationship between our spaces of algebraic automorphic forms and the space of automorphic forms on $G$. Write $S_{\lambda}(\mathbb{Q}_l)$ for the direct limit of the spaces $S_{\lambda}(U,\mathbb{Q}_l)$ over compact open subgroups $U$ as above (with the transition maps being the obvious inclusions $S_{\lambda}(U,\mathbb{Q}_l) \subset S_{\lambda}(V,\mathbb{Q}_l)$ whenever $V \subset U$). Concretely, $S_{\lambda}(\mathbb{Q}_l)$ is the set of functions

$$f : G(F^+)\backslash G(\mathbb{A}_{F}^{\infty}) \rightarrow W_{\lambda} \otimes_{O} \mathbb{Q}_l$$

such that there is a compact open subgroup $U$ of $G(\mathbb{A}_{F}^{\infty}) \times G(O_{F^+,l})$ with

$$f(gu) = u^{-1}f(g)$$

for all $u \in U, g \in G(\mathbb{A}_{F}^{\infty})$. This space has a natural left action of $G(\mathbb{A}_{F}^{\infty})$ via

$$(g \cdot f)(h) := gf(hg).$$

Fix an isomorphism $i : \mathbb{Q}_l \rightarrow \mathbb{C}$. For each embedding $\tau : F^+ \rightarrow \mathbb{R}$, there is a unique embedding $\hat{\tau} : F \hookrightarrow \mathbb{C}$ extending $\tau$ such that $i^{-1}\hat{\tau} \in \hat{I}_l$. Let $\sigma_{\lambda}$ denote the representation of $G(F^+_\mathbb{Q})$ given by $W_{\lambda} \otimes_{O} \mathbb{Q}_l \otimes_{\mathbb{Q}_l} \hat{\tau}$, with an element $g \in G(F^+_\mathbb{Q})$ acting via $\otimes_{\tau} \hat{\tau}(i(\tau))(g)$. Let $A$ denote the space of automorphic forms on $G(F^+)\backslash G(\mathbb{A}_{F}^{\infty})$. From the proof of Proposition 3.3.2 of [CHT08], one easily obtains the following.

**Lemma 2.1.1.** There is an isomorphism of $G(\mathbb{A}_{F}^{\infty})$-modules

$$S_{\lambda}(\mathbb{Q}_l) \cong \text{Hom}_{G(F^+_\mathbb{Q})}(\sigma_{\lambda}^\vee,A).$$

In particular, we note that $S_{\lambda}(\mathbb{Q}_l)$ is a semi-simple admissible $G(\mathbb{A}_{F}^{\infty})$-module.

Following [CHT08], we say that a cuspidal automorphic representation of $\text{GL}_{n}(\mathbb{A}_{F})$ is RACSDC (regular, algebraic, conjugate self dual, and cuspidal) if

- $\pi_{\infty}$ has the same infinitesimal character as some irreducible algebraic representation of $\text{Res}_{F/\mathbb{Q}} \text{GL}_{n}$, and
- $\pi^{c} \cong \pi^{\vee}$.

We say that $\pi$ has level prime to $l$ if $\pi_{v}$ is unramified for all $v|l$. If $\Omega$ is an algebraically closed field of characteristic $0$ we write $(\mathbb{Z}_{+}^{n})_{0}^{\text{Hom}(F,\Omega)}$ for the subset of elements $\lambda \in (\mathbb{Z}_{+}^{n})^{\text{Hom}(F,\Omega)}$ such that

$$\lambda_{\tau,i} + \lambda_{\text{rec},n+1-i} = 0$$

for all $\tau, i$.  


If $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(\mathbb{F}, \mathbb{C})}$ we write $\Sigma_\lambda$ for the irreducible algebraic representation of $\text{GL}(\mathbb{F}, \mathbb{C})$ given by the tensor product over $\tau$ of the irreducible representations with highest weights $\lambda_\tau$. We say that a RACSDC automorphic representation $\pi$ of $\text{GL}_n(\mathbb{A}_F)$ has weight $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(\mathbb{F}, \mathbb{C})}$ if $\pi_\infty$ has the same infinitesimal character as $\Sigma_\lambda$. If this is the case then necessarily $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(\mathbb{F}, \mathbb{C})}$.

**Theorem 2.1.2.** If $\pi$ is a RACSDC automorphic representation of $\text{GL}_n(\mathbb{A}_F)$ of weight $\lambda$, then there is a continuous semisimple representation

$$r_{l, \lambda}(\pi) : G_F \to \text{GL}_n(\overline{\mathbb{Q}}_l)$$

such that

1. $r_{l, \lambda}(\pi)^c \cong r_{l, \lambda}(\pi)^\vee \otimes \iota_l^{1-n}$.
2. The representation $r_{l, \lambda}(\pi)$ is de Rham, and is crystalline if $\pi$ has level prime to $l$. If $\tau : F \to \overline{\mathbb{Q}}_l$ then
   $$\text{HT}_\tau(r_{l, \lambda}(\pi)) = \{\lambda_{l\tau, 1} + n - 1, \ldots, \lambda_{l\tau, n}\}.$$
3. For each finite place $v$ of $l$, we have
   $$i\text{WD}(r_{l, \lambda}(\pi)|_{F_{v}})^{F-\text{ss}} \cong \text{rec}(\pi_v^\vee \otimes |\det|^{(1-n)/2}).$$

Here $\text{WD}(r_{l, \lambda}(\pi)|_{F_{v}})^{F-\text{ss}}$ denotes the Frobenius semisimplification of the Weil-Deligne representation associated to $r_{l, \lambda}(\pi)|_{F_{v}}$, as in section 1 of [TY07].

**Proof.** This follows at once from the main results of [Shi11], [CH13], [Car12a], [BLG14a] and [Car12b].

We say that a continuous irreducible representation $r : G_F \to \text{GL}_n(\overline{\mathbb{Q}}_l)$ (respectively $\tilde{r} : G_F \to \text{GL}_n(\mathbb{F}_l)$) is automorphic if $r \cong r_{l, \lambda}(\pi)$ (respectively $\tilde{r} \cong \tilde{r}_{l, \lambda}(\pi)$) for some RACSDC representation $\pi$ of $\text{GL}_n(\mathbb{A}_F)$. We say that a continuous irreducible representation $r : G_F \to \text{GL}_n(\overline{\mathbb{Q}}_l)$ is automorphic of weight $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(\mathbb{F}, \mathbb{C})}$ if $r \cong r_{l, \lambda}(\pi)$ for some RACSDC representation $\pi$ of $\text{GL}_n(\mathbb{A}_F)$ of weight $\lambda$.

The theory of base change gives a close relationship between automorphic representations of $G(\mathbb{A}_{F^+})$ and automorphic representations of $\text{GL}_n(\mathbb{A}_F)$. For example, one has the following consequences of Corollaire 5.3 and Théoréme 5.4 of [Lab09].

**Theorem 2.1.3.** Suppose that $\Pi$ is a RACSDC representation of $\text{GL}_n(\mathbb{A}_F)$ of weight $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(\mathbb{F}, \mathbb{C})}$. Then there is an automorphic representation $\pi$ of $G(\mathbb{A}_{F^+})$ such that

1. For each embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ and each $\check{\tau} : F \hookrightarrow \mathbb{C}$ extending $\tau$, we have $\pi_\tau \cong \Sigma_\lambda^{\check{\tau}} \circ \iota_\tau$.
2. If $v$ is a finite place of $F^+$ which splits as $ww^c$ in $F$, then $\pi_v \cong \Pi_w \circ \iota_w$.
3. If $v$ is a finite place of $F^+$ which is inert in $F$, and $\Pi_v$ is unramified, then $\pi_v$ has a fixed vector for some hyperspecial maximal compact subgroup of $G(F_v^\circ)$.

**Theorem 2.1.4.** Suppose that $\pi$ is an automorphic representation of $G(\mathbb{A}_{F^+})$. Then either:

1. There is an RACSDC automorphic representation $\Pi$ of $\text{GL}_n(\mathbb{A}_F)$ of some weight $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(\mathbb{F}, \mathbb{C})}$, or:
(2) There is a nontrivial partition \( n = n_1 + \cdots + n_r \) and cuspidal automorphic representations \( \Pi_i \) of \( \text{GL}_n(A_F) \) such that if \( \Pi := \Pi_1 \oplus \cdots \oplus \Pi_r \) is the isobaric direct sum of the \( \Pi_i \), then \( \Pi \) is regular, algebraic, and conjugate self-dual of some weight \( \lambda \in (\mathbb{Z}_+)^{\text{Hom}(F, C)} \) such that in either case

1. For each embedding \( \tau : F^+ \hookrightarrow \mathbb{R} \) and each \( \tau' \hookrightarrow \mathbb{C} \) extending \( \tau \), we have \( \pi_{\tau} \cong \sum \pi_{\lambda} \circ \iota_{\tau} \).
2. If \( v \) is a finite place of \( F^+ \) which splits as \( \pi_{\lambda} \circ v \) in \( F \), then \( \pi_{v} \cong \Pi_{w} \circ \iota_{w} \).
3. If \( v \) is a finite place of \( F^+ \) which is inert in \( F \), and \( \pi_{v} \) has a fixed vector for some hyperspecial maximal compact subgroup of \( G(F_v^+) \), then \( \Pi_v \) is unramified.

We now wish to define what it means for an irreducible representation \( \tilde{\pi} : G_F \to \text{GL}_n(F) \) to be modular of some weight. In order to do so, we return to the spaces of algebraic modular forms considered before. For each place \( w | l \) of \( F \), let \( k_w \) denote the residue field of \( F_w \). If \( w \) lies over a place \( v \) of \( F^+ \), write \( v = w v^c \). Let \( (\mathbb{Z}_+)^{\text{Hom}(k_{w,F})} \) denote the subset of \( (\mathbb{Z}_+)^{\text{Hom}(k_{w,F})} \) consisting of elements \( a \) such that for each \( w | l \), if \( \sigma \in \text{Hom}(k_w, F) \) and \( 1 \leq i \leq n \) then

\[
a_{\sigma,i} + a_{\sigma,n+1-i} = 0.
\]

We say that an element \( a \in (\mathbb{Z}_+)^{\text{Hom}(k_{w,F})} \) is a Serre weight if for each \( w | l \) and each \( \sigma \in \text{Hom}(k_w, F) \) we have

\[
l - 1 \geq a_{\sigma,i} - a_{\sigma,i+1}
\]

for all \( 1 \leq i \leq n - 1 \). Similarly, if \( \mathbb{F} \) is a finite extension of \( \mathbb{F}_l \), we say that an element \( a \in (\mathbb{Z}_+)^{\text{Hom}(\mathbb{F}, \mathbb{F}_l)} \) is a Serre weight if for each \( \sigma \in \text{Hom}(\mathbb{F}, \mathbb{F}_l) \) and each \( 1 \leq i \leq n - 1 \) we have

\[
l - 1 \geq a_{\sigma,i} - a_{\sigma,i+1}.
\]

Given any \( a \in \mathbb{Z}_+^n \) with \( l - 1 \geq a_i - a_{i+1} \) for all \( 1 \leq i \leq n - 1 \), we define the \( F \)-representation \( P_a \) of \( \text{GL}_n(F) \) to be the representation obtained by evaluating \( \text{Ind}_{G_{A,F}}^{\text{GL}_n(A,F)}(w_0 a) \) on \( F \), and let \( N_a \) be the irreducible sub-\( F \)-representation of \( P_a \) generated by the highest weight vector (that this is indeed irreducible follows for example from II.2.8(1) of [Jan03] and the appendix to [Her09]).

If \( a \in (\mathbb{Z}_+)^{\text{Hom}(\mathbb{F}, \mathbb{F}_l)} \) is a Serre weight then we define an irreducible \( \mathbb{F}_l \)-representation \( F_a \) of \( \text{GL}_n(F) \) by

\[
F_a := \otimes_{\tau \in \text{Hom}(\mathbb{F}, \mathbb{F}_l)} N_{a_{\tau}} \otimes_{F, \tau} \mathbb{F}_l.
\]

We will also consider the \( \mathbb{F}_l \)-representation \( P_a \) of \( \text{GL}_n(F) \) given by

\[
P_a := \otimes_{\tau \in \text{Hom}(\mathbb{F}, \mathbb{F}_l)} P_{a_{\tau}} \otimes_{F, \tau} \mathbb{F}_l.
\]

We say that two Serre weights \( a \) and \( b \) are equivalent if and only if \( F_a \cong F_b \) as representations of \( \text{GL}_n(F) \). This is equivalent to demanding that for each \( \sigma \in \text{Hom}(\mathbb{F}, \mathbb{F}_l) \), we have

\[
a_{\sigma,i} - a_{\sigma,i+1} = b_{\sigma,i} - b_{\sigma,i+1},
\]

for each \( 1 \leq i \leq n - 1 \), and the character \( F^x \to \mathbb{F}_l^\times \) given by

\[
x \mapsto \prod_{\sigma \in \text{Hom}(\mathbb{F}, \mathbb{F}_l)} \sigma(x)^{a_{\sigma,n} - b_{\sigma,n}}.
\]
is trivial. Every irreducible $\mathbb{F}_l$-representation of $\text{GL}_n(\mathbb{F})$ is of the form $F_a$ for some $a$ (see for example the appendix to [Her09]).

If $a \in (\mathbb{Z}_+^n)_0 \prod_{w} \text{Hom}(k_w, \mathbb{F}_l)$ is a Serre weight, we define an irreducible $\mathbb{F}_l$-representation $F_a$ of $G(\mathcal{O}_{F^+, l})$ as follows: we define

$$F_a = \otimes_{\mathbb{F}_l} F_{a,v},$$

an irreducible representation of $\prod_{v \in S_l} \text{GL}_n(k_v)$, and we let $G(\mathcal{O}_{F^+, l})$ act on $F_a$ by the composition of $\varphi_v$ and reduction modulo $l$. Again, we say that two Serre weights $a$ and $b$ are equivalent if and only if $F_a \cong F_b$ as representations of $G(\mathcal{O}_{F^+, l})$. This is equivalent to demanding that for each place $w|l$ and each $\sigma \in \text{Hom}(k_w, \mathbb{F}_l)$ and each $1 \leq i \leq n - 1$ we have

$$a_{\sigma, i} - a_{\sigma, i+1} = b_{\sigma, i} - b_{\sigma, i+1},$$

and the character $k_w^\times \to \mathbb{F}_l^\times$ given by

$$x \mapsto \prod_{\sigma \in \text{Hom}(k_w, \mathbb{F}_l)} \sigma(x)^{a_{\sigma, n} - b_{\sigma, n}}$$

is trivial.

Note that the representation $F_a$ is independent of the choice of $S_l$ (this follows easily from the condition that $a_{\sigma, n+1-i} = -a_{\sigma, i}$ and the relation $\iota_w(x) = t((t_u(x))^{-1})$).

For future use, if $a \in (\mathbb{Z}_+^n)_0 \prod_{w|l} \text{Hom}(k_w, \mathbb{F}_l)$ is a Serre weight, we also define an $\mathbb{F}_l$-representation $P_a$ of $G(\mathcal{O}_{F^+, l})$ as follows: we define

$$P_a = \otimes_{\mathbb{F}_l} P_{a,v},$$

a representation of $\prod_{v \in S_l} \text{GL}_n(k_v)$, and we let $G(\mathcal{O}_{F^+, l})$ act on $P_a$ by the composition of $\varphi_v$ and reduction modulo $l$. Note that $F_a$ is a subrepresentation of $P_a$.

We say that a weight $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(F, \overline{q}l)}$ is a lift of a Serre weight $a$ if for each $w|l$ and each $\sigma \in \text{Hom}(k_w, \mathbb{F}_l)$ there is an element $\tau \in \text{Hom}(F, \overline{q}l)$ lying over $w$ and lifting $\sigma$ such that $\lambda_{\sigma} = a_{\sigma}$, and for all other $\tau' \in \text{Hom}(F, \overline{q}l)$ lying over $w$ and lifting $\sigma$ we have $\lambda_{\tau'} = 0$. If $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(F, \overline{q}l)}$ and $w|l$ is a place of $F$, we let $\lambda_w \in (\mathbb{Z}_+^n)^{\text{Hom}(F_w, \overline{q}l)}$ be defined in the obvious way. If $L$ is a finite extension of $\mathbb{Q}_l$ with residue field $k_L$, we say that an element $\lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(L, \overline{q}l)}$ is a lift of an element $a \in (\mathbb{Z}_+^n)^{\text{Hom}(k_L, \overline{q}l)}$ if for each $\sigma \in \text{Hom}(k_L, \overline{q}l)$ there is an element $\tau \in \text{Hom}(L, \overline{q}l)$ lifting $\sigma$ such that $\lambda_{\tau} = a_{\sigma}$, and for all other $\tau' \in \text{Hom}(L, \overline{q}l)$ lifting $\sigma$ we have $\lambda_{\tau'} = 0$.

For the rest of this section, fix $K = \overline{q}l$.

**Definition 2.1.5.** We say that a compact open subgroup of $G(\mathbb{A}_{F^+}^\infty)$ is good if $U = \prod_v U_v$ with $U_v$ a compact open subgroup of $G(F_v^+)$ such that:

- $U_v \subset G(\mathcal{O}_{F_v^+})$ for all $v$ which split in $F$;
- $U_v = G(\mathcal{O}_{F_v^+})$ if $v|l$;
- $U_v$ is a hyperspecial maximal compact subgroup of $G(F_v^+)$ if $v$ is inert in $F$.

Let $U$ be a good compact open subgroup of $G(\mathbb{A}_{F^+}^\infty)$. Let $T$ be a finite set of finite places of $F^+$ which split in $F$, containing $S_l$ and all the places $v$ which split in $F$. 

for which \( U_v \neq G(\mathcal{O}_{F_v}) \). We let \( \mathbb{T}^T, \text{univ} \) be the commutative \( \mathcal{O} \)-polynomial algebra generated by formal variables \( T_w^{(j)} \) for all \( 1 \leq j \leq n \), \( w \) a place of \( F \) lying over a place \( v \) of \( F^+ \) which splits in \( F \) and is not contained in \( T \). For any \( \lambda \in (\mathbb{Z}_+^n)\tilde{I} \), the algebra \( \mathbb{T}^T, \text{univ} \) acts on \( S_\lambda(U, \mathcal{O}) \) via the Hecke operators

\[
T_w^{(j)} := \iota_w^{-1} \left[ \text{GL}_n(\mathcal{O}_{F_w}) \left( \begin{array}{cc} \omega_w^{1j} & 0 \\ 0 & 1_{n-j} \end{array} \right) \text{GL}_n(\mathcal{O}_{F_w}) \right]
\]

for \( w \notin T \) and \( \omega_w \) a uniformiser in \( \mathcal{O}_{F_w} \). Similarly, for any Serre weight \( \alpha \in (\mathbb{Z}_+^n)\prod_{\nu \mid I} \dim(k_{\nu}, \overline{F}_l) \), \( \mathbb{T}^T, \text{univ} \) acts on \( S(U, F_\alpha) \).

Suppose that \( \mathfrak{m} \) is a maximal ideal of \( \mathbb{T}^T, \text{univ} \) with residue field \( \overline{F}_l \) such that \( S_\lambda(U, \overline{Q}_l)_{\mathfrak{m}} \neq 0 \). Then (cf. Proposition 3.4.2 of [CHT08]) by Lemma 2.1.1, Theorem 2.1.4, and Theorem 2.1.2, there is a continuous semisimple representation

\[
\tilde{r}_\mathfrak{m} : G_F \to \text{GL}_n(\overline{F}_l)
\]

associated to \( \mathfrak{m} \), which is uniquely determined by the properties:

- \( \tilde{r}_\mathfrak{m} \cong \bar{r}_\mathfrak{m}^{(j)} \).
- For all finite places \( w \) of \( F \) not lying over \( T \), \( \tilde{r}_\mathfrak{m}|_{\text{G}_{F_w}} \) is unramified, and if \( w \) is a finite place of \( F \) which doesn’t lie over \( T \) and which splits over \( F^+ \), then the characteristic polynomial of \( \tilde{r}_\mathfrak{m}(\text{Frob}_w) \) is

\[
X^n - T_w^{(1)}X^{n-1} + \ldots + (-1)^j(\mathcal{W}_w)^{j(j-1)/2}T_w^{(j)}X^{n-j} + \ldots + (-1)^n(\mathcal{W}_w)^{n(n-1)/2}T_w^{(n)}.
\]

**Lemma 2.1.6.** Suppose that \( U \) is sufficiently small, and let \( \mathfrak{m} \) be a maximal ideal of \( \mathbb{T}^T, \text{univ} \) with residue field \( \overline{F}_l \). Suppose that \( \alpha \in (\mathbb{Z}_+^n)\prod_{\nu \mid I} \dim(k_{\nu}, \overline{F}_l) \) is a Serre weight, and that \( \lambda \in (\mathbb{Z}_+^n)\tilde{I} \) is a lift of \( \alpha \). Then

\[
S_\lambda(U, \overline{Q}_l)_{\mathfrak{m}} \neq 0
\]

if and only if for some Jordan-Hölder factor \( F \) of the \( G(\mathcal{O}_{F^+}) \)-representation \( P_\alpha \),

\[
S(U, F)_m \neq 0.
\]

In particular if \( S(U, F_\alpha)_{\mathfrak{m}} \neq 0 \) then \( S_\lambda(U, \overline{Q}_l)_{\mathfrak{m}} \neq 0 \).

**Proof.** We have \( S_\lambda(U, \overline{Q}_l)_{\mathfrak{m}} = S_\lambda(U, \mathcal{O}_{\overline{Q}_l})_{\mathfrak{m}} \otimes \overline{Q}_l \). Since \( U \) is sufficiently small, it follows that \( S_\lambda(U, \mathcal{O}_{\overline{Q}_l})_{\mathfrak{m}} \) is \( l \)-torsion free. Thus \( S_\lambda(U, \overline{Q}_l)_{\mathfrak{m}} \neq 0 \) if and only if \( S_\lambda(U, \mathcal{O}_{\overline{Q}_l})_{\mathfrak{m}} \neq 0 \). However, using the fact that \( U \) is sufficiently small again, we have \( S_\lambda(U, \overline{F}_l)_{\mathfrak{m}} \neq 0 \) if and only if \( S_\lambda(U, \mathcal{O}_{\overline{Q}_l})_{\mathfrak{m}} \neq 0 \). Thus, \( S_\lambda(U, \overline{Q}_l)_{\mathfrak{m}} \neq 0 \) if and only if \( S_\lambda(U, \overline{F}_l)_{\mathfrak{m}} \neq 0 \).

But \( S_\lambda(U, \overline{F}_l)_{\mathfrak{m}} = S(U, W_\lambda \otimes \overline{L} \overline{F}_l)_{\mathfrak{m}} \) is nonzero if and only if \( S(U, F)_{\mathfrak{m}} \) is nonzero for some Jordan-Hölder factor \( F \) of \( W_\lambda \otimes \overline{F}_l \). (This follows from the exactness of the functor \( F \mapsto S(U, F)_{\mathfrak{m}} \) which in turn follows from the fact that \( U \) is sufficiently small.) It then suffices to note that as an immediate consequence of the definitions, we have \( P_\alpha \cong W_\lambda \otimes \overline{L} \overline{F} \) and \( F_\alpha \) is a Jordan-Hölder factor of \( W_\lambda \otimes \overline{L} \overline{F} \). \qed

We have the following definitions.

**Definition 2.1.7.** If \( R \) is a commutative ring and \( r : G_F \to \text{GL}_n(R) \) is a representation, we say that \( r \) has **split ramification** if \( r|_{G_{F_w}} \) is unramified for any finite place \( w \in F \) which does not split over \( F^+ \).
Definition 2.1.8. If \( \pi \) is a RACSDC automorphic representation of \( \text{GL}_n(\mathbb{A}_F) \), we say that \( \pi \) has split ramification if \( \pi_w \) is unramified for any finite place \( w \in F \) which does not split over \( F^+ \).

Definition 2.1.9. Suppose that \( \bar{\rho} : G_F \to \text{GL}_n(\mathbb{F}_l) \) is a continuous irreducible representation. Then we say that \( \bar{\rho} \) is modular of weight \( a \in (\mathbb{Z}_l^+)^0 \bigoplus_{w|l} \text{Hom}(k_w, \mathbb{F}_l) \) if there is a good, sufficiently small level \( U \), a set of places \( T \) as above, and a maximal ideal \( \mathfrak{m} \) of \( \mathbb{T}_F^{\text{univ}} \) with residue field \( \mathbb{F}_l \) such that

- \( S(U, F_a)_{\mathfrak{m}} \neq 0 \), and
- \( \bar{\rho} \cong \bar{\rho}_{\mathfrak{m}} \).

(Note that \( \bar{\rho}_{\mathfrak{m}} \) exists by Lemma 2.1.6 and the remarks preceding it.) We say that \( \bar{\rho} \) is modular if it is modular of some weight.

Remark 2.1.10. Note that if \( \bar{\rho} : G_F \to \text{GL}_n(\mathbb{F}_l) \) is modular then \( \bar{\rho} \) must have split ramification, and \( \rho^c \cong \bar{\rho}^H \). Note also that this definition is independent of the choice of \( \mathfrak{S}_l \) (because \( F_q \) is independent of this choice). We need to restrict to split ramification and good level because a development of deformation theory for local Galois representations valued in the group \( G_n \) of [CHT08] is currently missing from the literature; in particular, this applies to the results that we use from [BLGGT14b].

Lemma 2.1.11. Suppose that \( \bar{\rho} : G_F \to \text{GL}_n(\mathbb{F}_l) \) is a continuous irreducible representation with split ramification. Let \( a \in (\mathbb{Z}_l^+)^0 \bigoplus_{w|l} \text{Hom}(k_w, \mathbb{F}_l) \) be a Serre weight, and let \( \lambda \in (\mathbb{Z}_l^+)^0 \bigoplus_{w|l} \text{Hom}(k_w, \mathbb{F}_l) \) be a lift of \( a \). Then if \( \bar{\rho} \) is modular of weight \( a \in (\mathbb{Z}_l^+)^0 \bigoplus_{w|l} \text{Hom}(k_w, \mathbb{F}_l) \), there is a RACSDC automorphic representation \( \pi \) of \( \text{GL}_n(\mathbb{A}_F) \) of weight \( \lambda \) and level prime to \( l \) which has split ramification, and which satisfies \( \bar{\rho}_{\lambda}(\pi) \cong \bar{\rho} \). Conversely, if there is a RACSDC automorphic representation \( \pi \) of \( \text{GL}_n(\mathbb{A}_F) \) of weight \( \lambda \) and level prime to \( l \) which has split ramification, and which satisfies \( \bar{\rho}_{\lambda}(\pi) \cong \bar{\rho} \), then \( \bar{\rho} \) is modular of weight \( b \in (\mathbb{Z}_l^+)^0 \bigoplus_{w|l} \text{Hom}(k_w, \mathbb{F}_l) \) for some \( b \) such that the \( G(\mathcal{O}_{F^+}) \)-representation \( P_b \) is a Jordan-Hölder factor isomorphic to \( F_b \).

Proof. Suppose firstly that \( \bar{\rho} \) is modular of weight \( a \). Then by definition there is a good \( U \) and a \( T \) as above with \( U \) sufficiently small, and a maximal ideal \( \mathfrak{m} \) of \( \mathbb{T}_F^{\text{univ}} \) with residue field \( \mathbb{F}_l \) such that

- \( S(U, F_a)_{\mathfrak{m}} \neq 0 \), and
- \( \bar{\rho} \cong \bar{\rho}_{\mathfrak{m}} \).

By Lemma 2.1.6, the first property implies that \( S_\lambda(U, \mathcal{O}_{\mathfrak{m}})_{\mathfrak{m}} \neq 0 \). Define a compact open subgroup \( U' = \prod_w U'_w \) of \( \text{GL}_n(\mathbb{A}_F^\infty) \) by

- \( U'_w = \text{GL}_n(\mathcal{O}_{F_w}) \) if \( w \) is not split over \( F^+ \),
- \( U'_w = I_w(U_w) \) if \( w \) splits over \( F^+ \).

By Lemma 2.1.1, Theorem 2.1.4, and Theorem 2.1.2, there is a RACSDC automorphic representation \( \pi \) of weight \( \lambda \) which satisfies \( \bar{\rho}_{\lambda}(\pi) \cong \bar{\rho} \), and \( \pi^w_{U'_w} \neq 0 \) for all finite places \( w \) of \( F \). Since \( U \) is good, we see that \( \pi \) has level prime to \( l \), and it has split ramification, as required.

Conversely, suppose that there is a RACSDC automorphic representation \( \pi \) of \( \text{GL}_n(\mathbb{A}_F) \) of weight \( \lambda \) which has split ramification and level prime to \( l \) with \( \bar{\rho}_{\lambda}(\pi) \cong \bar{\rho} \). Then there is a compact open subgroup \( U' = \prod_w U'_w \) of \( \text{GL}_n(\mathbb{A}_F^\infty) \) such that
for each finite place $w$ of $F$, $\pi_{w}^{L'} \neq 0$,
- $U_w' \subset \text{GL}_n(\mathcal{O}_{F_w})$ for all $w$,
- $U_w' = \text{GL}_n(\mathcal{O}_{F_w})$ for all $w \mid l$ and all $w$ which are not split over $F^+$,
- if $v = w$ is a place of $F^+$ which splits in $F$, then $U_w' = \pi(U_w^{-1})$,
- there is a finite place $w$ of $F$ which is split over $F^+$ such that
  - $w$ lies above a rational prime $p$ with $[F(\zeta_p) : F] > n$, and
  - $U_w' = \ker(\text{GL}_n(\mathcal{O}_w) \to \text{GL}_n(\mathcal{O}_w/\mathcal{w}_w))$.

Define a compact open subgroup $U = \prod_{v} U_v$ of $G(\mathbb{A}_F^\infty)$ by
- if $v$ is inert in $F$, then $U_v$ is hyperspecial, and
- if $v = w$ splits in $F$, then $U_v = \pi_w^{-1}(U_w')$ (which is well-defined by the fourth bullet point above).

By the final bullet point in the list of properties of $U'$ above, $U$ is sufficiently small. Then by Lemma 2.1.1 and Theorem 2.1.3 we have $S_{\lambda}(U, \overline{\mathbb{Q}}_{\ell})_{m} \neq 0$. The result now follows from Lemma 2.1.6.

### 3. A lifting theorem

3.1. We recall some terminology from [BLGGT14b], specialized to the crystalline (as opposed to potentially crystalline) case. Fix a prime $l$. Let $K$ be a finite extension of $\mathbb{Q}_l$, and $\mathcal{O}$ the ring of integers in a finite extension of $\mathbb{Q}_l$ inside $\mathbb{Q}_l$ with residue field $k$. Assume that for each continuous embedding $K \hookrightarrow \mathbb{Q}_l$, the image is contained in the field of fractions of $\mathcal{O}$.

Let $\overline{\mathcal{H}} : G_K \to \text{GL}_n(k)$ be a continuous representation, and let $R_{\mathcal{O}, \overline{\mathcal{H}}}$ be the universal $\mathcal{O}$-lifting ring. Let $\{H_{\tau}\}$ be a collection of $n$ element multisets of integers parametrized by $\tau \in \text{Hom}_{\mathbb{Q}_l}(K, \mathbb{Q}_l)$. Then $R_{\mathcal{O}, \overline{\mathcal{H}}}$ has a unique quotient $R_{\mathcal{O}, \pi, \{H_{\tau}\}, \text{cris}}$ which is reduced and without $l$-torsion and such that $\mathcal{H}$-point of $R_{\mathcal{O}, \overline{\mathcal{H}}}$ factors through $R_{\mathcal{O}, \pi, \{H_{\tau}\}, \text{cris}}$ if and only if it corresponds to a representation $\rho : G_K \to \text{GL}_n(\mathbb{Q}_l)$ which is crystalline and has $\text{HT}_{\tau}(\rho) = H_{\tau}$ for all $\tau : K \hookrightarrow \mathbb{Q}_l$. We will write $R_{\mathcal{O}, \pi, \{H_{\tau}\}, \text{cris}} \otimes \mathbb{Q}_l$ for $R_{\mathcal{O}, \pi, \{H_{\tau}\}, \text{cris}} \otimes \mathbb{Q}_l$. This definition is independent of the choice of $\mathcal{O}$. The scheme $\text{Spec}(R_{\mathcal{O}, \pi, \{H_{\tau}\}, \text{cris}} \otimes \mathbb{Q}_l)$ is formally smooth over $\text{Spec} \mathbb{Q}_l$. (See [Kis08].)

Let $\rho_1, \rho_2 : G_K \to \text{GL}_n(\mathcal{O}_{\mathbb{Q}_l})$ be continuous representations. We say that $\rho_1$ connects to $\rho_2$, which we denote $\rho_1 \sim \rho_2$, if and only if
- the reduction $\overline{\rho}_1 = \rho_1 \mod m_{\mathbb{Q}_l}$ is equivalent to the reduction $\overline{\rho}_2 = \rho_2 \mod m_{\mathbb{Q}_l}$;
- $\rho_1$ and $\rho_2$ are both crystalline;
- for each $\tau : K \hookrightarrow \mathbb{Q}_l$, we have $\text{HT}_{\tau}(\rho_1) = \text{HT}_{\tau}(\rho_2)$;
- and $\rho_1$ and $\rho_2$ define points on the same irreducible component of the scheme $\text{Spec}(R_{\mathcal{O}, \pi, \{\text{HT}_{\tau}(\rho_1)\}, \text{cris}} \otimes \mathbb{Q}_l)$.

We note that $\rho_1 \sim \rho_2$ in our sense if and only if both $\rho_1$ and $\rho_2$ are crystalline and $\rho_1 \sim \rho_2$ in the sense of [BLGGT14b]. As in section 2.3 of [BLGGT14b], we have the following:

1. The relation $\rho_1 \sim \rho_2$ does not depend on the equivalence chosen between the reductions $\overline{\rho}_1$ and $\overline{\rho}_2$, nor on the $\text{GL}_n(\mathcal{O}_{\mathbb{Q}_l})$-conjugacy class of $\rho_1$ or $\rho_2$.
2. $\sim$ is an equivalence relation.
3. If $K'/K$ is a finite extension and $\rho_1 \sim \rho_2$ then $\rho_1|_{G_{K'}} \sim \rho_2|_{G_{K'}}$. 
Let are two continuous representations. We define the notion that $\chi$ is an irreducible representation which satisfies the following properties.

Suppose $\rho_1$ is crystalline and $\mathfrak{p}_1$ is semisimple. Let $H$ be a finite subgroup of $GL_n$ such that $\rho_1|_{G_K}$ is diagonalizable. Note that if $K''/K$ is a finite extension and $\rho_1$ is diagonalizable (resp. potentially diagonalizable) then $\rho_1|_{G_{K''}}$ is diagonalizable (resp. potentially diagonalizable).

We call a crystalline representation $\rho : G_K \rightarrow GL_n(O_{\overline{\mathbb{Q}_l}})$ diagonal if it is of the form $\chi_1 \oplus \cdots \oplus \chi_n$ with $\chi_i : G_K \rightarrow O_{\overline{\mathbb{Q}_l}}^*$. We will call a crystalline representation $\rho : G_K \rightarrow GL_n(O_{\overline{\mathbb{Q}_l}})$ diagonalizable if it connects to some diagonal representation.

We will call a representation $\rho_1 : G_K \rightarrow GL_n(O_{\overline{\mathbb{Q}_l}})$ potentially diagonalizable if there is a finite extension $K'/K$ such that $\rho_1|_{G_{K'}}$ is diagonalizable. Note that if $K''/K$ is a finite extension and $\rho_1$ is diagonalizable (resp. potentially diagonalizable) then $\rho_1|_{G_{K''}}$ is diagonalizable (resp. potentially diagonalizable).

We call $H$ a finite subgroup of $GL_n$ adequate if the following conditions are satisfied.

1. $H$ has no non-trivial quotient of $l$-power order (i.e. $H^1(H, \mathbb{F}_l) = (0)$).
2. $l$ divides $n$.
3. The elements of $H$ with order coprime to $l$ span $M_{n \times n}(\mathbb{F}_l)$ over $\mathbb{F}_l$. (This implies that $\mathbb{F}_l$ is an irreducible representation of $H$.)
4. $H^1(H, gl_n(\mathbb{F}_l)) = (0)$.

In particular, we have the following useful result, an immediate consequence of Theorem 9 of [GHTT10].

**Theorem 3.1.2.** Suppose that $l \geq 2(n + 1)$, and that $H$ is a finite subgroup of $GL_n(\mathbb{F}_l)$ which acts irreducibly. Then $H$ is adequate.

Fix an isomorphism $\iota : \overline{\mathbb{Q}_l} \rightarrow \mathbb{C}$. Let $F$ be an imaginary CM field with maximal totally real subfield $F^+$.

**Theorem 3.1.3.** Let $l > 2$ be prime, and let $F$ be a CM field with maximal totally real subfield $F^+$, with $\zeta_l \notin F$. Assume that the extension $F/F^+$ is split at all places dividing $l$. Suppose that $\bar{r} : G_F \rightarrow GL_n(\mathbb{F}_l)$ is an irreducible representation which satisfies the following properties.

1. There is a RACSDC automorphic representation $\Pi$ of $GL_n(A_F)$ such that
   - $\bar{r} \cong \bar{r}_{\Pi}(\Pi)$ (so in particular, $\bar{r}^e \cong \bar{r}^e\Pi_{l-n}$).
   - For each place $w/l$ of $F$, $\bar{r}_{\Pi}(\Pi)|_{G_{F_w}}$ is potentially diagonalizable.
2. The image $\bar{r}(G_{F(\zeta_l)})$ is adequate.

Let $S$ be a finite set of finite places of $F^+$ which split in $F$. Assume that $S$ contains all the places of $F^+$ dividing $l$, and all places lying under a place of $F$ at which
\( \tilde{\rho} \) is ramified. For each \( v \in S \) choose a place \( \bar{v} \) of \( F \) above \( v \), and a lift \( \rho_{\bar{v}} : G_{F_{\bar{v}}} \to GL_n(\mathbb{Q}_l) \) of \( \tilde{\rho}|_{G_{F_{\bar{v}}}} \). Assume that if \( v \mid l \), then \( \rho_{\bar{v}} \) is crystalline and potentially diagonalizable, and if \( \tau : F_{\bar{v}} \hookrightarrow \overline{\mathbb{Q}}_l \) is any embedding, then \( HT_\tau(\rho_{\bar{v}}) \) consists of distinct integers.

Then there is a RACSDC automorphic representation \( \pi \) of \( GL_n(A_F) \) of level prime to \( l \) such that

- \( \tilde{\rho} \cong \tilde{\rho}_{l,1}(\pi) \).
- \( \pi_v \) is unramified for all \( w \) not lying over a place of \( S \), so that \( r_{l,1}(\pi_w) \) is unramified at all such \( w \).
- \( r_{l,1}(\pi)|_{G_{F_{\bar{v}}}} \sim \rho_{\bar{v}} \) for all \( v \in S \). In particular, for each place \( v|l \), \( r_{l,1}(\pi)|_{G_{F_{\bar{v}}}} \) is crystalline and for each embedding \( \tau : F_{\bar{v}} \hookrightarrow \overline{\mathbb{Q}}_l \), \( HT_\tau(r_{l,1}(\pi)|_{G_{F_{\bar{v}}}}) = HT_\tau(\rho_{\bar{v}}) \).

**Proof.** Let \( G_n \) be the group scheme over \( \mathbb{Z} \) defined in section 2.1 of [CHT08]. Then by the main result of [BC11], \( \tilde{\rho} \) extends to a representation \( \overline{\rho} : G_{F^+} \to G_n(\mathbb{F}_l) \) with multiplier \( \overline{\epsilon}_l^{1-n} \).

We will now apply Theorem A.4.1 of [BLGG13]. In fact, we need a slight strengthening of that theorem, where we remove the assumption that \( (\pi', \chi') \) is unramified outside of the set of primes above \( S \). (After replacing \( (\pi', \chi') \) with its base change to a finite solvable extension of \( F' \), we may then assume that \( (\pi', \chi') \) is unramified outside of a set of primes \( S' \) of \( F' \), which contains all primes above \( S \), and all of whose elements are split over \( (F')^+ \).) The proof of Theorem A.4.1 of [BLGG13] goes over essentially unchanged to prove this stronger result: the first (and longest) step in the proof is to show that after replacing \( F' \) by a finite solvable extension \( F_1/F \), the representation \( \pi' \) can be replaced by a representation \( \pi_1 \) with the property that \( \pi_1 \) is ordinary and \( r_{l,1}(\pi)|_{G_{F_{v}}} \sim r_{l,1}(\pi_1)|_{G_{F_{v}}} \) for all \( v \) not above \( l \). It is clear that the construction of \( \pi_1 \) can be carried out without reference to the subfield \( F \) of \( F' \). Once \( \pi_1 \) has been obtained, then Proposition 1.5.1(ii) and Theorems 2.3.1 and 2.3.2 of [BLGGT14b] are applied to produce an ordinary \( \pi_1' \) such that \( r_{l,1}(\pi_1')|_{G_{F_{v}}} \sim \rho|_{G_{F_{v}}} \) for all \( v \) not above \( l \). The proof then continues unchanged.

We now apply this strengthened version of Theorem A.4.1 of [BLGG13], with

- \( F', n \) and \( S \) as in the present setting.
- \( \tilde{\rho} \) our present \( \overline{\rho} \).
- \( \rho_{\bar{v}} \) our \( \rho_{\bar{v}} \).
- \( \mu = \epsilon_l^{1-n} \).
- \( F' = F \).

We conclude that \( \tilde{\rho} \) has a lift \( r : G_F \to GL_n(\overline{\mathbb{Q}}_l) \) (the restriction to \( G_F \) of the representation \( r \) of Theorem A.4.1 of [BLGG13]) such that

- \( r^c \cong r^\vee \epsilon_l^{1-n} \).
- if \( v \in S \) then \( r|_{G_{F_v}} \sim \rho_{\bar{v}} \).
- \( r \) is unramified outside \( S \).
- \( r \) is automorphic of level potentially prime to \( l \), say \( r \cong r_{l,1}(\pi) \).

By Theorem 2.1.2, we see that (since \( r|_{G_{F_w}} \) is crystalline for all \( w|l \), and unramified at all places \( w \) not lying over a place in \( \overline{S} \)) \( \pi_w \) is unramified for all \( w|l \) and all \( w \) not lying over a place in \( S \), as required.
4. Serre weight conjectures

4.1. We now briefly discuss Serre weight conjectures for $GL_n$. We refer the reader to the forthcoming [EGHS14] for a far more detailed discussion. In particular, in much of this section we restrict ourselves to the case that $l$ splits completely in $F$, both for simplicity of notation and because in this case we can prove theorems with cleaner conditions, as representations satisfying the Fontaine-Laffaille condition are always potentially diagonalizable.

Let $K$ be a finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}_K$ and residue field $k$. Let $\overline{\rho}: G_K \to GL_n(\overline{\mathbb{F}}_l)$ be a continuous representation. Then it is a folklore conjecture that for each such $\overline{\rho}$ there is a set $W(\overline{\rho})$ of Serre weights of $GL_n(k)$ for each $K$ and each $\overline{\rho}$ with the following property: if $F$ is a CM field, $\tilde{\rho}: G_F \to GL_n(\overline{\mathbb{F}}_l)$ is an irreducible modular representation (so in particular it is conjugate self-dual), $w|l$ is a place of $F$ and $\sigma_w$ is an irreducible $\overline{\mathbb{F}}_l$-representation of $GL_n(k_w)$, then $\tilde{\rho}$ is modular of Serre weight $\sigma_w \otimes \sigma_w^{-1}$ for some $\tilde{\sigma}$ if and only if $\sigma_w \in W(\tilde{\rho}|_{GL_n})$.

It is natural to believe that there is a description of $W(\overline{\rho})$ in terms of the existence of crystalline lifts with particular Hodge-Tate weights, as we now explain. This is one of the motivations for the general Serre weight conjectures explained in [EGHS14].

**Definition 4.1.1.** Let $K/\mathbb{Q}_l$ be a finite extension, let $\lambda \in (\mathbb{Z}_+^n)_{\text{Hom}(K,\overline{\mathbb{Q}}_l)}$, and let $\rho: G_K \to GL_n(\overline{\mathbb{Q}}_l)$ be a de Rham representation. Then we say that $\rho$ has Hodge type $\lambda$ if for each $\tau \in \text{Hom}(K,\overline{\mathbb{Q}}_l)$, we have $HT_{\tau}(\rho) = \{\lambda_{\tau,1} + (n-1), \lambda_{\tau,2} + (n-2), \ldots, \lambda_{\tau,n}\}$.

**Remark 4.1.2.** As an immediate consequence of this definition and of Theorem 2.1.2, we see that if $\pi$ is a RACSDC automorphic representation of weight $\lambda \in (\mathbb{Z}_+^n)_{\text{Hom}(F^+,\mathbb{C})}$, then for each place $w|l$, $r_{l,w}(\pi)|_{GL_n}$ has Hodge type $(i^{-1}\lambda)_w$.

**Lemma 4.1.3.** Let $n$ be a positive integer, and let $F$ be an imaginary CM field with maximal totally real subfield $F^+$, and suppose that $F/F^+$ is unramified at all finite places, that every place of $F^+$ dividing $l$ splits completely in $F$, and that if $n$ is even then $n|F^+/\mathbb{Q}_l/2$ is even. Suppose that $\tilde{\rho}: G_F \to GL_n(\overline{\mathbb{F}}_l)$ is an irreducible modular representation with split ramification. Let $a \in (\mathbb{Z}_+^n)_{\text{Hom}(k,\mathbb{F}_l)}$ be a Serre weight, and let $\lambda \in (\mathbb{Z}_+^n)_{\text{Hom}(F,\overline{\mathbb{Q}}_l)}$ be a lift of $a$. If $\tilde{\rho}$ is modular of weight $\lambda$, then for each place $w|l$ there is a continuous lift $r_w: G_{F_w} \to GL_n(\mathcal{O}_{\mathbb{Q}_l})$ of $\tilde{\rho}|_{GL_n}$ such that $r_w$ is crystalline of Hodge type $\lambda_w$.

**Proof.** By Lemma 2.1.11 there is a RACSDC automorphic representation $\pi$ of $GL_n(k_F)$, which has level prime to $l$ and weight $i\lambda$, such that $r_{l,w}(\pi) \cong \tilde{\rho}$. Then we may take $r_w := r_{l,w}(\pi)|_{GL_n}$, which has the required properties by Remark 4.1.2. \\

This suggests the following definition.

**Definition 4.1.4.** Let $K$ be a finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}_K$ and residue field $k$. Let $\overline{\rho}: G_K \to GL_n(\overline{\mathbb{F}}_l)$ be a continuous representation. Then we let $W^{\text{cris}}(\overline{\rho})$ be the set of Serre weights $\lambda \in (\mathbb{Z}_+^n)_{\text{Hom}(k,\mathbb{F}_l)}$ with the property that there is a crystalline representation $\rho: G_K \to GL_n(\mathcal{O}_{\mathbb{Q}_l})$ lifting $\overline{\rho}$, such that $\rho$ has Hodge type $\lambda$ for some lift $\lambda \in (\mathbb{Z}_+^n)_{\text{Hom}(k,\overline{\mathbb{Q}}_l)}$ of $a$.

The results of section 3 suggest the following definition.
Definition 4.1.5. Let $K$ be a finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}_K$ and residue field $k$. Let $\overline{\rho} : G_K \to \text{GL}_n(F_l)$ be a continuous representation. Then we let $W_{\text{diag}}(\overline{\rho})$ be the set of Serre weights $a \in (\mathbb{Z}_l^n)^{\text{Hom}(k,F_l)}$ with the property that there is a potentially diagonalizable crystalline representation $\rho : G_K \to \text{GL}_n(\mathbb{Q}_l)$ lifting $\overline{\rho}$, such that $\rho$ has Hodge type $\lambda$ for some lift $\lambda \in (\mathbb{Z}_l^n)^{\text{Hom}(K,\mathbb{Q}_l)}$ of $a$.

Remark 4.1.6. If $a$ and $b$ are equivalent Serre weights, then $a \in W_{\text{cris}}(\overline{\rho})$ (respectively $W_{\text{diag}}(\overline{\rho})$) if and only if $b \in W_{\text{cris}}(\overline{\rho})$ (respectively $W_{\text{diag}}(\overline{\rho})$). This is an easy consequence of Lemma 4.1.15 of [BLGG13], which provides a crystalline character with trivial reduction with which one can twist the crystalline Galois representations of Hodge type some lift of $a$ to obtain crystalline representations of Hodge type some lift of $b$. The same remarks apply to the set $W_{\text{ovb}}(\overline{\rho})$ defined below.

By definition we have $W_{\text{diag}}(\overline{\rho}) \subset W_{\text{cris}}(\overline{\rho})$. We “globalise” these definitions in the obvious way:

Definition 4.1.7. Let $\overline{\rho} : G_F \to \text{GL}_n(F_l)$ be a continuous representation with $\overline{\rho}^c \cong \overline{\rho}^\vee \overline{\rho}_l^{1-n}$. Then we let $W_{\text{cris}}(\overline{\rho})$ (respectively $W_{\text{diag}}(\overline{\rho})$) be the set of Serre weights $a \in (\mathbb{Z}_l^n)^{\text{Hom}(k,F_l)}$ such that for each place $w|l$, the corresponding Serre weight $a_w \in (\mathbb{Z}_l^n)^{\text{Hom}(k,F_l)}$ is an element of $W_{\text{cris}}(\overline{\rho}|_{G_{F_w}})$ (respectively $W_{\text{diag}}(\overline{\rho}|_{G_{F_w}})$).

The point of these definitions is the following Corollary and Theorem.

Corollary 4.1.8. Let $n$ be a positive integer, let $F$ be an imaginary CM field with maximal totally real subfield $F^+$, and suppose that $F/F^+$ is unramified at all finite places, that every place of $F^+$ dividing $l$ splits completely in $F$, and that if $n$ is even then $n[F^+ : \mathbb{Q}]/2$ is even. Suppose that $\overline{\rho} : G_F \to \text{GL}_n(F_l)$ is an irreducible modular representation with split ramification. Let $a \in (\mathbb{Z}_l^n)^{\text{Hom}(k,F_l)}$ be a Serre weight. If $\overline{\rho}$ is modular of weight $a$, then $a \in W_{\text{cris}}(\overline{\rho})$.

Proof. This is an immediate consequence of Lemma 4.1.3 and Definition 4.1.7. □

Theorem 4.1.9. Let $F$ be an imaginary CM field with maximal totally real subfield $F^+$, and suppose that $F/F^+$ is unramified at all finite places, that every place of $F^+$ dividing $l$ splits completely in $F$, and that if $n$ is even then $n[F^+ : \mathbb{Q}]/2$ is even. Assume that $\zeta_l \notin F$. Suppose that $l > 2$, and that $\overline{\rho} : G_F \to \text{GL}_n(F_l)$ is an irreducible representation with split ramification. Assume that

- There is a RACSDC automorphic representation $\Pi$ of $\text{GL}_n(A_F)$ such that
  - $\overline{\rho} \cong \overline{\rho}_l^{1-n}$ (so in particular, $\overline{\rho}^c \cong \overline{\rho}^\vee \overline{\rho}_l^{1-n}$).
  - For each place $w|l$ of $F$, $r_{l,t}(\Pi)|_{G_{F_w}}$ is potentially diagonalizable.
  - $\overline{\rho}(G_F(\zeta_l))$ is adequate.

Let $a \in (\mathbb{Z}_l^n)^{\text{Hom}(k,F_l)}$ be a Serre weight. Assume that $a \in W_{\text{diag}}(\overline{\rho})$. Then there is a Serre weight $b \in (\mathbb{Z}_l^n)^{\text{Hom}(k,F_l)}$ such that

- $\overline{\rho}$ is modular of weight $b$.
  - There is a Jordan-Hölder factor of the $G(F_{F^+,l})$ representation $P_a$ which is isomorphic to $F_b$.

Proof. By the assumption that $a \in W_{\text{diag}}(\overline{\rho})$, there is a lift $\lambda$ of $a$ such that for each $w|l$ there is a potentially diagonalizable crystalline lift $\rho_w : G_{F_w} \to \text{GL}_n(O_{\overline{Q}_l})$ of $\overline{\rho}|_{G_{F_w}}$ of Hodge type $\lambda_w$. 

By Theorem 3.1.3, there is a RACSDC automorphic representation \( \pi \) of \( \text{GL}_n(\mathbb{A}_F) \) of weight \( n \lambda \), of level prime to \( l \) and with split ramification, such that \( \tilde{r}_{l,i}(\pi) \cong \tilde{r} \).

The result follows from Lemma 2.1.11. \( \square \)

Since Fontaine–Laffaille representations are potentially diagonalizable, we obtain the following Corollary.

**Corollary 4.1.10.** Let \( F \) be an imaginary CM field with maximal totally real subfield \( F^+ \), and suppose that \( F/F^+ \) is unramified at all finite places, that every place of \( F^+ \) dividing \( l \) splits completely in \( F^+ \) and that if \( n \) is even then \( n[F^+:\mathbb{Q}]/2 \) is even. Suppose that \( l > 2 \), and that \( \tilde{r} : G_F \to \text{GL}_n(\mathbb{F}_l) \) is an irreducible representation with split ramification. Assume that

1. \( l \) is unramified in \( F \).
2. There is a RACSDC automorphic representation \( \Pi \) of \( \text{GL}_n(\mathbb{A}_F) \) of weight \( \mu \in (\mathbb{Z}_+)^n \) and level prime to \( l \) such that
   - \( \tilde{r} \cong \tilde{r}_{l,i}(\Pi) \) (so in particular, \( \tilde{r}^c \cong \tilde{r}^{c_1n-l} \)).
   - For each \( \tau \in \text{Hom}(F,\mathbb{C}), \mu_{\tau,1} - \mu_{\tau,n} \leq l - n \).
   - \( \tilde{r}(G_F(\mathfrak{q})) \) is adequate.

Let \( a \in (\mathbb{Z}_+)^n \) be a Serre weight. Assume that \( a \in W^{\text{diag}}(\tilde{r}) \). Then there is a Serre weight \( b \in (\mathbb{Z}_+) \) such that

- \( \tilde{r} \) is modular of weight \( b \).
- There is a Jordan–Hölder factor of the \( G(F_{l+1}) \) representation \( P_a \) which is isomorphic to \( F_b \).

**Proof.** By Theorem 4.1.9, it is enough to check that for each place \( w|l \) of \( F \), \( r_{l,i}(\Pi)|G_{F,w} \) is potentially diagonalizable. This follows from the main result of [GL12]. \( \square \)

As explained above, we now specialize to the case that \( l \) splits completely in \( F \). We further assume that \( \tilde{r}|G_{F,w} \) is semisimple for all \( w|l \), and specify a set \( W^{\text{obv}}(\tilde{r}) \) of Serre weights. These weights will have the property that if \( a \in W^{\text{obv}}(\tilde{r}) \), and \( \lambda \) is the unique lift of \( a \) to \( (\mathbb{Z}_+)^n \), then for each place \( w|l \), \( \tilde{r}|G_{F,w} \) has a potentially diagonalizable (indeed potentially diagonal) crystalline lift of Hodge type \( \lambda_w \).

Since the situation is purely local, we change notation and work with \( G_{Q_l} \). Let \( Q_{l,m} \) denote the unramified extension of \( Q_l \) of degree \( m \) inside \( Q_l^\times \), and let \( \omega_m : G_{Q_{l,m}} \to \mathbb{F}_l^\times \) denote a choice of fundamental character of nveau \( m \) (this is given by the action of \( G_{Q_{l,m}} \) on the \( (2^m-1) \)-st roots of \( l \)). Given \( \lambda \in \mathbb{F}_l^\times \) and an \( m \)-tuple of integers \( \xi = (c_0, \ldots, c_{m-1}) \), we consider the representation

\[
\overline{\rho}_{\lambda,\xi} := n_{\rho_{\lambda}} \otimes \text{Ind}^{G_{Q_{l,m}}}_{G_{Q_{l}}} \omega_m^{-c_0l_1 + c_1l_1 + \cdots + c_{m-1}l_1},
\]

where \( n_{\rho_{\lambda}} \) is the unramified character taking a geometric Frobenius to \( \lambda \). Given a partition \( n = n_1 + \cdots + n_r \), elements \( \lambda = (\lambda_1, \ldots, \lambda_r) \) of \( \mathbb{F}_l^\times \), and a tuple \( \xi = (\xi_1, \ldots, \xi_r) \) of tuples \( \xi_i = (c_{i,0}, \ldots, c_{i,n_{i}-1}) \) of integers, we define the representation

\[
\overline{\rho}_{\lambda,\xi} := \bigoplus_{i=1}^{r} \overline{\rho}_{\lambda_i,\xi_i}.
\]

Note that we can we can think of \( \xi \) as the element \( (c_{1,0}, c_{1,1}, \ldots, c_{r,n_r-1}) \) of \( \mathbb{Z}^n \), where \( n = n_1 + \cdots + n_r \).
Definition 4.1.11. Let \( \overline{\rho} : G_{Q_l} \to \text{GL}_n(F_l) \) be a semisimple representation. Let \( W^{\text{obv}}(\overline{\rho}) \) be the set of Serre weights \( a \in \mathbb{Z}^n_+ \) for which there exists a permutation \( \sigma \in S_n \), a partition \( \underline{a} \) of \( n \) and \( \underline{\lambda} \) as above such that
\[
\overline{\rho} \cong \overline{\rho}_{\underline{\lambda}, \underline{\sigma}} (\sigma(a_1 + n-1, a_2 + n-2, \ldots, a_n)).
\]

Lemma 4.1.12. If \( \overline{\rho} : G_{Q_l} \to \text{GL}_n(F_l) \) is a semisimple representation and \( a \in W^{\text{obv}}(\overline{\rho}) \), then \( \overline{\rho} \) has a potentially diagonalizable crystalline lift of Hodge type \( a \).

Proof. By the definition of “Hodge type \( a \)”, it is enough to show that each representation \( \overline{\rho}_{\lambda, \underline{\sigma}} : G_{Q_l} \to \text{GL}_m(F_l) \) defined above has a potentially diagonalizable crystalline lift with Hodge–Tate weights \( c_0, \ldots, c_{m-1} \) (note that the direct sum of potentially diagonalizable representations is again potentially diagonalizable). It thus suffices to show that the character \( \omega_m^{-(c_0+\ell c_1+\cdots+\ell^{m-1}c_{m-1})} \) of \( G_{Q_l} \) has a crystalline lift with Hodge–Tate weights \( c_0, \ldots, c_{m-1} \) (because the induction to \( G_{Q_l} \), of such a lift is certainly potentially diagonalizable). This follows at once from Lemma 6.2 of [GS11] (noting that the conventions on the sign of Hodge–Tate weights in [GS11] are the opposite of those of this paper). \( \square \)

Again we may globalise this definition in the obvious way.

Definition 4.1.13. Continue to assume that \( l \) splits completely in \( F \), and let \( \bar{\rho} : G_F \to \text{GL}_n(F_l) \) be a continuous representation with \( \bar{\rho}^c \cong \bar{\rho}^\dagger \bar{\rho}^{1-n} \) and such that \( \bar{\rho}|_{G_{F_w}} \) is semisimple for each \( w|l \). Then we let \( W^{\text{obv}}(\bar{\rho}) \) be the set of Serre weights \( a \in (\mathbb{Z}^n_+ \bigcup \bigcup_0^1 \text{Hom}(k_w, F_l)) \) such that for each place \( w|l \), the corresponding Serre weight \( a_w \in (\mathbb{Z}^n_+ \bigcup \bigcup_0^1 \text{Hom}(k_w, F_l)) \) is an element of \( W^{\text{obv}}(\bar{\rho}|_{G_{F_w}}) \).

Corollary 4.1.14. Let \( \bar{\rho} : G_F \to \text{GL}_n(F_l) \) be a continuous representation satisfying the assumptions of Definition 4.1.13. Then \( W^{\text{obv}}(\bar{\rho}) \subset W^{\text{diag}}(\bar{\rho}) \).

Proof. This follows immediately from Lemma 4.1.12. \( \square \)

In the case \( n = 2 \), which we explored more thoroughly in [BLGG13], \( W^{\text{obv}}(\bar{\rho}) \) is precisely the set of weights for which \( \bar{\rho} \) is modular. We do not conjecture this for \( n > 2 \); even for \( n = 3 \) one sees that the set of weights predicted in [Her09] is larger than \( W^{\text{obv}}(\bar{\rho}) \). In fact, we expect (see [EGHS14] for a much more detailed discussion) that the set of weights for which \( \bar{\rho} \) is modular is \( W^{\text{cris}}(\bar{\rho}) \), and it is easy to see that this set is typically larger than \( W^{\text{obv}}(\bar{\rho}) \). Indeed, by Lemma 2.1.11 and Theorem 2.1.2, if \( \bar{\rho} \) is modular of some Serre weight \( b \), then \( F_{b} \) is a Jordan-Hölder factor of \( F_{a} \) for some Serre weight \( a \), then \( a \in W^{\text{cris}}(\bar{\rho}) \). It is easy to find examples of \( a, b \) for which \( b \in W^{\text{obv}}(\bar{\rho}) \) but \( a \notin W^{\text{obv}}(\bar{\rho}) \). On the other hand, as explained in [EGHS14] we believe that \( W^{\text{cris}}(\bar{\rho}) \) is determined by \( W^{\text{obv}}(\bar{\rho}) \) and a simple combinatorial recipe, so that the weights in \( W^{\text{obv}}(\bar{\rho}) \) are in some sense fundamental.

4.2. Fontaine-Laffaille theory. In applications of our results it is often useful to have information in the opposite direction; namely one wishes to have information about \( \bar{\rho}|_{G_{F_w}} \) at places \( w|p \), given that \( \bar{\rho} \) is modular of some particular weight. In the case that \( l \) is unramified in \( F \) and the weight is sufficiently far inside the lowest alcove, this can be done by Fontaine–Laffaille theory. Again, we specialise to the case that \( l \) splits completely in \( F \).
Lemma 4.2.1. Let $F$ be an imaginary CM field with maximal totally real subfield $F^+$, and suppose that $F/F^+$ is unramified at all finite places, and that $l$ splits completely in $F$. If $n$ is even, assume that $|F^+: \mathbb{Q}|n/2$ is even. Suppose that $l > 2$, and that $\overline{r} : G_F \rightarrow \text{GL}_n(\mathbb{F}_l)$ is an irreducible modular representation with split ramification. Let $a \in (\mathbb{Z}_l^+)_0 \times \text{Hom}(k_w, \mathbb{F}_l)$ be a Serre weight, and $w|l$ is such that $a_{w,1} - a_{w,n} \leq l - n$, then $a_w \in W^{\text{obv}}(\overline{r}|_{\text{G}_{\mathbb{F}_w}})$.

Proof. This is a standard application of Fontaine–Laffaille theory. By Corollary 4.1.8, $\overline{r}|_{\text{G}_{\mathbb{F}_w}}$ has a crystalline lift with Hodge–Tate weights $a_{w,1} + n - 1, \ldots, a_{w,n}$. Since by assumption we have $a_{w,1} + n - 1 - a_{w,n} \leq l - 1$, the result follows immediately from, for example, Proposition 3 of [Wor02] (note that while this reference assumes that the crystalline representation has $\mathbb{Q}_l$-coefficients, the proof goes through unchanged with $\mathbb{Q}_l$-coefficients). \hfill \Box

5. Explicit results for $\text{GL}_3$

5.1. We now show how one can obtain cleaner results in the case $n = 3$, making use of the fact that the representation theory of $\text{GL}_3$, while more complicated than that of $\text{GL}_2$, is rather simpler than that of $\text{GL}_n$ for $n \geq 4$. The following Lemmas are key to our approach.

Lemma 5.1.1. Let $a \in \mathbb{Z}_l^+$ be a Serre weight for $\text{GL}_3(\mathbb{F}_l)$. Then

1. if $l - 1 \leq a_1 - a_3$ and $a_1 - a_2, a_2 - a_3 \leq l - 2$, then there is a short exact sequence

   $0 \rightarrow F_a \rightarrow P_a \rightarrow F_b \rightarrow 0$

   where $b = (a_3 + l - 2, a_2, a_1 - l + 2)$.

2. In all other cases, $P_a = F_a$.

Proof. This is Proposition 3.18 of [Her09]. \hfill \Box

Lemma 5.1.2. Suppose that $n = 3$, and that $a \in \mathbb{Z}_l^+$ is a Serre weight for $\text{GL}_3(\mathbb{F}_l)$. If $a \in W^{\text{obv}}(\overline{r})$ for some representation $\overline{r} : G_{\mathbb{Q}_l} \rightarrow \text{GL}_3(\mathbb{F}_l)$, then either $a_1 - a_3 = l - 1$ and

   \[ \overline{r}|_{I_{\mathbb{Q}_l}} \cong \omega^{-(a_1+1)} \otimes \omega^{-(a_2+1)} \otimes \omega^{-(a_3+1)}, \]

or there is a permutation $x, y, z$ of $-(a_1 + 2), -(a_2 + 1), -a_3$ such that $\overline{r}|_{I_{\mathbb{Q}_l}}$ is isomorphic to one of

   \[ \omega^x \otimes \omega^y \otimes \omega^z, \]

   \[ \omega^x \otimes \omega^{y+lz} \otimes \omega^{y+z}, \]

   \[ \omega^{x+y+lz} \otimes \omega^{y+lz+tx} \otimes \omega^{z+tx+ly}, \]

where in the second case we have $(l+1) \nmid ly + z$, and in the third case we have $(l^2 + l + 1) \nmid x + ly + l^2z$.

Proof. This is a simple calculation (it is immediate from the definition that $\overline{r}|_{I_{\mathbb{Q}_l}}$ is of the given form if one ignores the divisibility condition, so the only thing to check is when it can be the case that $ly + z$ is divisible by $l + 1$ or $x + ly + l^2z$ is divisible by $l^2 + l + 1$). \hfill \Box
Definition 5.1.3. Let \( a \in \mathbb{Z}_+^4 \) be a Serre weight for \( \text{GL}_3(F_l) \). Then we say that \( a \) is non-generic if it is in the upper alcove, and it is at distance exactly 1 from the boundary. More precisely, it is non-generic if one of the following three conditions hold: \( a_1 - a_3 = l - 1 \) and \( a_1 - a_2, a_2 - a_3 \leq l - 2 \); or \( a_2 - a_3 = l - 2 \) and \( a_1 - a_2 \geq 2 \); or \( a_1 - a_2 = l - 2 \) and \( a_2 - a_3 \geq 2 \). Otherwise we say that \( a \) is generic.

If \( l \) splits completely in \( F \) and \( a \in (\mathbb{Z}_+^4)_{\text{Hom}(F,F)} \) is a Serre weight, we say that \( a \) is generic if for each \( \tau \in \text{Hom}(F,F) \) the corresponding Serre weight \( a_{\tau} \in \mathbb{Z}_+^4 \) is generic.

We remark that this definition of generic is very mild; in particular, it is much less restrictive than the notion of generic used in [EGH13]. (See also Remark 5.1.5 below.)

Theorem 5.1.4. Let \( F \) be an imaginary CM field with maximal totally real subfield \( F^+ \), and suppose that \( F/F^+ \) is unramified at all finite places, and that \( l \) splits completely in \( F \). Suppose that \( l \geq 2 \), and that \( \tilde{r} : G_F \to \text{GL}_3(F_l) \) is an irreducible representation with split ramification. Assume that

1. There is a RACSDC automorphic representation \( \Pi \) of \( \text{GL}_3(A_F) \) of weight \( \mu \in (\mathbb{Z}_+^4)_{\text{Hom}(\Pi,F)} \) and level prime to \( l \) such that
   - \( \tilde{r} \cong \tilde{r}_{\Pi}(\Pi) \) (so in particular, \( \tilde{r}^c \cong \tilde{r}^{\Pi^{-2}} \)).
   - For each \( \tau \in \text{Hom}(F,F) \), \( \mu_{\tau,1} - \mu_{\tau,3} \leq l - 3 \).
   - \( \tilde{r}(G_{F_w}) \) is adequate.

Let \( a \in (\mathbb{Z}_+^4)_{\text{Hom}(F,w)} \) be a generic Serre weight. Assume that \( a \in W_{\text{obv}}(\tilde{r}) \) (so in particular, \( \tilde{r}|_{G_{F_w}} \) is semisimple for all \( w|l \)). Then \( \tilde{r} \) is modular of weight \( a \).

Remark 5.1.5. In fact, the proof below shows that it suffices to assume that \( a_w \) is generic for all places \( w|l \) for which \( \tilde{r}|_{G_{F_w}} \) has niveau 2, and that if \( \tilde{r}|_{G_{F_w}} \) has niveau 1, then we do not need to assume that \( a_{w_1} = a_{w_2} = a_{w_3} = l - 1 \) and \( a_1 - a_2, a_2 - a_3 \leq l - 2 \). In particular, if \( \tilde{r}|_{G_{F_w}} \) is irreducible for all places \( w|l \) (which is the situation considered in [EGH13]), then we do not need to assume that \( a \) is generic.

Proof of Theorem 5.1.4. By Corollaries 4.1.10 and 4.1.14, \( \tilde{r} \) is modular of weight \( b \) for some Serre weight \( b \) with the property that \( F_b \) is a Jordan-Hölder factor of \( F_l \). We wish to show that \( F_b \cong F_a \). Assume for the sake of contradiction that \( F_b \not\cong F_a \), so that there is a place \( w|l \) with \( F_{b_w} \not\cong F_{a_w} \). By Lemma 5.1.1, we must have \( l - 1 \leq a_{w_1} - a_{w_3} \) and \( a_{w_1} - a_{w_2}, a_{w_2} - a_{w_3} \leq l - 2 \), and \( b_{w_1} - b_{w_3} = 2l - 4 - (a_{w_1} - a_{w_3}) \leq l - 3 \).

Since \( l - 1 \leq a_{w_1} - a_{w_3} \), we have \( b_{w_1} - b_{w_3} = 2l - 4 - (a_{w_1} - a_{w_3}) \leq l - 3 \). Thus the assumption that \( \tilde{r} \) is modular of weight \( b \) together with Lemma 4.2.1 gives an explicit description of the possibilities for \( \tilde{r}|_{G_{F_w}} \) (which is assumed to be semisimple) in terms of \( b_w \), and hence in terms of \( a_w \). We also have another such description from the assumption that \( a \in W_{\text{obv}}(\tilde{r}) \). We will now compare these descriptions to obtain a contradiction.

It will be useful to note that since we are assuming that \( a_{w_1} - a_{w_2}, a_{w_2} - a_{w_3} \leq l - 2 \), and \( a_{w_1} - a_{w_3} \geq l - 1 \) we have

\[
1 \leq a_{w_1} - a_{w_2}, a_{w_2} - a_{w_3} \leq l - 2, \\
(5.1.1)
\]

\[
l - 1 \leq a_{w_1} - a_{w_3} \leq 2l - 4, \\
(5.1.2)
\]
so that
\[(5.1.3)\quad a_{w,1} \not\equiv a_{w,2} \pmod{l-1},\]
\[(5.1.4)\quad a_{w,2} \not\equiv a_{w,3} \pmod{l-1},\]
\[(5.1.5)\quad a_{w,3} \not\equiv a_{w,1} + 1 \pmod{l-1},\]
\[(5.1.6)\quad a_{w,1} - a_{w,3} \not\equiv l - 2 \pmod{l+1}.\]
If \(a_{w,1} - a_{w,2} = 1\) then the condition that \(a_{w,1} - a_{w,3} \geq l-1\) forces \(a_{w,2} - a_{w,3} = l-2\), so that \(a_{w}\) is not generic. Similarly if \(a_{w,2} - a_{w,3} = 1\) then \(a_{w}\) is not generic. Therefore if we assume that \(a_{w}\) is generic, we also have
\[(5.1.7)\quad a_{w,1} \not\equiv a_{w,2} + 1 \pmod{l-1},\]
\[(5.1.8)\quad a_{w,2} \not\equiv a_{w,3} + 1 \pmod{l-1}.\]
By the second and third conditions in the definition of genericity, we also have
\[(5.1.9)\quad a_{w,3} \not\equiv a_{w,2} + 1 \pmod{l-1},\]
\[(5.1.10)\quad a_{w,2} \not\equiv a_{w,1} + 1 \pmod{l-1}.
\]
**Niveau 1** Suppose firstly that \(\bar{r}|_{F_w}\) has niveau 1, i.e. that \(\bar{r}|_{F_w}\) is a direct sum of powers of the mod \(l\) cyclotomic character \(\omega\). Then since \(a \in W^\text{ov}(\bar{r})\) and \(a\) is generic, we see from Lemma 5.1.2 that
\[
\bar{r}|_{F_w} \cong \omega^{-(a_{w,1}+2)} \oplus \omega^{-(a_{w,2}+1)} \oplus \omega^{-(a_{w,3})}.
\]
By Lemma 4.2.1 (applied to \(F_b\)), we see that we also have
\[
\bar{r}|_{F_w} \cong \omega^{-(a_{w,3}+1)} \oplus \omega^{-(a_{w,2}+1)} \oplus \omega^{-(a_{w,1}+1)},
\]
Thus \(a_{w,3} \equiv a_{w,1} + 1 \pmod{l-1}\), contradicting (5.1.5).

**Niveau 2** Suppose next that \(\bar{r}|_{F_w}\) has niveau 2, i.e. that \(\bar{r}|_{F_w}\) is a direct sum of a power of the mod \(l\) cyclotomic character \(\omega\) and characters \(\omega^n\), \(\omega^n\), for some \(n\) with \((l+1) \nmid n\), where \(\omega^2\) is a choice of fundamental character of niveau 2. Then since \(a \in W^\text{ov}(\bar{r})\), we see from Lemma 5.1.2 that \(\bar{r}|_{F_w}\) is isomorphic to one of the following:
\[
\omega^{-(a_{w,1}+2)} \oplus \omega^{-(a_{w,2}+1)} \oplus \omega^{-(a_{w,3})} \oplus \omega^{-(a_{w,1}+2)+a_{w,3}},
\]
\[
\omega^{-(a_{w,2}+1)} \oplus \omega^{-(a_{w,1}+2)+a_{w,3}},
\]
\[
\omega^{-(a_{w,3})} \oplus \omega^{-(a_{w,1}+2)+a_{w,2}+1}.
\]
By Lemma 4.2.1 (applied to \(F_b\)), we see that we also have that \(\bar{r}|_{F_w}\) is isomorphic to one of the following:
\[
\omega^{-(a_{w,1}+1)} \oplus \omega^{-(a_{w,2}+1)} \oplus \omega^{-(a_{w,3}+1)},
\]
\[
\omega^{-(a_{w,2}+1)} \oplus \omega^{-(a_{w,1}+1)} \oplus \omega^{-(a_{w,3}+1)},
\]
\[
\omega^{-(a_{w,3})} \oplus \omega^{-(a_{w,1}+1)} \oplus \omega^{-(a_{w,2}+1+1)},
\]
Comparing the powers of \(\omega\) and using (5.1.3)–(5.1.10), the only possibility is that we simultaneously have
\[
\bar{r}|_{F_w} \cong \omega^{-(a_{w,2}+1)} \oplus \omega^{-(a_{w,1}+2)+a_{w,3}},
\]
\[
\bar{r}|_{F_w} \cong \omega^{-(a_{w,2}+1)} \oplus \omega^{-(a_{w,1}+1)+a_{w,3}},
\]
There are now two possibilities to examine. Firstly it could be the case that
\[ a_{w,1} + 2 + la_{w,3} \equiv a_{w,1} - l + 2 + l(a_{w,3} + l) \pmod{l^2 - 1}; \]
but this implies that \( l^2 - l \equiv 0 \pmod{l^2 - 1} \), a contradiction. So we must have
\[ a_{w,1} + 2 + la_{w,3} \equiv l(a_{w,1} - l + 2) + (a_{w,3} + l) \pmod{l^2 - 1}. \]

This simplifies to \( a_{w,1} - a_{w,3} \equiv l - 2 \pmod{l + 1} \), contradicting (5.1.6).

_Niveau 3_ Suppose finally that \( \overline{r}_{|F_w} \) has niveau 3, i.e. that \( \overline{r}_{|F_w} \) is isomorphic to one of the following:
\[ \begin{align*}
\omega_3^{-(a_{w,1}+2+l(a_{w,2}+1)+l^2a_{w,3})} & \oplus \omega_3^{-(a_{w,2}+1+la_{w,3}+l^2(a_{w,1}+2))} & \oplus \omega_3^{-(a_{w,3}+l(a_{w,1}+2)+l^2(a_{w,2}+1))} \\
\omega_3^{-(a_{w,1}+2+la_{w,3}+l^2(a_{w,1}+2))} & \oplus \omega_3^{-(a_{w,3}+l(a_{w,2}+1)+l^2(a_{w,1}+2))} & \oplus \omega_3^{-(a_{w,2}+1+l(a_{w,1}+2)+l^2a_{w,3})}
\end{align*} \]

On the other hand, by Lemma 4.2.1 (applied to \( F_5 \)) we also have that \( \overline{r}_{|F_w} \) is isomorphic to one of the following:
\[ \begin{align*}
\omega_3^{-(a_{w,1}+4+l(a_{w,2}+1)+l^2(a_{w,3}+l))} & \oplus \omega_3^{-(a_{w,2}+1+la_{w,3}+l^2(a_{w,1}+1)} & \oplus \omega_3^{-(a_{w,3}+l(a_{w,2}+1)+l^2(a_{w,1}+2))} \\
\omega_3^{-(a_{w,1}+4+la_{w,3}+l^2(a_{w,1}+2))} & \oplus \omega_3^{-(a_{w,3}+l(a_{w,2}+1)+l^2(a_{w,1}+2))} & \oplus \omega_3^{-(a_{w,2}+1+l(a_{w,1}+2)+l^2(a_{w,3}+l))}
\end{align*} \]

Examining the exponents in these expressions, we obtain 12 possible congruences (mod \( l^3 - 1 \)), each of which will now show yields a contradiction. In each case below we derive a congruence modulo \( l^2 + l + 1 \) or \( l^3 - 1 \), and it is easy to see in each case that the inequalities (5.1.1) and (5.1.2) imply that the congruence has no solutions.

1. \( a_{w,1} + 2 + l(a_{w,2} + 1) + l^2a_{w,3} \equiv a_{w,1} - l + 2 + l(a_{w,3} + l) \pmod{l^3 - 1}. \) This simplifies to \( l^2 - 1 \equiv 0 \pmod{l^3 - 1} \), a contradiction.

2. \( a_{w,1} + 2 + l(a_{w,2} + 1) + l^2a_{w,3} \equiv a_{w,1} - l + 2 + l(a_{w,3} + l) + l^2(a_{w,2} + 1) \pmod{l^3 - 1}. \) This simplifies to \( a_{w,2} - a_{w,3} + 2 \equiv 0 \pmod{l^2 + l + 1} \), a contradiction.

3. \( a_{w,1} + 2 + l(a_{w,2} + 1) + l^2a_{w,3} \equiv a_{w,2} + 1 + l(a_{w,1} - l + 2) + l^2(a_{w,3} + l) \pmod{l^3 - 1}. \) This simplifies to \( a_{w,1} - a_{w,2} \equiv l \pmod{l^2 + l + 1} \), a contradiction.

4. \( a_{w,1} + 2 + l(a_{w,2} + 1) + l^2a_{w,3} \equiv a_{w,2} + 1 + l(a_{w,3} + l) + l^2(a_{w,1} - l + 2) \pmod{l^3 - 1}. \) This simplifies to \( l(a_{w,1} - a_{w,3} + 3) + (a_{w,1} - a_{w,2} + 2) \equiv 0 \pmod{l^2 + l + 1} \), which is easily seen to be impossible.

5. \( a_{w,1} + 2 + l(a_{w,2} + 1) + l^2a_{w,3} \equiv a_{w,3} + l + l(a_{w,1} - l + 2) + l^2(a_{w,2} + 1) \pmod{l^3 - 1}. \) This simplifies to \( a_{w,1} - a_{w,3} + l(a_{w,2} - a_{w,3}) + 2 \equiv 0 \pmod{l^2 + l + 1} \), which is also impossible.

6. \( a_{w,1} + 2 + l(a_{w,2} + 1) + l^2a_{w,3} \equiv a_{w,3} + l + l(a_{w,2} + 1) + l^2(a_{w,1} - l + 2) \pmod{l^3 - 1}. \) This simplifies to \( (l+1)(a_{w,1} - a_{w,3}) + 2 \equiv 0 \pmod{l^2 + l + 1} \), which is impossible.

7. \( a_{w,1} + 2 + la_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,1} - l + 2 + l(a_{w,2} + 1) + l^2(a_{w,3} + l) \pmod{l^3 - 1}. \) This simplifies to \( l(a_{w,2} - a_{w,3} + 1) \equiv 0 \pmod{l^2 + l + 1} \), a contradiction.

8. \( a_{w,1} + 2 + la_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,1} - l + 2 + l(a_{w,3} + l) + l^2(a_{w,2} + 1) \pmod{l^3 - 1}. \) This simplifies to \( l^2 - l \equiv 0 \pmod{l^3 - 1} \), a contradiction.

9. \( a_{w,1} + 2 + la_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,2} + 1 + l(a_{w,1} - l + 2) + l^2(a_{w,3} + l) \pmod{l^3 - 1}. \) This simplifies to \( l(a_{w,2} - a_{w,3} + 2) \equiv a_{w,1} - a_{w,2} \pmod{l^2 + l + 1} \), which is easily seen to be impossible.
(10) \(a_{w,1} + 2 + la_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,2} + 1 + l(a_{w,3} + l) + l^2(a_{w,1} - l + 2) \pmod{l^3 - 1}\). This simplifies to \(a_{w,1} - a_{w,2} + 2 \equiv 0 \pmod{l^2 + l + 1}\), a contradiction.

(11) \(a_{w,1} + 2 + la_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,3} + l + l(a_{w,1} - l + 2) + l^2(a_{w,2} + 1) \pmod{l^3 - 1}\). This simplifies to \(a_{w,1} - a_{w,3} \equiv l - 2 \pmod{l^2 + l + 1}\), a contradiction.

(12) \(a_{w,1} + 2 + la_{w,3} + l^2(a_{w,2} + 1) \equiv a_{w,3} + l + l(a_{w,2} + 1) + l^2(a_{w,1} - l + 2) \pmod{l^3 - 1}\). This simplifies to \(l(a_{w,1} - a_{w,2} + 1) + a_{w,1} - a_{w,3} + 3 \equiv 0 \pmod{l^2 + l + 1}\), which is impossible.

As we have obtained a contradiction in every case, we see that \(F_b \cong F_a\), as required.

\[\square\]

References


E-mail address: tbl@brandeis.edu

DEPARTMENT OF MATHEMATICS, BRANDEIS UNIVERSITY

E-mail address: toby.gee@imperial.ac.uk

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON

E-mail address: david.geraghty@bc.edu

DEPARTMENT OF MATHEMATICS, BOSTON COLLEGE