ARTHUR'S MULTIPLICITY FORMULA FOR $\mathbf{GSp}_4$ AND RESTRICTION TO $\mathbf{Sp}_4$

TOBY GEE AND OLIVIER TAÏBI

Abstract. We prove the classification of discrete automorphic representations of $\mathbf{GSp}_4$ explained in [Art04], as well as a compatibility between the local Langlands correspondences for $\mathbf{GSp}_4$ and $\mathbf{Sp}_4$.

Contents

1. Introduction

1.1. In the paper [Art04], Arthur explained his classification of the discrete automorphic spectrum for classical groups in the particular case of $\mathbf{GSp}_4 \cong \mathbf{GSpin}_5$. Later, in [Art13] he proved this classification for quasi-split special orthogonal and symplectic groups of arbitrary rank, but now with trivial similitude factor. The classification stated in [Art04] is important for applications of the Langlands program to arithmetic. In particular, it is used in [Mok14] to associate Galois representations to Hilbert–Siegel modular forms, and these Galois representations have been used to prove modularity lifting theorems relating to abelian surfaces, for example in [BCGP]. It is therefore desirable to have an unconditional proof of this classification. While it is expected that the methods of [Art13] could be used to handle **GSpin** groups, the proofs involve a very complicated induction, which even in the case of $\mathbf{GSpin}_5$ would involve the use of groups of much higher rank, so there does not seem to be any way to give a (short) direct proof of the classification of [Art04] by following the arguments of [Art13].

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In this paper, we fill this gap in the literature by giving a proof of the classification announced in [Art04]. We also prove some new results concerning the compatibility of the local Langlands correspondences for $\mathbf{Sp}_4$ and $\mathbf{GSp}_4$. While, like Arthur, our main technique is the stable (twisted) trace formula, and we make substantial use of the results of [Art04] for the group $\mathbf{Sp}_4$, we also rely on a number of additional ingredients that are only available in the particular case of $\mathbf{GSp}_4$; in particular, we crucially use:

- the exterior square functoriality for $\mathbf{GL}_4$ proved in [Kim03] (and completed in [Hen09]);
- the results of [GT11a]: the local Langlands correspondence for $\mathbf{GSp}_4$ (established using theta correspondences), and the generic transfer to $\mathbf{GSp}_4$ (with local-global compatibility at all places) for essentially self dual cuspidal automorphic representations of $\mathbf{GL}_4$ of symplectic type;
- the results of [CG15], which check the compatibility of the local Langlands correspondence of [GT11b] with the predicted twisted endoscopic character relations of [Art04] in the tempered case.

We now briefly explain the strategy of our proof, and the structure of the paper. We begin in Section 2 with a precise statement of the results of [Art13] and of their conjectural extension to **GSpin** groups. Roughly speaking, these statements consist of:

(1) An assignment of global parameters (formal sums of essentially self-dual discrete automorphic representations of $\mathbf{GL}_n$) to discrete automorphic representations of classical groups.
(2) A description of packets of local representations in terms of local versions of the global parameters (which in particular gives the local Langlands correspondence for classical groups).
(3) A multiplicity formula, precisely describing which elements of global packets are automorphic, and the multiplicities with which they appear in the discrete spectrum.

In Arthur's work these statements are all proved together as part of a complicated induction, but in this paper (which of course uses Arthur's results for $\mathbf{Sp}_4$) we are able to prove the first two statements independently, and then use them as inputs to the proof of the third statement.

In section 3 we study the local packets. In the tempered case, the work has already been done in [CG15], and by again using that [Art13] has taken care of the cases where the similitude character is a square, we are reduced to constructing the local packets in two special non-tempered cases. We do this "by hand", following the much more general results proved in [MW06] and [AMgR18].

As a consequence of the stabilisation of the twisted trace formula [MW16a, MW16b], we can apply the twisted trace formula for $\mathbf{GL}_4 \times \mathbf{GL}_1$ to associate a global parameter to any discrete automorphic representation of $\mathbf{GSpin}_5$ (which is a twisted endoscopic group for $\mathbf{GL}_4 \times \mathbf{GL}_1$ endowed with the automorphism $g \mapsto {}^t g^{-1}$). We recall the details of this twisted trace formula in section 4, which we hope can serve as an introduction to the results of [MW16a, MW16b] for the reader not already familiar with them. In section 5 we briefly recall results about the restriction of representations to subgroups, which we apply to the case of restriction from $\mathbf{GSp}_4$ to $\mathbf{Sp}_4$.

In section 6 we show that the global parameter associated to a discrete automorphic representation of $\mathbf{GSp}_4$ by the stable twisted trace formula is of the form predicted by Arthur, by making use of the symplectic/orthogonal alternative for $\mathbf{GL}_2$ and $\mathbf{GL}_4$, the (known) description of automorphic representations of quasi-split forms of $\mathbf{GSpin}_4$ in terms of Asai representations, and the tensor product functoriality $\mathbf{GL}_2 \times \mathbf{GL}_2 \to \mathbf{GL}_4$ of [Ram00]. We also make use of [Art13] in two ways: if the similitude character is a square, then by twisting we can immediately reduce to the results of [Art13]. If the similitude character is not a square, then the possibilities for the parameter are somewhat constrained, and we are able to further constrain them by using the fact that by restricting to $\mathbf{Sp}_4$ and applying the results of [Art13], we know the possible forms of the exterior square of the parameter.

In section 7, we prove the global multiplicity formula in much the same way as [Art13], as a consequence of the stable (twisted) trace formulas for $\mathbf{GL}_4 \times \mathbf{GL}_1$ and $\mathbf{GSpin}_5$, together with the twisted endoscopic character relations already established.

Finally, in section 8 we show that the local Langlands correspondences for $\mathbf{Sp}_4$ established in [GT10] and [Art13] coincide. The correspondence of [GT10] was constructed by restricting the correspondence for $\mathbf{GSp}_4$ of [GT11a] to $\mathbf{Sp}_4$, which by the results of [CG15] is characterised using twisted endoscopy for $\mathbf{GL}_4 \times \mathbf{GL}_1$. The correspondence for $\mathbf{Sp}_4$ obtained in [Art13] is characterised using twisted endoscopy for $\mathbf{GL}_5$.

We postpone to the appendix two basic results concerning twisted endoscopy for $\mathbf{GL}_N \times \mathbf{GL}_1$ which are slight generalizations of results of Arthur for $\mathbf{GL}_N$: the classification of endoscopic data and the surjectivity of geometric endoscopic transfer for "simple" endoscopic data.

In the discrete case we prove this by a global argument, by realising the parameter as a local factor of a cuspidal automorphic representation, and using the exterior square functoriality for $\mathbf{GL}_4$ of [Kim03] and [Hen09]. In the remaining cases the parameter arises via parabolic induction, and we are able to treat it by hand. We are also able to use these arguments to give a precise description in terms of Arthur parameters of the restrictions to $\mathbf{Sp}_4$ of irreducible admissible tempered representations of $\mathbf{GSp}_4$ over a p-adic field.

We end this introduction with a small disclosure, and a comparison to other work. While we have said that the results of this paper are unconditional, they are only as unconditional as the results of [Art13] and [MW16a, MW16b]. In particular, they depend on cases of the twisted weighted fundamental lemma that were announced in [CL10], but whose proofs have not yet appeared in print, as well as on the references [A24], [A25], [A26] and [A27] in [Art13], which at the time of writing have not appeared publicly.

The strategy of using restriction to compare the representation theory of reductive groups related by a central isogeny is not a new one; indeed it goes back at least as far to the comparison of $\mathbf{GL}_2$ and $\mathbf{SL}_2$ in [LL79]. In the case of symplectic groups, there is the paper [GT10] mentioned above; while this does not make any use of trace formula techniques, we use some of its ideas in Section 8, when we compare the different constructions of the local Langlands correspondence.

More recently, there is the work of Xu, in particular [Xu17, Xu16], which also builds on [Art13], using the groups $\mathbf{GSp}_n$ and $\mathbf{GO}_n$ where we use the groups $\mathbf{GSpin}_n$

(of course, these cases overlap for $\mathbf{GSp}_4$). However, the emphasis of Xu's work is rather different, and is aimed at constructing "coarse L-packets" (which in the case of $\mathbf{GSp}_4$ are unions of L-packets lying over a common L-packet for $\mathbf{Sp}_4$), and proving a multiplicity formula for automorphic representations grouped together in a similar way. Xu's results are more general than ours in that they apply to groups of arbitrary rank, but are less precise in the special case of $\mathbf{GSp}_4$, and our proofs are independent.

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1.3. **Notation and conventions.**

1.3.1. *Algebraic groups.* We will use the boldface notation $\mathbf{G}$ for an algebraic group over a local field or a number field, and we use the Roman version G for reductive groups over $\mathbb{C}$, or their complex points. Thus for example if F is a number field, we will write $\mathbf{GL}_n$ for the general linear group over F, with Langlands dual group $\widehat{\mathbf{GL}}_n = \mathrm{GL}_n$, which we will also sometimes write as $\widehat{\mathbf{GL}}_n = \mathrm{GL}_n(\mathbb{C})$.

For a real connected reductive group $\mathbf{G}$, write $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Lie}\,(\mathbf{G}(\mathbb{R}))$, and let K be a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$. When working adelically we will sometimes abusively call $(\mathfrak{g}, K)$-modules "representations of $\mathbf{G}(\mathbb{R})$". This should cause no confusion as we will mostly be considering unitary representations in this global setting (see [Wal88, Theorem 3.4.11], [War72, Theorem 4.4.6.6]), and distinguish between $(\mathfrak{g}, K)$-modules and representations of $\mathbf{G}(\mathbb{R})$ when considering non-unitary representations.

1.3.2. *The local Langlands correspondence.* If K is a field of characteristic zero then we write Gal_K for its absolute Galois group $\mathrm{Gal}(\overline{K}/K)$. If K is a local or global field of characteristic zero, then we write W_K for its Weil group. If K is a local field of characteristic zero, then we write WD_K for its Weil–Deligne group, which is W_K if K is Archimedean, and $W_K \times \mathrm{SU}(2)$ otherwise.

If π is an irreducible admissible representation of $\mathbf{GL}_N(F)$ (F local) or $\mathbf{GL}_N(\mathbb{A}_F)$ (F global), then ω_π will denote its central character. We write rec for the local Langlands correspondence normalised as in [HT01], so that if F is a local field of characteristic zero, then $\mathrm{rec}(\pi)$ is an N-dimensional representation of WD_F. If F is p-adic then for this normalisation a uniformiser of F corresponds to the *geometric* Frobenius automorphism.

1.3.3. *The discrete spectrum.* Let $\mathbf{G}$ be a connected reductive group over a number field F. Write

$$\mathbf{G}(\mathbb{A}_F)^1 = \left\{ g \in \mathbf{G}(\mathbb{A}_F) \,\middle|\, \forall \beta \in X^*(\mathbf{G})^{\mathrm{Gal}_F}, |\beta(g)| = 1 \right\},$$

so that $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)^1$ has finite measure. Let $\mathbf{A}_{\mathbf{G}}$ be the biggest central split torus in $\mathrm{Res}_{F/\mathbb{Q}}(\mathbf{G})$, and let $\mathfrak{A}_{\mathbf{G}}$ be the vector group $\mathbf{A}_{\mathbf{G}}(\mathbb{R})^0$. Then $\mathbf{G}(\mathbb{A}_F) = \mathbf{G}(\mathbb{A}_F)^1 \times \mathfrak{A}_{\mathbf{G}}$. We write

$$\mathcal{A}^2(\mathbf{G}) = \mathcal{A}^2(\mathbf{G}(F)\mathfrak{A}_{\mathbf{G}}\backslash\mathbf{G}(\mathbb{A}_F)) = \mathcal{A}^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)^1)$$

for the space of square integrable automorphic forms. This decomposes discretely, i.e. it is canonically the direct sum, over the countable set $\Pi_{\mathrm{disc}}(\mathbf{G})$ of discrete

automorphic representations π for $\mathbf{G}$, of isotypical components

$$\mathcal{A}^2(\mathbf{G})_\pi$$

which have finite length.

If $\chi_{\mathbf{G}}$ is a character of $\mathfrak{A}_{\mathbf{G}}$, we could more generally consider the space of $\chi_{\mathbf{G}}$-equivariant square integrable automorphic forms

$$\mathcal{A}^2(\mathbf{G}) = \mathcal{A}^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G}}).$$

Since we can reduce to the case $\chi_{\mathbf{G}} = 1$ considered above by twisting, we will almost never use this more general definition.

2. Arthur's classification

2.1. **GSpin groups.** We now recall the results announced in [Art04] for $\mathbf{GSp}_4$, as well as those for $\mathbf{Sp}_4$ proved in [Art13]. In fact, for convenience we begin by recalling the conjectural extension of Arthur's results to **GSpin** groups of arbitrary rank, and then explain what is proved in [Art13].

We work with the following quasi-split groups over a local or global field F of characteristic zero:

- The split groups $\mathbf{GSpin}_{2n+1}$.
- The split groups $\mathbf{Sp}_{2n} \times \mathbf{GL}_1$.
- The quasi-split groups $\mathbf{GSpin}^\alpha_{2n}$.

Here we can define the groups $\mathbf{GSpin}_{2n+1}$ and $\mathbf{GSpin}^\alpha_{2n}$ as follows. If $\alpha \in F^\times/(F^\times)^2$, we have the quasi-split special orthogonal group $\mathbf{SO}^\alpha_{2n}$, which is defined as the special orthogonal group of the quadratic space given by the direct sum of $(n-1)$ hyperbolic planes and the plane $F[X]/(X^2-\alpha)$ equipped with the quadratic form equal to the norm. We have the spin double cover

$$0 \to \mu_2 \to \mathbf{Spin}^\alpha_{2n} \to \mathbf{SO}^\alpha_{2n} \to 0,$$

and we set

$$\mathbf{GSpin}^\alpha_{2n} := (\mathbf{Spin}^\alpha_{2n} \times \mathbf{GL}_1)/\mu_2$$

where μ_2 is embedded diagonally. Note that $\mathbf{GSpin}^\alpha_{2n}$ is split if and only if $\alpha = 1$. We define the split group $\mathbf{GSpin}_{2n+1}$ in the same way. This expedient definition is of course equivalent to the usual, more geometric one (see [Knu91, Ch. IV, §6]). The spinor norm is induced by $(g, \lambda) \mapsto \lambda^2$. It is convenient to let $\mathbf{GSpin}^1_0 = \mathbf{GSpin}_1 = \mathbf{GL}_1$.

The corresponding dual groups are as follows.

$\mathbf{G}$	$\widehat{\mathbf{G}}$
$\mathbf{GSpin}_{2n+1}$	$\mathrm{GSp}_{2n}(\mathbb{C})$
$\mathbf{Sp}_{2n} \times \mathbf{GL}_1$	$\mathrm{GSO}_{2n+1}(\mathbb{C}) = \mathrm{SO}_{2n+1}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$
$\mathbf{GSpin}^\alpha_{2n}$	$\mathrm{GSO}_{2n}(\mathbb{C})$

Let $\mu : \mathbf{GL}_1 \to \mathbf{Z}(\mathbf{G})$ be dual to the surjective "similitude factor" morphism $\widehat{\mu} : \widehat{\mathbf{G}} \to \mathrm{GL}_1(\mathbb{C})$. Note that in the case $\mathbf{G} = \mathbf{Sp}_{2n} \times \mathbf{GL}_1$, $\mu : \mathbf{GL}_1 \to \mathbf{Z}(\mathbf{G})$ is the map $x \mapsto (1, x^2)$, and it is the only case where it is not injective. Moreover the image of μ is $\mathbf{Z}(\mathbf{G})^0$ except in the case $\mathbf{G} = \mathbf{GSpin}^\alpha_2$.

We set ${}^L\mathbf{G} = \widehat{\mathbf{G}} \rtimes \mathrm{W}_F$, where the action of W_F on $\widehat{\mathbf{G}}$ is trivial except in the case that $\mathbf{G} = \mathbf{GSpin}^\alpha_{2n}$ with $\alpha \neq 1$, in which case the action of W_F factors through $\mathrm{Gal}(F(\sqrt{\alpha})/F) = \{1, \sigma\}$, and σ acts by outer conjugation on GSO_{2n}. More precisely, in this case we identify $\widehat{\mathbf{G}} \rtimes \mathrm{Gal}(F(\sqrt{\alpha})/F)$ with $\mathrm{GO}_{2n}(\mathbb{C})$ as follows: if

SO_{2n} is obtained from the symmetric bilinear form B on $\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{2n}$ given by $B(e_i, e_j) = \delta_{i,2n+1-j}$, then $1 \rtimes \sigma$ is the element of $\mathrm{O}_{2n}(\mathbb{C})$ which interchanges e_n and e_{n+1} and fixes the other e_i.

We have the standard representation

$$\mathrm{Std}_{\mathbf{G}} : {}^L\mathbf{G} \to \mathrm{GL}_N(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}),$$

where $N = N(\widehat{\mathbf{G}}) = 2n$ if $\mathbf{G} = \mathbf{GSpin}^{\alpha}_{2n}$ or $\mathbf{G} = \mathbf{GSpin}_{2n+1}$, and $N = 2n+1$ if $\mathbf{G} = \mathbf{Sp}_{2n} \times \mathbf{GL}_1$. In the first two cases the representation is trivial on W_F, and is given by the product of the standard N-dimensional representation of $\widehat{\mathbf{G}}$ and the similitude character. In the final case it is given by the product of the natural inclusion $\mathrm{O}_{2n+1}(\mathbb{C}) \subset \mathrm{GL}_{2n+1}(\mathbb{C})$ and the identity on $\mathrm{GL}_1(\mathbb{C})$. The standard representation realises $\mathbf{G}$ as an elliptic twisted endoscopic subgroup of $\mathbf{GL}_N \times \mathbf{GL}_1$, as we will explain below.

We set $\mathrm{sign}(\mathbf{G}) = 1$ if $\mathbf{G} = \mathbf{GSpin}^{\alpha}_{2n}$ or $\mathbf{GL}_1 \times \mathbf{Sp}_{2n}$, and $\mathrm{sign}(\mathbf{G}) = -1$ if $\mathbf{G} = \mathbf{GSpin}_{2n+1}$ (equivalently, we set $\mathrm{sign}(\mathbf{G}) = -1$ if and only if $\widehat{\mathbf{G}}$ is symplectic).

2.2. **Levi subgroups and dual embeddings.** As in our description of the dual group SO_{2n} above, we may realise the groups $\mathbf{SO}^{\alpha}_{2n}$ and $\mathbf{SO}_{2n+1}$ as matrix groups using an antidiagonal symmetric bilinear form (block antidiagonal with a 2×2 block in the middle for $\mathbf{SO}^{\alpha}_{2n}$ with $\alpha \neq 1$). Let $\mathbf{B}$ be the Borel subgroup consisting of upper diagonal elements (block upper diagonal in the case of $\mathbf{SO}^{\alpha}_{2n}$). Let $\mathbf{T}$ be the subgroup of diagonal (resp. block diagonal) elements. This Borel pair being given, we can now consider standard parabolic subgroups and standard Levi subgroups. (We recall that we only need to consider Levi subgroups up to conjugacy; indeed, given a Levi subgroup $\mathbf{L}$ of a parabolic $\mathbf{P}$, we obtain an L-embedding ${}^L\mathbf{L} \hookrightarrow {}^L\mathbf{G}$, which up to $\widehat{\mathbf{G}}$-conjugacy is independent of the choice of $\mathbf{P}$.)

It is well-known that the standard Levi subgroups are parametrised as follows. Consider ordered partitions $n = \sum_{i=1}^{r} n_i + m$, where $m > 0$ if $\mathbf{G} = \mathbf{SO}^{\alpha}_{2n}$ with $\alpha \neq 1$, and $m \neq 1$ if $\mathbf{G} = \mathbf{SO}^{1}_{2n}$. Such a partition yields a standard Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ isomorphic to $\mathbf{GL}_{n_1} \times \cdots \times \mathbf{GL}_{n_r} \times \mathbf{G}_m$ where $\mathbf{G}_m$ is a group of the same type as $\mathbf{G}$ of absolute rank m. Explicitly, an isomorphism is given by

$$(g_1, \ldots, g_r, h) \longmapsto \mathrm{diag}\left(g_1, \ldots, g_r, h, S_{n_r}^{-1}\, {}^t g_r^{-1} S_{n_r}, \ldots, S_{n_1}^{-1}\, {}^t g_1^{-1} S_{n_1}\right), \tag{2.2.1}$$

where S_n denotes the antidiagonal $n \times n$ matrix with 1's along the antidiagonal. For $\mathbf{G} = \mathbf{SO}^{1}_{2n}$ and $m = 0$ and $n_r > 1$, there are two standard Levi subgroups of $\mathbf{G}$ corresponding to the partition $n = \sum_{i=1}^{r} n_i$: the one described above and its image under the outer automorphism of $\mathbf{G}$. This completes the parameterisation of all standard Levi subgroups of special orthogonal groups. Standard Levi subgroups of $\mathbf{Sp}$ and $\mathbf{GSp}$ admit a similar description. In all three cases, two standard Levi subgroups are conjugated under $\mathbf{G}(F)$ if and only if they have the same associated family $(|\{i, \mid n_i = k\}|)_{k\geq 1}$ (i.e. same associated multi-set $\{n_1, \ldots, n_r\}$), except when $\mathbf{G} = \mathbf{SO}^{1}_{2n}$ and $m = 0$ and all n_i's are even, in which case there are two $\mathbf{G}(F)$-conjugacy classes of Levi subgroups of $\mathbf{G}(F)$ corresponding to the same multi-set, swapped by the non-trivial outer automorphism of $\mathbf{G}$.

Denote $\mathbf{G}' = \mathbf{GSpin}^{\alpha}_{2n}$ if $\mathbf{G} = \mathbf{SO}^{\alpha}_{2n}$ and $\mathbf{G}' = \mathbf{GSpin}_{2n+1}$ if $\mathbf{G} = \mathbf{SO}_{2n+1}$. Parabolic subgroups of $\mathbf{G}'$ correspond bijectively to parabolic subgroups of $\mathbf{G}$, and the same goes for their Levi subgroups. Consider $\mathbf{L}$ as above, and let $\mathbf{L}'$ be its preimage in $\mathbf{G}'$. An easy root-theoretic exercise shows that there exists a unique

isomorphism

$$\mathbf{GL}_{n_1} \times \cdots \times \mathbf{GL}_{n_r} \times \mathbf{G}'_m \simeq \mathbf{L}'$$

lifting (2.2.1) such that for any $1 \leq i \leq r$, the composition of the induced embedding of $\mathbf{GL}_{n_i}$ in $\mathbf{G}'$ with the spinor norm $\mathbf{G}' \to \mathbf{GL}_1$ is det. Alternatively, the embeddings $\mathbf{GL}_{n_i} \to \mathbf{GSpin}^1_{2n_i}$ can be constructed geometrically using the definition of **GSpin** groups via Clifford algebras (see [Knu91, Ch. IV, §6.6]), and the above parameterisation of $\mathbf{L}'$ easily follows. The conjugacy class of $\mathbf{L}'$ under $\mathbf{G}'(F)$ is determined by the multi-set $\{n_1, \ldots, n_r\}$.

Dually, this corresponds to identifying the dual Levi subgroup $\widehat{\mathbf{L}}$ of $\widehat{\mathbf{G}} = \mathrm{GSO}_{2n}$ or GSp_{2n} with $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r} \times \widehat{\mathbf{G}'_m}$ via the block diagonal embedding:

$$(g_1, \ldots, g_r, h) \mapsto \mathrm{diag}\left(g_1, \ldots, g_r, h, \widehat{\mu}(h) S_{n_r} {}^t g_r^{-1} S_{n_r}^{-1}, \ldots, \widehat{\mu}(h) S_{n_1} {}^t g_1^{-1} S_{n_1}^{-1}\right)$$

2.3. **Endoscopic groups and transfer.** Before stating the conjectural parameterisation, we need to recall some definitions and results about endoscopy. We begin by recalling that an endoscopic datum for a connected reductive group $\mathbf{G}$ over a local field F is a tuple $(\mathbf{H}, \mathcal{H}, s, \xi)$ (almost) as in [KS99, §2.1]:

- $\mathbf{H}$ is a quasi-split connected reductive group over F,
- $\xi : \widehat{\mathbf{H}} \to \widehat{\mathbf{G}}$ is a continuous embedding,
- $\mathcal{H}$ is a closed subgroup of ${}^L\mathbf{G}$ which surjects onto W_F with kernel $\xi(\widehat{\mathbf{H}})$, such that the induced outer action of W_F on $\xi(\widehat{\mathbf{H}})$ coincides with the usual one on $\widehat{\mathbf{H}}$ transported by ξ, and such that there exists a continuous splitting $W_F \to \mathcal{H}$,
- and $s \in \widehat{\mathbf{G}}$ is a semisimple element whose connected centraliser in $\widehat{\mathbf{G}}$ is $\xi(\widehat{\mathbf{H}})$ and such that the map $W_F \to \widehat{\mathbf{G}}$ induced by $h \in \mathcal{H} \mapsto shs^{-1}h^{-1}$ takes values in $Z(\widehat{\mathbf{G}})$ and is trivial in $H^1(W_F, Z(\widehat{\mathbf{G}}))$.

Note that we modified the notation slightly: in [KS99] $\mathcal{H}$ is not contained in ${}^L\mathbf{G}$ and instead ξ is an embedding of $\mathcal{H}$ in ${}^L\mathbf{G}$. We choose this convention because in contrast to the general case where z-extensions are a necessary complication, in all cases that we will consider the embedding $\xi : \widehat{\mathbf{H}} \to \widehat{\mathbf{G}}$ will admit a (non-unique) extension as ${}^L\xi : {}^L\mathbf{H} \to {}^L\mathbf{G}$. Of particular importance are the elliptic endoscopic data, which are those for which the identity component of $\xi(Z(\widehat{\mathbf{H}})^{\mathrm{Gal}_F})$ is contained in $Z(\widehat{\mathbf{G}})$.

For $\mathbf{G}$ belonging to the three families introduced in Section 2.1 the groups $\mathbf{H}$ will be products whose factors are either general linear groups, or quotients by $\mathbf{GL}_1$ of products of groups of the form considered in Section 2.1. At this level of generality we content ourselves with specifying the group $\mathbf{H}$, for each equivalence class of non-trivial ($s \notin Z(\widehat{\mathbf{G}})$) elliptic endoscopic datum of $\mathbf{G}$. They are as follows.

- If $\mathbf{G} = \mathbf{GSpin}_{2n+1}$, then $\mathbf{H} = (\mathbf{GSpin}_{2a+1} \times \mathbf{GSpin}_{2b+1})/\mathbf{GL}_1$ with $a+b = n$, $ab \neq 0$, and the quotient is by $\mathbf{GL}_1$ embedded as $z \mapsto (\mu(z), \mu(z)^{-1})$.
- If $\mathbf{G} = \mathbf{Sp}_{2n} \times \mathbf{GL}_1$, then $\mathbf{H} = (\mathbf{Sp}_{2a} \times \mathbf{GL}_1 \times \mathbf{GSpin}^{\alpha}_{2b})/\mathbf{GL}_1 \cong \mathbf{Sp}_{2a} \times \mathbf{SO}^{\alpha}_{2b} \times \mathbf{GL}_1$, where $a + b = n$, $ab \neq 0$, and $\alpha \neq 1$ if $b = 1$.
- If $\mathbf{G} = \mathbf{GSpin}^{\alpha}_{2n}$, then $\mathbf{H} = (\mathbf{GSpin}^{\beta}_{2a} \times \mathbf{GSpin}^{\gamma}_{2b})/\mathbf{GL}_1$, where $a + b = n$, $\beta\gamma = \alpha$, $\beta \neq 1$ if $a = 1$, and $\gamma \neq 1$ if $b = 1$.

In this paper we will also need one case of twisted endoscopy. Recall [MW16a, §I.1.1] that if F is a local field of characteristic zero (in the paper we will also take F to be a number field), and $\mathbf{G}$ is a connected reductive group defined over F, then

a twisted space $\widetilde{\mathbf{G}}$ for $\mathbf{G}$ is an algebraic variety over F which is simultaneously a left and right torsor for $\mathbf{G}$. Consider the split group $\mathbf{GL}_n \times \mathbf{GL}_1$ over a local or global field of characteristic zero F, and let θ be the automorphism of $\mathbf{GL}_n \times \mathbf{GL}_1$ given by $\theta(g,x) = (J^t g^{-1} J^{-1}, x \det g)$, where J is the antidiagonal matrix with alternating entries $-1, 1, -1, \dots$ (that is, $J_{ij} = (-1)^i \delta_{i,n+1-j}$). The reason for defining θ in this way is that it fixes the usual pinning $\mathcal{E}$ of $\mathbf{G}$ consisting of the upper-triangular Borel subgroup, the diagonal maximal torus and $((\delta_{i,a}\delta_{j,a+1})_{i,j})_{1\leq a\leq n-1}$. Then $\widetilde{\mathbf{G}} = \mathbf{GL}_n \times \mathbf{GL}_1 \rtimes \{\theta\}$ is a twisted space which happens to be a connected component of the non-connected reductive group $\mathbf{GL}_n \times \mathbf{GL}_1 \rtimes \{1, \theta\}$.

There is a notion of a twisted endoscopic datum $(\mathbf{H}, \mathcal{H}, s, \xi)$ for the pair $(\mathbf{GL}_n \times \mathbf{GL}_1, \theta)$, for which we again refer to [KS99, §2.1] (taking ω there to be equal to 1, as we will throughout this paper, and using the same convention as above for ξ) and [MW16b, §VI.3.1]. In Appendix A we classify twisted endoscopic data (up to isomorphism). We give slightly more details in the case $n = 4$ which is the main focus of this paper in Section 4.2 below. In the present section we shall only need the fact that if $\mathbf{H}$ is one of the groups considered in Section 2.1 (denoted $\mathbf{G}$ there), then $\mathbf{H}$ is part of an elliptic twisted endoscopic subgroup of $(\mathbf{GL}_{N(\widehat{\mathbf{H}})} \times \mathbf{GL}_1, \theta)$.

Remark 2.3.1. The definitions in [MW16a] and [MW16b], using twisted spaces rather than a fixed automorphism of $\mathbf{G}$ (not fixing a base point), are more general than those used in most of [KS99], due to an assumption in [KS99] that is only removed in (5.4) there. Note in particular the notion of twisted endoscopic space [MW16a, §I.1.7]. In the cases considered in this paper, where $\widetilde{\mathbf{G}}$ is either $\mathbf{G}$ (standard endoscopy) or $\mathbf{G} \rtimes \theta$ where $\theta \in \mathrm{Aut}(\mathbf{G})$ fixes a pinning $\mathcal{E}$ of $\mathbf{G}$ (defined over F, i.e. stable under Gal_F), this notion simplifies and we are under the assumption of [KS99, (3.1)]. Namely, the torsor $\mathcal{Z}(\widetilde{\mathbf{G}}, \mathcal{E})$ under $\mathcal{Z}(\mathbf{G}) := Z(\mathbf{G})/(1-\theta)Z(\mathbf{G})$ defined in [MW16a, I.1.2] is trivial with a natural base point $1 \rtimes \theta$, and so for any endoscopic datum $(\mathbf{H}, \mathcal{H}, \tilde{s}, \xi)$ for $\widetilde{\mathbf{G}}$, the twisted endoscopic space $\widetilde{\mathbf{H}} := \mathbf{H} \times_{\mathcal{Z}(\mathbf{G})} \mathcal{Z}(\widetilde{\mathbf{G}})$ is trivial with natural base point $1 \rtimes \theta$, where θ now acts trivially on $\mathbf{H}$. For this reason we can ignore twisted endoscopic spaces in the rest of the paper, and simply consider endoscopic groups as in most of [KS99].

We now very briefly recall the notion of (geometric) transfer in the setting of endoscopy. Suppose that F is a local field of characteristic zero, and that $(\mathbf{G}, \widetilde{\mathbf{G}})$ belongs to one of the four families of twisted spaces considered above, that is $\mathbf{G} = \mathbf{GSpin}_{2n+1}$, $\mathbf{Sp}_{2n} \times \mathbf{GL}_1$ or $\mathbf{G} = \mathbf{GSpin}^{\alpha}_{2n}$ with $\widetilde{\mathbf{G}} = \mathbf{G}$, or $\mathbf{G} = \mathbf{GL}_n \times \mathbf{GL}_1$ with $\widetilde{\mathbf{G}} = \mathbf{G} \rtimes \theta$. Given an endoscopic datum $\mathfrak{e} = (\mathbf{H}, \mathcal{H}, s, \xi)$ for $\widetilde{\mathbf{G}}$, and a choice of an extension ${}^L\xi : {}^L\mathbf{H} \to {}^L\mathbf{G}$ of the embedding ξ, Kottwitz and Shelstad defined *transfer factors* in [KS99], that is a function on the set of matching pairs of strongly regular semisimple $\mathbf{G}(F)$-conjugacy classes in $\widetilde{\mathbf{G}}(F)$ and regular semisimple stable conjugacy classes in $\mathbf{H}(F)$. In general such a function is only canonical up to $\mathbb{C}^\times$, but in all cases considered in this paper there is a Whittaker datum $\mathfrak{w} = (\mathbf{U}, \lambda)$ of $\mathbf{G}$ fixed by an element of $\widetilde{\mathbf{G}}(F)$ and this provides [KS99, §5.3] a normalisation of transfer factors, which we denote by $\Delta[\mathfrak{e}, {}^L\xi, \mathfrak{w}]$. To be more precise we use the transfer factors called Δ_D in [KS], corresponding to the normalisation of the local Langlands correspondence identifying uniformizers to *geometric* Frobenii. In all cases of ordinary endoscopy one can choose an arbitrary Whittaker datum of $\mathbf{G}$.

In the case that $\mathbf{G} = \mathbf{GSpin}^{\alpha}_{2n}$, there is an outer automorphism δ of $\mathbf{G}$ which preserves the Whittaker datum. This δ can be chosen to have order 2 and be induced

by an element of the orthogonal group having determinant -1; if F is Archimedean, for simplicity we can and do choose the maximal compact subgroup K of $\mathbf{G}(F)$ to be δ-stable. To treat all cases at once we let $\delta = 1$ if $\mathbf{G} = \mathbf{GSpin}_{2n+1}$ or $\mathbf{Sp}_{2n} \times \mathbf{GL}_1$.

In this paper we are particularly interested in the case $\mathbf{G} = \mathbf{GSpin}_5$. By Hilbert's theorem 90 the morphism $\mathbf{GSpin}_{2n+1}(F) \to \mathbf{SO}_{2n+1}(F)$ is surjective, so $\mathbf{GSpin}_{2n+1}$ is of adjoint type and there is up to conjugation by $\mathbf{GSpin}_{2n+1}(F)$ only one Whittaker datum in this case.

For $\widetilde{\mathbf{G}} = (\mathbf{GL}_n \times \mathbf{GL}_1) \rtimes \theta$ we choose for $\mathbf{U}$ the subgroup of unipotent upper triangular matrices in $\mathbf{GL}_n$ and $\lambda((g_{i,j})_{i,j}) = \kappa(\sum_{i=1}^{n-1} g_{i,i+1})$ where $\kappa : F \to S^1$ is a non-trivial continuous character. This is the Whittaker datum associated to $\mathcal{E}$ and κ. This Whittaker datum is fixed by θ (this is the reason for the choice of this particular θ in its $\mathbf{G}(F)$-orbit).

Definition 2.3.2. If F is p-adic, then we let $\mathcal{H}(\widetilde{\mathbf{G}})$ denote the space of smooth compactly supported distributions on $\widetilde{\mathbf{G}}(F)$ with $\mathbb{C}$-coefficients. Then $\mathcal{H}(\widetilde{\mathbf{G}}) = \varinjlim_K \mathcal{H}(\widetilde{\mathbf{G}}(F)//K)$ where the limit is over compact open subgroups of $\mathbf{G}(F)$ and $\mathcal{H}(\widetilde{\mathbf{G}}(F)//K)$ is the subspace of bi-K-invariant distributions. If F is Archimedean, then we fix a maximal compact subgroup K of $\mathbf{G}(F)$, and write $\mathcal{H}(\widetilde{\mathbf{G}})$ for the algebra of bi-K-finite smooth compactly supported distributions on $\widetilde{\mathbf{G}}(F)$ with $\mathbb{C}$-coefficients.

Under convolution, the space $\mathcal{H}(\widetilde{\mathbf{G}})$ is a bi-$\mathcal{H}(\mathbf{G})$-module, where $\mathcal{H}(\mathbf{G})$ is the usual (non-twisted) Hecke algebra for $\mathbf{G}$.

In the case that $\mathbf{G} = \mathbf{GSpin}_{2n}^{\alpha}$, we let $\widetilde{\mathcal{H}}(\mathbf{G})$ denote the subalgebra of $\mathcal{H}(\mathbf{G})$ consisting of δ-stable distributions, and otherwise we set $\widetilde{\mathcal{H}}(\mathbf{G}) = \mathcal{H}(\mathbf{G})$ and $\delta = 1$.

An admissible twisted representation of $\widetilde{\mathbf{G}}$ is by definition a pair $(\pi, \widetilde{\pi})$ consisting of an admissible representation π of $\mathbf{G}(F)$ and a map $\widetilde{\pi}$ from $\widetilde{\mathbf{G}}$ to the automorphism group of the underlying vector space of π, which satisfies

$$\widetilde{\pi}(g\gamma g') = \pi(g)\widetilde{\pi}(\gamma)\pi(g')$$

for all $g, g' \in \mathbf{G}(F)$, $\gamma \in \widetilde{\mathbf{G}}$. (This is the special case $\omega = 1$ of the notion of an ω-representation of a twisted space, which is defined in [MW16a].) If $F = \mathbb{R}$ or $\mathbb{C}$ there is an obvious notion of $(\mathfrak{g}, \widetilde{K})$-module where $\widetilde{K} \subset \mathbf{G}(F)$ is a torsor under K normalising K.

We will consider (invariant) linear forms on $\widetilde{\mathcal{H}}(\widetilde{\mathbf{G}})$. In particular, for each admissible representation π of $\mathbf{G}(F)$, there is the linear form

$$\mathrm{tr}(\pi(f(g)dg)) = \mathrm{tr}\left(\int_{G(F)} f(g)\pi(g)dg\right).$$

If F is Archimedean and π is an admissible $(\mathfrak{g}, \widetilde{K})$-module the action of $\widetilde{\mathcal{H}}(\widetilde{\mathbf{G}})$ is not obviously well-defined but it is so when π arises as the space of K-finite vectors of an admissible Banach representation of $\widetilde{\mathbf{G}}(F)$, independently of the choice of this realisation (see [War72, p. 326, Theorem 4.5.5.2]). In this paper all $(\mathfrak{g}, \widetilde{K})$-modules will naturally arise in this way, even with "Hilbert" instead of "Banach", although not all of them will be unitary.

We write $I(\widetilde{\mathbf{G}})$ for the quotient of $\widetilde{\mathcal{H}}(\widetilde{\mathbf{G}})$ by the subspace of those distributions $f(g)dg$ with the property that for any semisimple strongly regular $\gamma \in \widetilde{\mathbf{G}}(F)$,

the orbital integral $O_\gamma(f(g)dg)$ vanishes. There is a natural topology on $I(\widetilde{\mathbf{G}})$: see [MW16a, I.5.2]. Similarly, we write $SI(\widetilde{\mathbf{G}})$ for the quotient by the subspace for which the stable orbital integrals $SO_\gamma(f(g)dg)$ vanish. We say that a continuous linear form on $\widetilde{\mathcal{H}}(\mathbf{G})$ is stable if it descends to a linear form on $SI(\widetilde{\mathbf{G}})$.

Given an endoscopic datum $(\mathbf{H}, \mathcal{H}, s, \xi)$ for $\widetilde{\mathbf{G}}$, and our choice of Whittaker datum, there is a notion of transfer from $I(\widetilde{\mathbf{G}})$ to $SI(\mathbf{H})$ (see [KS99, §5.5], [MW16a, §I.2.4 and IV.3.4]); this transfer is defined by the property that it relates the values of orbital integrals on $\widetilde{\mathbf{G}}$ to stable orbital integrals on $\mathbf{H}$, using the transfer factors recalled above. Most importantly, this transfer *exists* ([Wal97], [Ngô10], [She12]). Dually, we may transfer *stable* continuous linear forms on $\widetilde{\mathcal{H}}(\mathbf{H})$ to continuous linear forms on $\widetilde{\mathcal{H}}(\mathbf{G})$.

In the twisted case where $\widetilde{\mathbf{G}} = (\mathbf{GL}_N \times \mathbf{GL}_1) \rtimes \theta$ over a p-adic field F, the chosen Whittaker datum yields a hyperspecial maximal compact subgroup K of $\mathbf{G}(F)$ (see [CS80]), which is stable under θ, so it is natural to consider the hyperspecial subspace (see [MW16a, §I.6]) $\widetilde{K} = K \rtimes \theta$ of $\widetilde{\mathbf{G}}(F)$. For any unramified endoscopic datum $(\mathbf{H}, \mathcal{H}, \tilde{s}, \xi)$ for $\widetilde{\mathbf{G}}$ (also defined in [MW16a, §I.6]), with the above trivialisation of $\widetilde{\mathbf{H}}$, the associated $\mathbf{H}_{\mathrm{ad}}(F)$-orbit of hyperspecial subspaces of $\widetilde{\mathbf{H}}$ is simply the obvious one, that is the set of $K' \rtimes \theta$ where K' is a hyperspecial maximal compact subgroup of $\mathbf{H}(F)$.

By the existence of transfer and [LMW15], [LW15] ([Hal95] in the case of standard endoscopy), the twisted fundamental lemma is now known for all elements of the unramified Hecke algebra, with no assumption on the residual characteristic. We formulate it in our situation, which is slightly simpler than the general case by the above remarks.

Theorem 2.3.3. *Let $\widetilde{\mathbf{G}}$ be a twisted group over a p-adic field F belonging to one of the four families introduced at the beginning of this section. Assume that $\mathbf{G}$ is unramified. Let $(\mathbf{H}, \mathcal{H}, \tilde{s}, \xi)$ be an unramified endoscopic datum for $\widetilde{\mathbf{G}}$. Choose an unramified L-embedding ${}^L\xi : {}^L\mathbf{H} \to {}^L\mathbf{G}$ extending ξ. Let $\widetilde{K}$ be the hyperspecial subspace of $\widetilde{\mathbf{G}}(F)$ associated to the chosen Whittaker datum for $\mathbf{G}$. Let $1_{\widetilde{K}}$ be the characteristic function of $\widetilde{K}$ multiplied by the $\mathbf{G}(F)$-invariant measure on $\widetilde{\mathbf{G}}(F)$ such that $\widetilde{K}$ has volume 1. Let $b : \mathcal{H}(\mathbf{G}(F_v)//K_v) \to \mathcal{H}(\mathbf{H}(F_v)//K'_v)$ be the morphism dual to*

$$\left(\widehat{\mathbf{H}} \rtimes \mathrm{Frob}\right)^{\mathrm{ss}} / \widehat{\mathbf{H}} - \mathrm{conj} \to \left(\widehat{\mathbf{G}} \rtimes \mathrm{Frob}\right)^{\mathrm{ss}} / \widehat{\mathbf{G}} - \mathrm{conj}$$

via the Satake isomorphisms (see [Bor79, §7]*). Then for any $f \in \mathcal{H}(\mathbf{G}(F_v)//K)$, $b(f)$ is a transfer of $f * 1_{\widetilde{K}}$.*

Remark 2.3.4. In the above setting, there is a natural notion of unramified twisted representation: extend an unramified representation (π, V) of $\mathbf{G}(F)$ which is isomorphic to its twist by $\widetilde{\mathbf{G}}(F)$ to a twisted representation by imposing that $\tilde{K}$ acts trivially on V^K.

2.4. **Local parameters.** Let F be a local field of characteristic zero. Let $\Psi^+(G)$ denote the set of $\widehat{\mathbf{G}}$-conjugacy classes of continuous morphisms

$$\psi : \mathrm{WD}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{G}$$

such that

- the composite with the projection ${}^L\mathbf{G} \to \mathrm{W}_F$ is the natural projection $\mathrm{WD}_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{W}_F$,
- for any $w \in \mathrm{WD}_F$, $\psi(w)$ is semisimple, and
- the restriction $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic.

We let $\Psi(\mathbf{G}) \subset \Psi^+(\mathbf{G})$ be the subset of bounded parameters.

Lemma 2.4.1. *Let $\mathbf{G} = \mathbf{GSpin}_{2n+1}$, $\mathbf{Sp}_{2n} \times \mathbf{GL}_1$ or $\mathbf{GSpin}^\alpha_{2n}$. Let $\widehat{\delta}$ be the automorphism of ${}^L\mathbf{G}$ dual to the involution δ of $\mathbf{G}$ defined in the previous section. Then composition with $\mathrm{Std}_{\mathbf{G}}$ induces an* injective *map $\{1, \widehat{\delta}\}\backslash\Psi^+(\mathbf{G}) \to \Psi^+(\mathbf{GL}_{N(\widehat{\mathbf{G}})} \times \mathbf{GL}_1)$.*

Proof. The case $\mathbf{G} = \mathbf{GSpin}_{2n+1}$ is proved in [GT11a, Lem. 6.1]. The proof in the other cases is almost identical. □

Let $\widetilde{\Psi}(G)$ and $\widetilde{\Psi}^+(\mathbf{G})$ be the set of $\{1, \widehat{\delta}\}$-orbits of parameters as above.

For $\psi \in \Psi^+(\mathbf{G})$ let φ_ψ be the Langlands parameter associated to ψ, that is ψ composed with the embedding

$$w \in \mathrm{WD}_F \longmapsto \Big(w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})\Big) \in \mathrm{WD}_F \times \mathrm{SL}_2(\mathbb{C}).$$

We write C_ψ for the centraliser of ψ in $\widehat{\mathbf{G}}$, $S_\psi = Z(\widehat{\mathbf{G}})C_\psi$, and

$$\mathcal{S}_\psi = \pi_0(S_\psi/Z(\widehat{\mathbf{G}})),$$

an abelian 2-group. We let $\mathcal{S}^\vee_\psi = \mathrm{Hom}(\mathcal{S}_\psi, \mathbb{C}^\times)$ be the character group of $\mathcal{S}_\psi$. Write s_ψ for the image in C_ψ of $-1 \in \mathrm{SL}_2(\mathbb{C})$.

We can now formulate the conjectures on local Arthur packets in terms of endoscopic transfer relations.

Conjecture 2.4.2. *Let $\mathbf{G} = \mathbf{GSpin}_{2n+1}$, $\mathbf{Sp}_{2n} \times \mathbf{GL}_1$ or $\mathbf{GSpin}^\alpha_{2n}$. Then there is a unique way to associate to each $(\psi) \in \widetilde{\Psi}(\mathbf{G})$ a multi-set Π_ψ of $\{1, \delta\}$-orbits of irreducible smooth unitary representations of $\mathbf{G}(F)$, together with a map $\Pi_\psi \to \mathcal{S}^\vee_\psi$, which we will denote by $\pi \mapsto \langle \cdot, \pi\rangle$, such that the following properties hold.*

(1) Let $\pi^{\mathbf{GL}}_\psi$ be the representation of $\mathbf{GL}_{N(\widehat{\mathbf{G}})}(F) \times \mathbf{GL}_1(F)$ associated to $(\mathrm{Std}_{\mathbf{G}} \circ \varphi_\psi)$ by the local Langlands correspondence for $\mathbf{GL}_{N(\widehat{\mathbf{G}})} \times \mathbf{GL}_1$, and let $\widetilde{\pi}^{\mathbf{GL}}_\psi$ be its extension to $\Big(\mathbf{GL}_{N(\widehat{\mathbf{G}})}(F) \times \mathbf{GL}_1(F)\Big) \rtimes \theta$ recalled in Section 3.2. Then $\sum_{\pi \in \Pi_\psi} \langle s_\psi, \pi\rangle \operatorname{tr} \pi$ is stable and its transfer to $\mathbf{GL}_{N(\widehat{\mathbf{G}})}(F) \times \mathbf{GL}_1(F) \rtimes \theta$ is $\operatorname{tr} \widetilde{\pi}^{\mathbf{GL}}_\psi$, i.e. for any $f \in I(\Big(\mathbf{GL}_{N(\widehat{\mathbf{G}})}(F) \times \mathbf{GL}_1(F)\Big) \rtimes \theta)$ having transfer $f' \in SI(\mathbf{G})$ we have

$$\operatorname{tr} \widetilde{\pi}^{\mathbf{GL}}_\psi(f) = \sum_{\pi \in \Pi_\psi} \langle s_\psi, \pi\rangle \operatorname{tr} \pi(f').$$

(2) Consider a semisimple $s \in C_\psi$ with image $\bar{s}$ in $\mathcal{S}_\psi$. The pair (ψ, s) determines an endoscopic datum $(\mathbf{H}, \mathcal{H}, s, \xi)$ for $\mathbf{G}$ (with $\mathcal{H} = \mathrm{Cent}\,(s, \widehat{\mathbf{G}})^0\psi(\mathrm{WD}_F)$), and if we fix an L-embedding ${}^L\xi : {}^L\mathbf{H} \to {}^L\mathbf{G}$ extending ξ we obtain $\psi' : \mathrm{WD}_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{H}$ such that $\psi = {}^L\xi \circ \psi'$. Then for any $f \in I(\mathbf{G})$ with transfer $f' \in SI(\mathbf{H})$, we have:

$$\sum_{\pi \in \Pi_\psi} \langle \bar{s}s_\psi, \pi\rangle \operatorname{tr} \pi(f) = \sum_{\pi' \in \Pi_{\psi'}} \langle s_{\psi'}, \pi'\rangle \operatorname{tr} \pi'(f').$$

(3) *If $\psi|_{\mathrm{SL}_2(\mathbb{C})} = 1$, then the elements of Π_ψ are tempered and Π_ψ is multiplicity free, and the map $\Pi_\psi \to \mathcal{S}_\psi^\vee$ is injective; if F is non-Archimedean, then it is bijective. Every tempered irreducible representation of $\mathbf{G}(F)$ belongs to exactly one such Π_ψ.*

Remark 2.4.3. Note that the uniqueness of the classification is clear from properties (1) and (2) and Proposition 2.4.4 below, as irreducible representations are determined by their traces. This Proposition is the generalization of [Art13, Cor. 2.1.2] from $\widetilde{\mathbf{GL}_N}$ to $\widetilde{\mathbf{GL}_N \times \mathbf{GL}_1}$. Now that [MW16a] has appeared, it is clearer to prove the Proposition following the constructions in [MW16a]. We give the proof in the Appendix (Section A.3).

Proposition 2.4.4. *In the situation of Conjecture 2.4.2, the transfer map*

$$I(\widetilde{\mathbf{GL}_{N(\widehat{\mathbf{G}})} \times \mathbf{GL}_1}) \to SI(\mathbf{G})^\delta$$

is surjective.

Remark 2.4.5. Part (3) of this conjecture gives the local Langlands correspondence for tempered representations of $\mathbf{G}(F)$ (up to outer conjugacy in case $\mathbf{G} = \mathbf{GSpin}_{2n}^\alpha$). It can be extended to give the local Langlands correspondence for all local parameters $\psi \in \Psi^+(\mathbf{G})$ with $\psi|_{\mathrm{SL}_2(\mathbb{C})} = 1$; indeed if Conjecture 2.4.2 is known for all $\mathbf{G}$, then a version can be deduced for $\Psi^+(\mathbf{G})$ using the Langlands classification (see [Lan89], [Sil78] and [SZ14]).

Remark 2.4.6. In the case where F is Archimedean and for an arbitrary reductive group the local Langlands correspondence was established by Langlands and Shelstad (see [She10], [She08]). Compatibility with twisted endoscopy was proved by Mezo [Mez16] (under a minor assumption, see (3.10) loc. cit., which is satisfied in all cases considered in the present article) up to a constant which a priori might depend on the parameter (see [AMgR18, Annexe C]).

Remark 2.4.7. If F is p-adic and $\mathbf{G}$ is unramified over F, then there is a unique $\mathbf{G}(F)$-conjugacy class of hyperspecial maximal compact subgroups of $\mathbf{G}(F)$ which is compatible with the Whittaker datum fixed above (in the sense of [CS80]), and we will say that a representation of $\mathbf{G}(F)$ is unramified if it is unramified with respect to a subgroup in this conjugacy class.

If $\psi \in \widetilde{\Psi}^+(G)$ and $\psi|_{\mathrm{WD}_F}$ is unramified, then assuming the conjecture the packet Ψ_ψ contains a unique unramified (orbit of) representation. It has Satake parameter φ_ψ (up to outer conjugation if $\mathbf{G} = \mathbf{GSpin}_{2n}^\alpha$) and corresponds to the trivial character on $\mathcal{S}_\psi$. This follows from the fundamental lemma (Theorem 2.3.3).

Remark 2.4.8. By [Mœg11] if F is p-adic and the conjecture holds then the packets Π_ψ are sets rather than multi-sets.

2.5. **Global parameters and the conjectural multiplicity formula.** Now let F be a number field, and fix a continuous unitary character $\chi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$. If π is a cuspidal automorphic representation of $\mathbf{GL}_N/F$ such that $\pi^\vee \otimes (\chi \circ \det) \cong \pi$, then we say that π is χ-self dual. Note that this implies that $\omega_\pi^2 = \chi^N$ (so in particular if N is odd, then $\chi = (\omega_\pi \chi^{(1-N)/2})^2$ is a square).

If π is χ-self dual and S is a big enough set of places of F then precisely one of the L-functions $L^S(s, \chi^{-1} \otimes \bigwedge^2(\pi))$ and $L^S(s, \chi^{-1} \otimes \mathrm{Sym}^2(\pi))$ has a pole at $s = 1$, and this pole is simple (see [Sha97]). In the former case we say that (π, χ) is of

symplectic type, and set $\mathrm{sign}(\pi,\chi) = -1$, and in the latter we say that it is of orthogonal type, and we set $\mathrm{sign}(\pi,\chi) = 1$.

We write $\Psi(\widetilde{\mathbf{GL}_N \times \mathbf{GL}_1}, \chi)$ for the set of formal unordered sums $\psi = \boxplus_i \pi_i[d_i]$, where the π_i are χ-self dual automorphic representations for $\mathbf{GL}_{N_i}/F$ and the $d_i \geq 1$ are integers (which are to be thought of as the dimensions of irreducible algebraic representations of $\mathrm{SL}_2(\mathbb{C})$), with the property that $\sum_i N_i d_i = N$. We refer to such a sum as a *parameter*, and say that it is *discrete* if the (isomorphism classes of) pairs (π_i, d_i) are pairwise distinct.

Remark 2.5.1.

(1) By the main result of [MW89], a discrete automorphic representation π of $\mathbf{GL}_N/F$ with $\pi^\vee \otimes (\chi \circ \det) \cong \pi$ gives rise to an element of $\Psi(\widetilde{\mathbf{GL}_N \times \mathbf{GL}_1}, \chi)$. Indeed, there is a natural bijection between such representations π and the elements of $\Psi(\widetilde{\mathbf{GL}_N \times \mathbf{GL}_1}, \chi)$ of the form $\pi[d]$ (that is, the elements where the formal sum consists of a single term). We will use this bijection without further comment below.

(2) The set of formal parameters $\Psi(\widetilde{\mathbf{GL}_N \times \mathbf{GL}_1}, \chi)$ that we consider does not contain all non-discrete χ-self-dual parameters, for example those containing a summand of the form $\pi \boxplus ((\chi \circ \det) \otimes \pi^\vee)$ for a non-χ-self-dual cuspidal automorphic representation π for $\mathbf{GL}_m$. Our ad hoc definition will turn out to be convenient when we will consider the discrete part of (the stabilisation of) trace formulas.

Definition 2.5.2. Let $\mathbf{G} = \mathbf{GSpin}_{2n+1}$, $\mathbf{Sp}_{2n} \times \mathbf{GL}_1$ or $\mathbf{GSpin}^\alpha_{2n}$ over F. We let $\widetilde{\Psi}_{\mathrm{disc}}(\mathbf{G}, \chi)$ be the subset of $\widetilde{\Psi}(\mathbf{GL}_{N(\mathbf{G})}, \chi)$ given by those $\psi = \boxplus_i \pi_i[d_i]$ with the properties that

- ψ is discrete,
- for each i, we have $\mathrm{sign}(\pi_i, \chi) = (-1)^{d_i - 1}\mathrm{sign}(\mathbf{G})$,
- if $\mathbf{G} = \mathbf{GSpin}^\alpha_{2n}$, then $\chi^{-n} \prod_i \omega_{\pi_i}^{d_i}$ is the quadratic character corresponding to the extension F_α/F.

(Conditions analogous to this last bullet point could be formulated for the other groups $\mathbf{G}$, but in fact they are conjecturally automatically satisfied.)

If $\mathbf{G} \neq \mathbf{GSpin}^\alpha_{2n}$ we also let $\Psi_{\mathrm{disc}}(\mathbf{G}, \chi) = \widetilde{\Psi}_{\mathrm{disc}}(\mathbf{G}, \chi)$. The reason for writing $\widetilde{\Psi}$ in the case of even **GSpin** groups is that this set only sees orbits of (substitutes for) Arthur-Langlands parameters under outer conjugation.

As a particular case of the above definition, for π a cuspidal automorphic representation for $\mathbf{GL}_N/F$ such that $(\chi \circ \det) \otimes \pi^\vee \simeq \pi$ there is a unique group $\mathbf{G}$ as above such that $N(\widehat{\mathbf{G}}) = N$ and $\pi[1] \in \widetilde{\Psi}_{\mathrm{disc}}(\mathbf{G})$.

Conjecture 2.5.3. *For π and $\mathbf{G}$ as above and for each place v of F, the representation $(\mathrm{rec}(\pi_v), \mathrm{rec}(\chi_v))$ factors through $\mathrm{Std}_{\mathbf{G}} : {}^L\mathbf{G} \to \mathrm{GL}_{N(\widehat{\mathbf{G}})}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$, so that by Lemma 2.4.1 we can regard (π_v, χ_v) as an element of $\widetilde{\Psi}^+(\mathbf{G}(F_v))$.*

Remark 2.5.4.

(1) This conjecture is the analogue of [Art13, Theorem 1.4.1] (reformulated using Theorem 1.5.3 loc. cit.). In particular it holds for $\mathbf{G} = \mathbf{Sp}_{2n} \times \mathbf{GL}_1$.

(2) Since we do not know the generalised Ramanujan conjecture for $\mathbf{GL}_n$, and do not wish to assume it, we can at present only hope to establish that the

local parameters ψ_v are elements of $\widetilde{\Psi}^+(\mathbf{G}_{F_v})$; they are, however, expected to be elements of $\widetilde{\Psi}(\mathbf{G}_{F_v})$.

Given a global parameter $\psi \in \widetilde{\Psi}_{\mathrm{disc}}(\mathbf{G}, \chi)$, we define groups C_ψ, S_ψ, $\mathcal{S}_\psi$ as follows. For each i, there is a unique group $\mathbf{G}_i$ of the kind we are considering for which $\pi_i \in \widetilde{\Psi}_{\mathrm{disc}}(\mathbf{G}_i, \chi)$. We let $\mathcal{L}_\psi$ denote the fibre product of the ${}^L\mathbf{G}_i$ over W_F. Then there is a map $\dot{\psi} : \mathcal{L}_\psi \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{G}$ such that $\mathrm{Std}_{\mathbf{G}} \circ \dot{\psi}$ is conjugate to $\oplus_i \mathrm{Std}_{\mathbf{G}_i} \otimes \nu_{d_i}$, where ν_{d_i} is the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension d_i. The map $\dot{\psi}$ is well-defined up to the action of $\mathrm{Aut}({}^L\mathbf{G})$. We let C_ψ be the centraliser of $\dot{\psi}$, and similarly define S_ψ and $\mathcal{S}_\psi$.

For each finite place v, under Conjecture 2.5.3 (applied to the π_i's) we may form a local Arthur-Langlands parameter $\psi_v^0 : \mathrm{WD}_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathcal{L}_\psi$. Composing with $\dot{\psi}$, we obtain $\psi_v \in \widetilde{\Psi}^+(\mathbf{G}_{F_v})$. The composition of ψ_v with $\mathrm{Std}_{\mathbf{G}}$ is given by

- χ_v on the GL_1 factor,
- the direct sum of the representations $\varphi_{\pi_i,v} \otimes \nu_{d_i}$ on the $\mathrm{GL}_{N(\widehat{\mathbf{G}})}$ factor, where $\varphi_{\pi_i,v} = \mathrm{rec}(\pi_{i,v})$.

Conjecture 2.5.6 below makes precise the expectation that the elements of the corresponding multi-sets Π_{ψ_v} of Conjecture 2.4.2 are the local factors of the discrete automorphic representations of $\mathbf{G}$ with multiplier χ. Before stating it, we need to introduce some more notation and terminology.

For each place v of F, write $\widetilde{\mathcal{H}}(\mathbf{G}_v)$ for the Hecke algebra defined after Definition 2.3.2, and write $\widetilde{\mathcal{H}}(\mathbf{G})$ for the restricted tensor product of the $\widetilde{\mathcal{H}}(\mathbf{G}_v)$. Assuming Conjecture 2.5.3, we have an obvious map $\mathcal{S}_\psi \to \mathcal{S}_{\psi_v}$ for each v, and we can associate to ψ a global packet (a multi-set) of representations of $\widetilde{\mathcal{H}}(\mathbf{G})$:

$$\widetilde{\Pi}_\psi := \{\otimes'_v \pi_v : \pi_v \in \Pi_{\psi_v} \text{ with } \pi_v \text{ unramified for all but finitely many } v\}.$$

For each $\pi \in \widetilde{\Pi}_\psi$, we have the associated character on $\mathcal{S}_\psi$,

$$\langle x, \pi \rangle := \prod_v \langle x_v, \pi_v \rangle$$

(note that by Remark 2.4.7, we have $\langle \cdot, \pi_v \rangle = 1$ for all but finitely many v, so this product makes sense).

Associated to each ψ is a character $\varepsilon_\psi : \mathcal{S}_\psi \to \{\pm 1\}$ which can be defined explicitly in terms of symplectic ε-factors. In the case $\chi = 1$ this is defined in [Art13, Theorem 1.5.2], and this definition can be extended to the case of general χ without difficulty. Since we will only need the case $\mathbf{G} = \mathrm{GSpin}_5$ in this paper, and in this case the characters ε_ψ are given explicitly in [Art04] and are recalled below in Remark 6.1.8, we do not give the general definition here.

Definition 2.5.5. $\widetilde{\Pi}_\psi(\varepsilon_\psi)$ is the subset of $\widetilde{\Pi}_\psi$ consisting of those elements for which $\langle \cdot, \pi \rangle = \varepsilon_\psi$.

This is the correct definition only because the groups $\mathcal{S}_{\psi_v}$ are all abelian.

Recall that we have fixed a maximal compact subgroup K_∞ of $\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$ in Section 2.3. Let $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Lie}\,(\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R}))$. We write $\mathcal{A}^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi)$ for the space of χ-equivariant (where the action of $\mathbb{A}_F^\times / F^\times$ is via μ) square integrable automorphic forms on $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)$. It decomposes discretely under the action of $\mathbf{G}(\mathbb{A}_{F,f}) \times (\mathfrak{g}, K_\infty)$.

Conjecture 2.5.6. *Assume that Conjectures 2.4.2 and 2.5.3 hold. Then there is an isomorphism of $\widetilde{\mathcal{H}}(\mathbf{G})$-modules*

$$\mathcal{A}^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F),\chi) \cong \bigoplus_{\psi\in\widetilde{\Psi}_{\mathrm{disc}}(\mathbf{G},\chi)} m_\psi \bigoplus_{\pi\in\widetilde{\Pi}_\psi(\varepsilon_\psi)} \pi,$$

where $m_\psi = 1$ unless $\mathbf{G} = \mathbf{GSpin}^\alpha_{2n}$, in which case $m_\psi = 2$ if and only if each $N_i d_i$ is even.

2.6. **The results of** [Art13]**.** As we have already remarked, the conjectures above are all proved in [Art13] in the case that $\chi = 1$. As we now explain, the case that χ is a square follows immediately by a twisting argument. The main results of this paper are a proof of Conjectures 2.4.2 (Theorem 3.1.1) and 2.5.6 (Theorem 7.4.1) in the case that $\mathbf{G} = \mathbf{GSpin}_5 \cong \mathbf{GSp}_4$ for general χ. Conjecture 2.5.3 for $\mathbf{G} = \mathbf{GSpin}_5$ is a consequence of [GT11a], see Proposition 7.3.1. The case that χ is a square will be a key ingredient in our arguments, as if χ is not a square, then it is easy to see that there are considerably fewer possibilities for the parameters ψ, and this will reduce the number of ad hoc arguments that we need to make. Moreover in the remaining cases, the statements pertaining to local tempered representations are covered by [CG15].

Theorem 2.6.1 (Arthur)**.** *If $\chi = \eta^2$ is a square, then Conjectures 2.4.2, 2.5.3 and 2.5.6 hold.*

Proof. Given a χ-self dual cuspidal automorphic representation π, the twist $\pi\otimes(\eta\circ\det)^{-1}$ is self dual. Similarly, we may twist the local parameters by the restriction to W_{F_v} of the character corresponding to η^{-1}, and we can also twist representations of $\mathbf{G}(F)$ and $\mathbf{G}(F_v)$ by η^{-1}. All of the conjectures are easily seen to be compatible with these twists, so we reduce to the case $\chi = 1$. In this case, representations of $\mathbf{GSpin}_{2n+1}$, (resp. $\mathbf{GSpin}^\alpha_{2n}$, resp. $\mathbf{Sp}_{2n}\times\mathbf{GL}_1$) with trivial similitude factor (recall that this was defined in Section 2.1 as the composition of the central character with μ) are equivalent to representations of $\mathbf{SO}_{2n+1}$, (resp. representations of $\mathbf{SO}^\alpha_{2n}$, resp. pairs given by a representation of $\mathbf{Sp}_{2n}$ and a character of $\mathbf{GL}_1$ of order 1 or 2), so the conjectures are equivalent to the main results of [Art13]. □

In particular, since in the case $\mathbf{G} = \mathbf{Sp}_{2n}\times\mathbf{GL}_1$ the character χ is always a square, Theorem 2.6.1 always holds in this case.

2.7. **Low rank groups.** If $N(\widehat{\mathbf{G}}) \le 3$ then Conjectures 2.4.2, 2.5.3 and 2.5.6 also hold unconditionally.

(1) If $N = 1$ the results are tautological.
(2) if $N = 2$ then $\mathbf{G} = \mathbf{GSpin}_3$ or $\mathbf{G} = \mathbf{GSpin}^\alpha_2$. In the first case $\mathbf{G} \simeq \mathbf{GL}_2$ and the results are also tautological. In the second case where $\mathbf{G} = \mathbf{GSpin}^\alpha_2 \simeq \mathrm{Res}_{F(\sqrt{\alpha})/F}(\mathbf{GL}_1)$ we are easily reduced to the well-known Theorem 2.7.1 below, the symplectic/orthogonal alternative for $\mathbf{GL}_2$.
(3) If $N = 3$ then $\mathbf{G} = \mathbf{Sp}_2\times\mathbf{GL}_1$ and we are reduced to a special case of Theorem 2.6.1. Note that the local Langlands correspondence and the multiplicity formula in this case go back to Labesse–Langlands [LL79] and [Ram00].

Theorem 2.7.1. *Let π be a χ-self dual cuspidal automorphic representation of $\mathbf{GL}_2$. Then either*

(1) $\chi = \omega_\pi$, *and* $L^S(s, \bigwedge^2(\pi) \otimes \chi^{-1})$ *has a pole at* $s = 1$*; or*
(2) $\omega_\pi\chi^{-1}$ *is the quadratic character given by some quadratic extension* E/F*,* π *is the automorphic induction of a character of* $\mathbb{A}_E^\times/E^\times$ *which is not fixed by the non-trivial element of* $\mathrm{Gal}(E/F)$*, and* $L^S(s, \mathrm{Sym}^2(\pi) \otimes \chi^{-1})$ *has a pole at* $s = 1$*.*

Proof. Certainly $L^S(s, \bigwedge^2(\pi) \otimes \chi^{-1}) = L^S(s, \omega_\pi\chi^{-1})$ has a pole at $s = 1$ if and only if $\chi = \omega_\pi$. So if $L^S(s, \mathrm{Sym}^2(\pi) \otimes \chi^{-1})$ has a pole at $s = 1$, we see that $\omega_\pi\chi^{-1}$ is a non-trivial quadratic character corresponding to an extension E/F. Since we always have $\pi^\vee \otimes (\omega_\pi \circ \det) \cong \pi$, this implies that $\pi \cong \pi \otimes (\omega_\pi\chi^{-1} \circ \det)$, and it follows (see [Lan80, end of §2]) that π is the automorphic induction of a character of $\mathbb{A}_E^\times/E^\times$ which is not fixed by the non-trivial element of $\mathrm{Gal}(E/F)$. □

2.8. **The local Langlands correspondence for $\mathbf{GSp}_4$.** Let F be a p-adic field. The local Langlands correspondence for $\mathbf{GSp}_4(F)$ was established in [GT11a], but was characterised by relations with γ-factors, rather than endoscopic character relations. The necessary endoscopic character relations were then proved in [CG15]. In particular, we have:

Theorem 2.8.1 (Chan–Gan)**.** *If* F *is a* p*-adic field then Conjecture 2.4.2 holds for* $\mathbf{GSpin}_5$ *and parameters* ψ *which are trivial on* $\mathrm{SL}_2(\mathbb{C})$*, i.e. tempered Langlands parameters.*

Proof. Parts (1) and (2) of Conjecture 2.4.2 are an immediate consequence of the main theorem of [CG15] (note that bounded parameters are automatically generic, in the sense that their adjoint L-functions are holomorphic at $s = 1$). Part (3) then follows from the main theorem of [GT11a]. □

Remark 2.8.2. Recall from Remark 2.4.6 that over an Archimedean field the local Langlands correspondence and (ordinary) endoscopic character relations are known in complete generality, and the twisted endoscopic character relations are known up to a constant (which might depend on the parameter).

If F is Archimedean and ψ is a tempered and non discrete Langlands parameter for $\mathbf{GSpin}_5$, then the twisted endoscopic character relation was verified in [CG15, §6], which amounts to saying that the above constant (the only ambiguity in Mezo's theorem) is 1. In Proposition 7.2.1 below we will show using a global argument as in [AMgR18, Annexe C] that this also holds for the discrete tempered ψ.

3. Construction of missing local Arthur packets for $\mathbf{GSpin}_5$

3.1. **Local packets.** Let F be a local field of characteristic zero. In this section we complete the proof of the following theorem, which completes the proof of Conjecture 2.4.2 for $\mathbf{GSpin}_5$.

Theorem 3.1.1. *Let* $\psi : \mathrm{WD}_F \times \mathrm{SL}_2 \to \mathrm{GSp}_4$ *be an element of* $\Psi(\mathbf{GSpin}_5)$*. Then there is a unique multi-set* Π_ψ *of irreducible smooth unitary representations of* $\mathbf{GSpin}_5(F)$*, together with a map* $\Pi_\psi \to \mathcal{S}_\psi^\vee$*, which we will simply denote by* $\pi \mapsto \langle \cdot, \pi \rangle$*, such that the following holds:*

(1) *Let* $\pi_\psi^{\mathbf{\Gamma}}$ *be the representation of* $\mathbf{\Gamma}(F)$ *associated to* $\mathrm{Std}_{\mathbf{GSpin}_5} \circ \varphi_\psi$ *by the local Langlands correspondence, and let* $\pi_\psi^{\widetilde{\mathbf{\Gamma}}}$ *be its extension to* $\widetilde{\mathbf{\Gamma}}(F)$ *(Whittaker-normalised as explained in Section 3.2). Then the linear form* $\sum_{\pi \in \Pi_\psi} \langle s_\psi, \pi \rangle \operatorname{tr} \pi$ *on* $I(\mathbf{GSpin}_5(F))$ *is stable and its transfer to* $\widetilde{\mathbf{\Gamma}}$ *is* $\operatorname{tr} \pi_\psi^{\widetilde{\mathbf{\Gamma}}}$*.*

(2) *Consider a semisimple* $s \in \mathrm{Cent}(\psi, \mathrm{GSp}_4)$*, and denote by* $\bar{s}$ *its image in* $\mathcal{S}_\psi$*. The pair* (ψ, s) *determines an endoscopic datum* $(\mathbf{H}, \mathcal{H}, s, \xi)$ *for* $\mathbf{GSpin}_5$*, as well as* $\psi' : \mathrm{WD}_F \times \mathrm{SL}_2 \to \widehat{\mathbf{H}}$ *such that* $\psi = \xi \circ \psi'$*. Then for any* $f \in I(\mathbf{GSpin}_5(F))$ *we have*

$$\sum_{\pi \in \Pi_\psi} \langle \bar{s} s_\psi, \pi \rangle \operatorname{tr} \pi(f) = \sum_{\pi' \in \Pi_{\psi'}} \langle s_{\psi'}, \pi' \rangle \operatorname{tr} \pi'(f').$$

Note that in the second point $\mathbf{H}$ is either $\mathbf{GSpin}_5$ or a quotient of a product of general linear groups by a split torus, and so $\Pi_{\psi'}$ is well-defined. In the latter case it is a singleton and $\mathcal{S}_{\psi'}$ is trivial.

As we recalled above (Theorems 2.6.1, 2.8.1 and Remark 2.8.2) this theorem is already known in the following cases:

- if $\widehat{\mu} \circ \psi$ is a square,
- if F is p-adic and $\psi|_{\mathrm{SL}_2} = 1$,
- if F is Archimedean, $\psi|_{\mathrm{SL}_2}$ and ψ is not discrete.

We will prove the case where F is Archimedean, ψ tempered discrete and χ not a square later in Proposition 7.2.1, since we will use a global argument using the stabilisation of the trace formula.

This section is devoted to the proof of Theorem 3.1.1 in the remaining cases, where $\psi|_{\mathrm{SL}_2}$ is not trivial and $\widehat{\mu} \circ \psi$ is not a square. It is easy to see that $\mathrm{Std}_{\mathbf{GSpin}_5} \circ \psi \simeq (\varphi[2], \chi)$, where $\varphi : \mathrm{WD}_F \to \mathrm{GL}_2$ is χ-self-dual of orthogonal type. Then φ factors through W_F and $\det \varphi / (\widehat{\mu} \circ \psi)$ has order 1 or 2. There are two cases to consider.

(1) If φ is irreducible then $\det \varphi / (\widehat{\mu} \circ \psi)$ has order 2. Let E/F be the corresponding quadratic extension and denote c the non-trivial element of $\mathrm{Gal}(E/F)$. We have $\varphi \simeq \mathrm{Ind}_{E/F} \mu$ for a character $\mu : E^\times \to \mathbb{C}^\times$ such that $\mu^c \neq \mu$ and $\mu|_{F^\times} = \chi$. Then $\mathrm{Cent}(\psi, \mathrm{GSp}_4) = Z(\mathrm{GSp}_4)$ and so we simply have to produce $\Pi_\psi = \{\pi\}$ such that $\operatorname{tr} \pi$ transfers to the trace of $\pi_\psi^{\widetilde{\mathbf{\Gamma}}}$.

(2) If φ is reducible then $\varphi = \eta_1 \oplus \eta_2$ with $\eta_1 \eta_2 = \chi$ and $\eta_1 \neq \eta_2$. Then $\mathrm{Cent}(\psi, \mathrm{GSp}_4) = \{\mathrm{diag}(u_1 I_2, u_2 I_2)\}$ and so we are led to define $\Pi_\psi = \{\mathrm{Ind}_{\mathbf{L}}^{\mathbf{GSpin}_5}((\mathrm{rec}(\eta_1) \circ \det) \otimes \mathrm{rec}(\chi))\}$ where $\mathbf{L} \simeq \mathbf{GL}_2 \times \mathbf{GSpin}_1$. Then the second point in Theorem 3.1.1 is automatically satisfied (see [CG15, §6.6]), and again we have to check that the twisted endoscopic character relation holds.

We will prove these two cases separately, distinguishing between the cases where F is p-adic, real, or complex (in which case only the second case occurs). Before doing so, we recall some material on Whittaker normalisations.

3.2. **Whittaker normalisation for general linear groups.** In this section F denotes a local field of characteristic zero, $\mathbf{G} = \mathbf{GL}_n \times \mathbf{GL}_1$ over F and $\widetilde{\mathbf{G}} = \mathbf{G} \rtimes \theta$. Following [MW06, §5], [Sha10], [AMgR18, §8] we briefly recall the Whittaker normalisation of extensions to $\widetilde{\mathbf{G}}(F)$ of irreducible representations of $\mathbf{G}(F)$ fixed by θ. Recall that we have fixed a θ-stable Whittaker datum $(\mathbf{U}, \lambda)$ for $\mathbf{G}$. If F is Archimedean for simplicity we choose the maximal compact subgroup K to be $\mathrm{O}_n(F) \times \{\pm 1\}$ (resp. $\mathrm{U}(n) \times \mathrm{U}(1)$) if F is real (resp. complex), so that $\theta(K) = K$.

First consider the case of essentially tempered representations. Let π be an essentially tempered (in particular, essentially unitary) irreducible representation of $\mathbf{G}(F)$. By [Sha74] there exists a continuous Whittaker functional Ω for π. If F

is p-adic this is just an element of the algebraic dual of the space π_K of smooth vectors. If F is Archimedean this is a continuous functional on the space π_∞ of smooth vectors for the topology defined by seminorms as in [Sha74, p. 183]. Now if π is fixed by θ, define $\tilde{\pi}(\theta)$ as the unique element $A \in \mathrm{Isom}(\pi, \pi^\theta)$ such that $\Omega \circ A = \Omega$. This does not depend on the choice of Ω. So we have an extension $\tilde{\pi}$ of π to a representation of $\widetilde{\mathbf{G}}(F)$, well-defined using the Whittaker datum $(\mathbf{U}, \lambda)$.

Next consider representations parabolically induced from a θ-stable parabolic subgroup. Fix the usual (diagonal) split maximal torus $\mathbf{T}$ of $\mathbf{G}$, as well as the usual (upper triangular) Borel subgroup $\mathbf{B} = \mathbf{TU}$ of $\mathbf{G}$. Both are θ-stable. Let $w_{\mathbf{G}}$ be the longest element of the Weyl group $W(\mathbf{T}, \mathbf{G})$. Let $\mathbf{P} = \mathbf{MN}$ be a standard parabolic subgroup of $\mathbf{G}$, with standard Levi subgroup $\mathbf{M} \supset \mathbf{T}$. Assume that $\mathbf{P}$ is θ-stable, which means that $\mathbf{M} = (\mathbf{GL}_{n_1} \times \cdots \times \mathbf{GL}_{n_r}) \times \mathbf{GL}_1$ (block diagonal) with $n_i = n_{r+1-i}$ for all i. Let σ be an irreducible admissible representation of $\mathbf{M}(F)$ fixed by θ, that is $\sigma \simeq (\sigma_1 \otimes \cdots \otimes \sigma_r) \otimes \chi$ with $(\chi \circ \det) \otimes \sigma_i^\vee \simeq \sigma_{r+1-i}$ for all i. Let $\mathbf{D}_{\mathbf{M}}$ be the largest split torus which is a quotient of $\mathbf{M}$, so that we have a canonical isogeny $\mathbf{A}_{\mathbf{M}} \to \mathbf{D}_{\mathbf{M}}$. In the present case we have a natural identification $\mathbf{D}_{\mathbf{M}} \simeq \mathbf{GL}_1^r \times \mathbf{GL}_1$ via the determinants $\mathbf{GL}_{n_i} \to \mathbf{GL}_1$. For $\nu \in X^*(\mathbf{D}_{\mathbf{M}}) \otimes \mathbb{C}$ inducing a character of $\mathbf{M}(F)$, consider the parabolically induced (normalised) representation $\pi_\nu := \mathrm{Ind}_{\mathbf{P}(F)}^{\mathbf{G}(F)} \sigma \otimes \nu$. We also assume that $\nu = (\nu_1, \ldots, \nu_r, \nu_0)$ is fixed by θ, i.e. $\nu_i + \nu_{r+1-i} = \nu_0$ for all i. Let $w_{\mathbf{M}}$ be the longest element of $W(\mathbf{T}, \mathbf{M})$ (for $\mathbf{B} \cap \mathbf{M}$) and $w = w_{\mathbf{G}} w_{\mathbf{M}}$. Let $\mathbf{P}^- = \mathbf{MN}^-$ be the parabolic subgroup of $\mathbf{G}$ opposite to $\mathbf{P}$ with respect to $\mathbf{M}$, and let $\mathbf{P}' = \mathbf{M}'\mathbf{N}' = w\mathbf{P}^- w^{-1} = w_{\mathbf{G}} \mathbf{P}^- w_{\mathbf{G}}^{-1}$ be the standard parabolic subgroup conjugated to $\mathbf{P}^-$. Choose a lift $\tilde{w}$ of w in $N_{\mathbf{G}(F)}(\mathbf{T})$. Let $\lambda_{\mathbf{M}}^{\tilde{w}} : (\mathbf{M} \cap \mathbf{U})(F) \to S^1$ be the generic character defined by $\lambda_{\mathbf{M}}^{\tilde{w}}(u) = \lambda(\tilde{w} u \tilde{w}^{-1})$. Assume that the space $\mathrm{Hom}_{(\mathbf{M}\cap\mathbf{U})(F)}(\sigma, \lambda_{\mathbf{M}}^{\tilde{w}})$ of Whittaker functionals for σ with respect to $\lambda_{\mathbf{M}}^{\tilde{w}}$ is non-zero and thus one-dimensional, and fix a basis Ω_σ of this line. In the p-adic case, according to a theorem of Rodier ([Rod73], [CS80], explained in [Sha10, §3.4]) we then have that $\mathrm{Hom}_{\mathbf{U}(F)}(\mathrm{Ind}_{\mathbf{P}(F)}^{\mathbf{G}(F)}(\sigma \otimes \nu), \lambda)$ also has dimension one. A basis Ω_{π_ν} can be made explicit: for f in the space of $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}} \sigma \otimes \nu$ whose support is contained in the big cell $\mathbf{P}(F) w^{-1} \mathbf{U}(F)$,

$$\Omega_{\pi_\nu}(f) := \int_{\mathbf{N}'(F)} \Omega_\sigma(f(\tilde{w}^{-1} n)) \lambda(n)^{-1} dn \tag{3.2.1}$$

is well-defined (the integrand is smooth and compactly supported). For arbitrary f the same formula holds with $\mathbf{N}'(F)$ replaced by large enough open compact subgroup which depends on f but not on ν (as usual realising the vector space underlying $\mathrm{Ind}_{\mathbf{P}(F)}^{\mathbf{G}(F)} \sigma \otimes \nu$ independently of ν by restriction to K), so that $\nu \mapsto \Omega_{\pi_\nu}(f)$ is holomorphic.

The Archimedean case is more subtle, since the notion of Whittaker functional requires a topology on the underlying space of the representation to be well-behaved (it is not defined directly on $(\mathfrak{g}, K)$-modules). So in this case one considers the smooth parabolically induced representation $\pi_\nu := \mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\sigma_\infty \otimes \nu)$, whose subspace $\pi_{\nu,K}$ of K-finite vectors is naturally isomorphic to the $(\mathfrak{g}, K)$-module algebraically induced from $\sigma_{\mathbf{M}(F)\cap K}$ (see [BW00, §III.7]). Assume that the central character of σ is unitary. Then the integral (3.2.1) is absolutely convergent for $\nu \in X^*(\mathbf{D}_{\mathbf{M}}) \otimes \mathbb{C}$ satisfying

$$\forall \alpha \in \Phi(\mathbf{T}, \mathbf{N}),\ \langle \alpha^\vee, \Re \nu \rangle > 0, \tag{3.2.2}$$

and extends analytically to $X^*(\mathbf{D_M}) \otimes \mathbb{C}$ ([Sha10, Theorem 3.6.4]). The proof of Theorem 3.6.7 in [Sha10] also shows uniqueness (up to a scalar) of a Whittaker functional for $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\sigma_\infty \otimes \nu)$ (note that the argument for uniqueness only involves the Jordan–Hölder factors of a principal series representation, and so one may replace $\mathbf{P}$ by another parabolic subgroup of $\mathbf{G}$ admitting $\mathbf{M}$ as a Levi factor and such that the opposite of (3.2.2) is satisfied, so that any generic subquotient of $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\sigma \otimes \nu)$ appears as a quotient).

We can now treat the p-adic and Archimedean cases together. Assume that ν is chosen so that $\mathrm{End}_{\mathbf{G}(F)}(\pi_\nu) = \mathbb{C}$. This is the case if the central character of σ is unitary and ν satisfies (3.2.2) (this follows from the fact that π_ν then has a unique irreducible quotient which occurs with multiplicity one in its composition series), or if $-\nu$ satisfies (3.2.2) (π_ν then has a unique irreducible subrepresentation). Then one can define the action of θ on π_ν to be the unique $A_\theta \in \mathrm{End}(\pi_\nu)$ such that $A_\theta \circ \pi_\nu(g) = \pi_\nu(\theta(g)) \circ A_\theta$ for all $g \in \mathbf{G}(F)$ and $\Omega_{\pi_\nu} \circ A = \Omega_{\pi_\nu}$. This can be made more explicit in the case at hand, see [MW06, §5.2]. The operator A_θ does not depend on the choice of $\tilde{w}$ made above.

For this definition we followed [AMgR18, §8]. As explained there, the resulting canonical extension of π_ν coincides with the extension defined by Arthur in [Art13, §2.2], by [MW06, §5.2] and analytic continuation (see [AMgR18, Remarque 8.3]).

Finally, consider an arbitrary irreducible smooth representation π of $\mathbf{G}(F)$ (admissible $(\mathfrak{g}, K)$-module in the Archimedean case). By the Langlands classification ([Lan89, Lemmas 3.14 and 4.2], [Sil78], [BW00, Chapter IV]), π is the unique irreducible quotient of $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\sigma \otimes \nu)$ (resp. unique irreducible subrepresentation of $\mathrm{Ind}_{\mathbf{P}^-}^{\mathbf{G}}(\sigma \otimes \nu)$) for $\nu \in X^*(\mathbf{D_M}) \otimes \mathbb{C}$ satisfying (3.2.2), with σ tempered (in particular, with unitary central character) and the pair $(\mathbf{P}, \sigma \otimes \nu)$ is well-defined up to conjugation. These two realisations of π as quotient (resp. subrepresentation) of a parabolically induced representation give two canonical extensions of π to $\widetilde{\mathbf{G}}$, by the above. In fact these two canonical extensions coincide: consider the composition

$$\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\sigma \otimes \nu) \to \pi \to \mathrm{Ind}_{\mathbf{P}^-}^{\mathbf{G}}(\sigma \otimes \nu)$$

which is clearly non-zero. From the properties of these induced representations mentioned above it follows that $\dim \mathrm{Hom}_{\mathbf{G}(F)}(\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}}(\sigma \otimes \nu), \mathrm{Ind}_{\mathbf{P}^-}^{\mathbf{G}}(\sigma \otimes \nu)) \leq 1$. Therefore the above composition coincides with the usual intertwining operator [Wal03, Théorème IV.1.1], [VW90] (up to a scalar and a normalising factor to make this intertwining operator holomorphic at ν). But this operator varies analytically if we vary ν, and generically it is an isomorphism between irreducible parabolically induced representations, thus generically it intertwines the two A_θ's, and by analytic continuation this also holds for the original ν.

3.3. **Proof of Theorem 3.1.1.** We now prove Theorem 3.1.1 in the cases described at the end of Section 3.1.

Proof in the first case for F p-adic. The proof is a very special case of the generalisation of [MW06, Théorème 4.7.1] to *essentially* self-dual representations. See also [Mœg06].

Let ρ be the supercuspidal representation of $\mathbf{GL}_2(F)$ such that $\mathrm{rec}(\rho) = \varphi$. Then $(\chi \circ \det) \otimes \rho^\vee \simeq \rho$. We will give an ad hoc definition of Π_ψ, using special cases of results of [MW06] to check compatibility with twisted endoscopy for $\mathbf{GL}_4 \times \mathbf{GL}_1$. In [MW06] Mœglin and Waldspurger consider self-dual parameters, and we will argue

that their arguments extend to the case at hand without substantial modification, the essential input being compatibility of local Langlands for $\mathbf{GSpin}_5$ for twisted endoscopy (and the same for $\mathbf{GSpin}_3$ and $\mathbf{GSpin}_1$, which is trivial).

Let Δ be the diagonal embedding $\mathrm{SU}(2) \hookrightarrow \mathrm{SU}(2) \times \mathrm{SL}_2(\mathbb{C})$, so that $\psi \circ \Delta$ is the essentially tempered Langlands parameter obtained by tensoring φ with the 2-dimensional irreducible representation of the factor $\mathrm{SU}(2)$ of WD_F. Then $\mathrm{Cent}\,(\psi \circ \Delta, \mathrm{GSp}_4) = Z(\mathrm{GSp}_4)$, and so $\Pi_{\psi\circ\Delta}$ (as defined by Gan–Takeda in [GT11a]) consists of a single irreducible discrete series representation $\pi_{\psi\circ\Delta}$ of $\mathbf{GSpin}_5(F)$. Let $\mathbf{P}$ be the standard parabolic subgroup of $\mathbf{GSpin}_5$ with Levi subgroup $\mathbf{L} \simeq \mathbf{GL}_2 \times \mathbf{GSpin}_1$ (conventions as in Section 2.2). Then $\mathrm{Jac}_{\mathbf{P}}(\pi_{\psi\circ\Delta}) = \rho|\det|^{1/2} \otimes \chi$ where Jac denotes the normalised Jacquet module. We briefly recall the proof. Let $\pi^{\mathbf{GL}}_{\psi\circ\Delta}$ be the (discrete series) representation of $\mathbf{GL}_4(F)$ corresponding to $\mathrm{pr}_1 \mathrm{Std} \circ \psi \circ \Delta : \mathrm{WD}_F \to \mathrm{GL}_4(\mathbb{C})$. Denoting by $\mathbf{P}^{\mathbf{GL}}$ the upper block triangular parabolic subgroup of $\mathbf{GL}_4$ with Levi subgroup $\mathbf{GL}_2 \times \mathbf{GL}_2$, it is well-known that $\mathrm{Jac}_{\mathbf{P}^{\mathbf{GL}}}\left(\pi^{\mathbf{GL}}_{\psi\circ\Delta}\right) = \rho|\det|^{1/2} \otimes \rho|\det|^{-1/2}$. Let $\pi^{\widetilde{\mathbf{\Gamma}}}_{\psi\circ\Delta}$ be the Whittaker-normalised (see Section 3.2 or [MW06, §5.1]) extension of $\pi^{\mathbf{GL}}_{\psi\circ\Delta} \otimes \chi$ to $\widetilde{\mathbf{\Gamma}}(F)$. By (iii) in the main theorem of [CG15] we have that $\mathrm{tr}\,\pi^{\widetilde{\mathbf{\Gamma}}}_{\psi\circ\Delta}$ is a transfer of $\mathrm{tr}\,\pi_{\psi\circ\Delta}$. The parabolic subgroup $\mathbf{P}^{\mathbf{GL}} \times \mathbf{GL}_1$ of $\mathbf{\Gamma}$ is stable under θ, write $\tilde{\mathbf{P}} = (\mathbf{P}^{\mathbf{GL}} \times \mathbf{GL}_1) \rtimes \theta$. By (an obvious generalisation of) [MW06, Lemme 4.2.1], $\mathrm{tr}\,\mathrm{Jac}_{\tilde{\mathbf{P}}}(\pi^{\widetilde{\mathbf{\Gamma}}}_{\psi\circ\Delta})$ is a transfer of $\mathrm{tr}\,\mathrm{Jac}_{\mathbf{P}}(\pi_{\psi\circ\Delta})$, and thus $\mathrm{Jac}_{\mathbf{P}}(\pi_{\psi\circ\Delta}) = \rho|\det|^{1/2} \otimes \chi$. By Frobenius reciprocity, $\pi_{\psi\circ\Delta}$ is naturally a subrepresentation of $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{GSpin}_5}\left(\rho|\det|^{1/2} \otimes \chi\right)$. By [BZ77, Theorem 2.8] this parabolic induction has length ≤ 2 and so the cokernel of

$$\pi_{\psi\circ\Delta} \hookrightarrow \mathrm{Ind}_{\mathbf{P}}^{\mathbf{GSpin}_5}\left(\rho|\det|^{-1/2} \otimes \chi\right)$$

is an irreducible Langlands quotient which we denote π_ψ. We let $\Pi_\psi = \{\pi_\psi\}$. Since $\mathrm{Cent}\,(\psi, \mathrm{GSp}_4(\mathbb{C})) = \mathbb{C}^\times$, we only have to check the twisted endoscopic character relation (Theorem 3.1.1 (1)). Following [MW06], this will be a consequence of comparing the short exact sequence

$$0 \to \pi_{\psi\circ\Delta} \to \mathrm{Ind}_{\mathbf{P}}^{\mathbf{GSpin}_5}\left(\rho|\det|^{1/2} \otimes \chi\right) \to \pi_\psi \to 0 \tag{3.3.1}$$

with a similar one for $\widetilde{\mathbf{\Gamma}}$.

We have a short exact sequence of representations of $\mathbf{\Gamma}(F) = \mathbf{GL}_4(F) \times \mathbf{GL}_1(F)$:

$$0 \to \pi^{\mathbf{GL}}_{\psi\circ\Delta} \otimes \chi \to \mathcal{E}_1(\pi^{\mathbf{GL}}_{\psi\circ\Delta}) \otimes \chi \to \pi^{\mathbf{GL}}_{\psi} \otimes \chi \to 0 \tag{3.3.2}$$

obtained as in [MW06, Prop. 3.1.2], by applying functorial constructions to $\pi^{\mathbf{GL}}_{\psi\circ\Delta}$ to get a resolution of $\pi^{\mathbf{GL}}_{\psi\circ\Delta}$ by sums of standard modules except possibly for the last term, which is defined as a cokernel and shown to be irreducible with Langlands parameter $(\psi \circ \Delta)^\sharp = \psi$ (the general definition of $\psi^\sharp$ is given in [MW06, §3.1.2]). The definition of the middle term is

$$\mathcal{E}_1(\pi^{\mathbf{GL}}_{\psi\circ\Delta}) := \mathrm{Ind}_{\mathbf{P}^{\mathbf{GL}}}^{\mathbf{GL}_4}\left(\mathrm{Jac}_{\mathbf{P}^{\mathbf{GL}}}(\pi_{\psi\circ\Delta})\right) \simeq \mathrm{Ind}_{\mathbf{P}^{\mathbf{GL}}}^{\mathbf{GL}_4}\left(\rho|\det|^{1/2} \otimes \rho|\det|^{-1/2}\right)$$

and in the present case Mœglin and Waldspurger's resolution does not involve any non-trivial "proj", so that the resolution actually goes back to [Aub95], [SS97]. Following Mœglin and Waldspurger one can extend $\pi^{\mathbf{GL}}_{\psi\circ\Delta} \otimes \chi$ from $\mathbf{\Gamma}(F)$ to $\mathbf{\Gamma}^+(F)$ by choosing an action of θ (see [MW06, §§1.7-1.9]), that we denote by θ_{MW}. The resolution (3.3.2) inherits an action of θ by functoriality (see [MW06, §3.2]), and

fortunately the resulting action on $\pi_\psi^{\mathbf{GL}} \otimes \chi$ happens to coincide with θ_{MW} (see [MW06, Lemma 3.2.2], in which we have $j(\psi) = 1$ and so $\beta(\psi \circ \Delta, \rho, \leq d) = +1$). Another way to choose an extension of $\pi_{\psi\circ\Delta}^{\mathbf{GL}} \otimes \chi$ (resp. $\pi_\psi^{\mathbf{GL}} \otimes \chi$) to θ is to use Whittaker functionals and the Langlands classification as we recalled in Section 3.2. Denote the resulting actions of θ by θ_W. In general θ_W and θ_{MW} differ by a sign, but here fortunately $\theta_W = \theta_{MW}$ on both $\pi_{\psi\circ\Delta}^{\mathbf{GL}} \otimes \chi$ and $\pi_\psi^{\mathbf{GL}} \otimes \chi$ (a special case of [MW06, Prop. 5.4.1]). Thus we have a well-defined extension

$$0 \to \left(\pi_{\psi\circ\Delta}^{\mathbf{GL}} \otimes \chi\right)^+ \to \left(\mathcal{E}_1(\pi_{\psi\circ\Delta}^{\mathbf{GL}}) \otimes \chi\right)^+ \to \left(\pi_\psi^{\mathbf{GL}} \otimes \chi\right)^+ \to 0 \tag{3.3.3}$$

of (3.3.2) to $\mathbf{\Gamma}^+(F)$. The trace of the left term is known to be the transfer of $\operatorname{tr} \pi_{\psi\circ\Delta}$. By compatibility of stable transfer with Jacquet modules [MW06, Lemme 4.2.1] and parabolic induction (a consequence of the explicit formula for parabolic induction ([vD72], [Clo84], [Lem10, §7.3, Corollaire 3])), the trace of the middle term is the transfer of the middle term of (3.3.1). So we can conclude that $\operatorname{tr}\left(\pi_\psi^{\mathbf{GL}} \otimes \chi\right)^+$ is the transfer of $\operatorname{tr} \pi_\psi$. □

Proof in the second case for p-adic F. This is similar to the previous case but now $\varphi : W_F \to \mathrm{GL}_2(\mathbb{C})$ is reducible and so it defines a principal series representation of $\mathbf{GL}_2(F_v)$. Write $\varphi \simeq \operatorname{rec}(\eta_1) \oplus \operatorname{rec}(\eta_2)$, so that $\chi = \eta_1\eta_2$. As explained above we can assume that $\eta_1 \neq \eta_2$. Define $\pi_\psi = \operatorname{Ind}_{\mathbf{P}}^{\mathbf{GSpin}_5}((\eta_1 \circ \det) \otimes \chi)$ where the standard parabolic subgroup $\mathbf{P}$ has Levi $\mathbf{L} \simeq \mathbf{GL}_2 \times \mathbf{GSpin}_2$ and $\Pi_\psi = \{\pi_\psi\}$. The representation π_ψ is certainly irreducible (see [Mœg11, §4.2]), but since this is not necessary to prove the Theorem we simply take the definition $\Pi_\psi = \{\pi_\psi\}$ to mean that Π_ψ is the multi-set of constituents of π_ψ.

Consider the parabolic induction for $\mathbf{GL}_4 \times \mathbf{GL}_1$

$$\pi_\psi^{\mathbf{\Gamma}} := \operatorname{Ind}_{\mathbf{P}^{\mathbf{GL}}}^{\mathbf{GL}_4}((\eta_1 \circ \det) \otimes (\eta_2 \circ \det)) \otimes \chi \tag{3.3.4}$$

where $\mathbf{P}^{\mathbf{GL}}$ is the standard parabolic subgroup of $\mathbf{GL}_4$ with Levi $\mathbf{GL}_2 \times \mathbf{GL}_2$. The twisted representation $\pi_\psi^{\widetilde{\mathbf{\Gamma}}}$ of $\widetilde{\mathbf{\Gamma}}(F)$ obtained from (3.3.4) using the canonical action of θ (defined as in [MW06, §1.3]) is such that its trace is the transfer of the trace of π_ψ, by compatibility of parabolic induction with transfer. This is almost the twisted endoscopic character relation, but again we need to be careful with the definition of Whittaker normalisation. The Whittaker-normalised action of θ on $\pi_\psi^{\mathbf{\Gamma}}$ is obtained by realising it as the Langlands quotient of

$$\operatorname{Ind}_{\mathbf{B}^{\mathbf{GL}}}^{\mathbf{GL}_4}\left(\eta_1|\cdot|^{1/2} \otimes \eta_2|\cdot|^{1/2} \otimes \eta_1|\cdot|^{-1/2} \otimes \eta_2|\cdot|^{-1/2}\right) \otimes \chi \tag{3.3.5}$$

where $\mathbf{B}^{\mathbf{GL}}$ is the standard Borel subgroup of $\mathbf{GL}_4$, which coincides with the canonical action of θ on this parabolic induction by (the obvious generalisation of) [MW06, Lemme 5.2.1].

Let us sketch the proof of the fact that these two actions of θ on $\pi_\psi^{\mathbf{\Gamma}}$ coincide. It will be convenient to denote $\sigma_1 \times \cdots \times \sigma_r$ for the parabolic induction (using the standard parabolic) of an admissible representation $\sigma_1 \otimes \cdots \otimes \sigma_r$ of $\mathbf{GL}_{n_1}(F) \times \cdots \times \mathbf{GL}_{n_r}(F)$ to $\mathbf{GL}_{n_1+\cdots+n_r}(F)$. Recall that for any $s \in C$ the parabolic induction $\eta_2|\cdot|^{1/2+s} \times \eta_1|\cdot|^{-1/2-s}$ is irreducible by [BZ76, Theorem 3], since the assumption that $\chi = \eta_1\eta_2$ is not a square implies that $\eta_1|_{\mathcal{O}_F^\times} \neq \eta_2|_{\mathcal{O}_F^\times}$. The intertwining operator

$$I_s : \eta_2|\cdot|^{1/2+s} \times \eta_1|\cdot|^{-1/2-s} \longrightarrow \eta_1|\cdot|^{-1/2-s} \times \eta_2|\cdot|^{1/2+s}$$

defined by the usual integral formula for $\Re(s) \gg 0$, is rational in q^{-s} (where q is the cardinality of the residue field of F) by [Wal03, Théorème IV.1.1], and so there is a polynomial $r(s)$ in q^{-s} such that $r(s)I_s$ is well-defined and non-zero for any s, and therefore an isomorphism. It induces an isomorphism $I_{s,\mathrm{norm}}$:

$$\eta_1|\cdot|^{1/2}\times\eta_2|\cdot|^{1/2+s}\times\eta_1|\cdot|^{-1/2-s}\times\eta_2|\cdot|^{-1/2} \to \eta_1|\cdot|^{1/2}\times\eta_1|\cdot|^{-1/2-s}\times\eta_2|\cdot|^{1/2+s}\times\eta_2|\cdot|^{-1/2}.$$

Denote $\pi_{1,s}$ (resp. $\pi_{2,s}$) the LHS (resp. RHS). Since $\eta_2|\cdot|^{-1/2} = \chi/\left(\eta_1|\cdot|^{1/2}\right)$ and $\eta_1|\cdot|^{-1/2-s} = \chi/\left(\eta_2|\cdot|^{1/2+s}\right)$, there is a canonical extension of $\pi_{1,s}\otimes\chi$ to $\mathbf{\Gamma}^+(F)$ (see [MW06, §1.3]). Denote by θ_1 this canonical action of θ on the space of $\pi_{1,s}\otimes\chi$ (one can easily check that it does not depend on s), so that for $s=0$ we recover the Whittaker normalisation on (3.3.5). The irreducible representation

$$\left(\left(\eta_1|\cdot|^{1/2}\times\eta_1|\cdot|^{-1/2-s}\right)\otimes\left(\eta_2|\cdot|^{1/2+s}\times\eta_2|\cdot|^{-1/2}\right)\right)\otimes\chi$$

of the θ-stable parabolic subgroup $\mathbf{P}\times\mathbf{GL}_1$ of $\mathbf{\Gamma}$ is also fixed by θ, and so $\pi_{2,s}\otimes\chi$ also admits a canonical extension to $\widetilde{\mathbf{\Gamma}}(F)$. Denote θ_2 this canonical action of θ on the space of $\pi_{2,s}\otimes\chi$, which for $s=0$ recovers the canonical action on the quotient (3.3.4). An easy computation that we skip shows that for $\Re(s)\gg 0$ we have $I_{s,\mathrm{norm}}\circ\theta_1 = \theta_2\circ I_{s,\mathrm{norm}}$, and the case of an arbitrary $s\in\mathbb{C}$ follows by analytic continuation. □

Proof in the first case for $F=\mathbb{R}$. This is similar to the first case for F a p-adic field except we now follow arguments of [AMgR18]. For $a\in\frac{1}{2}\mathbb{Z}_{\geq 0}$ let I_a be the tempered Langlands parameter $W_{\mathbb{R}}\to\mathrm{GL}_2(\mathbb{C})$ obtained by inducing the character $z\mapsto(z/\bar{z})^a := (z/|z|)^{2a}$ of $\mathbb{C}^\times$. Up to twisting we can assume that $\varphi = I_a$ with $a>0$ integral, with χ equal to the sign character sign of $W_{\mathbb{R}}$. Let $\pi_\psi^{\mathbf{GL}_4}$ be the irreducible unitary representation of $\mathbf{GL}_4(\mathbb{R})$ associated to φ_ψ. Let $\chi:\mathbf{GL}_1(\mathbb{R})\to\{\pm 1\}$ be the sign character, so that $(\chi\circ\det)\otimes(\pi_\psi^{\mathbf{GL}_4})^\vee\simeq\pi_\psi^{\mathbf{GL}_4}$. As in the p-adic case we have the Whittaker-normalised extension $\pi_\psi^{\widetilde{\mathbf{\Gamma}}}$ of $\pi_\psi^{\mathbf{\Gamma}} := \pi_\psi^{\mathbf{GL}_4}\otimes\chi$.

We have a (short) resolution from [Joh84] (see [AMgR18, §6.2] where this resolution is made completely explicit for $\mathbf{GL}_{2n}$ and parameters $I_w[n]$ for $w\in\frac{1}{2}\mathbb{Z}_{>0}$)

$$0\to\pi_\psi^{\mathbf{GL}_4}\to\pi^{\mathbf{GL}_2}_{I_a|\cdot|^{-1/2}}\times\pi^{\mathbf{GL}_2}_{I_a|\cdot|^{1/2}}\to\pi^{\mathbf{GL}_2}_{I_{a+1/2}}\times\pi^{\mathbf{GL}_2}_{I_{a-1/2}}\to 0$$

where $|\cdot|$ is the norm character of $W_{\mathbb{R}}$ (i.e. the square of the usual absolute value on $\mathbb{C}^\times$ and $|j|=1$) and we denoted parabolic induction for standard parabolic subgroups of $\mathbf{GL}$ as in the p-adic case. In [AMgR18, Lemme 9.9] only the first case occurs, so comparing normalisations (Whittaker and imposed by induction in Johnson's construction of the resolution) is particularly simple: we obtain the analogue of [AMgR18, Théorème 9.7] with $A_s = A_s^+$. □

Proof in the second case for $F=\mathbb{R}$ or $\mathbb{C}$. Up to twisting we can assume that $\varphi\simeq 1\oplus\chi$ with $\chi=\mathrm{sign}$ in the real case and $\chi(z) = (z/\bar{z})^a|z|^{it}$ with $a\in\frac{1}{2}\mathbb{Z}\smallsetminus\mathbb{Z}$ and $t\in\mathbb{R}$ in the complex case. The proof is identical to the p-adic case and we do not repeat the argument. Note that the complex case is the analogue of [MR15, Prop. 6.5]. □

4. Stabilisation of the twisted trace formula

We now state the stabilisation of the twisted trace formula proved by Mœglin and Waldspurger in [MW16a], [MW16b] following the case of ordinary (i.e. non-twisted) endoscopy proved by Arthur in [Art02], [Art01], [Art03] (also following [Lan83], [Kot86], [Lab99], and of course [LW13]). We recall some of the definitions needed to state the stabilisation, and mention some simplifications occurring in the cases at hand.

4.1. **The discrete part of the spectral side.** Consider a connected reductive group $\mathbf{G}$ over a number field F and an automorphism θ of $\mathbf{G}$ of finite order. Let $\widetilde{\mathbf{G}} = \mathbf{G} \rtimes \theta$. Let $\mathbf{A}_0$ be a maximal split torus in $\mathbf{G}$. We will only consider Levi subgroups of $\mathbf{G}$ which contain $\mathbf{A}_0$. Let $K = \prod_v K_v$ be a good maximal compact subgroup of $\mathbf{G}(\mathbb{A}_F)$ with respect to $\mathbf{A}_0$ as in [LW13, §3.1]. Choose a minimal parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}$ containing $\mathbf{A}_0$.

Following [MW16b, §X.5], let us recall the terms occurring in the discrete part of the spectral side of the twisted trace formula. To work with *discrete* automorphic spectra it is necessary to fix central characters (at least on a certain subgroup of the centre), and we follow [MW16b, §X.5.1]. We now elaborate on the notation for the discrete automorphic spectrum introduced in Section 1.3.3. Recall that $\mathfrak{A}_{\mathbf{G}}$ denotes the vector group $\mathbf{A}_{\mathbf{G}}(\mathbb{R})^0$ where $\mathbf{A}_{\mathbf{G}}$ is the biggest central split torus in $\mathrm{Res}_{F/\mathbb{Q}}(\mathbf{G})$. Then $\mathbf{G}(\mathbb{A}_F) = \mathbf{G}(\mathbb{A}_F)^1 \times \mathfrak{A}_{\mathbf{G}}$, where

$$\mathbf{G}(\mathbb{A}_F)^1 = \left\{ g \in \mathbf{G}(\mathbb{A}_F) \,\middle|\, \forall \beta \in X^*(\mathbf{G})^{\mathrm{Gal}_F},\ |\beta(g)| = 1 \right\},$$

so that $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)^1$ has finite measure. Let $\mathfrak{A}_{\widetilde{\mathbf{G}}} = \mathfrak{A}_{\mathbf{G}}^{\theta}$. Then $\mathfrak{A}_{\mathbf{G}} = (1-\theta)(\mathfrak{A}_{\mathbf{G}}) \times \mathfrak{A}_{\widetilde{\mathbf{G}}}$.

In the general definition of twisted endoscopy one considers a character ω of $\mathbf{G}(\mathbb{A}_F)$; in all cases considered in this paper we have $\omega = 1$. Mœglin and Waldspurger consider a character $\chi_{\mathbf{G}}$ of $\mathfrak{A}_{\mathbf{G}}$ which is trivial on $\mathfrak{A}_{\widetilde{\mathbf{G}}}$ and satisfies $\theta(\chi_{\mathbf{G}}) = \chi_{\mathbf{G}}\omega|_{\mathfrak{A}_{\mathbf{G}}}$; since we will always have $\omega = 1$ in this paper, we will have $\chi_{\mathbf{G}} = 1$.

Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$. Up to conjugating by $\mathbf{G}(F)$ we can assume that $\mathbf{L}$ is the standard Levi subgroup of a standard parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. There is a canonical splitting $\mathfrak{A}_{\mathbf{L}} = \mathfrak{A}_{\mathbf{G}} \times \mathfrak{A}_{\mathbf{L}}^{\mathbf{G}}$ (with $\mathfrak{A}_{\mathbf{L}}^{\mathbf{G}}$ included in the derived subgroup of $\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$), and we write $\chi_{\mathbf{G},\mathbf{L}}$ for the extension of $\chi_{\mathbf{G}}$ to $\mathfrak{A}_{\mathbf{L}}$ such that $\chi_{\mathbf{G},\mathbf{L}}|_{\mathfrak{A}_{\mathbf{L}}^{\mathbf{G}}} = 1$. As remarked above in all cases considered in this paper we simply have $\chi_{\mathbf{G},\mathbf{L}} = 1$. The space of square integrable automorphic forms $\mathcal{A}^2(\mathbf{L}(F)\backslash\mathbf{L}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}})$ decomposes discretely, i.e. it is canonically the direct sum, over the countable set $\Pi_{\mathrm{disc}}(\mathbf{L}, \chi_{\mathbf{G},\mathbf{L}})$ of discrete automorphic representations $\pi_{\mathbf{L}}$ for $\mathbf{L}$ such that $\pi_{\mathbf{L}}|_{\mathfrak{A}_{\mathbf{L}}} = \chi_{\mathbf{G},\mathbf{L}}$, of isotypical components

$$\mathcal{A}^2(\mathbf{L}(F)\backslash\mathbf{L}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}})_{\pi_{\mathbf{L}}}$$

which have finite length. Denote by $\mathbf{U}_{\mathbf{P}}$ the unipotent radical of $\mathbf{P}$. Recall [MW94, §I.2.17] the space $\mathcal{A}^2\left(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}}\right)$ of smooth K-finite functions ϕ on $\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F)$ such that for any $k \in K$,

$$x \mapsto \delta_{\mathbf{P}}(x)^{-1/2}(x)\phi(xk)$$

is an element of $\mathcal{A}^2(\mathbf{L}(F)\backslash\mathbf{L}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}})$. In other words,

$$\mathcal{A}^2\left(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}}\right) = \mathrm{Ind}_{\mathbf{P}(\mathbb{A}_F)}^{\mathbf{G}(\mathbb{A}_F)}\left(\mathcal{A}^2(\mathbf{L}(F)\backslash\mathbf{L}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}})\right)^{K-\mathrm{fin}}.$$

This space is endowed with the usual left action of $\mathcal{H}(\mathbf{G})$, which we will denote by $\rho_{\mathbf{P}}^{\mathbf{G}}$. If $\pi_{\mathbf{L}}$ is an irreducible admissible representation of $\mathbf{L}(\mathbb{A}_F)$ such that $\omega_{\pi_{\mathbf{L}}}|_{\mathfrak{A}_{\mathbf{L}}} = \chi_{\mathbf{G},\mathbf{L}}$, denote by

$$\mathcal{A}^2(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}})_{\pi_{\mathbf{L}}}$$

the sub-$\mathcal{H}(\mathbf{G})$-module of $\mathcal{A}^2(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}})$ consisting of functions ϕ such that for any $k \in K$,

$$\Big(x \mapsto \delta_{\mathbf{P}}(x)^{-1/2}(x)\phi(xk)\Big) \in \mathcal{A}^2(\mathbf{L}(F)\backslash\mathbf{L}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}})_{\pi_{\mathbf{L}}}.$$

Let $W(\mathbf{L}, \widetilde{\mathbf{G}}) = \mathrm{Norm}\,(\mathbf{L}, \widetilde{\mathbf{G}}(F))/\mathbf{L}(F)$, where the action of $\widetilde{\mathbf{G}}(F)$ on $\mathbf{G}$ is the adjoint action coming from the definition of a twisted space [MW16a, §I.1.1]. For $\tilde{w} \in W(\mathbf{L}, \widetilde{\mathbf{G}})$ and $f(\tilde{x})d\tilde{x} \in \mathcal{H}(\widetilde{\mathbf{G}})$, we have a map [MW16b, bottom of p. 1204]

(4.1.1)

$$\rho_{\mathbf{P},\tilde{w}}^{\widetilde{\mathbf{G}}}(f) : \mathcal{A}^2(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}}) \longrightarrow \mathcal{A}^2(\mathbf{U}_{\tilde{w}(\mathbf{P})}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}}),$$
$$\phi \longmapsto \left(g \mapsto \int_{\widetilde{\mathbf{G}}(\mathbb{A}_F)} \phi(\tilde{w}^{-1}g\tilde{x})f(\tilde{x})d\tilde{x}\right)$$

and for $f_1, f_3 \in \mathcal{H}(\mathbf{G})$ and $f_2 \in \mathcal{H}(\widetilde{\mathbf{G}})$ we have

$$\rho_{\mathbf{P},\tilde{w}}^{\widetilde{\mathbf{G}}}(f_1 * f_2 * f_3) = \rho_{\tilde{w}(\mathbf{P})}^{\mathbf{G}}(f_1) \circ \rho_{\mathbf{P},\tilde{w}}^{\widetilde{\mathbf{G}}}(f_2) \circ \rho_{\mathbf{P}}^{\mathbf{G}}(f_3).$$

If $\pi_{\mathbf{L}}$ is an irreducible admissible representation of $\mathbf{L}(\mathbb{A}_F)$ such that $\omega_{\pi_{\mathbf{L}}}|_{\mathfrak{A}_{\mathbf{L}}} = \chi_{\mathbf{G},\mathbf{L}}$, then for any $f \in \mathcal{H}(\widetilde{\mathbf{G}})$, $\rho_{\mathbf{P},\tilde{w}}^{\widetilde{\mathbf{G}}}(f)$ restricts to

$$\mathcal{A}^2(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}})_{\pi_{\mathbf{L}}} \longrightarrow \mathcal{A}^2(\mathbf{U}_{\tilde{w}(\mathbf{P})}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}})_{\tilde{w}(\pi_{\mathbf{L}})}$$

where $\tilde{w}(\pi_{\mathbf{L}}) = \pi_{\mathbf{L}} \circ \mathrm{Ad}(\tilde{w}^{-1})$.

By meromorphic continuation of the usual integral formula, there is an intertwining operator

$$M_{\mathbf{P}|\tilde{w}(\mathbf{P})}(0) : \mathcal{A}^2\left(\mathbf{U}_{\tilde{w}(\mathbf{P})}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}}\right) \to \mathcal{A}^2\left(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}}\right).$$

Since $\chi_{\mathbf{G},\mathbf{L}}$ is unitary, $M_{\mathbf{P}|\tilde{w}(\mathbf{P})}$ is well-defined (i.e. holomorphic) at 0, and is in fact unitary. Moreover for any irreducible admissible representation $\pi_{\mathbf{L}}$ of $\mathbf{L}(\mathbb{A}_F)$, $M_{\mathbf{P}|\tilde{w}(\mathbf{P})}(0)$ restricts to

$$\mathcal{A}^2(\mathbf{U}_{\tilde{w}(\mathbf{P})}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}})_{\pi_{\mathbf{L}}} \longrightarrow \mathcal{A}^2(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}})_{\pi_{\mathbf{L}}}.$$

Therefore for $f \in \mathcal{H}(\widetilde{\mathbf{G}})$ the composition $M_{\mathbf{P}|\tilde{w}(\mathbf{P})}(0) \circ \rho_{\mathbf{P},\tilde{w}}^{\widetilde{\mathbf{G}}}(f)$ maps

$$\mathcal{A}^2(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}})$$

to itself and restricts to

$$\mathcal{A}^2(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}})_{\pi_{\mathbf{L}}} \longrightarrow \mathcal{A}^2(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F), \chi_{\mathbf{G},\mathbf{L}})_{\tilde{w}(\pi_{\mathbf{L}})}.$$

We can finally recall the contribution of $\mathbf{L}$ to the discrete part of the spectral side of the twisted trace formula for $\widetilde{\mathbf{G}}$. For $f \in \mathcal{H}(\widetilde{\mathbf{G}})$, let

$$I_{\mathrm{disc}}^{\widetilde{\mathbf{G}},\mathbf{L}}(f) = |W(\mathbf{L},\mathbf{G})|^{-1} \sum_{\tilde{w} \in W(\mathbf{L},\widetilde{\mathbf{G}})_{\mathrm{reg}}} |\det\left(\tilde{w} - 1 \,|\, \mathfrak{A}_{\mathbf{L}}^{\mathbf{G}}\right)|^{-1} \mathrm{tr}\left(M_{\mathbf{P}|\tilde{w}(\mathbf{P})}(0) \circ \rho_{\mathbf{P},\tilde{w}}^{\widetilde{\mathbf{G}}}(f)\right)$$

where $W(\mathbf{L}, \widetilde{\mathbf{G}})_{\mathrm{reg}}$ is the set of $\tilde{w} \in W(\mathbf{L}, \widetilde{\mathbf{G}})$ such that $(\mathfrak{a}_{\mathbf{L}}^{\mathbf{G}})^{\tilde{w}} = 0$. As the notation suggests, $I_{\mathrm{disc}}^{\widetilde{\mathbf{G}},\mathbf{L}}(f)$ only depends on f and the $\mathbf{G}(F)$-conjugacy class of $\mathbf{L}$.

In fact it depends on f only via its image in $I(\widetilde{\mathbf{G}})$. The fact that the trace of $M_{\mathbf{P}|\tilde{w}(\mathbf{P})}(0)\circ\rho^{\widetilde{\mathbf{G}}}_{\mathbf{P},\tilde{w}}(f)$ on $\mathcal{A}^2(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F),\chi_{\mathbf{G},\mathbf{L}})$ is well-defined and equals the absolutely convergent sum

$$\sum_{\substack{\pi_{\mathbf{L}}\in\Pi_{\mathrm{disc}}(\mathbf{L},\chi_{\mathbf{G},\mathbf{L}})\\ \tilde{w}(\pi_{\mathbf{L}})\simeq\pi_{\mathbf{L}}}} \mathrm{tr}\left(M_{\mathbf{P}|\tilde{w}(\mathbf{P})}(0)\circ\rho^{\widetilde{\mathbf{G}}}_{\mathbf{P},\tilde{w}}(f)\,\middle|\,\mathcal{A}^2(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F),\chi_{\mathbf{G},\mathbf{L}})_{\pi_{\mathbf{L}}}\right)$$

is a consequence of work of Finis, Lapid and Müller, as explained in [LW13, §14.3] and [MW16b, §X.5.2 and X.5.3].

The most interesting case is of course for $\mathbf{L}=\mathbf{G}$, since $I^{\widetilde{\mathbf{G}},\mathbf{G}}_{\mathrm{disc}}(f)$ is simply the trace of f on the discrete automorphic spectrum for $\mathbf{G}$ and $\chi_{\mathbf{G}}$. We will recall below the refinement of discrete terms by infinitesimal character and Hecke eigenvalues following Arthur and Mœglin–Waldspurger, that allows one to forget about convergence issues and work with finite sums. But first we make explicit the condition $\tilde{w}(\pi_{\mathbf{L}})\simeq\pi_{\mathbf{L}}$ in the cases at hand.

(1) For $\mathbf{G}=\mathbf{GL}_N\times\mathbf{GL}_1$ and a standard (i.e. block diagonal) Levi $\mathbf{L}\simeq\left(\prod_{k\geq1}(\mathbf{GL}_k)^{n_k}\right)\times\mathbf{GL}_1$ (where $n_k=0$ for almost all k and $\sum_{k\geq1}kn_k=N$), there always exists an element of $\widetilde{\mathbf{G}}(F)$ normalising $\mathbf{L}$ (for example $\theta_0=(J^{-1},1)\rtimes\theta$, so that for any $(g,x)\in\mathbf{G}$ we have $\theta_0(g,x)\theta_0^{-1}=({}^tg^{-1},x\det g)$). Moreover there is a natural identification $W(\mathbf{L},\mathbf{G})\simeq\prod_{k\geq1}\mathfrak{S}_{n_k}$. For $\tilde{w}=(\sigma_k)_{k\geq1}\theta_0\in W(\mathbf{L},\widetilde{\mathbf{G}})$, $\tilde{w}$ is regular if and only if for every $k\geq1$, the decomposition of σ_k in cycles only involves cycles of odd length. For such a regular $\tilde{w}$ and if $\pi=\left(\bigotimes_{k\geq1}(\pi_{k,1}\otimes\cdots\otimes\pi_{k,n_k})\right)\otimes\chi$ is an irreducible admissible representation of $\mathbf{L}(\mathbb{A}_F)$, then $\tilde{w}(\pi)\simeq\pi$ if and only if each $\pi_{k,i}$ satisfies $\pi_{k,i}^\vee\otimes(\chi\circ\det)\simeq\pi_{k,i}$ *and* for every $k\geq1$, the isomorphism class of $(\pi_{k,i})_{1\leq i\leq n_k}$ is fixed by σ_k (i.e. $\pi_{k,i}\simeq\pi_{k,j}$ if i and j belong to the same cycle in the decomposition of σ_k).

(2) In the non-twisted cases $\mathbf{G}=\mathbf{GSpin}_{2n+1}$ or $\mathbf{GSpin}^\alpha_{2n}$, recall that in Section 2.2 we chose (non-uniquely) an isomorphism $\mathbf{L}\simeq\prod_{i\geq1}(\mathbf{GL}_i)^{r_i}\times\mathbf{G}_m$ where $m+\sum_{i\geq1}ir_i=n$ and $\mathbf{G}_m$ is a $\mathbf{GSpin}$ group of the same type as $\mathbf{G}$ of absolute rank m. There is a natural embedding $W(\mathbf{L},\mathbf{G})\hookrightarrow\prod_{i\geq1}(\{\pm1\}^{r_i}\rtimes\mathfrak{S}_{r_i})$ which is surjective unless $\mathbf{G}=\mathbf{GSpin}^1_{2n}$, $m=0$, and there exists an odd $i\geq1$ such that $r_i>0$, in which case it is of index two.

An element $w=((\varepsilon_{i,j})_{1\leq j\leq r_i}\rtimes\sigma_i)_{i\geq1}$ is regular if and only if for every $i\geq1$ and every cycle $(j_1\dots j_s)$ appearing in the decomposition of σ_i, $\prod_{l=1}^s\varepsilon_{i,j_l}=-1$. For such $w\in W(\mathbf{L},\mathbf{G})_{\mathrm{reg}}$ and $\pi_{\mathbf{L}}\simeq\bigotimes_{i\geq1}(\pi_{i,1}\otimes\cdots\otimes\pi_{i,r_i})\otimes\pi_{\mathbf{G}_m}$ an irreducible admissible representation of $\mathbf{L}(\mathbb{A}_F)$, we have $w(\pi_{\mathbf{L}})\simeq\pi_{\mathbf{L}}$ if and only

(a) for every $i\geq1$ and $1\leq j\leq r_i$, $\pi_{i,j}^\vee\otimes(\chi\circ\det)\simeq\pi_{i,j}$ where $\chi:\mathbb{A}_F^\times\to\mathbb{C}^\times$ is $\pi_{\mathbf{G}_m}\circ\mu$, and

(b) for every $i\geq1$ the isomorphism class of $(\pi_{i,j})_{1\leq j\leq r_i}$ is fixed by σ_i.

We now recall from [MW16b, p. 1212] the refinement of the discrete part of the spectral side of the twisted trace formula by infinitesimal characters (using Arthur's theory of multipliers) and families of Satake parameters.

Definition 4.1.2.

(1) Let $IC(\mathbf{G})$ be the set of semisimple conjugacy classes in the Lie algebra of the dual group (over $\mathbb{C}$) of $\mathrm{Res}_{F/\mathbb{Q}}(\mathbf{G})$. This is the set where infinitesimal characters for irreducible representations of $\mathbf{G}(F\otimes_{\mathbb{Q}}\mathbb{R})$ live. In the twisted case let $IC(\widetilde{\mathbf{G}})=IC(\mathbf{G})^{\hat\theta}$. For π_∞ an irreducible admissible representation of $\mathbf{G}(F\otimes_{\mathbb{Q}}\mathbb{R})$, denote by $\nu(\pi_\infty)\in IC(\mathbf{G})$ its infinitesimal character.

(2) Let S be a large enough (i.e. containing V_{ram} as in [MW16b, §VI.1.1]) finite set of places of F. Let $FS^S(\mathbf{G})=\prod_{v\notin S}\left(\widehat{\mathbf{G}}\rtimes\mathrm{Frob}_v\right)^{\mathrm{ss}}/\widehat{\mathbf{G}}$, and in the twisted case let $FS^S(\widetilde{\mathbf{G}})=\left(FS^S(\mathbf{G})\right)^{\hat\theta}$. Write also $FS(\mathbf{G})=\varinjlim_S FS^S(\mathbf{G})$ and in the twisted case $FS(\widetilde{\mathbf{G}})=\varinjlim_S FS^S(\widetilde{\mathbf{G}})$. If $\pi=\otimes'_v\pi_v$ is an irreducible admissible representation of $\mathbf{G}(\mathbb{A}_F)$, we will write $c(\pi)$ for the associated element of $FS(\mathbf{G})$ via the Satake isomorphisms.

(3) For $\nu\in IC(\widetilde{\mathbf{G}})$, S as above, $c^S=(c_v)_{v\notin S}\in FS^S(\widetilde{\mathbf{G}})$, and $\mathbf{L}$ a Levi subgroup of $\mathbf{G}$, let $\Pi_{\mathrm{disc}}(\mathbf{L},\chi_{\mathbf{G},\mathbf{L}})_{\nu,c^S}$ be the set of $\pi_{\mathbf{L}}\in\Pi_{\mathrm{disc}}(\mathbf{L},\chi_{\mathbf{G},\mathbf{L}})$ such that the infinitesimal character of $\pi_{\mathbf{L},\infty}$ maps to ν via $\mathrm{Lie}\left(\widehat{\mathrm{Res}_{F/\mathbb{Q}}(\mathbf{L})}\right)\to\mathrm{Lie}\left(\widehat{\mathrm{Res}_{F/\mathbb{Q}}(\mathbf{G})}\right)$, and for every $v\notin S$, $\pi_{\mathbf{L},v}$ is unramified for K_v and its Satake parameter maps to c_v via ${}^L\mathbf{L}\to{}^L\mathbf{G}$. For $f\in\bigotimes_{v\in S}\mathcal{H}(\widetilde{\mathbf{G}}(F_v))$, let

$$I^{\widetilde{\mathbf{G}},\mathbf{L}}_{\mathrm{disc},\nu,c^S}(f)=|W(\mathbf{L},\mathbf{G})|^{-1}\sum_{\tilde w\in W(\mathbf{L},\widetilde{\mathbf{G}})_{\mathrm{reg}}}\det\left(\tilde w-1\,|\,\mathfrak{A}^{\mathbf{G}}_{\mathbf{L}}\right)|^{-1}\sum_{\substack{\pi_{\mathbf{L}}\in\Pi_{\mathrm{disc}}(\mathbf{L},\chi_{\mathbf{G},\mathbf{L}})_{\nu,c^S}\\ \tilde w(\pi_{\mathbf{L}})\simeq\pi_{\mathbf{L}}}}\mathrm{tr}_{\pi_{\mathbf{L}}}(f),$$

where we write

$$\mathrm{tr}_{\pi_{\mathbf{L}}}(f)=\left(M_{\mathbf{P}|\tilde w(\mathbf{P})}(0)\circ\rho^{\widetilde{\mathbf{G}}}_{\mathbf{P},\tilde w}(f)\,\middle|\,\mathcal{A}^2(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F),\chi_{\mathbf{G},\mathbf{L}})_{\pi_{\mathbf{L}}}\right).$$

Finally let

$$I^{\widetilde{\mathbf{G}}}_{\mathrm{disc},\nu,c^S}(f)=\sum_{\mathbf{L}}I^{\widetilde{\mathbf{G}},\mathbf{L}}_{\mathrm{disc},\nu,c^S}(f)\tag{4.1.3}$$

where the sum is over $\mathbf{G}(F)$-conjugacy classes of Levi subgroups of $\mathbf{G}$.

Seeing this as a sum over triples $(\mathbf{L},\tilde w,\pi_{\mathbf{L}})$, all but finitely many terms vanish. Indeed, if we fix ν, S, c^S and an idempotent e of $\bigotimes_{\substack{v\in S\\ v\nmid\infty}}\mathcal{H}(\mathbf{G}(F_v))$, then there is a finite set $\Upsilon(\nu,S,c^S,e)$ of triples $(\mathbf{L},\tilde w,\pi_{\mathbf{L}})$ such that for any $f\in\bigotimes_{v\in S}\mathcal{H}(\widetilde{\mathbf{G}}(F_v))$ for which $e*f=f*e=f$, the terms corresponding to $(\mathbf{L},\tilde w,\pi_{\mathbf{L}})\notin\Upsilon(\nu,S,c^S,e)$ in the double sum defining $I^{\widetilde{\mathbf{G}}}_{\mathrm{disc},\nu,c^S}(f)$ all vanish.

Remark 4.1.4. (1) By [JS81] and [MW89], taking the image in $FS(\mathbf{GL}_N)$ is injective on formal sums of elements of $\Pi_{\mathrm{disc}}(\mathbf{GL}_{n_i},\chi)$ (note that it is essential that all of the summands are χ self-dual for the same character χ). For this reason we will often identify such formal sums and their image.

(2) In [MW16b] Mœglin–Waldspurger multiply (4.1.3) by $j(\widetilde{\mathbf{G}})^{-1}:=|\det(1-\theta|\mathcal{A}_{\mathbf{G}}/\mathcal{A}_{\widetilde{\mathbf{G}}})|^{-1}$, but this factor is also present in $\iota(\widetilde{\mathbf{G}},\mathbf{H})$ with their definition.

Definition 4.1.5. (1) We will say that $c^S\in FS(\widetilde{\mathbf{G}})$ *occurs* in $I^{\widetilde{\mathbf{G}},\mathbf{L}}_{\mathrm{disc}}$ if there exists $\nu\in IC(\widetilde{\mathbf{G}})$ and $f\in\mathcal{H}(\widetilde{\mathbf{G}})$ such that up to enlarging S we have $I^{\widetilde{\mathbf{G}},\mathbf{L}}_{\mathrm{disc},\nu,c^S}(f)\neq0$.

(2) Let $\mathbf{D}$ be an induced central torus in $\mathbf{G}$, so that there is a dual morphism ${}^L\mathbf{G} \to {}^L\mathbf{D}$. For $c^S \in FS(\widetilde{\mathbf{G}})$ occurring in $I_{\mathrm{disc}}^{\widetilde{\mathbf{G}},\mathbf{L}}$ we define the *central character* of c^S to be the (unique by weak approximation for $\mathbf{D}$ [PR94, Proposition 7.3]) character $\omega_c : \mathbf{A}_{\mathbf{G}}(\mathbb{A}_F)/\mathbf{A}_{\mathbf{G}}(F) \to \mathbb{C}^\times$ such that for almost all places v of F, the Langlands parameter of $(\omega_c)_v$ equals the image of c_v in ${}^L\mathbf{D}$.

Note that in all cases considered in this paper the connected centre of $\mathbf{G}$ is split and so one can take $\mathbf{D}$ to be the full connected centre.

Lemma 4.1.6. *Let $\mathbf{G} = \mathbf{GL}_N \times \mathbf{GL}_1$ and $\widetilde{\mathbf{G}} = \mathbf{G} \rtimes \theta$. If $c \in FS(\widetilde{\mathbf{G}})$ occurs in $I_{\mathrm{disc}}^{\widetilde{\mathbf{G}}}$ and χ is the central character of c, then there is a unique $\psi \in \Psi(\widetilde{\mathbf{G}}, \chi)$ such that c is associated to ψ.*

Proof. This simply follows from Remark 4.1.4 (1) and the above description in the case at hand of the pairs $(\tilde{w}, \pi_{\mathbf{L}})$ with $\tilde{w} \in W(\mathbf{L}, \widetilde{\mathbf{G}})_{\mathrm{reg}}$, $\pi_{\mathbf{L}} \in \Pi_{\mathrm{disc}}(\mathbf{L})$ and $\pi_{\mathbf{L}}^{\tilde{w}} \simeq \pi_{\mathbf{L}}$. □

Remark 4.1.7. Let $\mathbf{G} = \mathbf{GL}_N \times \mathbf{GL}_1$ and $\widetilde{\mathbf{G}} = \mathbf{G} \rtimes \theta$.

(1) For $\mathbf{P}$ a parabolic subgroup of $\mathbf{G}$ with Levi $\mathbf{L}$ and $\pi_{\mathbf{L}} \in \Pi_{\mathrm{disc}}(\mathbf{L}, \chi_{\mathbf{G},\mathbf{L}})$, the parabolically induced representation $\mathcal{A}^2(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash\mathbf{G}(\mathbb{A}_F))_{\pi_{\mathbf{L}}}$ is irreducible by [MW89] (implying multiplicity one for the discrete automorphic spectrum for $\mathbf{L}$) and [Ber84, §0.2], [Vog86, Theorem 17.6] (irreducibility of unitary parabolic induction for general linear groups).

(2) It follows from [JS81], [MW89] and Lemma 4.1.6 that for $c \in FS(\widetilde{\mathbf{G}})$ occurring in $I_{\mathrm{disc}}^{\widetilde{\mathbf{G}},\mathbf{L}}$, $\mathbf{L}$ is determined by c.

(3) For $S \subset S'$, the linear form $I_{\mathrm{disc},\nu,c^{S'}}^{\widetilde{\mathbf{G}}}$ on $I(\widetilde{\mathbf{G}}_{S'})$ is simply the tensor product of $I_{\mathrm{disc},\nu,c^S}^{\widetilde{\mathbf{G}}}$ with the unramified linear form on $I(\widetilde{\mathbf{G}}_{S'\smallsetminus S})$ corresponding to the Satake parameters $(c_v)_{v\in S'\smallsetminus S}$ (see Remark 2.3.4). This is particular to $\mathbf{GL}_n$ and is a direct consequence of strong multiplicity one. Also by strong multiplicity one for a given $c^S \in FS^S(\widetilde{\mathbf{G}})$ there is at most one $\nu \in IC(\widetilde{\mathbf{G}})$ such that $I_{\mathrm{disc},\nu,c^S} \neq 0$. By these remarks, for $c \in FS(\widetilde{\mathbf{G}})$ we have a well-defined linear form $I_{\mathrm{disc},c}^{\widetilde{\mathbf{G}}}$ on $I(\widetilde{\mathbf{G}})$, whose restriction to $I(\widetilde{\mathbf{G}}_S)$ (for large enough S) is $I_{\mathrm{disc},\nu,c^S}^{\widetilde{\mathbf{G}}}$ for the unique ν such that this is non-zero, or 0 if no such ν exists.

4.2. **Elliptic endoscopic groups.** Consider the split group $\mathbf{\Gamma} = \mathbf{GL}_4 \times \mathbf{GL}_1$ over F and its automorphism $\theta : (g, x) \mapsto (J^t g^{-1} J^{-1}, x \det g)$, where

$$J = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

was chosen so that the usual pinning of $\mathbf{GL}_4 \times \mathbf{GL}_1$ is stable under θ. Note that if (π, χ) is a representation of $\mathbf{\Gamma}(F_v)$ for some place v of F, then $(\pi, \chi) \circ \theta \simeq (\pi^\vee \otimes (\chi \circ \det), \chi)$. The dual group $\widehat{\mathbf{\Gamma}}$ is naturally identified with $\mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$, and $\widehat{\theta}(g, x) = (\widehat{J}^t g^{-1} \widehat{J}^{-1} x, x)$, where $\widehat{J} = J$ (but with coefficients in a different field). Denote $\widetilde{\mathbf{\Gamma}} = \mathbf{\Gamma} \rtimes \theta$ (that is, the non-identity connected component of $\mathbf{\Gamma} \rtimes \{1, \theta\}$). We consider twisted endoscopy with $\omega = 1$.

Then the elliptic endoscopic data $(\mathbf{H}, \mathcal{H}, s, \xi)$ for $\widetilde{\mathbf{\Gamma}}$ are easily seen to be of the following form.

(1) $\mathbf{H} = \mathbf{GSpin}_5$, dual $\widehat{\mathbf{H}} = \mathrm{GSp}_4$, for $s = 1$: The first projection identifies $\xi_1(\widehat{\mathbf{GSpin}_5}) = \widehat{\mathbf{\Gamma}}^{\widehat{\theta}}$ with the general symplectic group defined by $\widehat{J}$, and the "similitude factor" morphism $\widehat{\mathbf{GSpin}_5} \to \mathrm{GL}_1$ equals $\mathrm{pr}_2 \circ \xi_1|_{\widehat{\mathbf{GSpin}_5}}$. Both $\mathbf{\Gamma}$ and $\mathbf{GSpin}_5$ are split, so there is an obvious choice for ${}^L\xi : {}^L\mathbf{GSpin}_5 \to {}^L\mathbf{\Gamma}$.

(2) $\mathbf{GSpin}_4^\alpha$, with $\alpha \in F^\times/F^{\times,2}$, dual $\widehat{\mathbf{GSpin}_4^\alpha} = \mathrm{GSO}_4$ with action of $\mathrm{Gal}(E/F)$ if α is not a square, where $E = F(\sqrt{\alpha})$. Pick $s = \mathrm{diag}(-1,-1,1,1)$, then $\widehat{\mathbf{\Gamma}}^{\mathrm{Ad}(s)\circ\widehat{\theta}} = \mathrm{GO}_4$ for the Gram matrix
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$
If $\alpha = 1$ the group $\mathbf{GSpin}_4$ is split and we choose the obvious ${}^L\xi$. Otherwise let c be the non-trivial element of $\mathrm{Gal}(E/F)$, and define ${}^L\xi$ by mapping $1 \rtimes c$ to
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, 1.$$

(3) $\mathbf{R}^\alpha := (\mathbf{GSpin}_2^\alpha \times \mathbf{GSpin}_3)/\{(z, z^{-1}) | z \in \mathbf{GL}_1\}$, for non-trivial α. The dual $\widehat{\mathbf{R}^\alpha}$ is the subgroup of $\mathrm{GSO}_2 \times \mathrm{GSp}_2$ of pairs of elements with equal similitude factors, and $\mathrm{Gal}(E/F)$ acts on the first factor. Let $s = \mathrm{diag}(-1,1,1,1)$, so that
$$\xi(\widehat{\mathbf{R}^\alpha}) = \{\mathrm{diag}(x_1, A, x_2) \mid A \in \mathbf{GL}_2,\, x_1 x_2 = \det A\}.$$
Define ${}^L\xi$ by mapping $1 \rtimes c$ to
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, 1.$$

We also need to consider the elliptic endoscopic groups for $\mathbf{GSpin}_5$ and $\mathbf{GSpin}_4$. Let $\mathbf{H}_1$ be the unique non-trivial elliptic endoscopic group for $\mathbf{GSpin}_5$, so that $\mathbf{H}_1 \simeq \mathbf{GL}_2 \times \mathbf{GL}_2/\{(zI_2, z^{-1}I_2)\}$. Then $\widehat{\mathbf{H}}_1$ is the subgroup of $\mathrm{GSp}_2(\mathbb{C}) \times \mathrm{GSp}_2(\mathbb{C})$ of pairs of elements with equal similitude factors, so we have an obvious embedding of dual groups $\widehat{\mathbf{H}}_1 \to \widehat{\mathbf{GSpin}_5} = \mathrm{GSp}_4(\mathbb{C})$, inducing an embedding of L-groups ${}^L\xi' : {}^L\mathbf{H}_1 \to {}^L\mathbf{GSpin}_5$.

Let $\alpha \in F^\times/F^{\times,2}$ and let $\alpha_1, \alpha_2 \in F^\times/F^{\times,2} \smallsetminus \{1\}$ be such that $\alpha_1\alpha_2 = \alpha$. Let $\mathbf{H}_2^{\alpha_1,\alpha_2}$ be the elliptic endoscopic group for $\mathbf{GSpin}_4^\alpha$ associated to $\{\alpha_1, \alpha_2\}$, so that $\mathbf{H}_2^{\alpha_1,\alpha_2} \simeq \mathbf{GSpin}_2^{\alpha_1} \times \mathbf{GSpin}_2^{\alpha_2}/\{(z, z^{-1}) | z \in \mathbf{GL}_1\}$. Recall that $\mathbf{GSpin}_2^{\alpha_i}$ is naturally isomorphic to $\mathrm{Res}_{F(\sqrt{\alpha_i})/F}(\mathbf{GL}_1)$. Then $\widehat{\mathbf{H}_2^{\alpha_1,\alpha_2}}$ is the subgroup of $\mathrm{GSO}_2(\mathbb{C}) \times \mathrm{GSO}_2(\mathbb{C})$ consisting of pairs of elements with equal similitude factors, so we again have an obvious embedding of dual groups $\widehat{\mathbf{H}_2^{\alpha_1,\alpha_2}} \to \widehat{\mathbf{GSpin}_4} = \mathrm{GSO}_4(\mathbb{C})$. Let μ (resp. μ_1, μ_2) be the morphism $\mathrm{Gal}(\overline{F}/F) \to \mathbb{Z}/2\mathbb{Z}$ having kernel $\mathrm{Gal}(\overline{F}/F(\sqrt{\alpha}))$

(resp./ $\mathrm{Gal}(\overline{F}/F(\sqrt{\alpha_1}))$, $\mathrm{Gal}(\overline{F}/F(\sqrt{\alpha_2}))$). Denote $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. Writing $\mathrm{Gal}(F(\sqrt{\alpha})/F) = \{1, c\}$, define ${}^L\xi' : {}^L\mathbf{H}_2^\alpha \to {}^L\mathbf{GSpin}_4$ by mapping $1 \rtimes \sigma$ to

$$\mathrm{diag}(A^{\mu_1(\sigma)}, A^{\mu_2(\sigma)})\,\mathrm{diag}(1, A^{\mu(\sigma)}, 1) \rtimes \sigma.$$

4.3. **Stabilisation of the trace formula.** We will need to use the stabilisation of the (twisted) trace formula for $\widetilde{\mathbf{\Gamma}}$ and its elliptic endoscopic groups. Consider the latter first: let $(\mathbf{H}, \mathcal{H}, s, \xi)$ be an elliptic endoscopic datum for $(\mathbf{\Gamma}, \widetilde{\mathbf{\Gamma}})$. The stabilisation of the trace formula for $\mathbf{H}$ is as follows. Fix $\nu \in IC(\mathbf{H})$, S a big enough set of places, and $c \in FS^S(\mathbf{H})$. Choose representatives $(\mathbf{H}', \mathcal{H}', s, \xi)$ for the isomorphism classes of elliptic endoscopic data for $\mathbf{H}$, and for each representative choose ${}^L\xi' : {}^L\mathbf{H}' \to {}^L\mathbf{H}$ extending ξ (for example as in the previous section). It induces maps ${}^L\xi' : FS(\mathbf{H}') \to FS(\mathbf{H})$ and ${}^L\xi' : IC(\mathbf{H}') \to IC(\mathbf{H})$. Inductively define a linear form on $I(\mathbf{H}(F_S))$ by

$$S^{\mathbf{H}}_{\mathrm{disc},\nu,c}(f) := I^{\mathbf{H}}_{\mathrm{disc},\nu,c}(f) - \sum_{\substack{\mathfrak{e}'=(\mathbf{H}',\mathcal{H}',s',\xi') \\ \mathbf{H}'\neq\mathbf{H}}} \iota(\mathfrak{e}') \sum_{\substack{c'\mapsto c \\ \nu'\mapsto\nu}} S^{\mathbf{H}'}_{\mathrm{disc},\nu',c'}(f^{\mathbf{H}'}) \tag{4.3.1}$$

where the sum is over equivalence classes of nontrivial elliptic endoscopic data for $\mathbf{H}$, $f^{\mathbf{H}'}$ is a transfer of f (see Section 2.3), and the constants $\iota(\mathfrak{e}')$ are recalled after the following theorem.

Theorem 4.3.2 ([Art02, Global Theorems 2 and 2' and Lemma 7.3(b)])**.** *The linear form $S^{\mathbf{H}}_{\mathrm{disc},\nu,c}$ is stable, i.e. factors through $SI(\mathbf{H}(F_S))$.*

Note that in general (4.3.1) is only well-defined thanks to Theorem 4.3.2 applied to $\mathbf{H}'$. However, for the groups $\mathbf{H}$ considered here, and for any non-trivial endoscopic group $\mathbf{H}'$, the only elliptic endoscopic group for $\mathbf{H}'$ is $\mathbf{H}'$, and so $S^{\mathbf{H}'}_{\mathrm{disc}} = I^{\mathbf{H}'}_{\mathrm{disc}}$.

Let us recall the definition of $\iota(\mathfrak{e}')$, both for ordinary endoscopy and for twisted endoscopy. Assume that $\widetilde{\mathbf{G}}$ is a twisted space and $\mathfrak{e} = (\mathbf{H}, \mathcal{H}, s, \xi)$ is an elliptic endoscopic datum. Let

$$\iota(\mathfrak{e}) = \frac{\tau(\mathbf{G})}{\tau(\mathbf{H})} \frac{\left|\pi_0\left(Z(\widehat{\mathbf{G}})^{\mathrm{Gal}_F,0} \cap \widehat{\mathbf{T}}^{\hat{\theta},0}\right)\right|}{|\pi_0(\mathrm{Aut}(\mathfrak{e}))|}$$

where τ is the Tamagawa number and the superscript 0 denotes the identity component. We have not included the factor $|\det(1-\theta|\dots)|^{-1}$ from [MW16b, VI.5.1] because of Remark 4.1.4 (2); compare with the definition on p. 109 of [KS99] using [KS99, Lem. 6.4.B]. Recall [MW16b, p. 693] that there is a short exact sequence

$$1 \to \left(Z(\widehat{\mathbf{G}})/Z(\widehat{\mathbf{G}}) \cap \widehat{\mathbf{T}}^{\hat{\theta},0}\right)^{\mathrm{Gal}_F} \to \mathrm{Aut}(\mathfrak{e})/\widehat{\mathbf{H}} \to \mathrm{Out}\,(\mathfrak{e}) \to 1.$$

In the ordinary (non-twisted) case we have $\widehat{\mathbf{T}}^{\hat{\theta},0} = \widehat{\mathbf{T}} \supset Z(\widehat{\mathbf{G}})$ and thus $\iota(\mathfrak{e}) = \tau(\mathbf{G})\tau(\mathbf{H})^{-1}|\mathrm{Out}\,(\mathfrak{e})|^{-1}$. The only twisted case that we need in this paper is the case of $\widetilde{\mathbf{\Gamma}}$, when $\widehat{\mathbf{T}}^{\hat{\theta},0} = \{((t_1,\dots,t_4),x) | \forall i,\ t_i = t_{5-i}^{-1}x\}$ and so $Z(\widehat{\mathbf{G}}) \cap \widehat{\mathbf{T}}^{\hat{\theta},0} \simeq \mathbb{C}^\times$. Similarly it is easy to see that $Z(\widehat{\mathbf{G}})/Z(\widehat{\mathbf{G}}) \cap \widehat{\mathbf{T}}^{\hat{\theta},0} \simeq \mathbb{C}^\times$ with trivial action of Gal_F, so we can conclude that $\iota(\mathfrak{e}) = \tau(\mathbf{\Gamma})\tau(\mathbf{H})^{-1}|\mathrm{Out}\,(\mathfrak{e})|^{-1}$ for any elliptic endoscopic datum $\mathfrak{e} = (\mathbf{H}, \mathcal{H}, s, \xi)$ of $\widetilde{\mathbf{\Gamma}}$.

Let us make the constant $\iota(\mathfrak{e})$ explicit in the only two cases where it will be needed in this paper:

(1) For the elliptic endoscopic group $\mathbf{H}_1$ of $\mathbf{GSpin}_5$, $\iota(\mathfrak{e}) = 1/4$.
(2) For the elliptic endoscopic group $\mathbf{GSpin}_5$ of $\widetilde{\mathbf{\Gamma}}$, $\iota(\mathfrak{e}) = 1$.

We can finally state the stabilisation of the twisted trace formula for $(\mathbf{\Gamma}, \widetilde{\mathbf{\Gamma}})$. As in the case of ordinary endoscopy we fix representatives $\mathfrak{e} = (\mathbf{H}, \mathcal{H}, s, \xi)$ of isomorphism classes of elliptic endoscopic data for $\widetilde{\mathbf{\Gamma}}$ and for each $\mathfrak{e}$ we also choose an L-embedding ${}^L\xi : {}^L\mathbf{H} \to {}^L\mathbf{G}$ extending ξ (for example the ones defined in the previous section).

Theorem 4.3.3 ([MW16b, X.8.1])**.** *For any ν and c we have*

$$I^{\widetilde{\mathbf{\Gamma}}}_{\mathrm{disc},\nu,c}(f) = \sum_{\mathfrak{e}=(\mathbf{H},\mathcal{H},s,\xi)} \iota(\mathfrak{e}) \sum_{\substack{\nu' \mapsto \nu \\ c' \mapsto c}} S^{\mathbf{H}}_{\mathrm{disc},\nu',c'}(f^{\mathbf{H}})$$

where the first sum is over equivalence classes of elliptic endoscopic data for $\widetilde{\mathbf{\Gamma}}$.

5. Restriction of automorphic representations

5.1. **Restriction for general groups.** Let us recall a consequence of [HS12, §4] that we will need. Since in all cases considered in this paper the assumption of [Che18, Proposition 1 (iii)] will be satisfied, one can use the more precise result of [Che18] (which can be formally generalised from cuspidal to square-integrable forms) instead. Consider an injective morphism $\mathbf{G} \hookrightarrow \mathbf{G}'$ between connected reductive groups over a number field F such that $\mathbf{G}$ is normal in $\mathbf{G}'$ and $\mathbf{G}'/\mathbf{G}$ is a torus. Choose a maximal compact subgroup K'_∞ of $\mathbf{G}'(F \otimes_{\mathbb{Q}} \mathbb{R})$; then $K_\infty := \mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R}) \cap K'_\infty$ is a maximal compact subgroup of $\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$. Note that if π' is an irreducible unitary admissible $(\mathfrak{g}', K'_\infty) \times \mathbf{G}'(\mathbb{A}_{F,f})$-module then $\mathrm{Res}^{\mathbf{G}'}_{\mathbf{G}}(\pi')$ is a unitary admissible $(\mathfrak{g}, K_\infty) \times \mathbf{G}(\mathbb{A}_{F,f})$-module, but it has infinite length in general. We have a $(\mathfrak{g}, K_\infty) \times \mathbf{G}(\mathbb{A}_{F,f})$-equivariant map

$$\mathrm{res}^{\mathbf{G}'}_{\mathbf{G}} : \mathcal{A}^2(\mathfrak{A}_{\mathbf{G}'}\mathbf{G}'(F)\backslash\mathbf{G}'(\mathbb{A}_F)) \to \mathcal{A}^2(\mathfrak{A}_{\mathbf{G}}\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F))$$

obtained by restricting automorphic forms. The fact that $\mathrm{res}^{\mathbf{G}'}_{\mathbf{G}}$ takes values in $\mathcal{A}^2(\mathfrak{A}_{\mathbf{G}}\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F))$ is a routine verification, except for square-integrability which follows from the proof of [HS12, Lemma 4.19] (see also Remark 4.20 *op. cit.*). If $\pi' \in \Pi_{\mathrm{disc}}(\mathbf{G}')$ and $\iota : \pi' \hookrightarrow \mathcal{A}^2(\mathfrak{A}_{\mathbf{G}'}\mathbf{G}'(F)\backslash\mathbf{G}'(\mathbb{A}_F))$, then $\mathrm{res}^{\mathbf{G}'}_{\mathbf{G}}(\iota(\pi'))$ is naturally identified with a quotient of $\mathrm{Res}^{\mathbf{G}'}_{\mathbf{G}}(\pi')$. This quotient can be proper and of infinite length, but in any case it is *non-zero*. In particular there exists an irreducible constituent π of $\mathrm{Res}^{\mathbf{G}'}_{\mathbf{G}}(\pi')$ such that $\pi \in \Pi_{\mathrm{disc}}(\mathbf{G})$. In this situation we will say that π is an *automorphic restriction* of π'. Unsurprisingly, this notion of restriction is compatible with the Satake isomorphism at almost all places:

Lemma 5.1.1 (Satake)**.** *Suppose that $\pi \simeq \otimes'_v \pi_v \in \Pi_{\mathrm{disc}}(\mathbf{G})$ is an automorphic restriction of $\pi' \simeq \otimes'_v \pi'_v \in \Pi_{\mathrm{disc}}(\mathbf{G}')$, then for almost all places v of F the Satake parameter $c(\pi_v)$ of π_v is the image of $c(\pi'_v)$ under the natural map*

$$\left(\widehat{\mathbf{G}'} \rtimes \mathrm{Frob}_v\right)^{\mathrm{ss}} / \widehat{\mathbf{G}'} - \mathrm{conj} \longrightarrow \left(\widehat{\mathbf{G}} \rtimes \mathrm{Frob}_v\right)^{\mathrm{ss}} / \widehat{\mathbf{G}} - \mathrm{conj}.$$

Proof. For almost all places v, π_v is the unique unramified direct summand in $\mathrm{Res}^{\mathbf{G}'(F_v)}_{\mathbf{G}(F_v)}(\pi'_v)$. The result follows from [Sat63, §7.2] applied to $\mathbf{G} \times \mathbf{T} \to \mathbf{G}'$, where $\mathbf{T}$ is any central torus in $\mathbf{G}$ isogenous to $\mathbf{G}'/\mathbf{G}$, and the translation in terms of dual groups [Bor79, Prop. 6.7]. □

Let us now formulate a direct consequence of [HS12, Theorem 4.14], ignoring multiplicities.

Theorem 5.1.2 (Hiraga–Saito)**.** *The map* $\mathrm{res}_{\mathbf{G}}^{\mathbf{G}'}$ *is surjective, and so any discrete automorphic representation for* $\mathbf{G}$ *is an automorphic restriction of a discrete automorphic representation for* $\mathbf{G}'$*. In other words, there exists a surjective map*

$$\mathrm{ext}_{\mathbf{G}}^{\mathbf{G}'} : \Pi_{\mathrm{disc}}(\mathbf{G}) \longrightarrow \Pi_{\mathrm{disc}}(\mathbf{G}')/\left(\mathbf{G}'(\mathbb{A}_F)/\mathbf{G}(\mathbb{A}_F)\mathbf{G}(F)\mathfrak{A}_{\mathbf{G}'}\right)^{\vee}$$

such that for any $\pi' \in \mathrm{ext}_{\mathbf{G}}^{\mathbf{G}'}(\pi)$*,* π *is a subrepresentation of* $\mathrm{Res}_{\mathbf{G}}^{\mathbf{G}'}(\pi')$*.*

In general this map $\mathrm{ext}_{\mathbf{G}}^{\mathbf{G}'}$ is not uniquely determined.

We will mainly use this result for $\mathbf{Sp}_4 \hookrightarrow \mathbf{GSpin}_5$. This will be fruitful thanks to exterior square functoriality for $\mathbf{GL}_4$ [Kim03] and the commutativity of the following commutative diagram of dual groups:

$$\begin{array}{ccc} \widehat{\mathbf{GSpin}}_5 = \mathrm{GSp}_4 & \longrightarrow & \widehat{\mathbf{Sp}}_4 = \mathrm{SO}_5 \\ \big\downarrow {}^L\xi & & \big\downarrow \mathrm{Std}\oplus 1 \\ \mathrm{GL}_4 \times \mathrm{GL}_1 & \xrightarrow{\quad f \quad} & \mathrm{SL}_6 \end{array} \tag{5.1.3}$$

where $f := \bigwedge^2(\mathrm{pr}_1) \otimes \mathrm{pr}_2^{-1}$.

6. Global Arthur–Langlands parameters for $\mathbf{GSpin}_5$

6.1. **Classification of global parameters.** Let $\chi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ be a continuous unitary character. Recall the set $\Psi(\widetilde{\mathbf{\Gamma}}, \chi)$ of formal global parameters defined in Section 2.5. Recall that in Section 4.2 we fixed a representative $(\mathbf{H}, \mathcal{H}, s, \xi)$ for each equivalence class of elliptic endoscopic data for $\widetilde{\mathbf{\Gamma}}$, and in each case an L-embedding ${}^L\xi : {}^L\mathbf{H} \to {}^L\mathbf{\Gamma} = \widehat{\mathbf{\Gamma}} \times W_F$. We also fixed, for each $\mathbf{H}$ as above, a representative $(\mathbf{H}', \mathcal{H}', s', \xi')$ for each equivalence class of elliptic endoscopic data for $\mathbf{H}$, as well as an L-embedding ${}^L\xi' : {}^L\mathbf{H}' \to {}^L\mathbf{H}$. Throughout this section we will use this generic notation.

A difficulty in using the twisted trace formula is to separate contributions from different endoscopic groups. In our case the problematic endoscopic groups are $\mathbf{GSpin}_5$ and $\mathbf{GSpin}_4$. For this reason we begin with parameters of orthogonal type, although they will not be our main focus.

There are two natural conjugacy classes of morphisms of complex algebraic groups $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GSO}_4$: up to scalars there is on $\mathbb{C}^2$ a unique non-degenerate alternating bilinear form, so that we have $\mathrm{GL}_2 = \mathrm{GSp}_2$, and the tensor product bilinear form on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is symmetric. A simple computation by restriction to a maximal torus shows that the composition of any such morphism $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GSO}_4$ with the standard representation of GSO_4 is isomorphic to $(\mathrm{Sym}^2 \mathrm{Std} \otimes \det) \oplus (\det \otimes \mathrm{Sym}^2 \mathrm{Std})$. Dually we get two conjugacy classes of embeddings $\iota : \mathbf{GSpin}_4 \to \mathbf{GL}_2 \times \mathbf{GL}_2$, with image the distinguished subgroup $\{(g_1, g_2) \mid \det g_1 = \det g_2\}$. Note that the equality $\mathbf{GL}_2 = \mathbf{GSp}_2$ is reflected by the fact that for any irreducible representation π of $\mathbf{GL}_2(F_v)$ we have $(\omega_\pi \circ \det) \otimes \pi^\vee \simeq \pi$. Ramakrishnan [Ram00, Theorem M] defined the "tensor product" functoriality $\mathbf{GL}_2 \times \mathbf{GL}_2 \to \mathbf{GL}_4$, which we will denote by $(\pi_1, \pi_2) \mapsto \pi_1 \boxtimes \pi_2$, for cuspidal π_i's. This is an isobaric automorphic representation for $\mathbf{GL}_4$, i.e. we may see it as formal sum of cuspidal automorphic representations for $\mathbf{GL}_{n_i}$ with $\sum_i n_i = 4$. If $\omega_{\pi_1}\omega_{\pi_2}$ is unitary then $\pi_1 \boxtimes \pi_2$ is a formal sum of cuspidal representations having *unitary* central character.

(Note that this condition is put in the definition of isobaric automorphic representations in [Ram00], as is apparent in §2.2 loc. cit., contrary to the original definition in [Lan79].) We will only need this lift in the weak sense, i.e. compatibility with Satake parameters at all but finitely many places. This transfer is easily extended to discrete representations:

- if $\pi_1 = \eta[2]$ for some character η and π_2 is cuspidal, then $\pi_1 \boxtimes \pi_2 = ((\eta \circ \det) \otimes \pi_2)[2]$.
- if $\pi_1 = \eta_1[2]$ and $\pi_2 = \eta_2[2]$, then $\pi_1 \boxtimes \pi_2 = \eta_1\eta_2 \boxplus \eta_1\eta_2[3]$.

Proposition 6.1.1. *For any discrete automorphic representations π_1, π_2 for $\mathbf{GL}_2$ such that $\omega_{\pi_1}\omega_{\pi_2} = \chi$ we have $\pi_1 \boxtimes \pi_2 \in \Psi_{\mathrm{disc}}(\mathbf{GSpin}_4, \chi)$.*

Proof. If π_1 or π_2 is discrete but not cuspidal this is clear, so we may assume that both are cuspidal. We have $\pi_1 \boxtimes \pi_2 = \pi_1' \boxplus \cdots \boxplus \pi_r'$ and this decomposition is χ-self-dual, so that each factor π_i' is either χ-self-dual or occurs together with $\pi_j' \simeq (\chi \circ \det) \otimes \pi_i'^\vee$ for some $j \neq i$. For any large enough S we have

$$L^S(s, \bigwedge^2(\pi_1 \boxtimes \pi_2) \otimes \chi^{-1}) = \prod_{i=1}^r L^S(s, \bigwedge^2(\pi_i') \otimes \chi^{-1}) \prod_{1 \leq i < j \leq r} L^S(s, \pi_i' \times \pi_j' \times \chi^{-1}).$$

Each factor is meromorphic and does not vanish at $s = 1$ [Sha81, Theorem 5.2], and $L^S(s, \pi_i' \times \pi_j' \times \chi^{-1})$ has a pole at $s = 1$ if and only if $\pi_j' \simeq (\chi \circ \det) \otimes \pi_i'^\vee$ [JS81, Proposition 3.6]. So to prove that each factor π_i' is χ-self-dual of orthogonal type and occurs with multiplicity one it is enough to prove that the L-function $L^S(s, \bigwedge^2(\pi_1 \boxtimes \pi_2) \otimes \chi^{-1})$ is holomorphic at $s = 1$. We have

$$L^S(s, \bigwedge^2(\pi_1 \boxtimes \pi_2) \otimes (\omega_{\pi_1}\omega_{\pi_2})^{-1}) = L^S(s, \mathrm{ad}^0(\pi_1))L^S(s, \mathrm{ad}^0(\pi_2))$$

where $\mathrm{ad}^0(\pi_i)$ is the Gelbart-Jacquet lift [GJ78] of π_i. The Gelbart-Jacquet lift decomposes as follows (see Theorem 9.3 and Remark 9.9 loc. cit.). Recall that any continuous character $\eta : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ such that $\pi_i \otimes (\eta \circ \det) \simeq \pi_i$ satisfies $\eta^2 = 1$. Denote $\Sigma(\pi_i) = \{\eta \,|\, \pi_i \otimes (\eta \circ \det) \simeq \pi_i\}$.

(1) If $\Sigma(\pi_i) = \{1\}$ then $\mathrm{ad}^0(\pi_i)$ is a self-dual cuspidal automorphic representation for $\mathbf{GL}_3$ with trivial central character.
(2) If $\Sigma(\pi_i) = \{1, \eta\}$ for some non-trivial character η then letting E be the quadratic extension of F corresponding to η, there exists a continuous character $\gamma : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$ such that π_i corresponds to $\mathrm{Ind}_{\mathbb{A}_E^\times / E^\times}^{W_F}(\gamma)$, and denoting $\mathrm{Gal}(E/F) = \{1, c\}$ we have $(\gamma/\gamma^c)^2 \neq 1$. In particular $\omega_{\pi_i} = \gamma|_{\mathbb{A}_F^\times / F^\times}\eta$. Then $\mathrm{ad}^0(\pi_i) = \eta \boxplus \sigma$ where σ is the self-dual cuspidal automorphic representation for $\mathbf{GL}_2$ of orthogonal type corresponding to $\mathrm{Ind}_{\mathbb{A}_E^\times / E^\times}^{W_F}(\gamma/\gamma^c)$.
(3) The last possibility is that $\Sigma(\pi_i)$ has 4 elements, say $\Sigma(\pi_i) = \{1, \eta_1, \eta_2, \eta_1\eta_2\}$. Then $\mathrm{ad}^0(\pi_i) = \eta_1 \boxplus \eta_2 \boxplus \eta_1\eta_2$. This case is similar to the previous one except that E/F is not unique, γ/γ^c is a quadratic character and σ is not cuspidal.

In any case no factor of $\mathrm{ad}^0(\pi_i)$ is the trivial representation of $\mathbb{A}_F^\times / F^\times$ and so $L^S(s, \mathrm{ad}^0(\pi_i))$ is holomorphic at $s = 1$. □

When we state the symplectic/orthogonal alternative (Proposition 6.1.7 below), we will for completeness include the non-split case in the statement. For this reason we next state a special case of the non-split analogue of Proposition 6.1.1.

Let $E = F(\sqrt{\alpha})$ be a quadratic extension of F. We now consider the analogue of Proposition 6.1.1 for $\mathbf{GSpin}_4^\alpha$. Any one of the two conjugacy classes of embeddings with distinguished image $\mathbf{GSpin}_4^\alpha \times_F E \to \mathbf{GL}_{2,E} \times \mathbf{GL}_{2,E}$ descends to a unique conjugacy class of embeddings $\iota^\alpha : \mathbf{GSpin}_4^\alpha \to \mathrm{Res}_{E/F}\,\mathbf{GL}_2$. Dually this corresponds to two morphisms ${}^L(\mathrm{Res}_{E/F}\,\mathbf{GL}_2) \to {}^L\mathbf{GSpin}_4$. The composition with the embedding ${}^L\xi : {}^L\mathbf{GSpin}_4^\alpha \to {}^L\mathbf{\Gamma}$ gives the dual of the inclusion $\mathbf{GL}_1 \hookrightarrow \mathrm{Res}_{E/F}\,\mathbf{GL}_1 = \mathbf{Z}(\mathrm{Res}_{E/F}\,\mathbf{GL}_2)$ on the second factor, and one of the two Asai representations on the first factor. Recall that the two Asai representations ${}^L(\mathrm{Res}_{E/F}\,\mathbf{GL}_2) \to \mathrm{GL}_4$ are the two representations extending the representation $\mathrm{Std} \otimes \mathrm{Std} \otimes 1$ of the index two subgroup $\mathrm{GL}_2 \times \mathrm{GL}_2 \times W_E$; in particular they are twists of each other by the quadratic character $W_F \to \mathrm{Gal}(E/F) \simeq \{\pm 1\}$. We will not need to distinguish these Asai representations, since we will only use the fact that their composition with the exterior square morphism $\mathrm{GL}_4 \to \mathrm{GL}_6$ is the unique representation of ${}^L(\mathrm{Res}\,\mathbf{GL}_2) = (\mathrm{GL}_2 \times \mathrm{GL}_2) \rtimes W_F$ which coincides with $(\mathrm{Sym}^2\,\mathrm{Std} \otimes \det \otimes 1) \oplus (\det \otimes \mathrm{Sym}^2\,\mathrm{Std} \otimes 1)$ on $\mathrm{GL}_2 \times \mathrm{GL}_2 \times W_E$, i.e. the induction of either factor. For a cuspidal automorphic representation π for $\mathrm{Res}_{E/F}\,\mathbf{GL}_2$, its Asai lift $\mathrm{As}(\pi)$ was constructed in [Ram02] and [Kri03] and is an isobaric automorphic representation for $\mathbf{GL}_4$ which is a sum of cuspidal representations having unitary central character. As in the previous case we will only need compatibility at unramified places. This construction is trivially extended to non-cuspidal discrete automorphic representations for $\mathrm{Res}_{E/F}\,\mathbf{GL}_2$: for a continuous character γ of $\mathbb{A}_E^\times/E^\times$, one of the two Asai lifts of $\gamma[2]$ is $\gamma|_{\mathbb{A}_F^\times/F^\times}\eta_{E/F} \boxplus \gamma|_{\mathbb{A}_F^\times/F^\times}[3]$, where $\eta_{E/F}$ is the quadratic character of $\mathbb{A}_F^\times/F^\times$ corresponding to E.

Proposition 6.1.2. *As above let $E = F(\sqrt{\alpha})$ be a quadratic extension of F. Let π be a discrete automorphic representation for $\mathrm{Res}_{E/F}\,\mathbf{GL}_2$. Assume that the restriction of ω_π to $\mathbb{A}_F^\times/F^\times$ equals χ. Then (Π, χ) is of orthogonal type, i.e. for any large enough S the L-function $L^S(s, \mathrm{Sym}^2(\Pi) \otimes \chi^{-1})$ has a pole at $s = 1$.*

Proof. As in the proof of Proposition 6.1.1 the case where π is not cuspidal is trivial, and in the cuspidal case it is enough to prove that the L-function $L^S(s, \bigwedge^2 \mathrm{As}(\pi) \otimes \chi^{-1})$ is holomorphic at $s = 1$. We have

$$L^S(s, \bigwedge^2 \mathrm{As}(\pi) \otimes \chi^{-1}) = L^S(s, \mathrm{Ind}_E^F(\mathrm{ad}^0\,\pi)).$$

As we recalled in the previous proof the Gelbart-Jacquet lift $\mathrm{ad}^0(\pi)$ may decompose into one, two or three cuspidal representations but in any case the trivial representation of $\mathbb{A}_E^\times/E^\times$ does not occur. Thus the trivial representation of $\mathbb{A}_F^\times/F^\times$ does not occur in the automorphic induction $\mathrm{Ind}_E^F(\mathrm{ad}^0\,\pi)$. □

Remark 6.1.3. In [Kri12, Appendix A] the precise decomposition of the tensor product and Asai lifts are given.

Corollary 6.1.4. *Let $\alpha \in F^\times/F^{\times,2}$. Let Π be a discrete automorphic representation for $\mathbf{GSpin}_4^\alpha$ and let $c(\Pi) \in FS(\mathbf{GSpin}_4^\alpha)$ be the associated family of Satake parameters. Assume that Π has central character χ, i.e. $\widehat{\mu}(c(\Pi)) = c(\chi)$. Then ${}^L\xi(c(\Pi)) \in \widetilde{\Psi}_{\mathrm{disc}}(\mathbf{GSpin}_4^\alpha, \chi)$.*

Proof. We use an embedding ι^α as introduced before Propositions 6.1.1 and 6.1.2 and Theorem 5.1.2 to realize Π as an automorphic restriction of some discrete automorphic representation for $\mathrm{Res}_{F[\sqrt{\alpha}]/F}\mathbf{GL}_2$. The fact that (π,χ) is of orthogonal type then follows from Propositions 6.1.1 and 6.1.2. □

Proposition 6.1.6 below shows that we may associate a parameter in the set $\Psi(\widetilde{\mathbf{\Gamma}},\chi)$ to each discrete automorphic representation of $\mathbf{GSpin}_5$ with central character χ. We will refine this in Proposition 6.1.9 to show that these parameters are in fact contained in the subset $\Psi_{\mathrm{disc}}(\mathbf{GSpin}_5,\chi)$. The following elementary remark will help us to distinguish parameters coming from different endoscopic subgroups for $\widetilde{\mathbf{\Gamma}}$.

Remark 6.1.5. For any $\alpha\in F^\times/F^{\times,2}\smallsetminus\{1\}$ the sets

$$\left({}^L\xi(FS(\mathbf{GSpin}_5))\cup{}^L\xi(FS(\mathbf{GSpin}_4))\right)\text{ and }\left({}^L\xi(FS(\mathbf{GSpin}_4^\alpha))\cup{}^L\xi(FS(\mathbf{R}^\alpha))\right)$$

are pairwise disjoint, because we can recover α as follows (by the definition of ${}^L\xi$): for $\mathbf{H}=\mathbf{GSpin}_4^\alpha$ or $\mathbf{H}=\mathbf{R}^\alpha$, $c^S\in FS(\mathbf{H})$ and $(g^S,x^S)={}^L\xi(c^S)$, for any $v\notin S$, then v splits in $F(\sqrt{\alpha})$ if and only if $\det g_v=x_v^2$. On the other hand if $\mathbf{H}=\mathbf{GSpin}_5$ or $\mathbf{H}=\mathbf{GSpin}_4$ then we always have $\det g_v=x_v^2$.

Proposition 6.1.6.

(1) *For $\mathbf{L}$ a proper Levi subgroup of $\mathbf{GSpin}_5$, any $c\in FS(\mathbf{GSpin}_5)$ occurring in $I_{\mathrm{disc}}^{\mathbf{GSpin}_5,\mathbf{L}}$ such that $\widehat{\mu}(c)=c(\chi)$ satisfies ${}^L\xi(c)\in\Psi(\widetilde{\mathbf{\Gamma}},\chi)$ and this element of $\Psi(\widetilde{\mathbf{\Gamma}},\chi)$ is not discrete.*

(2) *Let $\mathbf{H}=(\mathbf{GL}_2\times\mathbf{GL}_2)/\{(zI_2,z^{-1}I_2\,|\,z\in\mathbf{GL}_1\}$ be the unique non-trivial elliptic endoscopic group for $\mathbf{GSpin}_5$. Then any $c\in FS(\mathbf{H})$ occurring in $I_{\mathrm{disc}}^{\mathbf{H}}=S_{\mathrm{disc}}^{\mathbf{H}}$ and such that $\widehat{\mu}(c)=c(\chi)$ satisfies $({}^L\xi\circ{}^L\xi')(c)\in\Psi(\widetilde{\mathbf{\Gamma}},\chi)$.*

(3) *Let $\alpha\in F^\times/F^{\times,2}$. Let $\mathbf{H}'$ be a non-trivial elliptic endoscopic group for $\mathbf{GSpin}_4^\alpha$. Then any $c\in FS(\mathbf{H}')$ occurring in $I_{\mathrm{disc}}^{\mathbf{H}'}=S_{\mathrm{disc}}^{\mathbf{H}'}$ and such that $\widehat{\mu}(c)=c(\chi)$ satisfies $({}^L\xi\circ{}^L\xi')(c)\in\Psi(\widetilde{\mathbf{\Gamma}},\chi)$ and each one of its factors is of orthogonal type with respect to χ.*

(4) *Let $\alpha\in F^\times/F^{\times,2}$. For $\mathbf{L}$ a Levi subgroup of $\mathbf{GSpin}_4^\alpha$, any $c\in FS(\mathbf{GSpin}_4^\alpha)$ occurring in $I_{\mathrm{disc}}^{\mathbf{GSpin}_4^\alpha,\mathbf{L}}$ and such that $\widehat{\mu}(c)=c(\chi)$ satisfies ${}^L\xi(c)\in\Psi(\widetilde{\mathbf{\Gamma}},\chi)$. If $\mathbf{L}\neq\mathbf{GSpin}_4^\alpha$ then ${}^L\xi(c)$ is not discrete.*

(5) *Any $c\in FS(\mathbf{GSpin}_4)$ occurring in $S_{\mathrm{disc}}^{\mathbf{GSpin}_4}$ and such that $\widehat{\mu}(c)=c(\chi)$ satisfies ${}^L\xi(c)\in\Psi(\widetilde{\mathbf{\Gamma}},\chi)$, and if this parameter is discrete then each one of its factors is of orthogonal type with respect to χ.*

(6) *Any $c\in FS(\mathbf{GSpin}_5)$ occurring in $S_{\mathrm{disc}}^{\mathbf{GSpin}_5}$ and such that $\widehat{\mu}(c)=c(\chi)$ satisfies ${}^L\xi(c)\in\Psi(\widetilde{\mathbf{\Gamma}},\chi)$.*

(7) *Any $c\in FS(\mathbf{GSpin}_5)$ associated to a discrete automorphic representation for $\mathbf{GSpin}_5$ with central character χ satisfies ${}^L\xi(c)\in\Psi(\widetilde{\mathbf{\Gamma}},\chi)$.*

Proof. We use repeatedly the description of $I_{\mathrm{disc}}^{\widetilde{\mathbf{G}},\mathbf{L}}$ explained in Section 4.1, namely that if $c\in FS(\widetilde{\mathbf{G}})$ occurs in $I_{\mathrm{disc}}^{\widetilde{\mathbf{G}},\mathbf{L}}$, then there is a regular element $\tilde{w}\in W(\mathbf{L},\widetilde{\mathbf{G}})$, and $\pi_{\mathbf{L}}\in\Pi_{\mathrm{disc}}(\mathbf{L})$ such that $\pi^{\tilde{w}}\simeq\pi$ and $c(\pi_{\mathbf{L}})$ maps to c via ${}^L\mathbf{L}\to{}^L\mathbf{G}$.

(1) The possible proper Levi subgroups $\mathbf{L}$ and the embeddings ${}^L\mathbf{L}\to{}^L\mathbf{GSpin}_5$ are listed in Section 2.2. In the case at hand, the possibilities are
   (a) $\mathbf{GL}_1\times\mathbf{GSpin}_3\cong\mathbf{GL}_1\times\mathbf{GL}_2$,

(b) $\mathbf{GL}_2 \times \mathbf{GSpin}_1 \cong \mathbf{GL}_2 \times \mathbf{GL}_1$, and
(c) $\mathbf{GL}_1 \times \mathbf{GL}_1 \times \mathbf{GSpin}_1 \cong \mathbf{GL}_1 \times \mathbf{GL}_1 \times \mathbf{GL}_1$.

In the first case we find that the corresponding parameter is of the form $\eta \boxplus \pi \boxplus \eta$, where π is a unitary discrete automorphic representation of $\mathbf{GL}_2(\mathbb{A}_F)$ with $\omega_\pi = \chi$ and $\eta^2 = \chi$; in the second case, that the parameter is of the form $\pi \boxplus \pi$, where π is a unitary discrete automorphic representation of $\mathbf{GL}_2(\mathbb{A}_F)$ such that $\pi^\vee \otimes (\chi \circ \det) \simeq \pi$; and in the third case that the parameter is of the form $\eta_1 \boxplus \eta_2 \boxplus \eta_2 \boxplus \eta_1$ with $\eta_1^2 = \eta_2^2 = \chi$.

(2) By the description of $\mathbf{H}$ as a quotient, c corresponds to a pair (π_1, π_2) with each π_i either an element of $\Pi_{\mathrm{disc}}(\mathbf{GL}_2)$ with $\omega_{\pi_i} = \chi$ or $\eta \boxplus \eta$, with $\eta^2 = \chi$. It is easy to check that $({}^L\xi \circ {}^L\xi')(c) = (c(\pi_1) \oplus c(\pi_2), c(\chi))$, so that the corresponding parameter is $\pi_1 \boxplus \pi_2$.

(3) This is similar to the previous two parts. Write $\mathbf{H}' = \mathbf{H}_2^{\alpha_1,\alpha_2}$ as in Section 4.2, so that an element of $\Pi_{\mathrm{disc}}(\mathbf{H}')$ is given by a pair of automorphic representations ρ_1, ρ_2 for the tori $\mathbf{GSpin}_2^{\alpha_i} \simeq \mathrm{Res}_{E_i/F}(\mathbf{GL}_1)$ (here $E_i = F(\sqrt{\alpha})$) whose restrictions to $\mathbf{GL}_1$ are equal to χ. Then via the natural embedding ${}^L\mathbf{GSpin}_2^{\alpha_i} = \mathrm{GSO}_2 \rtimes \mathrm{Gal}(E_i/F)) \to \mathrm{GL}_2$, we have $({}^L\xi \circ {}^L\xi')(c) = (c(\pi_1) \oplus c(\pi_2), c(\chi))$ where π_1 and π_2 are the cuspidal automorphic representations for $\mathbf{GL}_2$ with central character χ automorphically induced (for E_i/F) from ρ_1 and ρ_2 seen as unitary characters of $\mathbb{A}_{E_i}^\times / E_i^\times$.

(4) We use the embedding ι introduced before Proposition 6.1.1 (or rather one of the two possible embeddings, the choice being irrelevant as before). If c is discrete automorphic, i.e. it occurs in $I_{\mathrm{disc}}^{\mathbf{GSpin}_4,\mathbf{GSpin}_4}$, then by Theorem 5.1.2 it comes from the automorphic restriction of some $(\pi_1, \pi_2) \in \Pi_{\mathrm{disc}}(\mathbf{GL}_2 \times \mathbf{GL}_2)$, with $c(\omega_{\pi_1})c(\omega_{\pi_2}) = c(\chi)$ and so $\omega_{\pi_1}\omega_{\pi_2} = \chi$. Then ${}^L\xi(c) = (c(\pi_1) \otimes c(\pi_2), c(\chi))$, and the corresponding parameter is $\pi_1 \boxtimes \pi_2$, considered in Proposition 6.1.1.

Otherwise c occurs in $I_{\mathrm{disc}}^{\mathbf{GSpin}_4,\mathbf{L}}$ for some proper Levi subgroup. By the description given in Section 2.2, we see that $\mathbf{L}$ is isomorphic to $\mathbf{GL}_2 \times \mathbf{GSpin}_0^1 \cong \mathbf{GL}_2 \times \mathbf{GL}_1$ or to $\mathbf{GL}_1 \times \mathbf{GL}_1 \times \mathbf{GSpin}_0^1 \cong \mathbf{GL}_1 \times \mathbf{GL}_1 \times \mathbf{GL}_1$. In either case we can compute explicitly as in (1), and we find that we obtain parameters of the form $\pi \boxplus \pi$, where π is a discrete automorphic representation of $\mathbf{GL}_2(\mathbb{A}_F)$ such that $\pi^\vee \otimes (\chi \circ \det) \simeq \pi$, and parameters of the form $\eta_1 \boxplus \eta_2 \boxplus \eta_2 \boxplus \eta_1$ with $\eta_1^2 = \eta_2^2 = \chi$.

(5) This follows immediately from the stable trace formula (4.3.1) for $\mathbf{GSpin}_4$ and the two previous points.

(6) This follows from the stable twisted trace formula of Theorem 4.3.3 and Remark 6.1.5. Observe that we can associate an element of $\Psi(\widetilde{\mathbf{\Gamma}}, \chi)$ to any family of Satake parameters occurring in $S_{\mathrm{disc}}^{\mathbf{GSpin}_4}$ or to $I_{\mathrm{disc}}^{\widetilde{\mathbf{\Gamma}}}$; in the former case this is the content of (5), and in the latter case it follows from Lemma 4.1.6.

(7) This follows as in (6), this time using the stable trace formula for $\mathbf{GSpin}_5$, and applying parts (2) and (6). □

Note that points 4 and 5 in Proposition 6.1.6 could be proved for $\mathbf{GSpin}_4^\alpha$ in a similar way, but we will not need this case in the sequel.

We can now prove the symplectic/orthogonal alternative for $\mathbf{GL}_4$. This is well known, and can also be proved using the theta correspondence or converse theorems; indeed, [AS14, Thm. 4.26] proves a corresponding result for **GSpin** groups of arbitrary rank, showing that a χ-self dual cuspidal automorphic representation π of $\mathbf{GL}_n$ arises as the transfer of a globally generic representation of a **GSpin** group which is uniquely determined by the data of which of the corresponding symmetric and alternating square L-functions has a pole, together with the central character of π.

However, our emphasis here is slightly different (we wish to determine which representations have Satake parameters which occur in the discrete spectrum of $\mathbf{GSpin}_5$), and in any case we find it instructive to show how this follows from the trace formula together with Kim's exterior square transfer [Kim03].

Proposition 6.1.7. *Let π be a χ-self dual cuspidal automorphic representation for $\mathbf{GL}_4$, and let S be a finite set of places of F containing all Archimedean places and all non-Archimedean places where π is ramified.*

(1) There is a unique (up to isomorphism) elliptic endoscopic datum $(\mathbf{H}, \mathcal{H}, s, \xi)$ for $\widetilde{\mathbf{\Gamma}}$ such that there exists $c' \in FS(\mathbf{H})$ satisfying ${}^L\xi(c') = (c(\pi), c(\xi))$. Moreover $\mathbf{H}$ is not isomorphic to $\mathbf{R}^\alpha$, and for any c' as above we have, for any large enough finite set S of places of F and any $\nu \in IC(\mathbf{H})$,

$$S^{\mathbf{H}}_{\mathrm{disc},\nu,(c')^S} = I^{\mathbf{H}}_{\mathrm{disc},\nu,(c')^S} = I^{\mathbf{H},\mathbf{H}}_{\mathrm{disc},\nu,(c')^S}.$$

(2) (a) If $\mathbf{H} \simeq \mathbf{GSpin}_4^\alpha$ for some $\alpha \in F^\times/F^{\times,2}$ then (π, χ) is of orthogonal type, i.e. for any large enough S the L-function $L^S(s, \mathrm{Sym}^2(\pi) \otimes \chi^{-1})$ has a pole at $s = 1$.

(b) If $\mathbf{H} \simeq \mathbf{GSpin}_5$ then (π, χ) is of symplectic type, i.e. for any large enough S the L-function $L^S(s, \bigwedge^2(\pi) \otimes \chi^{-1})$ has a pole at $s = 1$.

Proof. (1) By Remark 4.1.7 (2) we know that $(c(\pi), c(\chi))$ does not occur in $I^{\widetilde{\mathbf{\Gamma}},\mathbf{L}}_{\mathrm{disc}}$ for any proper Levi subgroup $\mathbf{L}$ of $\mathbf{\Gamma}$. Since (π, χ) occurs with multiplicity one in the discrete automorphic spectrum for $\mathbf{\Gamma}$, the automorphic extension $\widetilde{\pi}$ of π to $\widetilde{\mathbf{\Gamma}}$ (provided by (4.1.1) for $\mathbf{L} = \mathbf{G}$, with $\tilde{w} = \theta$) has non-vanishing trace (see [Lem10, Proposition A.5] for the p-adic case, the Archimedean case is proved similarly). Therefore $(c(\pi), c(\chi))$ occurs in $I^{\widetilde{\mathbf{\Gamma}}}_{\mathrm{disc}}$. In the stabilisation of the twisted trace formula (Theorem 4.3.3) this contribution comes from at least one elliptic endoscopic datum, i.e. there is an elliptic endoscopic group $\mathbf{H}$ and $c' \in FS(\mathbf{H})$ occurring in $S^{\mathbf{H}}_{\mathrm{disc}}$ such that ${}^L\xi(c') = (c(\pi), c(\chi))$. Again using [JS81] we see that $\mathbf{H} \simeq \mathbf{R}^\alpha$ would contradict cuspidality of π. By (2), (3) and (4) in Proposition 6.1.6 we also know that c' cannot come from a proper Levi subgroup of $\mathbf{H}$ or from a non-trivial endoscopic datum for $\mathbf{H}$. We have proved every statement of the first point except for uniqueness of $\mathbf{H}$, which will follow from the second point.

(2) Let $\mathbf{H}$ and c' be as above. The previous point shows that there exists a discrete automorphic representation Π for $\mathbf{H}$ such that ${}^L\xi(c(\Pi)) = (c(\Pi), c(\chi))$.

(a) If $\mathbf{H} \simeq \mathbf{GSpin}_4^\alpha$ the result follows from Corollary 6.1.4.

(b) If $\mathbf{H} \simeq \mathbf{GSpin}_5$, let Π' be an automorphic restriction (in the sense of Section 5) of Π to $(\mathbf{GSpin}_5)_{\mathrm{der}} \simeq \mathbf{Sp}_4$. Then Π' is a discrete automorphic representation for $\mathbf{Sp}_4$, and Arthur associates a discrete

parameter $\psi' \in \Psi_{\mathrm{disc}}(\mathbf{Sp}_4)$ to Π' (see Theorem 2.6.1). Now $\bigwedge^2(c(\pi)) \otimes c(\chi)^{-1} = 1 \oplus c(\psi')$ (see the commutative diagram (5.1.3)) and so $L^S(s, \bigwedge^2(\pi) \otimes \chi^{-1}) = \zeta_F^S(s) L^S(s, \psi')$. Moreover by [Kim03, Thm. 5.3.1], $1 \oplus c(\psi')$ is associated to a (unique by [JS81, Thm. 4.4]) isobaric sum of *unitary* cuspidal representations, and so the same holds for ψ'. This implies that $L^S(s, \psi')$ does not vanish on the line $\Re(s) = 1$, by the main result of [JS77]. □

Remark 6.1.8. Thanks to Theorem 2.7.1 we see that $\Psi_{\mathrm{disc}}(\mathbf{GSpin}_5, \chi)$ is the subset of $\Psi(\widetilde{\mathbf{\Gamma}}, \chi)$ consisting of pairs (ψ, χ) with ψ of the following kinds. (We have labelled them in the same way as in [Art04]. The groups $\mathcal{S}_\psi$ are easy to compute; for the values of ε_ψ, see [Art13, (1.5.6)].)

(a) cuspidal automorphic representations π of $\mathbf{GL}_4$ such that $\pi^\vee \otimes (\chi \circ \det) \simeq \pi$ and $L^S(s, \chi^{-1} \otimes \bigwedge^2 \pi)$ has a pole at $s = 1$. (General type, $\mathcal{S}_\psi = 1$, $\varepsilon_\psi = 1$.)
(b) $\pi_1 \boxplus \pi_2$ where π_i are cuspidal automorphic representations of $\mathbf{GL}_2$, $\omega_{\pi_1} = \omega_{\pi_2} = \chi$ and $\pi_1 \not\simeq \pi_2$. (Yoshida type, $\mathcal{S}_\psi = \mathbb{Z}/2\mathbb{Z}$, $\varepsilon_\psi = 1$.)
(c) $\pi[2]$ for π a cuspidal automorphic representation for $\mathbf{GL}_2$ such that ω_π/χ has order 2 (i.e. (π, χ) is of orthogonal type, which means that π is automorphically induced from a character $\eta : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$ for the quadratic extension E/F corresponding to ω_π/χ, such that $\eta^c \neq \eta$ and $\eta|_{\mathbb{A}_F^\times / F^\times} = \chi$). (Soudry type, $\mathcal{S}_\psi = 1$, $\varepsilon_\psi = 1$.)
(d) $\pi \boxplus \eta[2]$ with π cuspidal for $\mathbf{GL}_2$ and $\omega_\pi = \eta^2 = \chi$. (Saito–Kurokawa type, $\mathcal{S}_\psi = \mathbb{Z}/2\mathbb{Z}$, $\varepsilon_\psi = \mathrm{sgn}$ if $\varepsilon(1/2, \pi \otimes \eta^{-1}) = -1$, and $\varepsilon_\psi = 1$ otherwise.)
(e) $\eta_1[2] \boxplus \eta_2[2]$ with $\eta_1^2 = \eta_2^2 = \chi$ and $\eta_1 \neq \eta_2$. (Howe–Piatetski-Shapiro type, $\mathcal{S}_\psi = \mathbb{Z}/2\mathbb{Z}$, $\varepsilon_\psi = 1$.)
(f) $\eta[4]$ with $\eta^2 = \chi$. (One dimensional type, $\mathcal{S}_\psi = 1$, $\varepsilon_\psi = 1$.)

Proposition 6.1.9. *For $c \in FS(\mathbf{GSpin}_5)$ associated to a discrete automorphic representation Π of $\mathbf{GSpin}_5$ with central character χ, the associated element of $\Psi(\widetilde{\mathbf{\Gamma}}, \chi)$ (by Proposition 6.1.6) belongs to the subset $\Psi_{\mathrm{disc}}(\mathbf{GSpin}_5, \chi)$.*

Proof. As in the proof of Proposition 6.1.7, we use an automorphic restriction Π' of Π to $\mathbf{Sp}_4$, and the associated parameter ψ', which we know to be discrete. We also know that $1 \oplus c(\psi') = \bigwedge^2(c(\psi)) \otimes c(\chi)^{-1}$.

By Theorem 2.6.1, we can and do assume that χ is not a square. In particular, this implies that ψ does not have a summand of the form η, $\eta[2]$ or $\eta[4]$ (as the condition that η is χ-self dual forces $\eta^2 = \chi$). In addition, if $\psi = \psi_1 \boxplus \psi_1$, then $c(\psi') = \left(\bigwedge^2(c(\psi_1)) \otimes c(\chi)^{-1}\right)^{\oplus 2} \oplus \mathrm{ad}^0(c(\psi_1))$, which contradicts the discreteness of ψ'. Thus we have the following possibilities for ψ.

(1) $\psi = \psi_1 \boxplus \psi_2$ where ψ_i is a cuspidal automorphic representation for $\mathbf{GL}_2$ such that $\psi_i^\vee \otimes (\chi \circ \det) \simeq \psi_i$ and $\psi_1 \not\simeq \psi_2$. We need to show that $\omega_{\pi_i} = \chi$, i.e. that (π_i, χ) is of symplectic type. Suppose not. We have $\omega_{\pi_i}^2 = \chi^2$, and by Remark 6.1.5 we also have $\omega_{\pi_1}\omega_{\pi_2} = \chi^2$ and so $\omega_{\pi_1} = \omega_{\pi_2}$. Then we find that $\bigwedge^2(\psi) \otimes \chi^{-1} = (\omega_{\pi_1}/\chi) \boxplus (\omega_{\pi_2}/\chi) \boxplus (\chi^{-1}\pi_1 \boxtimes \pi_2)$. Since $\omega_{\pi_1}/\chi = \omega_{\pi_2}/\chi$ is a non-trivial quadratic character, this cannot be written as $1 \boxplus \psi'$ with ψ' discrete, a contradiction.
(2) $\psi = \pi[2]$, where π is a cuspidal automorphic representation for $\mathbf{GL}_2$ such that $\pi^\vee \otimes (\chi \circ \det) \simeq \pi$. In this case we need to check that ω_π/χ has order

2, i.e. is non-trivial. But if $\chi = \omega_\pi$ then $\psi' = \bigwedge^2(\pi[2]) \otimes \omega_\pi^{-1} = \mathrm{ad}^0(\pi) \boxplus [3]$, which cannot be written as an isobaric sum of 1 and discrete automorphic representations for general linear groups, a contradiction.

(3) $\psi = \pi[1]$ where π is a cuspidal automorphic representation for $\mathbf{GL}_4$ such that $\pi^\vee \otimes (\chi \circ \det) \simeq \pi$. This case was considered in Proposition 6.1.7. □

7. Multiplicity formula

In this section we prove the multiplicity theorem for $\mathbf{GSpin}_5$ (Theorem 7.4.1), which describes the discrete automorphic spectrum in terms of the packets $\Pi_\psi(\varepsilon_\psi)$ defined in Definition 2.5.5. We begin with some preliminaries.

7.1. **Canonical global normalisation versus Whittaker normalisation.** Recall from Remark 4.1.7 that for $\mathbf{G} = \mathbf{GL}_N \times \mathbf{GL}_1$ and $\widetilde{\mathbf{G}} = \mathbf{G} \rtimes \theta$, for a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ and $\pi_{\mathbf{L}} \in \Pi_{\mathrm{disc}}(\mathbf{L})$ the parabolically induced representation $\mathcal{A}^2(\mathbf{U}_{\mathbf{P}}(\mathbb{A}_F)\mathbf{L}(F)\backslash \mathbf{G}(\mathbb{A}_F))_{\pi_{\mathbf{L}}}$ is irreducible. For $\tilde{w} \in W(\mathbf{L}, \widetilde{\mathbf{G}})$ we have a canonical ("automorphic") extension of this representation of $\mathbf{G}(\mathbb{A}_F)$ to $\widetilde{\mathbf{G}}$, denoted $M_{\mathbf{P}|\tilde{w}(\mathbf{P})}(0) \circ \rho^{\mathbf{G}}_{\mathbf{P},\tilde{w}}$ in Section 4. We have another canonical normalisation of this extension, namely the Whittaker normalisation recalled in Section 3.2.

Lemma 7.1.1 (Arthur)**.** *These two extensions coincide.*

Proof. The proof of [Art13, Lemma 4.2.3] readily extends to the case at hand. □

7.2. **The twisted endoscopic character relation for real discrete tempered parameters.**

Proposition 7.2.1. *Let $\varphi : W_{\mathbb{R}} \to \mathrm{GSp}_4$ be a discrete parameter. Then the twisted endoscopic character relation holds for Π_φ (as defined by Langlands in* [Lan89]*), i.e. part 1 of Theorem 3.1.1 holds.*

Recall that for φ such that $\widehat{\mu} \circ \varphi$ is a square, this twisted endoscopic character relation is a direct consequence of [Mez16] and [AMgR18, Annexe C].

Proof. We use a global argument similar to (but simpler than) [AMgR18, Annexe C]. Up to twisting we can assume that $\mathrm{Std}_{\mathbf{GSpin}_5} \circ \varphi \simeq (I_{a_1} \oplus I_{a_2}, \mathrm{sign}^{2a_1})$, where $a_1, a_2 \in \frac{1}{2}\mathbb{Z}_{>0}$ are such that $a_1 - a_2 \in \mathbb{Z}_{>0}$ (and as before, $I_a = \mathrm{Ind}_{\mathbb{C}^\times}^{W_{\mathbb{R}}}(z \mapsto (z/\bar{z})^a)$). Fix a continuous character $\chi : \mathbb{A}^\times/\mathbb{R}_{>0}\mathbb{Q}^\times \to \mathbb{C}^\times$ such that $\chi|_{\mathbb{R}^\times} = \mathrm{sign}^{2a_1}$. There are cuspidal automorphic representations π_1, π_2 for $\mathbf{GL}_2/\mathbb{Q}$ with central characters $\omega_{\pi_1} = \omega_{\pi_2} = \chi$ and such that $\mathrm{rec}(\pi_{i,\infty}) = I_{a_i}$ (apply [Ser97, Proposition 4] with $n = 1$, $k = 2a_i + 1$ fixed and N of the form $\ell\mathrm{cond}(\chi)$ where $\mathrm{cond}(\chi)$ is the conductor of χ and $\ell \to +\infty$ prime). Let $\psi = \pi_1 \boxplus \pi_2 \in \Psi_{\mathrm{disc}}(\mathbf{GSpin}_5, \chi)$, so that $\psi_\infty = \varphi$.

By [Mez16] there is $z(\varphi) \in \mathbb{C}^\times$ such that for any $f_\infty \in I(\widetilde{\mathbf{\Gamma}}\mathbb{R})$ we have

$$\mathrm{tr}\, \pi_\varphi^{\widetilde{\mathbf{\Gamma}}}(f_\infty) = z(\varphi)\left(\mathrm{tr}\, \pi_\infty^+(f') + \mathrm{tr}\, \pi_\infty^-(f'_\infty)\right)$$

where π_∞^+ (resp. π_∞^-) is the generic (resp. non-generic) element of Π_φ, i.e. $\langle \cdot, \pi_\infty^+ \rangle$ (resp. $\langle \cdot, \pi_\infty^- \rangle$) is the trivial (resp. non-trivial) character of $\mathcal{S}_\varphi$. We need to show that $z(\varphi) = 1$. Recall that for any finite prime p the twisted endoscopic character relation

$$\mathrm{tr}\, \pi_{\psi_p}^{\widetilde{\mathbf{\Gamma}}}(f_p) = \sum_{\pi_p \in \Pi_{\psi_p}} \mathrm{tr}\, \pi_p(f'_p)$$

holds by the main theorem of [CG15].

In the discrete part of the trace formula for $\widetilde{\mathbf{\Gamma}}$, the contribution $I^{\widetilde{\mathbf{\Gamma}}}_{\mathrm{disc},c(\psi)}$ of $c(\psi)$ only comes from $\mathbf{L} = \mathbf{GL}_2 \times \mathbf{GL}_2$ and $\tilde{w} = \theta_0$, using notation as in the discussion preceding Definition 4.1.2. By Lemma 7.1.1 and since $\det(\tilde{w} - 1|\mathfrak{A}^{\mathbf{\Gamma}}_{\mathbf{L}}) = 2$ this contribution is (on $I(\widetilde{\mathbf{\Gamma}}_S)$ for S containing ∞ and all places where π_1 or π_2 ramify)

$$\prod_{v\in S} h_v \longmapsto \frac{1}{2}\prod_v \operatorname{tr}\pi^{\widetilde{\mathbf{\Gamma}}}_{\psi_v}(h_v)$$

where $\pi^{\widetilde{\mathbf{\Gamma}}}_{\psi_v}$ is the Whittaker-normalised extension to $\widetilde{\mathbf{\Gamma}}(F_v)$ of the irreducible parabolically induced representation $\pi_{1,v}\times\pi_{2,v}$. Thus we get for $h = \prod_{v\in S} h_v \in I(\widetilde{\mathbf{\Gamma}}_S)$

$$I^{\widetilde{\mathbf{\Gamma}}}_{\mathrm{disc},c(\psi)}(h) = \frac{z(\varphi)}{2}\prod_{v\in S}\sum_{\pi_v\in\Pi_{\psi_v}} \operatorname{tr}\pi_v(h_v^{\mathbf{GSpin}_5}). \tag{7.2.2}$$

By the stabilisation of the twisted trace formula (Theorem 4.3.3), and using Remark 6.1.5 and Proposition 6.1.6 5 which imply that the endoscopic groups $\mathbf{GSpin}_4^\alpha$ (for $\alpha \in F^\times/F^{\times,2}$) and $\mathbf{R}^\alpha$ (for $\alpha \in F^\times/F^{\times,2}\smallsetminus\{1\}$) have vanishing contributions corresponding to $c(\psi)^S$, (7.2.2) equals

$$S_{\mathrm{disc},\nu(\varphi),c(\psi)^S}(h^{\mathbf{GSpin}_5}).$$

By surjectivity of the transfer map $h\mapsto h^{\mathbf{GSpin}_5}$ (Proposition 2.4.4), this determines the stable linear form $S^{\mathbf{GSpin}_5}_{\mathrm{disc},\nu(\psi),c(\psi)^S}$. Let

$$\mathbf{H} = (\mathbf{GL}_2\times\mathbf{GL}_2)/\{(zI_2, z^{-1}I_2 \mid z\in\mathbf{GL}_1\}$$

be the unique non-trivial elliptic endoscopic group for $\mathbf{GSpin}_5$. The $(\nu(\psi),c(\psi)^S)$-part of the stabilisation of the trace formula (Theorem 4.3.2) for $\mathbf{GSpin}_5$ now reads, for $f=\prod_{v\in S} f_v \in I(\mathbf{GSpin}_5)$,

$$I^{\mathbf{GSpin}_5}_{\mathrm{disc},\nu(\psi),c(\psi)^S}(f) = \frac{z(\varphi)}{2}\prod_{v\in S}\sum_{\pi_v\in\Pi_{\psi_v}}\operatorname{tr}\pi_v(f_v) + \frac{1}{4}\sum_{\substack{\nu'\mapsto\nu(\psi)\\ c'^S\mapsto c(\psi)^S}} S^{\mathbf{H}}_{\mathrm{disc},\nu',c'^S}(f^{\mathbf{H}}).$$

Now $S^{\mathbf{H}}_{\mathrm{disc},\nu',c'^S} = I^{\mathbf{H}}_{\mathrm{disc},\nu',c'^S}$ is non-vanishing if and only if (ν',c'^S) is associated to (π_1,π_2) or to (π_2,π_1), in which case it equals $\operatorname{tr}(\pi_1\otimes\pi_2)$ or $\operatorname{tr}(\pi_2\otimes\pi_1)$. By the endoscopic character relations, in either case we have

$$S^{\mathbf{H}}_{\mathrm{disc},\nu',c'^S}(f^{\mathbf{H}}) = \prod_{v\in S}\sum_{\pi_v\in\Pi_{\psi_v}}\langle s,\pi_v\rangle\operatorname{tr}\pi_v(f_v),$$

where s is the non-trivial element of $\mathcal{S}_\psi$. Thus we obtain

$$I^{\mathbf{GSpin}_5}_{\mathrm{disc},\nu(\psi),c(\psi)^S}(f) = \sum_{(\pi_v)_v\in\prod_{v\in S}\Pi_{\psi_v}} \frac{z(\varphi)+\prod_{v\in S}\langle s,\pi_v\rangle}{2}\prod_{v\in S}\operatorname{tr}\pi_v(f_v).$$

By Proposition 6.1.6 (1) the left-hand side simply equals the trace of f in the $(\nu(\psi),c(\psi)^S)$-part of the discrete automorphic spectrum for $\mathbf{GSpin}_5$. Varying S, the above equality means that the multiplicity of $\pi=\otimes'_v\pi_v\in\Pi_\psi$ in $\mathcal{A}^2(\mathbf{GSpin}_5)$ equals $(z(\varphi)+\langle s,\pi\rangle)/2$. Comparing with [CG15, Theorem 3.1] (which relies on the theta correspondence and not trace formulas) for any π we finally obtain $z(\varphi)=1$. □

Remark 7.2.3. Arguing as in Lemma C.1 of [AMgR18] one could certainly prove the Proposition without using [CG15, Theorem 3.1], since $|z(\psi)| = 1$ and $(z(\psi) - 1)/2 \in \mathbb{Z}_{\geq 0}$ imply $z(\psi) = 1$ (consider the multiplicity of $\pi_\infty^- \otimes \bigotimes_p' \pi_p$ where $\langle s, \pi_p \rangle = +1$ for all p).

7.3. **Local parameters.** In this section we obtain Arthur's multiplicity formula for $\mathbf{GSpin}_5$, by formally using the stable twisted trace formula and twisted endoscopic character relations to get the desired expression for $S^{\mathbf{GSpin}_5}_{\mathrm{disc},c}$ for c corresponding to $\psi \in \Psi_{\mathrm{disc}}(\mathbf{GSpin}_5)$, and then the stable trace formula for $\mathbf{GSpin}_5$.

We begin with the following important point, which is Conjecture 2.5.3 for $\mathbf{G} = \mathbf{GSpin}_5$.

Proposition 7.3.1. *If π is a χ-self dual cuspidal automorphic representation of $\mathbf{GL}_4(\mathbb{A}_F)$ of symplectic type, then for any place v of F, the pair $(\mathrm{rec}(\pi_v), \mathrm{rec}(\chi_v))$ is of symplectic type, i.e. factors through $\mathrm{GSp}_4(\mathbb{C})$.*

Proof. This follows from [GT11a, Thm. 12.1], which shows that π arises as the transfer of a (globally generic) automorphic representation Π of $\mathbf{GSp}_4(\mathbb{A}_F)$, and that at each place v, the pair $(\mathrm{rec}(\pi_v), \mathrm{rec}(\chi_v))$ is obtained from the L-parameter associated to Π_v by the main theorem of [GT11a]. □

Remark 7.3.2. There are at least two alternative ways of proving Proposition 7.3.1. One is to use the main results of [Kim03] and [Hen09], which imply in particular that for each place v the representation $\bigwedge^2 \mathrm{rec}(\pi_v) \otimes \mathrm{rec}(\chi_v)^{-1}$ contains the trivial representation, together with a case by case analysis. The other is to follow the argument of [Art13, §8.1].

7.4. **The global multiplicity formula.** Given Proposition 6.1.9, the multiplicity formula is morally equivalent to the following formula for any $\psi \in \Psi_{\mathrm{disc}}(\mathbf{GSpin}_5)$, $f \in \mathcal{H}(\mathbf{GSpin}_5)$ and S large enough:

$$S^{\mathbf{GSpin}_5}_{\mathrm{disc},\nu,c(\psi)^S} = \begin{cases} \frac{\varepsilon_\psi(s_\psi)}{|\mathcal{S}_\psi|} \sum_{\pi \in \Pi_\psi} \langle s_\psi, \pi \rangle \operatorname{tr} \pi & \text{if } \nu = \nu(\psi) \\ 0 & \text{otherwise.} \end{cases}$$

This is the simplification (for discrete parameters) of the general stable multiplicity formula (see [Art13, Theorem 4.1.2]).

We now prove the multiplicity formula; the following theorem is Conjecture 2.5.6, specialised to the case $\mathbf{G} = \mathbf{GSpin}_5$. We write $\Pi_\psi(\varepsilon_\psi)$ for the set of representations defined in 2.5.5 (with no tilde, since we are working with $\mathbf{GSpin}_5$).

Theorem 7.4.1. *There is an isomorphism of $\mathcal{H}(\mathbf{GSpin}_5)$-modules*

$$\mathcal{A}^2(\mathbf{GSpin}_5) \cong \bigoplus_{\substack{\chi : \mathbb{A}_F^\times / F^\times \mathbb{R}_{>0} \to \mathbb{C}^\times \\ \psi \in \Psi_{\mathrm{disc}}(\mathbf{GSpin}_5, \chi) \\ \pi \in \Pi_\psi(\varepsilon_\psi)}} \pi \tag{7.4.2}$$

where χ runs over the continuous (automatically unitary) characters.

Proof. Fix a continuous character $\chi : \mathbb{A}_F^\times / F^\times \mathbb{R}_{>0} \to \mathbb{C}^\times$, and write

$$\mathcal{A}^2(\mathbf{GSpin}_5, \chi)$$

for the space of χ-equivariant square-integrable automorphic forms on which $\mathbb{A}_F^\times/F^\times$ acts via χ. For any $\nu \in IC(\mathbf{G})$ and $c \in FS(\mathbf{G})$, write

$$\mathcal{A}^2(\mathbf{GSpin}_5,\chi)_{\nu,c} := \varinjlim_S \mathcal{A}^2(\mathbf{GSpin}_5,\chi)_{\nu,c^s}.$$

Then we have

$$\begin{aligned}\mathcal{A}^2(\mathbf{GSpin}_5,\chi) &= \bigoplus_{\substack{\nu\in IC(\mathbf{G})\\ c\in FS(\mathbf{G})}} \mathcal{A}^2(\mathbf{GSpin}_5,\chi)_{\nu,c}\\ &= \bigoplus_{\substack{\nu\in IC(\mathbf{G})\\ \psi\in\Psi_{\mathrm{disc}}(\mathbf{GSpin}_5,\chi)}} \mathcal{A}^2(\mathbf{GSpin}_5)_{\nu,c(\psi)}.\end{aligned}$$

Indeed, it follows from Proposition 6.1.9 that for any c with $\mathcal{A}^2(\mathbf{GSpin}_5,\chi)_c \neq 0$, there is some $\psi \in \Psi_{\mathrm{disc}}(\mathbf{GSpin}_5,\chi)$ such ${}^L\xi(c(\pi)) = c(\psi)$. It follows that we are reduced to showing that for each $\psi \in \Psi_{\mathrm{disc}}(\mathbf{GSpin}_5,\chi)$, we have

$$\mathcal{A}^2(\mathbf{GSpin}_5)_{\nu,c(\psi)} \cong \begin{cases} \bigoplus_{\pi\in\Pi_\psi(\varepsilon_\psi)} \pi & \text{if } \nu = \nu(\psi)\\ 0 & \text{if } \nu \neq \nu(\psi).\end{cases} \tag{7.4.3}$$

Fix $\nu \in IC(\mathbf{G})$ and $\psi \in \Psi_{\mathrm{disc}}(\mathbf{GSpin}_5,\chi)$. If χ is a square, then we are done by Theorem 2.6.1 (that is, by reducing to $\mathbf{SO}_5$, already proved by Arthur). So we only have to consider the following cases:

(1) Cuspidal π for $\mathbf{GL}_4$ such that $\pi^\vee \otimes (\chi\circ\det) \simeq \pi$ and (π,χ) is of symplectic type.
(2) $\pi_1 \boxplus \pi_2$ where the π_i's are distinct cuspidal automorphic representations for $\mathbf{GL}_2$ with $\omega_{\pi_i} = \chi$ (Yoshida type).
(3) $\pi[2]$ where π is a cuspidal automorphic representation for $\mathbf{GL}_2$ such that ω_π/χ is a quadratic character, i.e. $\pi^\vee \otimes (\chi\circ\det) \simeq \pi$ and (π,χ) is of orthogonal type (Soudry type).

In case (2), the multiplicity formula is a special case of [CG15, Theorem 3.1], proved using the global theta correspondence. So we can and do assume that we are in case (1) or case (3), so that in particular $\mathcal{S}_\psi = 1$ and $\varepsilon_\psi = 1$. Furthermore, in either case we know that for any place v, the parameter ψ_v is of symplectic type, i.e. factors through GSp_4 (in case (1) this is Proposition 7.3.1, and in case (3) it follows from Theorem 2.7.1).

We will prove (7.4.3) by computing $I^{\mathbf{GSpin}_5,\mathbf{GSpin}_5}_{\mathrm{disc},\nu,c(\psi)}(f)$ for each $f \in \mathcal{H}(\mathbf{GSpin}_5)$, which by definition is the trace of f on the left hand side of (7.4.3) (note that this is well-defined, and equal to $I^{\mathbf{GSpin}_5,\mathbf{GSpin}_5}_{\mathrm{disc},\nu,c(\psi)^S}(f)$ for any sufficiently large S). To this end, note firstly that by Proposition 6.1.6 (1), we know that for any proper Levi $\mathbf{L}$ of $\mathbf{GSpin}_5$, and for any $c \in FS(\mathbf{GSpin}_5)$ occurring in $I^{\mathbf{GSpin}_5,\mathbf{L}}_{\mathrm{disc},\nu}$, with central character χ, we have ${}^L\xi(c) \in \Psi(\widetilde{\mathbf{\Gamma}},\chi) \smallsetminus \Psi_{\mathrm{disc}}(\mathbf{GSpin}_5,\chi)$. Consequently, we see that for any $\psi \in \Psi_{\mathrm{disc}}(\mathbf{GSpin}_5,\chi)$, we have

$$I^{\mathbf{GSpin}_5}_{\mathrm{disc},\nu,c(\psi)} = I^{\mathbf{GSpin}_5,\mathbf{GSpin}_5}_{\mathrm{disc},\nu,c(\psi)}. \tag{7.4.4}$$

Denoting as usual the unique non-trivial elliptic endoscopic group of $\mathbf{GSpin}_5$ by $\mathbf{H}$, we have that $S^{\mathbf{H}}_{\mathrm{disc},\nu',c'}$ vanishes identically for any $\nu' \in IC(\mathbf{H})$ and any $c' \in FS(\mathbf{H})$ such that ${}^L\xi'(c') = c(\psi)$ (because the proof of Proposition 6.1.6 (2) shows that any c' occurring in $S^{\mathbf{H}}_{\mathrm{disc}}$ is such that ${}^L\xi\circ{}^L\xi'(c')$ is a sum of at least two

discrete automorphic representations of general linear groups). It follows that we have

$$I^{\mathbf{GSpin}_5}_{\mathrm{disc},\nu,c(\psi)} = S^{\mathbf{GSpin}_5}_{\mathrm{disc},\nu,c(\psi)}. \tag{7.4.5}$$

By Proposition 6.1.6 5, for any c' occurring in $S^{\mathbf{GSpin}_4}_{\mathrm{disc}}$ we have ${}^L\xi(c') \neq c(\psi)$, so that (using also Remark 6.1.5) the contribution of ψ to the stabilisation of the twisted trace formula for $\widetilde{\mathbf{\Gamma}}$ simply reads

$$I^{\widetilde{\mathbf{\Gamma}}}_{\mathrm{disc},\nu,c(\psi)}(h) = S^{\mathbf{GSpin}_5}_{\mathrm{disc},\nu,c(\psi)}(h^{\mathbf{GSpin}_5}) \tag{7.4.6}$$

where on the right-hand side $c(\psi)$ denotes the unique element of $FS(\mathbf{GSpin}_5)$ which is the preimage of $c(\psi) \in FS(\widetilde{\mathbf{\Gamma}})$ by ${}^L\xi$, and similarly for ν seen as an element of $IC(\mathbf{GSpin}_5)$. By surjectivity of $h \mapsto h^{\mathbf{GSpin}_5}$ (see Proposition 2.4.4), and Remark 4.1.7, this implies that $S^{\mathbf{GSpin}_5}_{\mathrm{disc},\nu,c(\psi)}$ vanishes identically if $\nu \neq \nu(\psi)$. In the definition of $I^{\widetilde{\mathbf{\Gamma}}}_{\mathrm{disc},\nu,c(\psi)}$ as a sum over Levi subgroups, the only non-vanishing summand corresponds to $\mathbf{L} = \mathbf{GL}_4$. By Lemma 7.1.1 we have for $h = \prod_v h_v \in I(\widetilde{\mathbf{\Gamma}})$

$$I^{\widetilde{\mathbf{\Gamma}}}_{\mathrm{disc},\nu(\psi),c(\psi)}(h) = \prod_v \operatorname{tr} \pi^{\widetilde{\mathbf{\Gamma}}}_{\psi_v}(h_v).$$

Applying Theorem 3.1.1 (1) (or rather its extension to parameters in $\Psi^+(\mathbf{GSpin}_5)$ via parabolic induction; see [Art13, §1.5]) to the right-hand side of this equality and using (7.4.6) we obtain

$$S^{\mathbf{GSpin}_5}_{\mathrm{disc},\nu(\psi),c(\psi)}(\prod_v f_v) = \prod_v \sum_{\pi_v \in \Pi_{\psi_v}} \operatorname{tr} \pi_v(f_v).$$

Combining this with (7.4.4) and (7.4.5), we conclude that

$$I^{\mathbf{GSpin}_5,\mathbf{GSpin}_5}_{\mathrm{disc},\nu,c(\psi)}(\prod_v f_v) = \begin{cases} \prod_v \sum_{\pi_v \in \Pi_{\psi_v}} \operatorname{tr} \pi_v(f_v) & \text{if } \nu = \nu(\psi) \\ 0 & \text{if } \nu \neq \nu(\psi) \end{cases}$$

Recalling that $\mathcal{S}_\psi = 1$ and $\varepsilon_\psi = 1$, this is equivalent to 7.4.3, so we are done. □

Remark 7.4.7. A consequence of the multiplicity formula and [AS14] is that for any discrete automorphic representation π for $\mathbf{GSpin}_5$ which is formally tempered (i.e. of general or Yoshida type), there exists a *globally generic* discrete automorphic representation π' for $\mathbf{GSpin}_5$ such that for any place v of F, π_v and π'_v have the same Langlands parameter. Indeed letting $\psi \in \Psi_{\mathrm{disc}}(\mathbf{GSpin}_5, \chi)$ be the parameter of π (well-defined by the multiplicity formula), Shahidi's conjecture (proved in [GT11a]) implies that there is a unique representation in Π_ψ which is generic at each place. In fact the multiplicity formula asserts that it is automorphic with multiplicity one. By (the converse part of) [AS14, Theorem 4.26] there exists a globally generic discrete (even cuspidal) automorphic representation π' for $\mathbf{GSpin}_5$ such that $\pi'_v \simeq \pi_v$ for almost all v. In particular π' has parameter ψ, and for any place v of F, π'_v is generic.

Note that in the case $\chi = 1$, Arthur used the the analogue of [AS14] in order to prove Shahidi's conjecture: see [Art13, Proposition 8.3.2]. More precisely, he used the descent theorem of Ginzburg, Rallis and Soudry (and thus indirectly the converse theorem of Cogdell, Kim, Piatestski-Shapiro and Shahidi).

Remark 7.4.8. Let $\mathbf{G}$ be an inner form of $\mathbf{GSpin}_5$ over a number field F. Using the stabilisation of the trace formula for $\mathbf{G}$ qualitatively (i.e. only considering families of Satake parameters), we see that for any $\pi \in \Pi_{\mathrm{disc}}(\mathbf{G}, \chi)$, there is a well-defined $\psi \in \Psi(\widetilde{\mathbf{\Gamma}}, \chi)$ such that $c(\pi) = (c(\psi), c(\chi))$. Moreover if ψ is discrete then $\psi \in \Psi_{\mathrm{disc}}(\mathbf{GSpin}_5, \chi)$. If $\psi \in \Psi_{\mathrm{disc}}(\mathbf{GSpin}_5, \chi)$ is tempered (i.e. either of general type or of Yoshida type) then using the stabilisation of the trace formula quantitatively and the endoscopic character relations proved in [CG15] for inner forms as well, one could certainly prove the multiplicity formula for the part of the discrete automorphic spectrum for $\mathbf{G}$ corresponding to $(c(\psi), c(\chi)) \in FS(\mathbf{G})$. The proof would be similar to those of Proposition 7.2.1 and Theorem 7.4.1. Note however that to even state the multiplicity formula, one has to fix a normalisation of local transfer factors satisfying a product formula. This normalisation was achieved in [Kal] and used in [Taï] to prove the multiplicity formula for certain inner forms of symplectic groups. It would thus be necessary to compare Kaletha's normalisation of local transfer factors for the non-split inner form of $\mathbf{GSp}_4$ realised as a rigid inner twist with Chan–Gan's ad hoc normalisation [CG15, §4.3].

8. Compatibility of the local Langlands correspondences for $\mathbf{Sp}_4$ and $\mathbf{GSpin}_5$

In this section, we study the compatibility of the local Langlands correspondence with restriction from $\mathbf{GSp}_4(F) \simeq \mathbf{GSpin}_5(F)$ to $\mathbf{Sp}_4(F)$, where F is a p-adic field. We do not consider the case of Archimedean places, which could certainly be done by a careful examination of the Langlands–Shelstad correspondence.

8.1. **Compatibility with restriction.** Let F be a p-adic field. The proof of the existence of the local Langlands correspondence for $\mathbf{GSp}_4(F) \simeq \mathbf{GSpin}_5(F)$ in [GT11a] used the theta correspondence, and its compatibility with the correspondence stated in [Art04] (characterised by (twisted) endoscopic character relations) was proved in [CG15]. In the paper [GT10], a local Langlands correspondence for $\mathbf{Sp}_4(F)$ was deduced from the correspondence for $\mathbf{GSp}_4(F)$ by restriction. This correspondence is uniquely characterised by the commutativity of the diagram

$$\begin{array}{ccc} \Pi(\mathbf{GSpin}_5) & \longrightarrow & \Phi(\mathbf{GSpin}_5) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \Pi(\mathbf{Sp}_4) & \longrightarrow & \Phi(\mathbf{Sp}_4) \end{array} \tag{8.1.1}$$

where $\Pi(\mathbf{GSpin}_5)$ (resp. $\Pi(\mathbf{Sp}_4)$) is the set of equivalence classes of irreducible admissible representations of $\mathbf{GSpin}_5(F)$ (resp. $\mathbf{Sp}_4(F)$), $\Phi(\mathbf{GSpin}_5)$ (resp. $\Phi(\mathbf{Sp}_4)$) is the set of equivalence classes of continuous semisimple representations of WD_F valued in $\mathrm{GSp}_4(\mathbb{C})$ (resp. $\mathrm{SO}_5(\mathbb{C})$), the horizontal arrows are the local Langlands correspondences, and pr is the projection $\mathrm{GSp}_4(\mathbb{C}) \to \mathrm{PGSp}_4(\mathbb{C}) \cong \mathrm{SO}_5(\mathbb{C})$. The left hand vertical arrow is not in fact a map at all, but a correspondence, given by taking any restriction of an element of $\Pi(\mathbf{GSpin}_5)$ to $\mathbf{Sp}_4(F)$.

Of course, [Art13] gives another definition of the local Langlands correspondence for $\mathbf{Sp}_4$, which is characterised by twisted endoscopy for $(\mathbf{GL}_5, g \mapsto {}^t g^{-1})$. It is not obvious that this correspondence agrees with that of [GT10]; this amounts to proving the commutativity of the diagram (8.1.1), where now the horizontal arrows are the correspondences characterised by twisted endoscopy. Proving this

is the main point of this section; we will also prove a refinement, describing the constituents of the restrictions of representations of $\mathbf{GSpin}_5(F)$ to $\mathbf{Sp}_4(F)$ in terms of the parameterisation of L-packets.

We begin by recalling some results about restriction of admissible representations, most of which go back to [GK82], and are explained in the context of $\mathbf{GSp}_{2n}$ in [GT10]. They are also proved in [Xu16], which we refer to as a self-contained reference. If $\widetilde{\pi}$ is an irreducible admissible representation of $\mathbf{GSpin}_5(F)$, then $\widetilde{\pi}|_{\mathbf{Sp}_4(F)}$ is a direct sum of finitely many irreducible representations of $\mathbf{Sp}_4(F)$ ([Xu16, Lem. 6.1]), and these representations are pairwise non-isomorphic ([AP06, Thm. 1.4]). Furthermore if π is an irreducible admissible representation of $\mathbf{Sp}_4(F)$, then there exists an irreducible representation $\widetilde{\pi}$ of $\mathbf{GSpin}_5(F)$ whose restriction to $\mathbf{Sp}_4(F)$ contains π, and $\widetilde{\pi}$ is unique up to twisting by characters ([Xu16, Cor. 6.3, 6.4]). There is also an analogue of these statements for L-parameters, which is that L-parameters for $\mathbf{Sp}_4$ may be lifted to $\mathbf{GSpin}_5$, and such lifts are unique up to twist; see [GT10, Prop. 2.8] (see also [Lab85, Théorème 7.1] for a more general lifting result).

In particular, it follows that if $\pi \in \Pi(\mathbf{Sp}_4)$, and $\widetilde{\pi}$ lifts π, with L-parameter $\varphi_{\widetilde{\pi}}$, then $\mathrm{pr} \circ \varphi_{\widetilde{\pi}}$ depends only on π (because $\varphi_{\widetilde{\pi}}$ is well-defined up to twist, as $\widetilde{\pi}$ itself is); we need to show that it is equal to the L-parameter of π defined by the local Langlands correspondence of [Art13].

Theorem 8.1.2. *The diagram* (8.1.1) *commutes, where the horizontal arrows are the correspondences of* [Art13, Art04] *determined by twisted endoscopy; that is, the local Langlands correspondences for* $\mathbf{Sp}_4$ *of* [GT10] *and* [Art13] *coincide.*

Proof. By the preceding discussion, it suffices to show that for each irreducible admissible representation π, there is some lift $\widetilde{\pi}$ of π such that $\varphi_\pi = \mathrm{pr} \circ \varphi_{\widetilde{\pi}}$.

We begin with the case that π is a discrete series representation. Then by [Clo86, Thm. 1B] and Krasner's lemma, we can find a totally real number field K, a finite place v of K, and a discrete automorphic representation Π of $\mathbf{Sp}_4(\mathbb{A}_K)$, such that:

(1) $K_v \cong F$ (so we identify K_v with F from now on).
(2) $\Pi_v \simeq \pi$.
(3) at each infinite place w of K, Π_w is a discrete series representation.
(4) for some finite place w of K, Π_w is a discrete series representation whose parameter is irreducible when composed with $\mathrm{Std}_{\mathbf{Sp}_4} : \mathrm{SO}_5 \to \mathrm{GL}_5$ (for example the parameter which is trivial on W_{K_w} and the "principal SL_2" on $\mathrm{SU}(2)$).

By Theorem 5.1.2, there is a discrete automorphic representation $\widetilde{\Pi}$ of $\mathbf{GSpin}_5(\mathbb{A}_K)$ such that $\widetilde{\Pi}|_{\mathbf{Sp}_4(\mathbb{A}_K)}$ contains Π. We can and do assume that the infinitesimal character of Π is sufficiently regular, so that in particular the parameter ψ of Π is tempered. By (4) above, ψ is just a self-dual representation for $\mathbf{GL}_5/K$ with trivial central character. Write $\widetilde{\psi}$ for the parameter of $\widetilde{\Pi}$.

As in the proof of Proposition 6.1.7 2 (i.e. comparing at the unramified places using (5.1.3)), we see that $1 \boxplus \psi = \bigwedge^2(\widetilde{\psi}) \otimes \omega_{\widetilde{\psi}}^{-1}$. Given the possibilities in Remark 6.1.8 we see (using [GJ78] to rule out the case $\widetilde{\psi} = \pi[2]$, see the proof of Proposition 6.1.7 (1)) that $\widetilde{\psi}$ is necessarily tempered. If $\widetilde{\psi} = \pi_1 \boxplus \pi_2$ was of Yoshida type then we would have $\psi = 1 \boxplus (\pi_1 \boxtimes \pi_2^\vee)$, a contradiction with the fourth property of Π above. Therefore $\widetilde{\psi}$ is of general type, i.e. a χ-self-dual cuspidal automorphic

representation for $\mathbf{GL}_4/K$ of symplectic type for some character χ of $\mathbb{A}_K^\times/K^\times$. By the main results of [Kim03] and [Hen09], the Langlands parameter of $1 \boxplus \psi$ at v equals $\bigwedge^2(\mathrm{rec}(\widetilde{\psi}_v)) \otimes \mathrm{rec}(\omega_{\widetilde{\psi}})^{-1}$, which implies that $\varphi_{\Pi_v} = \mathrm{pr} \circ \varphi_{\widetilde{\Pi}_v}$. Taking $\widetilde{\pi} = \widetilde{\Pi}_v$, we are done in this case.

We now treat the case that the parameter φ_π is not discrete, but is bounded modulo centre. Recall that a minimal Levi subgroup ${}^L\mathbf{M}$ of ${}^L\mathbf{Sp}_4$ such that $\varphi_\pi(\mathrm{WD}_F) \subset {}^L\mathbf{M}$ is unique up to conjugation by $\mathrm{Cent}\,(\varphi_\pi, \widehat{\mathbf{Sp}_4})$ [Bor79, Proposition 3.6]. Then φ_π factors through a well-defined discrete parameter $\varphi_{\mathbf{M}} : \mathrm{WD}_F \to {}^L\mathbf{M}$. Since $\mathbf{Sp}_4$ is quasi-split we have a natural identification of ${}^L\mathbf{M}$ with the L-group of a Levi subgroup $\mathbf{M}$ of $\mathbf{GSp}_4$ (well-defined up to conjugation by normalisers in $\mathbf{Sp}_4$, resp. $\widehat{\mathbf{Sp}_4}$). Since φ_π is assumed to be non-discrete we have ${}^L\mathbf{M} \neq {}^L\mathbf{Sp}_4$. It follows from the construction in [Art13] (see the proof of Proposition 2.4.3 loc. cit., in particular (2.4.13)) that π is a constituent of the parabolic induction $\mathrm{Ind}_{\mathbf{P}(F)}^{\mathbf{G}(F)} \pi_{\mathbf{M}}$, where $\mathbf{P}$ is any parabolic subgroup of $\mathbf{Sp}_4$ with Levi $\mathbf{M}$, and $\pi_{\mathbf{M}}$ is in the L-packet of $\varphi_{\mathbf{M}}$. Recall that this L-packet is defined via the natural identification $\mathbf{M}$ with a product of copies of $\mathbf{GL}$ groups with $\mathbf{Sp}_{2a}$ for some $0 \leq a < 2$, using rec for the $\mathbf{GL}$ factors and Arthur's local Langlands correspondence for the $\mathbf{Sp}$ factor.

Write $\mathbf{M} = \widetilde{\mathbf{M}} \cap \mathbf{Sp}_4$ where $\widetilde{\mathbf{M}}$ is a Levi subgroup of $\mathbf{GSp}_4$, and similarly $\mathbf{P} = \widetilde{\mathbf{P}} \cap \mathbf{Sp}_4$. Let $\widetilde{\pi_{\mathbf{M}}}$ be an essentially discrete series representation of $\widetilde{\mathbf{M}}(F)$ whose restriction to $\mathbf{M}(F)$ contains $\pi_{\mathbf{M}}$. Then there is an irreducible constituent $\widetilde{\pi}$ of the (semisimple) parabolic induction $\mathrm{Ind}_{\widetilde{\mathbf{P}}(F)}^{\mathbf{GSpin}_5(F)} \widetilde{\pi_{\mathbf{M}}}$ such that π is a restriction of $\widetilde{\pi}$. We will prove that $\varphi_\pi = \mathrm{pr} \circ \varphi_{\widetilde{\pi}}$. Note that for non-discrete parameters, the local Langlands correspondence for $\mathbf{GSpin}_5(F)$ of [GT11a] is also compatible with parabolic induction (see [CG15, §6.6] and [GT11b, Prop. 13.1]), i.e. the parameter of $\widetilde{\pi}$ is $\varphi_{\widetilde{\pi_{\mathbf{M}}}}$ (the Langlands parameter of $\widetilde{\pi_{\mathbf{M}}}$) composed with ${}^L\widetilde{\mathbf{M}} \subset {}^L\mathbf{GSpin}_5$. Note that $\widetilde{\mathbf{M}}$ is isomorphic to a product of $\mathbf{GL}$ and for such a group the (bijective) local Langlands correspondence is well-defined, i.e. it does not depend on the choice of an isomorphism. This follows from compatibility of rec with twisting, central characters and duality. The same argument shows that any morphism with normal image between two such groups is also compatible with the local Langlands correspondence. We have a commutative diagram

$$\begin{array}{ccc} {}^L\widetilde{\mathbf{M}} & \longrightarrow & {}^L\mathbf{GSpin}_5 \\ \big\downarrow \mathrm{pr} & & \big\downarrow \mathrm{pr} \\ {}^L\mathbf{M} & \longrightarrow & {}^L\mathbf{Sp}_4 \end{array}$$

so that to conclude that $\varphi_\pi = \mathrm{pr} \circ \varphi_{\widetilde{\pi}}$ it is enough to show that $\varphi_{\mathbf{M}} = \mathrm{pr} \circ \varphi_{\widetilde{\pi_{\mathbf{M}}}}$, which is simply a compatibility of local Langlands correspondences for $\mathbf{M}$ and $\widetilde{\mathbf{M}}$. There are three cases to consider. We write the standard parabolic subgroups of $\mathbf{GSpin}_5$ and $\mathbf{Sp}_4$ as in Section 2.2. We do not justify the embedding $\mathbf{M} \hookrightarrow \widetilde{\mathbf{M}}$, as this is a simple but tedious exercise in root data.

- $\widetilde{\mathbf{M}} = \mathbf{GL}_1^2 \times \mathbf{GSpin}_1$, $\mathbf{M} = \mathbf{GL}_1^2$, the embedding $\mathbf{M} \hookrightarrow \widetilde{\mathbf{M}}$ is $(x_1, x_2) \mapsto (x_1 x_2, x_1/x_2, x_1^{-1})$. This case is trivial.
- $\widetilde{\mathbf{M}} = \mathbf{GL}_2 \times \mathbf{GSpin}_1$, $\mathbf{M} = \mathbf{Sp}_2 \times \mathbf{GL}_1$, the embedding $\mathbf{M} \hookrightarrow \widetilde{\mathbf{M}}$ is $(g, x_1) \mapsto (gx_1, x_1^{-1})$. This case is not formal as for the factor $\mathbf{Sp}_2$ the local Langlands correspondence that is used is Arthur's from [Art13] and it is not

obvious that it is compatible with rec for $\mathbf{GL}_2$, in other words that Arthur's local Langlands correspondence for $\mathbf{Sp}_2 \simeq \mathbf{SL}_2$ (characterised by twisted endoscopy for $\mathbf{GL}_3$) coincides with Labesse-Langlands [LL79]. Fortunately Arthur verified this compatibility in [Art13, Lemma 6.6.2].

- $\widetilde{\mathbf{M}} = \mathbf{GL}_1 \times \mathbf{GSpin}_3$, $\mathbf{M} = \mathbf{GL}_1 \times \mathbf{GL}_2$, the embedding $\mathbf{M} \hookrightarrow \widetilde{\mathbf{M}}$ is $g \mapsto (\det g, g/\det g)$ where we have identified $\mathbf{GSpin}_3$ with $\mathbf{GL}_2$. This case also follows from the above remark about the local Langlands correspondence for groups isomorphic to a product of $\mathbf{GL}$.

Finally, we must treat the case that φ is not bounded modulo centre. The description of the L-packets in this case is again in terms of parabolic inductions from Levi subgroups ("Langlands classification"). This is well-known and completely general (see [Sil78], [SZ14]). The argument is similar to the above reduction, except that $\mathbf{P}$ and $\widetilde{\mathbf{P}}$ are uniquely determined by a positivity condition and that π and $\widetilde{\pi}$ are unique quotients of standard modules and not arbitrary constituents. We do not repeat the argument. □

We now examine the restriction from $\mathbf{GSpin}_5(F)$ to $\mathbf{Sp}_4(F)$ more closely, proving a slight refinement of the results of [GT10]. In [GT10, App. A], a detailed qualitative description of the constituents of $\widetilde{\pi}|_{\mathbf{Sp}_4(F)}$ is given, which is obtained by examining the local Langlands correspondence (see [GT10, §5, 6] for the corresponding calculations with L-parameters). However, since the local Langlands correspondence of [GT10] is not characterised in terms of twisted and ordinary endoscopic character relations, they cannot describe precisely which elements of the L-packets for $\mathbf{Sp}_4(F)$ arise as the restrictions of given elements of the L-packets for $\mathbf{GSpin}_5(F)$.

Theorem 8.3.2 below answers this question. In its proof, we need to make use of the results of Section 5 for $\mathbf{SO}_4 \hookrightarrow \mathbf{H}$ where

$$\mathbf{H} = (\mathbf{GL}_2 \times \mathbf{GL}_2) / \{(zI_2, z^{-1}I_2 \mid z \in \mathbf{GL}_1\}$$

is the non-trivial elliptic endoscopic group of $\mathbf{GSpin}_5$. Here $\mathbf{SO}_4$ is identified with the subgroup of pairs (a, b) with $(\det a)(\det b) = 1$. Indeed, $\mathbf{H}$ may be identified with the subgroup $\mathbf{GSO}_4$ of $\mathbf{GO}_4$ given by the elements for which $\det = \nu^2$, where ν is the similitude factor.

Note that $\mathbf{SO}_4$ is an elliptic endoscopic group for $\mathbf{Sp}_4$ and that we have the following commutative diagram:

$$\begin{array}{ccc} \widehat{\mathbf{H}} & \longrightarrow & \widehat{\mathbf{SO}_4} = \mathrm{SO}_4 \\ \downarrow{\scriptstyle {}^L\xi'} & & \downarrow{\scriptstyle {}^L\xi'} \\ \widehat{\mathbf{GSpin}_5} = \mathrm{GSp}_4 & \longrightarrow & \widehat{\mathbf{Sp}_4} = \mathrm{SO}_5 \end{array} \tag{8.1.3}$$

8.2. **Multiplicity one.** In studying restriction from $\mathbf{H}$ to $\mathbf{SO}_4$ we will make use of the following variant of the results of [AP06]. In fact, we could prove the special case that we need in a simpler but more ad-hoc fashion by using the description of $\mathbf{H}$ in terms of $\mathbf{GL}_2$, but it seems worthwhile to prove this more general result.

Proposition 8.2.1. *Let $n \geq 1$, and let V be a vector space of dimension $2n$ over F endowed with a non-degenerate quadratic form q. Let π be an irreducible admissible representation of $\mathrm{GSO}(V, q) = \mathbf{GSO}(V, q)(F)$. Then the irreducible constituents of the restriction $\pi|_{\mathrm{SO}(V,q)}$ are pairwise non-isomorphic.*

Proof. By [AP06, Theorem 2.3], it suffices to show that there is an algebraic anti-involution τ of $\mathbf{GSO}(V,q)$ which preserves $\mathbf{SO}(V,q)$ and takes each $\mathrm{SO}(V,q)$-conjugacy class in $\mathrm{GSO}(V,q)$ to itself. To define τ, we set $\tau(g) = \nu(g)\delta^n g^{-1}\delta^{-n}$ where $\delta \in \mathrm{O}(V,q)$ is an involution with $\det\delta = -1$. This obviously preserves $\mathbf{SO}(V,q)$, so we need only check that it also preserves $\mathrm{SO}(V,q)$-conjugacy classes in $\mathrm{GSO}(V,q)$.

To see this, we claim that it is enough to show that we can write $g = xy$ with $x \in \mathrm{O}(V,q)$, $y \in \mathrm{GO}(V,q)$ (so $\nu(y) = \nu(g)$) satisfying $x^2 = 1$, $\det(x) = (-1)^n$, $y^2 = \nu(y)$. Indeed, we then have

$$\tau(g) = \nu(g)\delta^n g^{-1}\delta^{-n} = \delta^n \nu(y) y^{-1}x^{-1}\delta^{-n} = \delta^n yx\delta^{-n} = \delta^n x^{-1}(xy)x\delta^{-n} = (x\delta^{-n})^{-1}g(x\delta^{-n}),$$

as required. The result then follows from Lemma 8.2.2 below, which is a slight refinement of [RV18, Thm. A]. □

Lemma 8.2.2. *Let $n \geq 0$, let K be a field of characteristic 0, and let V be a vector space of dimension $2n$ over K endowed with a non-degenerate quadratic form q. If $g \in \mathrm{GSO}(V,q)$ then we can write $g = xy$ with $x \in \mathrm{O}(V,q)$, $y \in \mathrm{GO}(V,q)$ satisfying $x^2 = 1$, $\det(x) = (-1)^n$, $y^2 = \nu(y)$.*

Proof. We argue by induction on n, the case $n = 0$ being trivial. Suppose now that $n > 0$. By [RV18, Thm. A], we can write $g = xy$ with $x \in \mathrm{O}(V,q)$, $y \in \mathrm{GO}(V,q)$ satisfying $x^2 = 1$, $y^2 = \nu(y) = \nu(g)$. If $\det(x) = (-1)^n$ then we are done, so suppose that $\det(x) = (-1)^{n+1}$ and so $\det(y) = (-1)^{n+1}\nu(y)^n$.

Since $y^2 = \nu(y)$, any eigenvalue (in an extension of K) of y is a square root of $\nu(y)$. Since $\det(y) = (-1)^{n+1}\nu(y)^n$, we see that the two eigenspaces of y do not have equal dimension. It follows that $\nu(y)$ is a square, as otherwise the characteristic polynomial of y would be a power of the irreducible polynomial $X^2 - \nu(y)$. So the eigenvalues of y are in K, and up to dividing g and y by one of these eigenvalues we can assume that $g \in \mathrm{SO}(V,q)$ and $y \in \mathrm{O}(V,q)$ with $\det(y) = (-1)^{n+1}$. Then y has an eigenspace (for an eigenvalue ± 1) of dimension at least $n+1$. The same analysis applies to x, and it follows that there is a subspace W (the intersection of these eigenspaces for x and y) of dimension at least 2 of V on which g acts by a scalar which is ± 1.

Up to replacing g by $-g$ and y by $-y$, we can assume that $\ker(g-1)$ has dimension at least 2. We have a canonical g-stable decomposition of V as the direct sum of $\ker((g-1)^{2n})$ and its orthogonal complement, and they both have even dimension over K since $g \in \mathrm{SO}(V,q)$ with $\dim_K V$ even. If g is not unipotent, we conclude using the induction hypothesis for the restriction of g to $\ker((g-1)^{2n})$ and to its orthogonal complement.

Suppose for the rest of the proof that g is unipotent. If $n = 1$ the conclusion is trivial, so assume that $n > 1$, so that $\mathbf{SO}(V,q)$ is semisimple. By Jacobson–Morozov (see for example [Bou05, Ch. VIII §11]) there is an algebraic morphism $\mathbf{SL}_2 \to \mathbf{SO}(V,q)$ mapping $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to g, unique up to conjugation by the centraliser of g in the subgroup $\mathrm{Aut}_e(\mathfrak{so}(V,q))$ of $\mathrm{SO}(V,q)/\{\pm 1\}$ where Aut_e is the subgroup of automorphisms of the Lie algebra generated by exponentials of nilpotent elements. For $d \geq 1$ fix an irreducible representation U_d of $\mathbf{SL}_2$ of dimension d as well as a non-degenerate $(-1)^{d-1}$-symmetric $\mathbf{SL}_2$-invariant pairing B_d on U_d. We have a canonical decomposition

$$V = \bigoplus_{d \geq 1} U_d \otimes V_d$$

where $V_d = (V \otimes_K U_d^*)^{\mathbf{SL}_2}$. The quadratic form q corresponds to an element of

$$(\mathrm{Sym}^2 V^*)^{\mathbf{SL}_2} = \bigoplus_{d\geq 1 \text{ odd}} \mathrm{Sym}^2(V_d^*) \oplus \bigoplus_{d\geq 2 \text{ even}} \bigwedge^2 V_d^*$$

and non-degeneracy of q is equivalent to non-degeneracy of each factor. Writing each V_d for d odd (resp. even) as an orthogonal direct sum of quadratic lines (resp. planes endowed with a non-degenerate alternate form), we are left to prove a decomposition $g' = x'y'$ in the following cases.

(1) V' has odd dimension $2m+1$ and is endowed with a non-degenerate quadratic form q' and a unipotent automorphism g'. Applying [RV18, Thm. A] we obtain $g' = x'y'$ with x', y' involutions in $\mathrm{O}(V, q)$. Up to replacing (x', y') by $(-x', -y')$ we can assume that $\det(x')$ is ± 1 as we may desire.
(2) $V' = U_{2m} \otimes V'''$ where V''' is 2-dimensional and endowed with a non-degenerate alternating form B''', and $g' = g'' \otimes \mathrm{Id}_{V'''} \in \mathrm{SO}(V', q')$ for q' the quadratic form corresponding to the symmetric bilinear form $B' = B_{2m} \otimes B'''$ and g'' a unipotent element of $\mathrm{Sp}(U_{2m}, B_{2m})$. Applying [RV18, Thm. A] again we can write $g'' = x''y''$ where x'', y'' are involutions in $\mathrm{GSp}(U_{2m}, B_{2m})$ having similitude factor -1. Similarly write $\mathrm{Id}_{V'''} = x'''y'''$ where x''', y''' are involutions in $\mathrm{GSp}(V''', B''')$ having similitude factor -1. Then $g' = (x'' \otimes x''')(y'' \otimes y''')$ is the desired decomposition as a product of involutions in $\mathrm{SO}(V', q')$. □

8.3. **Restriction of local Arthur packets.** We now give our description of the restriction of representations of $\mathbf{GSpin}_5(F)$. Recall that if $\varphi : \mathrm{WD}_F \to \mathrm{GSp}_4$ is a bounded parameter, then the corresponding component group $\mathcal{S}_\varphi$ is either trivial or is $\mathbb{Z}/2\mathbb{Z} = \{1, s\}$. In the former case, the L-packet Π_φ associated to φ is a singleton, and in the latter case it is a pair $\{\pi^+, \pi^-\}$, where $\pi^\pm$ is characterised by the fact that $\mathrm{tr}\,\pi^+ - \mathrm{tr}\,\pi^-$ is the transfer to $\mathbf{GSpin}_5(F)$ of $\mathrm{tr}\,\pi_{\varphi_{\mathbf{H}}}$ where $\varphi_{\mathbf{H}} \in \Phi(\mathbf{H})$ is the parameter mapping to (φ, s) via ${}^L\xi'$. In either case, if we write $\varphi' = \mathrm{pr} \circ \varphi$, then by [GT10, Prop. 2.8], we have

$$\bigoplus_{\pi \in \Pi_\varphi} \pi|_{\mathbf{Sp}_4(F)} \cong \bigoplus_{\pi' \in \Pi_{\varphi'}} \pi'. \tag{8.3.1}$$

(Indeed, this follows from Theorem 8.1.2, the fact that lifts of representations of $\mathbf{Sp}_4(F)$ to $\mathbf{GSp}_4(F)$ are unique up to twist, and the fact that the restrictions of representations of $\mathbf{GSp}_4(F)$ to $\mathbf{Sp}_4(F)$ are semisimple and multiplicity free.) The following theorem improves on this result by giving a precise description of the restrictions of the individual elements of Π_φ.

Theorem 8.3.2. *Let φ be a bounded L-parameter, and write $\varphi' = \mathrm{pr} \circ \varphi$, so that $\mathcal{S}_\varphi \hookrightarrow \mathcal{S}_{\varphi'}$. Write Π_φ and $\Pi_{\varphi'}$ for the respective L-packets. If $\mathcal{S}_\varphi$ is trivial, and $\Pi_\varphi = \{\pi\}$, then*

$$\pi|_{\mathbf{Sp}_4(F)} \cong \bigoplus_{\pi' \in \Pi_{\varphi'}} \pi'.$$

If $\mathcal{S}_\varphi = \mathbb{Z}/2\mathbb{Z} = \{1, s\}$, and $\Pi_\varphi = \{\pi^+, \pi^-\}$ as above, then

$$\pi^\pm|_{\mathbf{Sp}_4(F)} \cong \bigoplus_{\substack{\pi' \in \Pi_{\varphi'} \\ \langle s, \pi' \rangle = \pm 1}} \pi'.$$

Proof. In the case that $\mathcal{S}_\varphi$ is trivial, this is (8.3.1), so we may suppose that $\mathcal{S}_\varphi$ is non-trivial, so that φ is endoscopic. We can write $\varphi = \varphi_1 \oplus \varphi_2$ where $\varphi_1, \varphi_2 : \mathrm{WD}_F \to \mathrm{GL}_2$ are bounded with same determinant; that is, $\varphi = {}^L\xi' \circ \varphi_{\mathbf{H}}$, where $\varphi_{\mathbf{H}} = \varphi_1 \times \varphi_2 : \mathrm{WD}_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{\mathbf{H}}$. Via ${}^L\xi'$ we can see s as the non-trivial element of $Z(\widehat{\mathbf{H}})/Z(\widehat{\mathbf{GSpin}}_5)$, i.e. the image of $(1,-1) \in \widehat{\mathbf{H}} \subset \mathrm{GL}_2 \times \mathrm{GL}_2$. Then by Conjecture 2.4.2 (2) for $\mathbf{GSpin}_5$ (i.e. the main theorem of [CG15]), we have an equality of traces

$$\operatorname{tr} \pi^+(f) - \operatorname{tr} \pi^-(f) = \sum_{\pi_{\mathbf{H}} \in \Pi_{\varphi_{\mathbf{H}}}} \operatorname{tr} \pi_{\mathbf{H}}(f^{\mathbf{H}}).$$

Applying Conjecture 2.4.2 (2) (or rather Theorem 2.6.1) for $\mathbf{Sp}_4$, and writing $\varphi'_{\mathbf{H}}$ for the composite of $\varphi_{\mathbf{H}}$ and the natural map $\widehat{\mathbf{H}} \to \widehat{\mathbf{SO}}_4$, we also have an equality of traces

$$\sum_{\substack{\pi' \in \Pi_{\varphi'} \\ \langle s, \pi' \rangle = 1}} \operatorname{tr} \pi'(f) - \sum_{\substack{\pi' \in \Pi_{\varphi'} \\ \langle s, \pi' \rangle = -1}} \operatorname{tr} \pi'(f) = \sum_{\pi'_{\mathbf{SO}_4} \in \Pi_{\varphi'_{\mathbf{H}}}} \operatorname{tr} \pi'_{\mathbf{SO}_4}(f').$$

The result now follows from (8.3.1) and Theorem 8.3.3 below. □

We end with a result on the restriction of representations from $\mathbf{H} \simeq \mathbf{GSO}_4$ to $\mathbf{SO}_4$ that we used in the course of the proof of Theorem 8.3.2. The arguments are very similar to those for $\mathbf{GSpin}_5$, but are rather simpler, as $\mathbf{H}$ has no non-trivial elliptic endoscopic groups. Since $\mathbf{H}$ is isomorphic to the quotient of $\mathbf{GL}_2 \times \mathbf{GL}_2$ by a split torus, the local Langlands correspondence for $\mathbf{H}$, and the corresponding endoscopic character identities, are easily deduced from those for $\mathbf{GL}_2$. The correspondence and endoscopic character identities for $\mathbf{SO}_4$ are of course proved in [Art13] (up to the outer automorphism δ).

By Proposition 8.2.1, if π is an irreducible admissible representation of $\mathbf{H}(F)$, then $\pi|_{\mathbf{SO}_4(F)}$ is a direct sum of representations occurring with multiplicity one. The proof of [GT10, Lem. 2.6] goes through unchanged and shows that $\pi_1|_{\mathbf{SO}_4(F)}$, $\pi_2|_{\mathbf{SO}_4(F)}$ have a common constituent if and only if π_1, π_2 differ by a twist by a character. By [GT10, Lem. 2.7], the analogous statement is also true for L-parameters: every L-parameter $\varphi' : \mathrm{WD}_F \to \widehat{\mathbf{SO}}_4(\mathbb{C})$ arises from some $\varphi : \mathrm{WD}_F \to \widehat{\mathbf{H}}(\mathbb{C})$, which is unique up to twist.

Theorem 8.3.3. *Let $\varphi : \mathrm{WD}_F \to \widehat{\mathbf{H}}(\mathbb{C})$ be a bounded L-parameter, and let $\varphi' : \mathrm{WD}_F \to \widehat{\mathbf{SO}}_4(\mathbb{C})$ be the parameter obtained from* (8.1.3)*. Let π be the tempered irreducible representation of $\mathbf{H}$ associated to φ. Then*

$$\pi|_{\widetilde{\mathcal{H}}(\mathbf{SO}_4(F))} \cong \bigoplus_{\pi' \in \Pi_{\varphi'}} \pi'.$$

Proof. By the preceding discussion, we need to show that for each bounded L-parameter $\varphi' : \mathrm{WD}_F \to \widehat{\mathbf{SO}}_4(\mathbb{C})$ (up to outer conjugacy), and each $\pi' \in \Pi_{\varphi'}$, there is some π lifting π' (or π'^{δ}) whose L-parameter φ lifts φ'.

Suppose firstly that φ' is discrete. As in the proof of Theorem 8.1.2, by Krasner's lemma and [Clo86, Thm. 1B], we can find a totally real number field K, a finite place v of K, and a discrete automorphic representation Π' of $\mathbf{SO}_4(\mathbb{A}_K)$, such that:

- $K_v \cong F$ (so we identify K_v with F from now on).
- $\Pi'_v = \pi'$.

- at each infinite place w of F, Π'_w is a discrete series representation.

By Theorem 5.1.2, there is a discrete automorphic representation Π of $\mathbf{H}(\mathbb{A}_K)$ such that $\Pi|_{\mathbf{SO}_4(\mathbb{A}_K)}$ contains Π'. Then Π corresponds to a pair π_1, π_2 of discrete automorphic representations of $\mathbf{GL}_2(\mathbb{A}_K)$ with equal central characters. The condition that Π'_w is a discrete series representation at an infinite place w of K implies that π_1 and π_2 are cuspidal.

We now consider the following commutative diagram of dual groups:

$$\begin{array}{ccc} \widehat{\mathbf{H}} & \twoheadrightarrow & \widehat{\mathbf{SO}_4} = \mathrm{SO}_4 \\ \downarrow & & \downarrow \\ \mathrm{GL}_2 \times \mathrm{GL}_2 & \longrightarrow & \mathrm{GL}_4 \end{array} \tag{8.3.4}$$

where the vertical arrows are the natural inclusions, and the lower horizontal arrow is given by $(g, h) \mapsto (\det g)^{-1} g \otimes h$. Since the functorial transfer from $\mathbf{GL}_2 \times \mathbf{GL}_2$ to $\mathbf{GL}_4$ exists (as we recalled at the beginning of Section 6), we may compare at the unramified places and then use strong multiplicity one to compare at the ramified places, and we obtain that the composite $\mathrm{WD}_F \xrightarrow{\varphi'} \widehat{\mathbf{H}} \to \mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_4$ is given by $\varphi_{1,v} \otimes \varphi_{2,v}^\vee$, where $\varphi_{1,v}$, $\varphi_{2,v}$ are the L-parameters of $\pi_{1,v}$ and $\pi_{2,v}$ respectively. Since the L-parameter of Π_v is $\varphi_{1,v} \oplus \varphi_{2,v}$, we can take $\pi = \Pi_v$, so we are done in the case that φ' is discrete.

Suppose now that φ' is not discrete. Then one can argue as in the proof of 8.1.2, since both local Langlands correspondences for $\mathbf{H}$ and $\mathbf{SO}_4$ are compatible with parabolic induction. In fact the proof is simpler since all proper Levi subgroups are simply products of $\mathbf{GL}$, and we do not repeat the argument. □

Remark 8.3.5. Theorem 8.3.2 (or rather its straightforward extension from tempered to generic parameters) gives the complete spectral description of the automorphic restriction map of Section 5 for $\mathbf{Sp}_4 \subset \mathbf{GSpin}_5$ *for formally tempered global parameters*. This is the analogue of the results of Labesse–Langlands [LL79] (ignoring inner forms) and the multiplicity one theorem of Ramakrishnan for $\mathbf{SL}_2$ [Ram00]. It would perhaps be interesting to extend this to parameters which are not formally tempered, but in the interests of brevity we do not consider this question here.

Appendix A. Classification of endoscopic data and surjectivity of transfer

In this appendix we denote $\mathbf{\Gamma} = \mathbf{GL}_N \times \mathbf{GL}_1$ over a local or global field F of characteristic zero. Let J be the anti-diagonal $N \times N$ matrix with $J_{i,N+1-i} = (-1)^i$. Let θ be the automorphism of $\mathbf{\Gamma}$ given by $\theta(g, x) = (J^t g^{-1} J^{-1}, x \det g)$. The matrix J was chosen so that the standard pinning $(\mathbf{B}, \mathbf{T}, ((E_{i,i+1}, 0))_{1 \le i \le N-1})$, where $\mathbf{T}$ is the diagonal torus and $\mathbf{B}$ the upper triangular Borel, is fixed by θ. A basis of $X_*(\mathbf{T})$ is given by $(e_1^*, \dots, e_N^*, z^*)$ where $e_i^*(x) = (\mathrm{diag}(1, \dots, x, \dots, 1), 1)$ (x is the i-th term) and $z^*(x) = (1, x)$. Let $(e_1, \dots, e_N, z)$ be the dual basis of $X^*(\mathbf{T})$. Then the roots of $\mathbf{T}$ are $e_i - e_j$ for $i \ne j$, the positive ones (with respect to $\mathbf{B}$) being those for which $i < j$. The Langlands dual group $\widehat{\mathbf{\Gamma}}$ is also isomorphic to $\mathrm{GL}_N \times \mathrm{GL}_1$ (now over $\mathbb{C}$), and we also fix the usual (upper triangular and diagonal) Borel pair $(\mathcal{B}, \mathcal{T})$ of $\widehat{\mathbf{\Gamma}}$. To make the identification explicit, for $y \in \mathbb{C}^\times$ we have $e_i(y) = (\mathrm{diag}(1, \dots, y, \dots, 1), 1)$ and $z(y) = (1, y)$. We also fix the usual pinning consisting of the elements $(E_{i,i+1}, 0)$ of $\mathrm{Lie}\, \mathcal{B}$.

Let $\widetilde{\mathbf{\Gamma}}$ be the twisted space $\mathbf{\Gamma} \rtimes \theta$. A simple computation shows that the automorphism $\widehat{\theta}$ of $\widehat{\mathbf{\Gamma}}$ dual to θ (preserving the chosen pinning of $\widehat{\mathbf{\Gamma}}$) is

$$(g, x) \mapsto (\widehat{J}^t g^{-1} \widehat{J}^{-1} x, x)$$

where $\widehat{J}$ is J (but now over $\mathbb{C}$). It extends to an automorphism ${}^L\theta$ of ${}^L\mathbf{\Gamma} = \widehat{\mathbf{\Gamma}} \times W_F$ which acts trivially on W_F. Recall that an endoscopic datum of $\widetilde{\mathbf{\Gamma}}$ is a quadruple $(\mathbf{G}, \mathcal{G}, s, \xi)$ where

- $\mathbf{G}$ is a quasi-split connected reductive group over F,
- $\xi : \widehat{\mathbf{G}} \to \widehat{\mathbf{\Gamma}}$ is a continuous embedding,
- $\mathcal{G}$ is a closed subgroup of ${}^L\mathbf{\Gamma}$ which surjects onto W_F with kernel $\xi(\widehat{\mathbf{G}})$, such that the induced outer action of W_F on $\xi(\widehat{\mathbf{G}})$ coincides with the usual one on $\widehat{\mathbf{G}}$ transported by ξ, and such that there exists a continuous splitting $W_F \to \mathcal{G}$,
- and $s \in \widehat{\mathbf{\Gamma}}$ is such that $(\mathrm{Ad}s) \circ \widehat{\theta}$ is quasi-semisimple (i.e. it stabilizes a Borel pair of $\widehat{\mathbf{\Gamma}}$), $(\widehat{\mathbf{\Gamma}}^{(\mathrm{Ad}s)\circ\widehat{\theta}})^0 = \xi(\widehat{\mathbf{G}})$ and such that the map $W_F \to \widehat{\mathbf{\Gamma}}$ induced by $h \in \mathcal{G} \mapsto s\,{}^L\theta(h) s^{-1} h^{-1}$ takes values in $Z(\widehat{\mathbf{\Gamma}})$ and defines an element of $H^1(W_F, Z(\widehat{\mathbf{\Gamma}}))$ which is trivial at every place of F.

Instead of giving s one could also give $\tilde{s} = s \rtimes \widehat{\theta}$ which belongs to the twisted space ${}^L\widetilde{\mathbf{\Gamma}} := {}^L\mathbf{\Gamma} \rtimes \widehat{\theta}$ for the group ${}^L\mathbf{\Gamma}$. The action of $\mathrm{Gal}(\overline{F}/F)$ on $Z(\widehat{\mathbf{\Gamma}})$ is trivial so this cocycle $W_F \to Z(\widehat{\mathbf{\Gamma}})$ is in fact trivial and we simply have $\mathcal{G} \subset \widehat{\mathbf{\Gamma}}^{(\mathrm{Ad}s)\circ\widehat{\theta}} \times W_F$. It is clear that the endoscopic datum $(\mathbf{G}, \mathcal{G}, s, \xi)$ can be recovered from s and the locally constant morphism $\alpha : \mathrm{Gal}(\overline{F}/F) \to \pi_0(\widehat{\mathbf{\Gamma}}^{(\mathrm{Ad}s)\circ\widehat{\theta}})$ such that

$$\xi(\mathcal{G}) = \{g \times \sigma \in \widehat{\mathbf{\Gamma}}^{(\mathrm{Ad}s)\circ\widehat{\theta}} \times W_F \,|\, g \in \alpha(\sigma)\}.$$

So to classify endoscopic data up to isomorphism it is enough to classify $\widehat{\mathbf{\Gamma}}$-conjugacy classes of elements $s \rtimes \widehat{\theta} \in \widehat{\mathbf{\Gamma}} \rtimes \widehat{\theta}$ such that $(\mathrm{Ad}s) \circ \widehat{\theta}$ is quasi-semisimple and to determine $\pi_0(\widehat{\mathbf{\Gamma}}^{(\mathrm{Ad}s)\circ\widehat{\theta}})$.

A.1. **Conjugacy classes in $\widehat{\mathbf{\Gamma}} \rtimes \widehat{\theta}$ and centralizers.** Let us first consider conjugation by $\widehat{\mathbf{\Gamma}}$ in $\widehat{\mathbf{\Gamma}} \rtimes \widehat{\theta}$. For $(y, t) \in \widehat{\mathbf{\Gamma}}$ and $(s_N, s_1) \in \widehat{\mathbf{\Gamma}}$ we compute

$$(y, t)((s_N, s_1) \rtimes \widehat{\theta})(y^{-1}, t^{-1}) = (y s_N \widehat{J}^t y \widehat{J}^{-1} t^{-1}, s_1) \rtimes \widehat{\theta}.$$

Thus the map $s \rtimes \widehat{\theta} \mapsto (\widehat{J}^{-1} s_N^{-1}, s_1)$ is a bijection $\widehat{\mathbf{\Gamma}} \rtimes \widehat{\theta} \simeq \mathrm{GL}_N \times \mathrm{GL}_1$ which intertwines the conjugation action of $\widehat{\mathbf{\Gamma}}$ on $\widehat{\mathbf{\Gamma}} \rtimes \widehat{\theta}$ with the action on $\mathrm{GL}_N \times \mathrm{GL}_1$ given by the formula

$$(y, t) \cdot (h, u) = ({}^t y^{-1} h y^{-1} t, u).$$

In particular, denoting $h = \widehat{J}^{-1} s_N^{-1}$ we see that $\widehat{\mathbf{\Gamma}}^{(\mathrm{Ad}s)\circ\widehat{\theta}}$ equals

$$\mathrm{GAut}(h) = \{(y, t) \in \widehat{\mathbf{\Gamma}} \,|\, {}^t y h y = t h\}.$$

Denote by ν_h the morphism $\mathrm{GAut}(h) \to \mathrm{GL}_1$, $(y, t) \mapsto t$. Denote $h_{\mathrm{sym}} = (h + {}^t h)/2$ for the symmetric part and $h_{\mathrm{anti}} = (h - {}^t h)/2$ for the antisymmetric part of h. Note that h defines a bilinear form $B : (X, Y) \mapsto {}^t X h Y$ on $V := \mathbb{C}^N$, and that the similarly defined bilinear form B_{sym} (resp. B_{anti}) associated to h_{sym} (resp. h_{anti}) is symmetric (resp. antisymmetric). The decomposition $h = h_{\mathrm{sym}} + h_{\mathrm{anti}}$ is canonical. In particular we have $\mathrm{GAut}(h) = \{g \in \mathrm{GAut}(h_{\mathrm{sym}}) \cap \mathrm{GAut}(h_{\mathrm{anti}}) \,|\, \nu_{h_{\mathrm{sym}}}(g) = \nu_{h_{\mathrm{anti}}}(g)\}$.

Lemma A.1.1. *(1) Let $V_{\mathrm{sym}} = \ker B_{\mathrm{anti}}$ and $V_{\mathrm{anti}} = \ker B_{\mathrm{sym}}$. Then $B_{\mathrm{sym}}|_{V_{\mathrm{sym}}}$ and $B_{\mathrm{anti}}|_{V_{\mathrm{anti}}}$ are non-degenerate.*

(2) Let $V_{\mathrm{sym}}^{\perp,B_{\mathrm{sym}}}$ be the orthogonal of V_{sym} in V with respect to B_{sym}. Let $V_{\mathrm{anti}}^{\perp,B_{\mathrm{anti}}}$ be the orthogonal of V_{anti} in V with respect to B_{anti}. Let $V_{\mathrm{both}} = V_{\mathrm{sym}}^{\perp,B_{\mathrm{sym}}} \cap V_{\mathrm{anti}}^{\perp,B_{\mathrm{anti}}}$. Then $V = V_{\mathrm{sym}} \oplus V_{\mathrm{anti}} \oplus V_{\mathrm{both}}$ and this decomposition is orthogonal with respect to B_{sym} and B_{anti}.

Proof. The condition $h \in \mathrm{GL}_N$ implies that $V_{\mathrm{sym}} \cap V_{\mathrm{anti}} = 0$, and that the restriction of B_{sym} to V_{sym} (resp. of B_{anti} to V_{anti}) is non-degenerate. The second point follows easily. □

This decomposition is clearly canonical. Both $B_{\mathrm{sym}}|_{V_{\mathrm{both}}}$ and $B_{\mathrm{anti}}|_{V_{\mathrm{both}}}$ are non-degenerate. Let φ be the endomorphism of V_{both} defined by $B_{\mathrm{anti}}(x, y) = B_{\mathrm{sym}}(x, \varphi(y))$ for all $x, y \in V_{\mathrm{both}}$.

Lemma A.1.2. *For any $\lambda \in \mathbb{C}$ we have $\ker(\varphi - \lambda) = (\mathrm{im}(\varphi - \lambda))^{\perp,B_{\mathrm{sym}}}$. The set of eigenvalues of φ is contained in $\mathbb{C} \smallsetminus \{-1, 0, 1\}$ and stable under $\lambda \mapsto -\lambda$.*

Proof. Easy. □

From now on we assume that $(\mathrm{Ad}s) \circ \varphi$ is quasi-semisimple.

Lemma A.1.3. *The endomorphism φ of V_{both} is semisimple.*

Proof. The hypothesis means that up to conjugating $s \rtimes \widehat{\theta}$ by an element of $\widehat{\mathbf{\Gamma}}$, we may assume that $s \in \mathcal{T}$. Then h is antidiagonal, say

$$h = \begin{pmatrix} & & h_1 \\ & \iddots & \\ h_N & & \end{pmatrix}. \tag{A.1.4}$$

There is a natural partition $\{1, \dots, N\} = I_{\mathrm{sym}} \sqcup I_{\mathrm{anti}} \sqcup I_{\mathrm{both}}$ where

$$\begin{aligned} I_{\mathrm{sym}} &= \{i \,|\, h_i = h_{N+1-i}\}, \\ I_{\mathrm{anti}} &= \{i \,|\, h_i = -h_{N+1-i}\}, \\ I_{\mathrm{both}} &= \{i \,|\, h_i^2 \neq h_{N+1-i}^2\}. \end{aligned}$$

Let e_i be the standard basis of $\mathbb{C}^N$. For $? \in \{\mathrm{sym}, \mathrm{anti}, \mathrm{both}\}$ the family $(e_i)_{i \in I_?}$ is a basis of $V_?$. In this basis of V_{both} the matrices of B_{sym} and B_{anti} are antidiagonal and so the matrix of φ is diagonal. □

In particular we have a canonical orthogonal (with respect to B_{sym} and B_{anti}) decomposition

$$V_{\mathrm{both}} = \bigoplus_{\mu \in \mathbb{C} \smallsetminus \{0,1\}}^{\perp} \bigoplus_{\substack{\lambda \in \mathbb{C} \\ \lambda^2 = \mu}} \ker(\varphi - \lambda)$$

and each $\ker(\varphi - \lambda)$ is totally isotropic for B_{sym} and B_{anti} and in perfect duality with $\ker(\varphi + \lambda)$ (using either bilinear form). Let R be a set of representatives for the action of $\{\pm 1\}$ on the set of eigenvalues of φ.

We obtain a canonical (except for the choice of R) identification

$$\mathrm{GAut}(h) \simeq \Big\{(g_{\mathrm{sym}}, g_{\mathrm{anti}}, (g_\lambda)_{\lambda\in R}) \;\Big|\; g_\lambda \in \mathrm{GL}(\ker(\varphi-\lambda)), g_{\mathrm{sym}} \in \mathrm{GAut}(V_{\mathrm{sym}}, B_{\mathrm{sym}}), \\ g_{\mathrm{anti}} \in \mathrm{GAut}(V_{\mathrm{anti}}, B_{\mathrm{anti}}) \text{ satisfying } \nu(g_{\mathrm{sym}}) = \nu(g_{\mathrm{anti}})\Big\}$$

obtained by restricting to the stable subspaces V_{sym}, V_{anti} and $\ker(\varphi-\lambda) \subset V_{\mathrm{both}}$ for $\lambda \in R$. To go from the right to the left, define for $\lambda \in R$ the element $g_\lambda \in \mathrm{GL}(\ker(\varphi+\lambda))$ determined by the relation

$$B_{\mathrm{sym}}(g_{-\lambda}(x), g_\lambda(y)) = \nu(g_{\mathrm{sym}}) B_{\mathrm{sym}}(x,y)$$

for all $(x,y) \in \ker(\varphi+\lambda) \times \ker(\varphi-\lambda)$.

A.2. **Endoscopic data.** Let S be the finite subset of $\mathbb{C}\smallsetminus\{0,1\}$ such that $\lambda^2 \in S$ if and only if $\ker(\varphi-\lambda) \neq 0$. Let $N_{\mathrm{sym}} = \dim V_{\mathrm{sym}}$, $N_{\mathrm{anti}} = \dim V_{\mathrm{anti}}$, and for $\mu \in S$ let $N_\mu = \dim \ker(\varphi-\lambda)$ (for either of the two λ such that $\lambda^2 = \mu$). We have $N = N_{\mathrm{sym}} + N_{\mathrm{anti}} + 2\sum_{\mu\in S} N_\mu$, in particular $N_{\mathrm{sym}} \equiv N \mod 2$. The group $\pi_0(\mathrm{GAut}(h))$ has one or two elements, and it has two if and only if $N_{\mathrm{sym}} > 0$. The characteristic polynomial of φ is clearly an invariant of the conjugacy class of $s\rtimes\widehat{\theta}$. We have associated a quintuple $(N_{\mathrm{sym}}, N_{\mathrm{anti}}, S, (N_\mu)_{\mu\in S}, \alpha)$ to any endoscopic datum for $\widetilde{\mathbf{\Gamma}}$. It is easy to check that two endoscopic data are isomorphic if and only if the associated quintuples are equal.

Conversely if we give ourselves:

- a finite set $S \subset \mathbb{C}\smallsetminus\{0,1\}$,
- a partition $N = N_{\mathrm{sym}} + N_{\mathrm{anti}} + 2\sum_{\mu\in S} N_\mu$ with $N_{\mathrm{sym}} \geq 0$, $N_{\mathrm{anti}} \geq 0$ even and $N_\mu > 0$ for all $\mu \in S$,
- a continuous morphism $\alpha : \mathrm{Gal}(\overline{F}/F) \to \{\pm 1\}$ which is trivial if $N_{\mathrm{sym}} = 0$,

it is not difficult to exhibit an endoscopic datum such that the associated quintuple is $(N_{\mathrm{sym}}, N_{\mathrm{anti}}, S, (N_\mu)_{\mu\in S}, \alpha)$. We have thus proved the first part of the following classification result.

Proposition A.2.1. *(1) Isomorphism classes of endoscopic data of $\widetilde{\mathbf{\Gamma}}$ are parametrized by tuples $(N_{\mathrm{sym}}, N_{\mathrm{anti}}, S, (N_\mu)_{\mu\in S}, \alpha)$ as above.*

(2) An endoscopic datum is elliptic if and only if the corresponding tuple $(N_{\mathrm{sym}}, N_{\mathrm{anti}}, S, (N_\mu)_{\mu\in S}, \alpha)$ satisfies:

- *$S = \emptyset$, and*
- *α is non-trivial if $N_{\mathrm{sym}} = 2$.*

Proof of the second part. If S is not empty then the center of $\mathrm{GAut}(h)$ contains a torus isomorphic to $\mathrm{GL}_1^{|S|}$ which is not included in the center of $\widehat{\mathbf{\Gamma}}$, and so the endoscopic datum cannot be elliptic.

If S is empty then the connected center of $\mathrm{GAut}(h)^0$ is $1\times\mathrm{GL}_1 \subset \widehat{\mathbf{\Gamma}}$ except in the case where $N_{\mathrm{sym}} = 2$, in which case it is isomorphic to $\mathrm{SO}_2 \times \mathrm{GL}_1$. The action of Gal_F on the factor SO_2 has kernel $\mathrm{Gal}_{F(\sqrt{\alpha})}$, so $\mathrm{SO}_2^{\mathrm{Gal}_F,0}$ is contained in $Z(\mathrm{GL}_N)$ (the first factor of $\widehat{\mathbf{\Gamma}}$) if and only if $\alpha \neq 1$. □

Let $\mathfrak{e} = (\mathbf{G}, \mathcal{G}, s, \xi)$ be an elliptic endoscopic datum for $\widetilde{\mathbf{\Gamma}}$, corresponding to $(N_{\mathrm{sym}}, N_{\mathrm{anti}}, \alpha)$ as above. Since the standard N-dimensional representation of $\widehat{\mathbf{G}}$ (obtained by composing ξ with the first projection $\widehat{\mathbf{\Gamma}} \to \mathrm{GL}_N$) is irreducible, we have an embedding $\mathrm{Out}\,(\mathfrak{e}) \subset \mathrm{Out}\,(\widehat{\mathbf{G}})_0$, where $\mathrm{Out}\,(\widehat{\mathbf{G}})_0 \subset \mathrm{Out}\,(\widehat{\mathbf{G}})$ is the subgroup

of elements acting trivially on $\{(\lambda I_N, \lambda) \,|\, \lambda \in \mathrm{GL}_1\} \subset \widehat{\mathbf{G}}$. If $N_{\mathrm{sym}} = 0$ or if N_{sym} is odd we simply have $\mathrm{Out}\,(\widehat{\mathbf{G}})_0 = 1$. If $N_{\mathrm{sym}} > 0$ is even then $\mathrm{Out}\,(\widehat{\mathbf{G}})_0 = \mathbb{Z}/2\mathbb{Z}$, and there is a non-trivial element in $\mathrm{Out}\,(\mathfrak{e})$. Indeed, we can assume that h is antidiagonal and that $I_{\mathrm{sym}} = \{(N - N_{\mathrm{sym}})/2 + 1, \dots, (N + N_{\mathrm{sym}})/2\}$, and in this situation the element

$$\mathrm{diag}\left(I_{N_{\mathrm{anti}}/2},\, I_{N_{\mathrm{sym}}/2-1},\, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\, I_{N_{\mathrm{sym}}/2-1},\, I_{N_{\mathrm{anti}}/2}\right) \in \widehat{\mathbf{\Gamma}}$$

belongs to $\mathrm{Aut}(\mathfrak{e}) \smallsetminus \xi(\widehat{\mathbf{G}})Z(\widehat{\mathbf{\Gamma}})$.

A.3. **Surjectivity of transfer.** We now assume that F is local and consider the particular case of elliptic endoscopic data $\mathfrak{e} = (\mathbf{G}, \mathcal{G}, s, \xi)$ for $\widetilde{\mathbf{\Gamma}}$ satisfying $N_{\mathrm{sym}}N_{\mathrm{anti}} = 0$ (the analogous ones in [Art13] were called "simple" endoscopic data), i.e. the case where $\mathbf{G}$ is not a product of two non-trivial groups, and prove Proposition 2.4.4. We simply follow the strategy of the proof of Proposition I.4.11 in [MW16a], observing a few facts which are particular to our situation (in particular Key Fact A.3.3 below). For simplicity we fix ${}^L\xi : {}^L\mathbf{G} \simeq \mathcal{G}$ extending ξ, avoiding the use of arbitrary auxiliary datum which is necessary in general (see §I.2.5 loc. cit.). Note that since the action of Gal_F on $Z(\widehat{\mathbf{\Gamma}})$ is trivial, $Z(\widehat{\mathbf{\Gamma}})$ is a subgroup of $\mathrm{Aut}(\mathfrak{e})$ which acts trivially on $SI(\mathbf{G})$, and so the action of $\mathrm{Aut}(\mathfrak{e})$ on $SI(\mathbf{G})$ factors through $\mathrm{Out}\,(\mathbf{G}) := \mathrm{Aut}(\mathfrak{e})/Z(\widehat{\mathbf{\Gamma}})\xi(\widehat{\mathbf{G}}) = \{1, \delta\}$ (with δ defined as in Section 2.3, so that this group has cardinality dividing 2).

First we need to recall basic facts about Levi subgroups. There is an injection from the set of $\mathbf{G}(F)$-conjugacy classes of Levi subgroups of $\mathbf{G}$ to the set of $\widehat{\mathbf{G}}$-conjugacy classes of Levi subgroups of ${}^L\mathbf{G}$, and similarly for $\widetilde{\mathbf{\Gamma}}$ and ${}^L\widetilde{\mathbf{\Gamma}}$ (see [MW16a, §I.3.1] for the notion of Levi subspace of ${}^L\widetilde{\mathbf{\Gamma}} := {}^L\mathbf{\Gamma}\widehat{\theta}$). More precisely, for any Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ (resp. Levi subspace $\widetilde{\mathbf{M}}$ of $\widetilde{\mathbf{\Gamma}}$ with associated Levi subgroup $\mathbf{M}$ of $\mathbf{\Gamma}$) there is a well-defined $\widehat{\mathbf{G}}$-conjugacy class (resp. $\widehat{\mathbf{\Gamma}}$-conjugacy class) of L-embeddings $\iota_{\mathbf{L}} : {}^L\mathbf{L} \hookrightarrow {}^L\mathbf{G}$ (resp. $\iota_{\widetilde{\mathbf{M}}} : {}^L\widetilde{\mathbf{M}} \hookrightarrow {}^L\widetilde{\mathbf{\Gamma}}$), and $\iota_{\mathbf{L}}({}^L\mathbf{L})$ (resp. $\iota_{\widetilde{\mathbf{M}}}({}^L\widetilde{\mathbf{M}})$) is a Levi subgroup of ${}^L\mathbf{G}$ (resp. Levi subspace of ${}^L\widetilde{\mathbf{\Gamma}}$). It is well known (see [Bor79, §I.3]) that a choice of parabolic subgroup of $\mathbf{G}$ admitting $\mathbf{L}$ as a Levi factor induces such an embedding, and the extension to the twisted case is straightforward. The fact that the $\widehat{\mathbf{G}}$-conjugacy class (resp. $\widehat{\mathbf{\Gamma}}$-conjugacy class) does not depend on the choice of a parabolic subgroup (resp. subspace) can be checked using the Springer section (in particular [Spr98, Prop. 9.3.5]). In the case at hand since $\mathbf{G}$ (resp. $\mathbf{\Gamma}$) is quasi-split the map on conjugacy classes $\mathbf{L} \mapsto \iota_{\mathbf{L}}({}^L\mathbf{L})$ (resp. $\widetilde{\mathbf{M}} \mapsto \iota_{\widetilde{\mathbf{M}}}({}^L\widetilde{\mathbf{M}})$) is also surjective. By [MW16a, I.3.1 (8)], we also have an identification between $W(\mathbf{L}, \mathbf{G}) := \mathrm{Norm}\,(\mathbf{L}, \mathbf{G}(F))/\mathbf{L}(F)$ and $W(\iota_{\mathbf{L}}({}^L\mathbf{L}), \widehat{\mathbf{G}}) := \mathrm{Norm}\,(\iota_{\mathbf{L}}({}^L\mathbf{L}), \widehat{\mathbf{G}})/\iota_{\mathbf{L}}(\widehat{\mathbf{L}})$ (resp. between $W(\widetilde{\mathbf{M}}, \mathbf{\Gamma}) := \mathrm{Norm}\,(\widetilde{\mathbf{M}}, \mathbf{\Gamma}(F))/\mathbf{M}(F)$ and $W(\iota_{\widetilde{\mathbf{M}}}({}^L\widetilde{\mathbf{M}}), \widehat{\mathbf{\Gamma}}) := \mathrm{Norm}\,(\iota_{\widetilde{\mathbf{M}}}({}^L\widetilde{\mathbf{M}}), \widehat{\mathbf{\Gamma}})/\iota_{\widetilde{\mathbf{M}}}\mathbf{L}(\widetilde{\mathbf{M}})$). As explained loc. cit. these identifications depend on choices of parabolic subgroups, but it is easy to check that the embeddings $\iota_{\mathbf{L}}$ and $\iota_{\widetilde{\mathbf{M}}}$ also pin them down. Finally, recall that we can recover $\mathbf{L}$ (resp. $\widetilde{\mathbf{M}}$) as the centralizer of $\mathbf{A}_{\mathbf{L}}$ in $\mathbf{G}$ (resp. of $\mathbf{A}_{\widetilde{\mathbf{M}}}$ in $\widetilde{\mathbf{\Gamma}}$), where $\mathbf{A}_{\mathbf{L}}$ (resp. $\mathbf{A}_{\widetilde{\mathbf{M}}}$) is the largest split torus in $\mathbf{G}$ (resp. $\mathbf{\Gamma}$) centralizing $\mathbf{L}$ (resp. $\widetilde{\mathbf{M}}$). On the dual side we similarly have

$$\iota_{\mathbf{L}}({}^L\mathbf{L}) = \mathrm{Cent}\,(Z(\iota_{\mathbf{L}}({}^L\mathbf{L}))^0, \widehat{\mathbf{G}}) = \mathrm{Cent}\,(\iota_{\mathbf{L}}(Z(\widehat{\mathbf{L}})^{\mathrm{Gal}_F,0}), \widehat{\mathbf{G}}) \quad \text{and}$$

$$\iota_{\widetilde{\mathbf{M}}}({}^L\widetilde{\mathbf{M}}) = \mathrm{Cent}\left(\mathrm{Cent}\left(\iota_{\widetilde{\mathbf{M}}}({}^L\widetilde{\mathbf{M}}), \widehat{\mathbf{\Gamma}}\right)^0, {}^L\widetilde{\mathbf{\Gamma}}\right) = \mathrm{Cent}\left(\iota_{\widetilde{\mathbf{M}}}(Z(\widehat{\mathbf{M}})^{\mathrm{Gal}_F,\widehat{\theta},0}), {}^L\widetilde{\mathbf{\Gamma}}\right).$$

We now recall a construction from [MW16a, §I.3.4]. Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$. We fix an embedding $\iota_{\mathbf{L}}$ as above. Let $\mathcal{M}^0$ (resp. $\mathcal{M}$, resp. $\widetilde{\mathcal{M}}$) be the centralizer of $\xi(\iota_{\mathbf{L}}(Z(\widehat{\mathbf{L}})^{\mathrm{Gal}_F,0}))$ in $\widehat{\mathbf{\Gamma}}$ (resp. ${}^L\mathbf{\Gamma}$, resp. ${}^L\widetilde{\mathbf{\Gamma}}$). Then $\widetilde{\mathcal{M}}$ is a Levi subspace of ${}^L\widetilde{\mathbf{\Gamma}}$ and it contains $\tilde{s} := s\widehat{\theta}$. Since $\mathbf{\Gamma}$ is quasi-split, there exists a Levi subspace $\widetilde{\mathbf{M}}$ and an isomorphism $\iota_{\widetilde{\mathbf{M}}} : {}^L\widetilde{\mathbf{M}} \simeq \widetilde{\mathcal{M}}$, which identifies $\widehat{\mathbf{M}}$ (resp. ${}^L\mathbf{M}$) with $\mathcal{M}^0$ (resp. $\mathcal{M}$). Let $\mathcal{L} = \mathcal{G} \cap \mathcal{M}$, then

$$\mathfrak{e}_{\mathrm{Levi}} = \left(\mathbf{L},\ \iota_{\widetilde{\mathbf{M}}}^{-1}(\mathcal{L}),\ \iota_{\widetilde{\mathbf{M}}}^{-1}(s),\ \iota_{\widetilde{\mathbf{M}}}^{-1} \circ \xi \circ \iota_{\mathbf{L}}|_{\widehat{\mathbf{L}}}\right)$$

is an elliptic endoscopic datum for $\widetilde{\mathbf{M}}$. In particular $\dim \mathbf{A}_{\widetilde{\mathbf{M}}} = \dim \mathbf{A}_{\mathbf{L}}$. The pair $(\widetilde{\mathbf{M}}, \mathfrak{e}_{\mathrm{Levi}})$ is only well-defined up to the action of $\mathrm{Aut}(\widetilde{\mathbf{M}}, \mathfrak{e}_{\mathrm{Levi}})$, the group of $g \in \widehat{\mathbf{\Gamma}}$ normalizing $\mathcal{L}$ such that $g\tilde{s}g^{-1} \in \iota_{\widetilde{\mathbf{M}}}(Z(\widehat{\mathbf{M}}))\tilde{s}$. In particular any $g \in \mathrm{Aut}(\widetilde{\mathbf{M}}, \mathfrak{e}_{\mathrm{Levi}})$ normalizes $Z(\mathcal{L}) \cap \mathcal{L}^0 = \xi(\iota_{\mathbf{L}}(Z(\widehat{\mathbf{L}})^{\mathrm{Gal}_F,0}))$ and thus also $\widetilde{\mathcal{M}}$. We have the following commutative diagram with exact rows (by definition of all objects in the right column) and where all vertical arrows are injective.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \xi(\iota_{\mathbf{L}}(\widehat{\mathbf{L}})) & \longrightarrow & \xi\left(\mathrm{Norm}\left(\iota_{\mathbf{L}}({}^L\mathbf{L}), \widehat{\mathbf{G}}\right)\right) & \longrightarrow & W(\iota_{\mathbf{L}}({}^L\mathbf{L}), \widehat{\mathbf{G}}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z(\widehat{\mathbf{\Gamma}})\xi(\iota_{\mathbf{L}}(\widehat{\mathbf{L}})) & \longrightarrow & \mathrm{Norm}\left({}^L\xi(\iota_{\mathbf{L}}({}^L\mathbf{L})), \mathrm{Aut}(\mathfrak{e})\right) & \longrightarrow & W(\iota_{\mathbf{L}}({}^L\mathbf{L}), \mathfrak{e}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \iota_{\widetilde{\mathbf{M}}}(\mathrm{Aut}(\mathfrak{e}_{\mathrm{Levi}})) & \longrightarrow & \mathrm{Aut}(\widetilde{\mathbf{M}}, \mathfrak{e}_{\mathrm{Levi}}) & \longrightarrow & W(\widetilde{\mathbf{M}}, \mathfrak{e}_{\mathrm{Levi}}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{M}^0 & \longrightarrow & \mathrm{Norm}\left(\widetilde{\mathcal{M}}, \widehat{\mathbf{\Gamma}}\right) & \longrightarrow & W(\widetilde{\mathcal{M}}, \widehat{\mathbf{\Gamma}}) & \longrightarrow & 1
\end{array}$$

We now make all these objects explicit in the cases at hand. In Section 2.2 we recalled that to a $\mathbf{G}(F)$-conjugacy class of Levi subgroups of $\mathbf{G}$ is associated a family $(r_i)_{i\geq 1}$ with $r_i \in \mathbb{Z}_{\geq 0}$ satisfying $2\sum_i ir_i \leq N$ with strict inequality if $\mathbf{G}$ is not split and $2\sum_i ir_i \neq N-2$ if $N = N_{\mathrm{sym}}$ is even and $\mathbf{G}$ is split, and any such family occurs. This family determines the conjugacy class except when $N = N_{\mathrm{sym}}$ is even, $\mathbf{G}$ is split, $2\sum_i ir_i = N$ and $r_i = 0$ for all odd i, in which case there are two conjugacy classes corresponding to $(r_i)_i$, swapped by the non-trivial outer automorphism δ of $\mathbf{G}$. We may assume that the element h introduced in the previous section is antidiagonal, say

$$h = \begin{pmatrix} & & & h_1 \\ & & \iddots & \\ & \iddots & & \\ h_N & & & \end{pmatrix}$$

and that $h_1 = \cdots = h_{\lfloor N/2 \rfloor} = 1$. Let k be the smallest integer ≥ 0 such that $r_i = 0$ for any $i > k$, $\hat{S}_i$ is the i-dimensional antidiagonal complex square matrix with 1's

on the antidiagonal, and

$$h' = \begin{pmatrix} & & h_{1+\sum_i ir_i} \\ & \iddots & \\ h_{N-\sum_i ir_i} & & \end{pmatrix} \in \mathrm{GL}_{N-2\sum_i ir_i}(\mathbb{C}).$$

Then the (conjugacy class of a) Levi subgroup $\mathbf{L}$ corresponding to $(r_i)_i$ is characterized by the fact that there exists a Levi embedding $\iota_{\mathbf{L}} : {}^L\mathbf{L} \hookrightarrow {}^L\mathbf{G}$ such that $\xi \circ \iota_{\mathbf{L}}({}^L\mathbf{L})$ is either an open subgroup of

(A.3.1)
$$\Big\{(\mathrm{diag}(g_{k,1},\dots,g_{k,r_k},\dots,g_{1,1},g_{1,r_1},x,\lambda\hat{S}_1^{-1t}g_{1,r_1}^{-1}\hat{S}_1,\dots,\lambda\hat{S}_k^{-1t}g_{k,1}^{-1}\hat{S}_k),\lambda)\ \Big|\ g_{i,j}\in\mathrm{GL}_i(\mathbb{C}),\ (x,\lambda)\in\mathrm{GAut}(h')\Big\}\times W_F\subset {}^L\mathbf{\Gamma},$$

or, if $N = N_{\mathrm{sym}}$ is even, $\mathbf{G}$ is split, $\sum_i ir_i = N/2$ and $r_i = 0$ for all odd i, an open subgroup of the conjugate of (A.3.1) by $\mathrm{diag}(I_{N/2-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I_{N/2-1})$. In the rest of the argument we shall refer to this case as the exceptional case. We fix $\mathbf{L}$ and such an embedding $\iota_{\mathbf{L}}$. There is a natural embedding $W(\iota_{\mathbf{L}}({}^L\mathbf{L}), {}^L\mathbf{G}) \hookrightarrow \prod_{i\geq 1}\{\pm 1\}^{r_i} \rtimes \mathfrak{S}_{r_i}$, which is surjective unless $N = N_{\mathrm{sym}}$ is even, $\mathbf{G}$ is split, $\sum_i ir_i = N/2$ and there exists an odd $i \geq 1$ such that $r_i > 0$, in which case the image of this embedding has index two.

To be explicit, $\mathcal{M}^0$ is the diagonal Levi subgroup

$$\big(\mathrm{GL}_k^{r_k} \times \cdots \times \mathrm{GL}_1^{r_1} \times \mathrm{GL}_{N-2\sum_i ir_i} \times \mathrm{GL}_1^{r_1} \times \cdots \times \mathrm{GL}_k^{r_k}\big) \times \mathrm{GL}_1 \tag{A.3.2}$$

of $\widehat{\mathbf{\Gamma}}$, $\mathcal{M} = \mathcal{M}^0 \times W_F$ and $\widetilde{\mathcal{M}} = \mathcal{M}\widehat{\theta}$, except in the exceptional case where the situation is conjugated by (A.3.2) under $\mathrm{diag}(I_{N/2-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I_{N/2-1})$. In particular $W(\widetilde{\mathcal{M}}, \widehat{\mathbf{\Gamma}}) = W(\mathcal{M}^0, \widehat{\mathbf{\Gamma}})^{\widehat{\theta}}$ in the non-exceptional cases, and in any case $W(\widetilde{\mathcal{M}}, \widehat{\mathbf{\Gamma}})$ is identified to $\prod_{i\geq 1}\{\pm 1\}^{r_i} \rtimes \mathfrak{S}_{r_i}$. Thus:

(1) If $N_{\mathrm{sym}} = 0$ or if $N = N_{\mathrm{sym}}$ is odd, we simply have $W(\iota_{\mathbf{L}}({}^L\mathbf{L}), \widehat{\mathbf{G}}) = W(\widetilde{\mathcal{M}}, \widehat{\mathbf{\Gamma}})$ and $\mathrm{Out}\,(\mathfrak{e}) = 1$.
(2) If $N = N_{\mathrm{sym}}$ is even and $\sum_i ir_i < N/2$, we again have $W(\iota_{\mathbf{L}}({}^L\mathbf{L}), \widehat{\mathbf{G}}) = W(\widetilde{\mathcal{M}}, \widehat{\mathbf{\Gamma}})$, and there exists an element of $\iota_{\widetilde{\mathbf{M}}}(\mathrm{Aut}(\mathfrak{e}_{\mathrm{Levi}})) \cap \mathrm{Aut}(\mathfrak{e})$ mapping to the non-trivial element of $\mathrm{Out}\,(\mathfrak{e})$.
(3) If $N = N_{\mathrm{sym}}$ is even, $\sum_i ir_i = N/2$ (this implies that $\mathbf{G}$ is split) and there exists an odd i such that $r_i > 0$, then $W(\iota_{\mathbf{L}}({}^L\mathbf{L}), \widehat{\mathbf{G}})$ has index two in $W(\iota_{\mathbf{L}}({}^L\mathbf{L}), \mathfrak{e}) = W(\widetilde{\mathcal{M}}, \widehat{\mathbf{\Gamma}})$, and there exists an element of $\mathrm{Norm}\,(\xi(\iota_{\mathbf{L}}({}^L\mathbf{L})), \mathrm{Aut}(\mathfrak{e}))$ which maps to the non-trivial element of $\mathrm{Out}\,(\mathfrak{e})$.
(4) Finally in the exceptional case we have $W(\iota_{\mathbf{L}}({}^L\mathbf{L}), \widehat{\mathbf{G}}) = W(\widetilde{\mathcal{M}}, \widehat{\mathbf{\Gamma}})$ and δ does not fix the $\mathbf{G}(F)$-conjugacy class of $\mathbf{L}$.

We also observe the following.

Key fact A.3.3. *The $\mathbf{\Gamma}(F)$-conjugacy class of $\widetilde{\mathbf{M}}$ determines the $\mathrm{Out}\,(\mathfrak{e})$-orbit of the $\mathbf{G}(F)$-conjugacy class of $\mathbf{L}$, i.e. each fiber of $\mathbf{L} \mapsto \widetilde{\mathbf{M}}$ consists of (at most) one $\mathrm{Out}\,(\mathfrak{e})$-orbit.*

Now start with an arbitrary Levi subspace $\widetilde{\mathbf{M}}$ of $\widetilde{\mathbf{\Gamma}}$. Denote by $I_{\mathrm{cusp}}(\widetilde{\mathbf{M}})$ the subspace of $I(\widetilde{\mathbf{M}})$ consisting of all functions whose orbital integrals at non-elliptic semi-simple regular elements vanish. The endoscopic transfer induces an isomorphism (see [MW16a, §I.4.12 p.97], as well as §IV.3.5 loc. cit. to deduce the K-finite case if F is Archimedean)

$$I_{\mathrm{cusp}}(\widetilde{\mathbf{M}})^{W(\widetilde{\mathbf{M}},\mathbf{\Gamma})} \simeq \left(\bigoplus_{\mathfrak{e}'} SI_{\mathrm{cusp}}(\mathbf{L})^{\mathrm{Aut}(\mathfrak{e}')}\right)^{W(\widetilde{\mathbf{M}},\mathbf{\Gamma})} = \bigoplus_{\mathfrak{e}'} SI_{\mathrm{cusp}}(\mathbf{L})^{\mathrm{Aut}(\widetilde{\mathbf{M}},\mathfrak{e}')},$$

where the middle sum is over equivalence classes of elliptic endoscopic data $\mathfrak{e}' = (\mathbf{L}, \mathcal{L}, \tilde{s}', \xi')$ for $\widetilde{\mathbf{M}}$, the sum on the right-hand side is over $W(\widetilde{\mathbf{M}}, \mathbf{\Gamma})$-orbits of such equivalence classes, and SI_{cusp} is defined similarly to I_{cusp}, replacing "orbital integrals" by "stable orbital integrals". Note that in the case $F = \mathbb{R}$ the above isomorphism only holds for a K-space for $\widetilde{\mathbf{M}}$ (see §I.1.11 loc. cit.), but since $H^1(F, \mathbf{M}_{\mathrm{sc}}) = 1$ the space $\widetilde{\mathbf{M}}$ is a K-space. By the Key Fact A.3.3 and using a straightforward argument in each of the four cases detailed above, the natural map

$$\bigoplus_{\mathfrak{e}'} SI_{\mathrm{cusp}}(\mathbf{L})^{\mathrm{Aut}(\widetilde{\mathbf{M}},\mathfrak{e}')} \longrightarrow \left(\bigoplus_{\mathbf{L}} SI_{\mathrm{cusp}}(\mathbf{L})^{W(\mathbf{L},\mathbf{G})}\right)^{\mathrm{Aut}(\mathfrak{e})}, \tag{A.3.3}$$

where the sum on the right-hand side is over conjugacy classes of Levi subgroups of $\mathbf{G}$ mapping to the conjugacy class of $\widetilde{\mathbf{M}}$, is surjective. This is the crucial step in the proof of Proposition 2.4.4, and to conclude the proof it simply remains to follow the strategy of [MW16a, §I.4.12], using natural filtrations on $I(\widetilde{\mathbf{\Gamma}})$ and $SI(\mathbf{G})^{\mathrm{Aut}(\mathfrak{e})}$.

To this end we now recall compatibility properties of endoscopic transfer for Levi subgroups. As above we consider a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$, and a corresponding Levi subspace $\widetilde{\mathbf{M}}$ of $\widetilde{\mathbf{\Gamma}}$. It follows easily from [BT65, Théorème 4.13] that the maps $H^1(F, \mathbf{L}) \to H^1(F, \mathbf{G})$ and $H^1(F, \mathbf{M}) \to H^1(F, \mathbf{\Gamma})$ are injective. Thus for any $\widetilde{\mathbf{\Gamma}}$-regular $\gamma \in \widetilde{\mathbf{M}}(F)$, the natural map from the set of $\mathbf{M}(F)$-conjugacy classes in $\widetilde{\mathbf{M}}(F)$ stably conjugated to γ to the set of $\mathbf{\Gamma}(F)$-conjugacy classes in $\widetilde{\mathbf{\Gamma}}(F)$ stably conjugated to γ is bijective, and similarly for $\mathbf{L} \subset \mathbf{G}$. As explained in [MW16a, §I.3.1, p.57], this implies that the "constant term map" $I(\mathbf{G}) \to I(\mathbf{L})^{W(\mathbf{L},\mathbf{G})}$ induces a well-defined map $SI(\mathbf{G}) \to SI(\mathbf{L})^{W(\mathbf{L},\mathbf{G})}$. Moreover the restriction of a transfer factor for the endoscopic datum $\mathfrak{e}$ to $\widetilde{\mathbf{\Gamma}}$-strongly regular matching pairs in $\mathbf{L}(F) \times \widetilde{\mathbf{M}}(F)$ coincides with the restriction of a unique transfer factor for $\mathfrak{e}_{\mathrm{Levi}}$. This is seen by choosing a parabolic subgroup $\mathbf{P}_{\mathbf{L}}$ of $\mathbf{G}$ admitting $\mathbf{L}$ as a Levi factor (resp. a parabolic subspace $\widetilde{\mathbf{P}}_{\widetilde{\mathbf{M}}}$ of $\widetilde{\mathbf{\Gamma}}$ admitting $\widetilde{\mathbf{M}}$ as a Levi factor), which gives corresponding parabolic subgroups in Langlands dual groups, and following the constructions in [MW16a, §I.2.2] using Borel subgroups contained in these parabolic subgroups and choosing χ-data which is trivial on asymmetric Galois orbits. Using such a transfer factor for $\mathfrak{e}_{\mathrm{Levi}}$, it is straightforward to check that the diagram

$$\begin{array}{ccc} I(\widetilde{\mathbf{\Gamma}}) & \longrightarrow & SI(\mathbf{G}) \\ \downarrow & & \downarrow \\ I(\widetilde{\mathbf{M}})^{W(\widetilde{\mathbf{M}},\mathbf{\Gamma})} & \longrightarrow & SI(\mathbf{L})^{W(\mathbf{L},\mathbf{G})} \end{array}$$

is commutative, where the vertical arrows are "constant term" maps and the horizontal arrows are transfers maps.

Recall from [MW16a, §I.4.2] that there is a filtration $(\mathrm{Fil}^n I(\widetilde{\mathbf{\Gamma}}))_{n\geq -1}$ such that the "constant term" maps identify $\mathrm{Gr}^n I(\widetilde{\mathbf{\Gamma}})$ with

$$\bigoplus_{\widetilde{\mathbf{M}} \text{ s.t. } \dim \mathbf{A}_{\widetilde{\mathbf{M}}}=n} I_{\mathrm{cusp}}(\widetilde{\mathbf{M}})^{W(\widetilde{\mathbf{M}},\mathbf{\Gamma})}$$

where the sum is over $\mathbf{\Gamma}(F)$-conjugacy classes of Levi subspaces in $\widetilde{\mathbf{\Gamma}}$. There is an analogous filtration $(\mathrm{Fil}^n SI(\mathbf{G}))_{n\geq -1}$ of $SI(\mathbf{G})$, which by §I.4.15 loc. cit. is simply the image of the natural filtration of $I(\mathbf{G})$, such that $\mathrm{Gr}^n SI(\mathbf{G})$ is identified with

$$\bigoplus_{\mathbf{L} \text{ s.t. } \dim \mathbf{A}_{\mathbf{L}}=n} SI_{\mathrm{cusp}}(\mathbf{L})^{W(\mathbf{L},\mathbf{G})}$$

This filtration is clearly stable under $\mathrm{Aut}(\mathfrak{e})$, and a straightforward induction allows one to deduce Proposition 2.4.4 from the surjectivity of (A.3.3).

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E-mail address: toby.gee@imperial.ac.uk

Department of Mathematics, Imperial College London, London SW7 2AZ, UK

E-mail address: olivier.taibi@ens-lyon.fr

CNRS et Unité de Mathématiques Pures et Appliquées, ENS de Lyon