CRYSTALLINE EXTENSIONS AND THE WEIGHT PART OF SERRE'S CONJECTURE

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Abstract. Let $p > 2$ be prime. We complete the proof of the weight part of Serre's conjecture for rank two unitary groups for mod p representations in the totally ramified case, by proving that any Serre weight which occurs is a predicted weight. This completes the analysis begun in [BLGG11], which proved that all predicted Serre weights occur. Our methods are a mixture of local and global techniques, and in the course of the proof we use global techniques (as well as local arguments) to establish some purely local results on crystalline extension classes. We also apply these local results to prove similar theorems for the weight part of Serre's conjecture for Hilbert modular forms in the totally ramified case.

Contents

1. Introduction

The weight part of generalisations of Serre's conjecture has seen significant progress in recent years, particularly for (forms of) GL_2. Conjectural descriptions of the set of Serre weights were made in increasing generality by [BDJ10], [Sch08] and [GHS11], and cases of these conjectures were proved in [Gee11] and [GS11a]. Most recently, significant progress was made towards completely establishing the conjecture for rank two unitary groups in [BLGG11]. We briefly recall this result. Let $p > 2$ be prime, let F be a CM field, and let $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a modular representation (see [BLGG11] for the precise definition of "modular", which is in terms of automorphic forms on compact unitary groups). There is a conjectural set $W^?(\bar{r})$ of Serre weights in which $\bar{r}$ is predicted to be modular, which is defined in Section 2 below, following [GHS11]. Then the main result of [BLGG11] is that under mild technical hypotheses, $\bar{r}$ is modular of every weight in $W^?(\bar{r})$. We note

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that this result is rather more general than anything that has been proved for inner forms of GL_2 over totally real fields, where there is a parity obstruction due to the unit group; algebraic Hilbert modular forms must have paritious weight. This problem does not arise for the unitary groups considered here, which is why we use them, rather than making use of the more obvious choice of an inner form. In the absence of a mod p functoriality principle, it is not known that the results for inner and outer forms of GL_2 are equivalent, and at present the theory for outer forms is in a better state.

It remains to show that if $\bar{r}$ is modular of some Serre weight, then this weight is contained in $W^?(\bar{r})$. It had been previously supposed that this was the easier direction; indeed, just as in the classical case, the results of [BLGG11] reduce the weight part of Serre's conjecture for these unitary groups to a purely local problem in p-adic Hodge theory. However, this problem has proved to be difficult, and so far only fragmentary results are known. In the present paper we resolve the problem in the totally ramified case, so that in combination with [BLGG11] we resolve the weight part of Serre's conjecture in this case, proving the following Theorem (see Theorem 6.1.2).

Theorem A. *Let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, that $\zeta_p \notin F$, and that $[F^+ : \mathbb{Q}]$ is even. Suppose that $p > 2$, and that $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is an irreducible modular representation with split ramification such that $\bar{r}(G_{F(\zeta_p)})$ is adequate. Assume that for each place $w|p$ of F, $F_w/\mathbb{Q}_p$ is totally ramified.*

Let $a \in (\mathbb{Z}^2_+)^S_0$ be a Serre weight. Then $a_w \in W^?(\bar{r}|_{G_{F_w}})$ if and only if $\bar{r}$ is modular of weight a.

(See the body of the paper, especially Section 2.2, for any unfamiliar notation and terminology.) While [BLGG11] reduced this result to a purely local problem, our methods are not purely local; in fact we use the main result of [BLGG11], together with potential automorphy theorems, as part of our proof.

In the case that $\bar{r}|_{G_{F_w}}$ is semisimple for each place $w|p$, the result was established (in a slightly different setting) in [GS11a]. The method of proof was in part global, making use of certain potentially Barsotti-Tate lifts to obtain conditions on $\bar{r}|_{G_{F_w}}$. We extend this analysis in the present paper to the case that $\bar{r}|_{G_{F_w}}$ is reducible but non-split, obtaining conditions on the extension classes that can occur; we show that (other than in one exceptional case) they lie in a certain set L_{flat}, defined in terms of finite flat models. We are also able to apply our final local results to improve on the global theorems proved in [GS11a]; see Theorem 6.1.3 below.

In the case that $\bar{r}|_{G_{F_w}}$ is reducible the definition of $W^?$ also depends on the extension class; it is required to lie in a set L_{crys}, defined in terms of reducible crystalline lifts with specified Hodge-Tate weights. To complete the proof, we show that $L_{\mathrm{crys}} = L_{\mathrm{flat}}$, except in one exceptional case that we handle separately in Proposition 5.2.9. An analogous result was proved in generic unramified cases in section 3.4 of [Gee11] by means of explicit calculations with Breuil modules; our approach here is less direct, but has the advantage of working in non-generic cases, and requires far less calculation.

We use a global argument to show that $L_{\mathrm{crys}} \subset L_{\mathrm{flat}}$. Given a class in L_{crys}, we use potential automorphy theorems to realise the corresponding local representation as part of a global modular representation, and then apply the main result of [BLGG11] to show that this representation is modular of the expected weight.

Standard congruences between automorphic forms then show that this class is also contained in L_{flat}.

To prove the converse inclusion, we make a study of different finite flat models to show that L_{flat} is contained in a vector space of some dimension d. A standard calculation shows that L_{crys} contains a space of dimension d, so equality follows. As a byproduct, we show that both L_{flat} and L_{crys} are vector spaces. We also show that various spaces defined in terms of crystalline lifts are independent of the choice of lift (see Corollary 5.2.8). The analogous property was conjectured in the unramified case in [BDJ10].

It is natural to ask whether our methods could be extended to handle the general case, where $F_w/\mathbb{Q}_p$ is an arbitrary extension. Unfortunately, this does not seem to be the case, because in general the connection between being modular of some Serre weight and having a potentially Barsotti-Tate lift of some type is less direct. We expect that our methods could be used to reprove the results of section 3.4 of [Gee11], but we do not see how to extend them to cover the unramified case completely. In particular, we are unsure as to when the equality $L_{\mathrm{flat}} = L_{\mathrm{crys}}$ holds in general.

We now explain the structure of the paper. In Section 2 we recall the definition of $W^?$, and the global results from [BLGG11] that we will need. In Section 3 we recall (and give a concise proof of) a potential automorphy result from [GK11], allowing us to realise a local mod p representation globally. Section 4 contains the definitions of the spaces L_{crys} and L_{flat} and the proof that $L_{\mathrm{crys}} \subset L_{\mathrm{flat}}$, and in Section 5 we carry out the necessary calculations with Breuil modules to prove our main local results. All of these results are in the reducible case, the irreducible case being handled in [GS11a]. Finally, in section 6 we combine our local results with the techniques of [GS11a] and the main result of [BLGG11] to prove Theorem A, and we deduce a similar result in the setting of [GS11a].

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1.1. **Notation.** If M is a field, we let G_M denote its absolute Galois group. Let ϵ denote the p-adic cyclotomic character, and $\bar{\epsilon}$ the mod p cyclotomic character. If M is a global field and v is a place of M, let M_v denote the completion of M at v. If M is a finite extension of $\mathbb{Q}_l$ for some l, we write I_M for the inertia subgroup of G_M. If R is a local ring we write $\mathfrak{m}_R$ for the maximal ideal of R.

Let K be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$ and residue field k. We write $\mathrm{Art}_K : K^\times \to W_K^{\mathrm{ab}}$ for the isomorphism of local class field theory, normalised so that uniformisers correspond to geometric Frobenius elements. For each $\sigma \in \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$ we define the fundamental character ω_σ corresponding to σ to be the composite

$$I_K \longrightarrow W_K^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_K^{-1}} \mathcal{O}_K^\times \longrightarrow k^\times \xrightarrow{\sigma} \overline{\mathbb{F}}_p^{\times}.$$

In the case that $k \cong \mathbb{F}_p$, we will sometimes write ω for ω_σ. Note that in this case we have $\omega^{[K:\mathbb{Q}_p]} = \bar{\epsilon}$.

We fix an algebraic closure $\overline{K}$ of K. If W is a de Rham representation of G_K over $\overline{\mathbb{Q}}_p$ and τ is an embedding $K \hookrightarrow \overline{\mathbb{Q}}_p$ then the multiset $\mathrm{HT}_\tau(W)$ of Hodge-Tate weights of W with respect to τ is defined to contain the integer i with multiplicity

$$\dim_{\overline{\mathbb{Q}}_p}(W \otimes_{\tau,K} \widehat{\overline{K}}(-i))^{G_K},$$

with the usual notation for Tate twists. Thus for example $\mathrm{HT}_\tau(\epsilon) = \{1\}$.

2. Serre weight conjectures: definitions

2.1. **Local definitions.** We begin by recalling some generalisations of the weight part of Serre's conjecture. We begin with some purely local definitions. Let K be a finite totally ramified extension of $\mathbb{Q}_p$ with absolute ramification index e, and let $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation.

Definition 2.1.1. A *Serre weight* is an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_2(\mathbb{F}_p)$. Up to isomorphism, any such representation is of the form

$$F_a := \det{}^{a_2} \otimes \mathrm{Sym}^{a_1 - a_2}\, \overline{\mathbb{F}}_p^2$$

where $0 \leq a_1 - a_2 \leq p-1$. We also use the term Serre weight to refer to the pair $a = (a_1, a_2)$.

We say that two Serre weights a and b are *equivalent* if and only if $F_a \cong F_b$ as representations of $\mathrm{GL}_2(\mathbb{F}_p)$. This is equivalent to demanding that we have $a_1 - a_2 = b_1 - b_2$ and $a_2 \equiv b_2 \pmod{p-1}$.

We write $\mathbb{Z}^2_+$ for the set of pairs of integers (n_1, n_2) with $n_1 \geq n_2$, so that a Serre weight a is by definition an element of $\mathbb{Z}^2_+$. We say that an element $\lambda \in (\mathbb{Z}^2_+)^{\mathrm{Hom}_{\mathbb{Q}_p}(K,\overline{\mathbb{Q}}_p)}$ is a *lift* of a Serre weight $a \in \mathbb{Z}^2_+$ if there is an element $\tau \in \mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$ such that $\lambda_\tau = a$, and for all other $\tau' \in \mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$ we have $\lambda_{\tau'} = (0,0)$.

Definition 2.1.2. Let $K/\mathbb{Q}_p$ be a finite extension, let $\lambda \in (\mathbb{Z}^2_+)^{\mathrm{Hom}_{\mathbb{Q}_p}(K,\overline{\mathbb{Q}}_p)}$, and let $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a de Rham representation. Then we say that ρ has *Hodge type* λ if for each $\tau \in \mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$ we have $\mathrm{HT}_\tau(\rho) = \{\lambda_{\tau,1} + 1, \lambda_{\tau,2}\}$.

In particular, we will say that ρ has "Hodge type $\underline{0}$" if its Hodge-Tate weights are $(0,1)$ with respect to each embedding. Following [GHS11] (which in turn follows [BDJ10] and [Sch08]), we define an explicit set of Serre weights $W^?(\overline{\rho})$.

Definition 2.1.3. If $\overline{\rho}$ is reducible, then a Serre weight $a \in \mathbb{Z}^2_+$ is in $W^?(\overline{\rho})$ if and only if $\overline{\rho}$ has a crystalline lift of the form

$$\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

which has Hodge type λ for some lift $\lambda \in (\mathbb{Z}^2_+)^{\mathrm{Hom}_{\mathbb{Q}_p}(K,\overline{\mathbb{Q}}_p)}$ of a. In particular, if $a \in W^?(\overline{\rho})$ then by Lemma 6.2 of [GS11a] it is necessarily the case that there is a decomposition $\mathrm{Hom}(\mathbb{F}_p, \overline{\mathbb{F}}_p) = J \coprod J^c$ and an integer $0 \leq \delta \leq e-1$ such that

$$\overline{\rho}|_{I_K} \cong \begin{pmatrix} \omega^\delta \prod_{\sigma \in J} \omega_\sigma^{a_1+1} \prod_{\sigma \in J^c} \omega_\sigma^{a_2} & * \\ 0 & \omega^{e-1-\delta} \prod_{\sigma \in J^c} \omega_\sigma^{a_1+1} \prod_{\sigma \in J} \omega_\sigma^{a_2}. \end{pmatrix}$$

We remark that this definition in terms of crystalline lifts is hard to work with concretely, and this is the reason for most of the work in this paper. We also remark that while it may seem strange to consider the single element set $\mathrm{Hom}(\mathbb{F}_p, \overline{\mathbb{F}}_p)$, this notation will be convenient for us (note that we always assume that the residue field of K is $\mathbb{F}_p$).

Definition 2.1.4. Let K' denote the quadratic unramified extension of K inside $\overline{K}$, with residue field k' of order p^2.

If $\overline{\rho}$ is irreducible, then a Serre weight $a \in \mathbb{Z}_+^2$ is in $W^?(\overline{\rho})$ if and only if there is a subset $J \subset \mathrm{Hom}(k', \overline{\mathbb{F}}_p)$ of size 1, and an integer $0 \leq \delta \leq e-1$ such that if we write $\mathrm{Hom}(k', \overline{\mathbb{F}}_p) = J \coprod J^c$, then

$$\overline{\rho}|_{I_K} \cong \begin{pmatrix} \prod_{\sigma \in J} \omega_\sigma^{a_1+1+\delta} \prod_{\sigma \in J^c} \omega_\sigma^{a_2+e-1-\delta} & 0 \\ 0 & \prod_{\sigma \in J^c} \omega_\sigma^{a_1+1+\delta} \prod_{\sigma \in J} \omega_\sigma^{a_2+e-1-\delta} \end{pmatrix}.$$

We remark that by Lemma 4.1.19 of [BLGG11], if $a \in W^?(\overline{\rho})$ and $\overline{\rho}$ is irreducible then $\overline{\rho}$ necessarily has a crystalline lift of Hodge type λ for any lift $\lambda \in (\mathbb{Z}_+^2)^{\mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)}$ of a. Note also that if a and b are equivalent and $a \in W^?(\overline{\rho})$ then $b \in W^?(\overline{\rho})$.

Remark 2.1.5. Note that if $\overline{\theta} : G_K \to \overline{\mathbb{F}}_p^\times$ is an unramified character, then $W^?(\bar{r}) = W^?(\bar{r} \otimes \overline{\theta})$.

2.2. **Global conjectures.** The point of the local definitions above is to allow us to formulate global Serre weight conjectures. Following [BLGG11], we work with rank two unitary groups which are compact at infinity. As we will not need to make any arguments that depend on the particular definitions made in [BLGG11], and our main results are purely local, we simply recall some notation and basic properties of the definitions, referring the reader to [BLGG11] for precise formulations.

We emphasise that our conventions for Hodge-Tate weights are the opposite of those of [BLGG11]; for this reason, we must introduce a dual into the definitions.

Fix an imaginary CM field F, and let F^+ be its maximal totally real subfield. We assume that each prime of F^+ over p has residue field $\mathbb{F}_p$ and splits in F. We define a global notion of Serre weight by taking a product of local Serre weights in the following way.

Definition 2.2.1. Let S denote the set of places of F above p. If $w \in S$ lies over a place v of F^+, write $v = ww^c$. Let $(\mathbb{Z}_+^2)_0^S$ denote the subset of $(\mathbb{Z}_+^2)^S$ consisting of elements $a = (a_w)_{w \in S}$ such that $a_{w,1} + a_{w^c,2} = 0$ for all $w \in S$. We say that an element $a \in (\mathbb{Z}_+^2)_0^S$ is a *Serre weight* if for each $w|p$ we have

$$p - 1 \geq a_{w,1} - a_{w,2}.$$

Let $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous irreducible representation. Definition 2.1.9 of [BLGG11] states what it means for $\bar{r}$ to be modular, and more precisely for $\bar{r}$ to be modular of some Serre weight a; roughly speaking, $\bar{r}$ is modular of weight a if there is a cohomology class on some unitary group with coefficients in the local system corresponding to a whose Hecke eigenvalues are determined by the characteristic polynomials of $\bar{r}$ at Frobenius elements. Since our conventions for Hodge-Tate weights are the opposite of those of [BLGG11], we make the following definition.

Definition 2.2.2. Suppose that $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous irreducible modular representation. Then we say that $\bar{r}$ *is modular of weight* $a \in (\mathbb{Z}_+^2)_0^S$ if $\bar{r}^\vee$ is modular of weight a in the sense of Definition 2.1.9 of [BLGG11].

We remark that if $\bar{r}$ is modular then $\bar{r}^c \cong \bar{r}^\vee \otimes \bar{\epsilon}$. We globalise the definition of the set $W^?(\overline{\rho})$ in the following natural fashion.

Definition 2.2.3. If $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous representation, then we define $W^?(\bar{r})$ to be the set of Serre weights $a \in (\mathbb{Z}^2_+)^S_0$ such that for each place $w|p$ the corresponding Serre weight $a_w \in \mathbb{Z}^2_+$ is an element of $W^?(\bar{r}|_{G_{F_w}})$.

One then has the following conjecture.

Conjecture 2.2.4. *Suppose that $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous irreducible modular representation, and that $a \in (\mathbb{Z}^2_+)^S_0$ is a Serre weight. Then $\bar{r}$ is modular of weight a if and only if $a \in W^?(\bar{r})$.*

If $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous representation, then we say that $\bar{r}$ has *split ramification* if any finite place of F at which $\bar{r}$ is ramified is split over F^+. We will frequently place ourselves in the following situation.

Hypothesis 2.2.5. Let F be an imaginary CM field with maximal totally real subfield F^+, and let $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Assume that:

- $p > 2$,
- $[F^+ : \mathbb{Q}]$ is even,
- F/F^+ is unramified at all finite places,
- $F_w/\mathbb{Q}_p$ is totally ramified for each place $w|p$ of F, and
- $\bar{r}$ is an irreducible modular representation with split ramification.

We point out that the condition that any place above p in F^+ splits in F, which is assumed throughout [BLGG11], is implied by the third and fourth conditions above. The following result is Theorem 5.1.3 of [BLGG11], one of the main theorems of that paper, specialised to the case of interest to us where $F_w/\mathbb{Q}_p$ is totally ramified for each place $w|p$ of F. (Note that in [BLGG11], the set of Serre weights $W^?(\bar{r})$ is referred to as $W^{\mathrm{explicit}}(\bar{r})$.)

Theorem 2.2.6. *Suppose that Hypothesis 2.2.5 holds. Suppose further that $\zeta_p \notin F$ and $\bar{r}(G_{F(\zeta_p)})$ is adequate. Let $a \in (\mathbb{Z}^2_+)^S_0$ be a Serre weight. Assume that $a \in W^?(\bar{r})$. Then $\bar{r}$ is modular of weight a.*

Here *adequacy* is a group-theoretic condition, introduced in [Tho10], that for subgroups of $\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ with $p > 5$ is equivalent to the usual condition that $\bar{r}|_{G_{F(\zeta_p)}}$ is irreducible. For a precise definition we refer the reader to Definition A.1.1 of [BLGG11]. We also remark that the hypotheses that F/F^+ is unramified at all finite places, that every place of F^+ dividing p splits in F, and that $[F^+ : \mathbb{Q}]$ is even, are in fact part of the definition of "modular" made in [BLGG11]. .

Theorem 2.2.6 establishes one direction of Conjecture 2.2.4, and we are left with the problem of "elimination," i.e., the problem of proving that if $\bar{r}$ is modular of weight a, then $a \in W^?(\bar{r})$. We believe that this problem should have a purely local resolution, as we now explain.

The key point is the relationship between being modular of weight a, and the existence of certain de Rham lifts of the local Galois representations $\bar{r}|_{G_{F_w}}$ with $w|p$. The link between these properties is provided by local-global compatibility for the Galois representations associated to the automorphic representations under consideration; rather than give a detailed development of this connection, for which see [BLGG11], we simply summarise the key results from [BLGG11] that we will use. The following is Corollary 4.1.8 of [BLGG11].

Proposition 2.2.7. *Suppose that Hypothesis 2.2.5 holds. Let $a \in (\mathbb{Z}^2_+)^S_0$ be a Serre weight. If $\bar{r}$ is modular of weight a, then for each place $w|p$ of F, there is a crystalline representation $\rho_w : G_{F_w} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ lifting $\bar{r}|_{G_{F_w}}$, such that ρ_w has Hodge type λ_w for some lift $\lambda_w \in (\mathbb{Z}^2_+)^{\mathrm{Hom}_{\mathbb{Q}_p}(F_w,\overline{\mathbb{Q}}_p)}$ of a.*

We stress that Proposition 2.2.7 does not already complete the proof of Conjecture 2.2.4, because the representation ρ_w may be irreducible (compare with Definition 2.1.3). However, in light of this result, it is natural to make the following purely local conjecture, which together with Theorem 2.2.6 would essentially resolve Conjecture 2.2.4.

Conjecture 2.2.8. *Let $K/\mathbb{Q}_p$ be a finite totally ramified extension, and let $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Let $a \in \mathbb{Z}^2_+$ be a Serre weight, and suppose that for some lift $\lambda \in (\mathbb{Z}^2_+)^{\mathrm{Hom}_{\mathbb{Q}_p}(K,\overline{\mathbb{Q}}_p)}$, there is a continuous crystalline representation $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ lifting $\overline{\rho}$, such that ρ has Hodge type λ.*

Then $a \in W^?(\bar{r})$.

We do not know how to prove this conjecture, and we do not directly address the conjecture in the rest of this paper. Instead, we proceed more indirectly. Proposition 2.2.7 is a simple consequence of lifting automorphic forms of weight a to forms of weight λ; we may also obtain non-trivial information by lifting to forms of weight 0 and non-trivial type. In this paper, we will always consider principal series types. Recall that if $K/\mathbb{Q}_p$ is a finite extension the *inertial type* of a potentially semistable Galois representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is the restriction to I_K of the corresponding Weil-Deligne representation. In this paper we normalise this definition as in the appendix to [CDT99], so that for example the inertial type of a finite order character is just the restriction to inertia of that character. We refer the reader to Definition 2.1.2 and the discussion immediately following it for our definition of "Hodge type $\underline{0}$."

Proposition 2.2.9. *Suppose that Hypothesis 2.2.5 holds. Let $a \in (\mathbb{Z}^2_+)^S_0$ be a Serre weight. If $\bar{r}$ is modular of weight a, then for each place $w|p$ of F, there is a continuous potentially semistable representation $\rho_w : G_{F_w} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ lifting $\bar{r}|_{G_{F_w}}$, such that ρ_w has Hodge type $\underline{0}$ and inertial type $\widetilde{\omega}^{a_1} \oplus \widetilde{\omega}^{a_2}$. (Here $\widetilde{\omega}$ is the Teichmüller lift of ω.) Furthermore, ρ_w is potentially crystalline unless $a_1 - a_2 = p - 1$ and $\bar{r}|_{G_{F_w}} \cong \begin{pmatrix} \overline{\chi\epsilon} & * \\ 0 & \overline{\chi} \end{pmatrix}$ for some character $\overline{\chi}$.*

Proof. This may be proved in exactly the same way as Lemma 3.4 of [GS11a], working in the setting of [BLGG11] (cf. the proof of Lemma 3.1.1 of [BLGG11]). Note that if ρ_w is not potentially crystalline, then it is necessarily a twist of an extension of the trivial character by the cyclotomic character. □

3. Realising local representations globally

3.1. We now recall a result from the forthcoming paper [GK11] which allows us to realise local representations globally, in order to apply the results of Section 2.2 in a purely local setting.

Theorem 3.1.1. *Suppose that $p > 2$, that $K/\mathbb{Q}_p$ is a finite extension, and let $\bar{r}_K : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Then there is an imaginary*

CM field F and a continuous irreducible representation $\bar{r}: G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ such that, if F^+ denotes the maximal totally real subfield of F,

- *each place $v|p$ of F^+ splits in F and has $F^+_v \cong K$,*
- *for each place $v|p$ of F^+, there is a place $\widetilde{v}$ of F lying over F^+ with $\bar{r}|_{G_{F_{\widetilde{v}}}}$ isomorphic to an unramified twist of $\bar{r}_K$,*
- *$\zeta_p \notin F$,*
- *$\bar{r}$ is unramified outside of p,*
- *$\bar{r}$ is modular in the sense of* [BLGG11], *and*
- *$\bar{r}(G_{F(\zeta_p)})$ is adequate.*

Proof. We give a brief (but complete) proof; a more detailed version will appear in [GK11]. The argument is a straightforward application of potential modularity techniques. First, an application of Proposition 3.2 of [Cal10] supplies a totally real field L^+ and a continuous irreducible representation $\bar{r}: G_{L^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ such that

- for each place $v|p$ of L^+, $L^+_v \cong K$ and $\bar{r}|_{L^+_v} \cong \bar{r}_K$,
- for each place $v|\infty$ of L^+, $\det \bar{r}(c_v) = -1$, where c_v is a complex conjugation at v, and
- there is a non-trivial finite extension $\mathbb{F}/\mathbb{F}_p$ such that $\bar{r}(G_{L^+}) = \mathrm{GL}_2(\mathbb{F})$.

By a further base change one can also arrange that $\bar{r}|_{G_{L^+_v}}$ is unramified at each finite place $v \nmid p$ of L^+.

By Lemma 6.1.6 of [BLGG10] and the proof of Proposition 7.8.1 of [Sno09], $\bar{r}_K$ admits a potentially Barsotti-Tate lift, and one may then apply Proposition 8.2.1 of [Sno09] to deduce that there is a finite totally real Galois extension F^+/L^+ in which all primes of L^+ above p split completely, such that $\bar{r}|_{G_{F^+}}$ is modular in the sense that it is congruent to the Galois representation associated to some Hilbert modular form of parallel weight 2.

By the theory of base change between GL_2 and unitary groups (*cf.* section 2 of [BLGG11]), it now suffices to show that there is a totally imaginary quadratic extension F/F^+ and a character $\overline{\theta}: G_F \to \overline{\mathbb{F}}_p^\times$ such that $\bar{r}|_{G_F} \otimes \overline{\theta}$ has multiplier $\bar{\epsilon}^{-1}$ and such that for each place $v|p$ of F^+, there is a place $\widetilde{v}$ of F lying over v with $\overline{\theta}|_{G_{F_{\widetilde{v}}}}$ unramified. The existence of such a character is a straightforward exercise in class field theory, and follows for example from Lemma 4.1.5 of [CHT08]. □

4. Congruences

4.1. Having realised a local mod p representation globally, we can now use the results explained in Section 2 to deduce non-trivial local consequences.

Proposition 4.1.1. *Let $p > 2$ be prime, let $K/\mathbb{Q}_p$ be a finite totally ramified extension, and let $\overline{\rho}: G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Let $a \in W^?(\overline{\rho})$ be a Serre weight. Then there is a continuous potentially semistable representation $\rho: G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ lifting $\overline{\rho}$, such that ρ has Hodge type $\underline{0}$ and inertial type $\widetilde{\omega}^{a_1} \oplus \widetilde{\omega}^{a_2}$. Furthermore, ρ is potentially crystalline unless $a_1 - a_2 = p - 1$ and $\overline{\rho} \cong \begin{pmatrix} \overline{\chi\epsilon} & * \\ 0 & \overline{\chi} \end{pmatrix}$ for some character $\overline{\chi}$.*

Proof. By Theorem 3.1.1, there is an imaginary CM field F and a modular representation $\bar{r}: G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ such that

- for each place $v|p$ of F^+, v splits in F as $\widetilde{v}\widetilde{v}^c$, and we have $F_{\widetilde{v}} \cong K$, and $\bar{r}|_{G_{F_{\widetilde{v}}}}$ is isomorphic to an unramified twist of $\overline{\rho}$,

- $\bar{r}$ is unramified outside of p,
- $\zeta_p \notin F$, and
- $\bar{r}(G_{F(\zeta_p)})$ is adequate.

Now, since the truth of the result to be proved is obviously unaffected by making an unramified twist (if $\overline{\rho}$ is replaced by a twist by an unramified character $\overline{\theta}$, one may replace ρ by a twist by an unramified lift of $\overline{\theta}$), we may without loss of generality suppose that $\bar{r}|_{G_{F_w}} \cong \overline{\rho}$. Let $b \in (\mathbb{Z}^2_+)^S_0$ be the Serre weight such that $b_{\widetilde{v}} = a$ for each place $v|p$ of F^+, where S denotes the set of places of F above p. By Remark 2.1.5, $b \in W^?(\bar{r})$. Then by Theorem 2.2.6, $\bar{r}$ is modular of weight b. The result now follows from Proposition 2.2.9. □

4.2. **Spaces of crystalline extensions.** We now specialise to the setting of Definition 2.1.3. As usual, we let $K/\mathbb{Q}_p$ be a finite totally ramified extension with residue field $k = \mathbb{F}_p$, ramification index e, and uniformiser π. We fix a Serre weight $a \in \mathbb{Z}^2_+$. Note that all the subsequent constructions that we make (such as the definitions of the spaces L_{flat} and L_{crys} below) will depend on this choice. We fix a continuous representation $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, and we assume that there is:

- a decomposition $\mathrm{Hom}(\mathbb{F}_p, \overline{\mathbb{F}}_p) = J \coprod J^c$, and
- an integer $0 \le \delta \le e-1$ such that

$$\overline{\rho}|_{I_K} \cong \begin{pmatrix} \omega^\delta \prod_{\sigma \in J} \omega_\sigma^{a_1+1} \prod_{\sigma \in J^c} \omega_\sigma^{a_2} & * \\ 0 & \omega^{e-1-\delta} \prod_{\sigma \in J^c} \omega_\sigma^{a_1+1} \prod_{\sigma \in J} \omega_\sigma^{a_2}. \end{pmatrix}$$

Note that in general there might be several choices of J, δ. Consider pairs of characters χ_1, $\chi_2 : G_K \to \overline{\mathbb{Q}}_p^\times$ with the properties that:

(1) $\overline{\rho} \cong \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix}$,
(2) χ_1 and χ_2 are crystalline, and
(3) if we let S denote the set of $\mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$, then there exist J, δ as above such that either
    (i) J is non-empty, and there is one embedding $\tau \in S$ with $\mathrm{HT}_\tau(\chi_1) = a_1 + 1$ and $\mathrm{HT}_\tau(\chi_2) = a_2$, there are δ embeddings $\tau \in S$ with $\mathrm{HT}_\tau(\chi_1) = 1$ and $\mathrm{HT}_\tau(\chi_2) = 0$, and for the remaining $e-1-\delta$ embeddings $\tau \in S$ we have $\mathrm{HT}_\tau(\chi_1) = 0$ and $\mathrm{HT}_\tau(\chi_2) = 1$, or
    (ii) $J = \emptyset$, and there is one embedding $\tau \in S$ with $\mathrm{HT}_\tau(\chi_1) = a_2$ and $\mathrm{HT}_\tau(\chi_2) = a_1 + 1$, there are δ embeddings $\tau \in S$ with $\mathrm{HT}_\tau(\chi_1) = 1$ and $\mathrm{HT}_\tau(\chi_2) = 0$, and for the remaining $e-1-\delta$ embeddings $\tau \in S$ we have $\mathrm{HT}_\tau(\chi_1) = 0$ and $\mathrm{HT}_\tau(\chi_2) = 1$.

Note that these properties do not uniquely determine the characters χ_1 and χ_2, even in the unramified case, as one is always free to twist either character by an unramified character which is trivial mod p. We point out that the Hodge type of any de Rham extension of χ_2 by χ_1 will be a lift of a. Conversely, by Lemma 6.2 of [GS11a] any χ_1, χ_2 satisfying (1) and (2) such that the Hodge type of $\chi_1 \oplus \chi_2$ is a lift of a will satisfy (3) for a valid choice of J and δ (unique unless $a = 0$).

Suppose now that we have fixed two such characters χ_1 and χ_2, and we now allow the (line corresponding to the) extension class of $\overline{\rho}$ in $\mathrm{Ext}_{G_K}(\overline{\chi}_2, \overline{\chi}_1)$ to vary. We naturally identify $\mathrm{Ext}_{G_K}(\overline{\chi}_2, \overline{\chi}_1)$ with $H^1(G_K, \overline{\chi}_1\overline{\chi}_2^{-1})$ from now on.

Definition 4.2.1. Let L_{χ_1,χ_2} be the subset of $H^1(G_K, \overline{\chi}_1\overline{\chi}_2^{-1})$ such that the corresponding representation $\overline{\rho}$ has a crystalline lift ρ of the form

$$\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}.$$

We have the following variant of Lemma 3.12 of [BDJ10].

Lemma 4.2.2. *L_{χ_1,χ_2} is an $\overline{\mathbb{F}}_p$-vector subspace of $H^1(G_K, \overline{\chi}_1\overline{\chi}_2^{-1})$ of dimension $|J| + \delta$, unless $\overline{\chi}_1 = \overline{\chi}_2$, in which case it has dimension $|J| + \delta + 1$.*

Proof. Let $\chi = \chi_1\chi_2^{-1}$. Recall that $H^1_f(G_K, \overline{\mathbb{Z}}_p(\chi))$ is the preimage of $H^1_f(G_K, \overline{\mathbb{Q}}_p(\chi))$ under the natural map $\eta : H^1(G_K, \overline{\mathbb{Z}}_p(\chi)) \to H^1(G_K, \overline{\mathbb{Q}}_p(\chi))$, so that L_{χ_1,χ_2} is the image of $H^1_f(G_K, \overline{\mathbb{Z}}_p(\chi))$ in $H^1(G_K, \overline{\chi})$. The kernel of η is precisely the torsion part of $H^1(G_K, \overline{\mathbb{Z}}_p(\chi))$. Since $\chi \neq 1$, e.g. by examining Hodge-Tate weights, this torsion is non-zero if and only if $\overline{\chi} = 1$, in which case it has the form $\kappa^{-1}\overline{\mathbb{Z}}_p/\overline{\mathbb{Z}}_p$ for some $\kappa \in \mathfrak{m}_{\overline{\mathbb{Z}}_p}$. (To see this, note that if $\chi \neq 1$ is defined over E, then the long exact sequence associated to $0 \to \mathcal{O}_E(\chi) \to \mathcal{O}_E(\chi) \to k_E(\overline{\chi}) \to 0$ identifies $k_E(\overline{\chi})^{G_K}$ with the ϖ-torsion in $\ker(\eta)$.)

By Proposition 1.24(2) of [Nek93] we see that $\dim_{\overline{\mathbb{Q}}_p} H^1_f(G_K, \overline{\mathbb{Q}}_p(\chi)) = |J| + \delta$, again using $\chi \neq 1$. Since $H^1(G_K, \overline{\mathbb{Z}}_p(\chi))$ is a finitely generated $\overline{\mathbb{Z}}_p$-module, the result follows. ∎

Definition 4.2.3. If $\overline{\chi}_1$ and $\overline{\chi}_2$ are fixed, we define L_{crys} to be the subset of $H^1(G_K, \overline{\chi}_1\overline{\chi}_2^{-1})$ given by the union of the L_{χ_1,χ_2} over all χ_1 and χ_2 as above.

Note that L_{crys} is a union of subspaces of possibly varying dimensions, and as such it is not clear that L_{crys} is itself a linear subspace. Note also that the representations $\overline{\rho}$ corresponding to elements of L_{crys} are by definition precisely those for which $a \in W^?(\overline{\rho})$. Note also that L_{crys} depends only on $\overline{\rho}^{\text{ss}}$ and a.

Definition 4.2.4. Let L_{flat} be the subset of $H^1(G_K, \overline{\chi}_1\overline{\chi}_2^{-1})$ consisting of classes with the property that if $\overline{\rho} \cong \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix}$ is the corresponding representation, then there is a finite field $k_E \subset \overline{\mathbb{F}}_p$ and a finite flat k_E-vector space scheme over $\mathcal{O}_{K(\pi^{1/(p-1)})}$ with generic fibre descent data to K of type $\omega^{a_1} \oplus \omega^{a_2}$ (see Definition 5.1.1) whose generic fibre is $\overline{\rho}$.

Note that L_{flat} depends only on $\overline{\rho}^{\text{ss}}$ and a.

Proposition 4.2.5. *Provided that $a_1 - a_2 \neq p-1$ or that $\overline{\chi}_1\overline{\chi}_2^{-1} \neq \overline{\epsilon}$, $L_{\text{crys}} \subset L_{\text{flat}}$.*

Proof. Take a class in L_{crys}, and consider the corresponding representation $\overline{\rho} \cong \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix}$. As remarked above, $a \in W^?(\overline{\rho})$, so by Proposition 4.1.1, $\overline{\rho}$ has a crystalline lift of Hodge type $\underline{0}$ and inertial type

$$\widetilde{\omega}^{a_1} \oplus \widetilde{\omega}^{a_2},$$

and this representation can be taken to have coefficients in the ring of integers $\mathcal{O}_E$ of a finite extension $E/\mathbb{Q}_p$. Let ϖ be a uniformiser of $\mathcal{O}_E$, and k_E the residue field. Such a representation corresponds to a p-divisible $\mathcal{O}_E$-module with generic fibre descent data, and taking the ϖ-torsion gives a finite flat k_E-vector space scheme with generic fibre descent data whose generic fibre is $\overline{\rho}$. By Corollary 5.2 of [GS11b] this descent data has type $\omega^{a_1} \oplus \omega^{a_2}$. ∎

In the next section we will make calculations with finite flat group schemes in order to relate L_{flat} and L_{crys}.

5. Finite flat models

5.1. We work throughout this section in the following setting:

- $K/\mathbb{Q}_p$ is a finite extension with ramification index e, ring of integers $\mathcal{O}_K$, uniformiser π and residue field $\mathbb{F}_p$.
- $\overline{\chi}_1$, $\overline{\chi}_2$ are characters $G_K \to \overline{\mathbb{F}}_p^{\times}$.
- $a \in \mathbb{Z}_+^2$ is a Serre weight.
- There is a decomposition $\mathrm{Hom}(\mathbb{F}_p, \overline{\mathbb{F}}_p) = J \coprod J^c$, and an integer $0 \le \delta \le e-1$ such that

$$\overline{\chi}_1|_{I_K} = \omega^{\delta} \prod_{\sigma \in J} \omega^{a_1+1} \prod_{\sigma \in J^c} \omega^{a_2},$$

$$\overline{\chi}_2|_{I_K} = \omega^{e-1-\delta} \prod_{\sigma \in J^c} \omega^{a_1+1} \prod_{\sigma \in J} \omega^{a_2}.$$

Note in particular that $(\overline{\chi}_1\overline{\chi}_2)|_{I_K} = \omega^{a_1+a_2+e}$.

Let $K_1 := K(\pi^{1/(p-1)})$. Let k_E be a finite extension of $\mathbb{F}_p$ such that $\overline{\chi}_1, \overline{\chi}_2$ are defined over k_E; for the moment k_E will be fixed, but eventually it will be allowed to vary.

We wish to consider the representations $\overline{\rho} \cong \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix}$ such that there is a finite flat k_E-vector space scheme $\mathcal{G}$ over $\mathcal{O}_{K_1}$ with generic fibre descent data to K of type $\omega^{a_1} \oplus \omega^{a_2}$ (see Definition 5.1.1), whose generic fibre is $\overline{\rho}$. In order to do so, we will work with Breuil modules with descent data from K_1 to K. We recall the necessary definitions from [GS11b].

Fix π_1, a $(p-1)$-st root of π in K_1. Write $e' = e(p-1)$. The category $\mathrm{BrMod}_{\mathrm{dd}}$ consists of quadruples $(\mathcal{M}, \mathrm{Fil}^1\mathcal{M}, \phi_1, \{\widehat{g}\})$ where:

- $\mathcal{M}$ is a finitely generated free $k_E[u]/u^{e'p}$-module,
- $\mathrm{Fil}^1\mathcal{M}$ is a $k_E[u]/u^{e'p}$-submodule of $\mathcal{M}$ containing $u^{e'}\mathcal{M}$,
- $\phi_1 : \mathrm{Fil}^1\mathcal{M} \to \mathcal{M}$ is k_E-linear and ϕ-semilinear (where $\phi : \mathbb{F}_p[u]/u^{e'p} \to \mathbb{F}_p[u]/u^{e'p}$ is the p-th power map) with image generating $\mathcal{M}$ as a $k_E[u]/u^{e'p}$-module, and
- $\widehat{g} : \mathcal{M} \to \mathcal{M}$ for each $g \in \mathrm{Gal}(K_1/K)$ are additive bijections that preserve $\mathrm{Fil}^1\mathcal{M}$, commute with the ϕ_1-, and k_E-actions, and satisfy $\widehat{g}_1 \circ \widehat{g}_2 = \widehat{g_1 \circ g_2}$ for all $g_1, g_2 \in \mathrm{Gal}(K_1/K)$; furthermore $\widehat{1}$ is the identity, and if $a \in k_E$, $m \in \mathcal{M}$ then $\widehat{g}(au^i m) = a((g(\pi)/\pi)^i)u^i\widehat{g}(m)$.

The category $\mathrm{BrMod}_{\mathrm{dd}}$ is equivalent to the category of finite flat k_E-vector space schemes over $\mathcal{O}_{K_1}$ together with descent data on the generic fibre from K_1 to K (this equivalence depends on π_1); see [Sav08], for instance. We obtain the associated G_K-representation (which we will refer to as the generic fibre) of an object of $\mathrm{BrMod}_{\mathrm{dd},K_1}$ via the covariant functor $T^K_{\mathrm{st},2}$ (which is defined immediately before Lemma 4.9 of [Sav05]).

Definition 5.1.1. Let $\mathcal{M}$ be an object of $\mathrm{BrMod}_{\mathrm{dd}}$ such that the underlying k_E-module has rank two. We say that the finite flat k_E-vector space scheme corresponding to $\mathcal{M}$ *has descent data of type* $\omega^{a_1} \oplus \omega^{a_2}$ if $\mathcal{M}$ has a basis e_1, e_2 such that

$\widehat{g}(e_i) = \omega^{a_i}(g)e_i$. (Here we abuse notation by identifying an element of G_K with its image in $\mathrm{Gal}(K_1/K)$.)

We now consider a finite flat group scheme with generic fibre descent data $\mathcal{G}$ as above. By a standard scheme-theoretic closure argument, $\overline{\chi}_1$ corresponds to a finite flat subgroup scheme with generic fibre descent data $\mathcal{H}$ of $\mathcal{G}$, so we begin by analysing the possible finite flat group schemes corresponding to characters.

Suppose now that $\mathcal{M}$ is an object of $\mathrm{BrMod}_{\mathrm{dd}}$. The rank one objects of $\mathrm{BrMod}_{\mathrm{dd}}$ are classified as follows.

Proposition 5.1.2. *With our fixed choice of uniformiser π, every rank one object of $\mathrm{BrMod}_{\mathrm{dd}}$ has the form:*

- $\mathcal{M} = (k_E[u]/u^{e'p}) \cdot v$,
- $\mathrm{Fil}^1\mathcal{M} = u^{x(p-1)}\mathcal{M}$,
- $\phi_1(u^{x(p-1)}v) = cv$ *for some* $c \in k_E^{\times}$, *and*
- $\widehat{g}(v) = \omega(g)^k v$ *for all* $g \in \mathrm{Gal}(K_1/K)$,

where $0 \leq x \leq e$ and $0 \leq k < p-1$ are integers.

Then $T^K_{\mathrm{st},2}(\mathcal{M}) = \omega^{k+x} \cdot \mathrm{ur}_{c^{-1}}$, where $\mathrm{ur}_{c^{-1}}$ is the unramified character taking an arithmetic Frobenius element to c^{-1}.

Proof. This is a special case of Proposition 4.2 and Corollary 4.3 of [GS11b]. □

Let $\mathcal{M}$ (or $\mathcal{M}(x)$) be the rank one Breuil module with k_E-coefficients and descent data from K_1 to K corresponding to $\mathcal{H}$, and write $\mathcal{M}$ in the form given by Proposition 5.1.2. Since $\mathcal{G}$ has descent data of type $\omega^{a_1} \oplus \omega^{a_2}$, we must have $\omega^k \in \{\omega^{a_1}, \omega^{a_2}\}$.

5.2. **Extensions.** Having determined the rank one objects, we now go further and compute the possible extension classes. By a scheme-theoretic closure argument, the Breuil module $\mathcal{P}$ corresponding to $\mathcal{G}$ is an extension of $\mathcal{N}$ by $\mathcal{M}$, where $\mathcal{M}$ is as in the previous section, and $\mathcal{N}$ (or $\mathcal{N}(y)$) is defined by

- $\mathcal{N} = (k_E[u]/u^{e'p}) \cdot w$,
- $\mathrm{Fil}^1\mathcal{N} = u^{y(p-1)}\mathcal{N}$,
- $\phi_1(u^{y(p-1)}v) = dw$ for some $d \in k_E^{\times}$, and
- $\widehat{g}(v) = \omega(g)^l v$ for all $g \in \mathrm{Gal}(K_1/K)$,

where $0 \leq y \leq e$ and $0 \leq l < p-1$ are integers. Now, as noted above, the descent data for $\mathcal{G}$ is of type $\omega^{a_1} \oplus \omega^{a_2}$, so we must have that either $\omega^k = \omega^{a_1}$ and $\omega^l = \omega^{a_2}$, or $\omega^k = \omega^{a_2}$ and $\omega^l = \omega^{a_1}$. Since by definition we have $(\overline{\chi}_1\overline{\chi}_2)|_{I_K} = \omega^{a_1+a_2+e}$, we see from Proposition 5.1.2 that

$$x + y \equiv e \pmod{p-1}.$$

We have the following classification of extensions of $\mathcal{N}$ by $\mathcal{M}$.

Proposition 5.2.1. *Every extension of $\mathcal{N}$ by $\mathcal{M}$ is isomorphic to exactly one of the form*

- $\mathcal{P} = (k_E[u]/u^{e'p}) \cdot v + (k_E[u]/u^{e'p}) \cdot w$,
- $\mathrm{Fil}^1\mathcal{P} = (k_E[u]/u^{e'p}) \cdot u^{x(p-1)}v + (k_E[u]/u^{e'p}) \cdot (u^{y(p-1)}w + \nu v)$,
- $\phi_1(u^{x(p-1)}v) = cv$, $\phi_1(u^{y(p-1)}w + \nu v) = dw$,
- $\widehat{g}(v) = \omega^k(g)v$ *and* $\widehat{g}(w) = \omega^l(g)w$ *for all* $g \in \mathrm{Gal}(K_1/K)$,

where $\nu \in u^{\max\{0,(x+y-e)(p-1)\}} k_E[u]/u^{e'p}$ *has all nonzero terms of degree congruent to* $l-k$ *modulo* $p-1$*, and has all terms of degree less than* $x(p-1)$*, unless* $\overline{\chi}_1 = \overline{\chi}_2$ *and* $x \geq y$*, in which case it may additionally have a term of degree* $px - y$*.*

Proof. This is a special case of Theorem 7.5 of [Sav04] with the addition of k_E-coefficients. When K (in the notation of *loc. cit.*) is totally ramified, the proof of *loc. cit.* is argued in precisely the same manner when coefficients are added, taking care to note the following changes. (Note that *loc. cit.* uses l instead of p, so let $l = p$ in what follows.)

- Replace Lemma 7.1 of *loc. cit.* (i.e., Lemma 5.2.2 of [BCDT01]) with Lemma 5.2.4 of [BCDT01] (with $k' = k_E$ and $k = \mathbb{F}_p$ in the notation of that Lemma). In particular replace t^l with $\phi(t)$ wherever it appears in the proof, where ϕ is the k_E-linear endomorphism of $k_E[u]/u^{e'l}$ sending u^i to u^{li}.
- Instead of applying Lemma 4.1 of [Sav04], note that the cohomology group $H^1(\mathrm{Gal}(K_1/K), k_E[u]/u^{e'l})$ vanishes because $\mathrm{Gal}(K_1/K)$ has prime-to-l order while $k_E[u]/u^{e'l}$ has l-power order.
- Every occurrence of T_i^l in the proof (for any subscript i) should be replaced with T_i.
- The coefficients of h, t are permitted to lie in k_E (i.e., they are not constrained to lie in any particular proper subfield).

$\square$

The formulas for $(\mathcal{P}, \mathrm{Fil}^1\mathcal{P}, \phi_1, \{\widehat{g}\})$ in the statement of Proposition 5.2.1 define a Breuil module with descent data provided that $\mathrm{Fil}^1\mathcal{P}$ contains $u^{e'}\mathcal{P}$ and is preserved by each $\widehat{g}$. This is the case as long as ν lies in $u^{\max\{0,(x+y-e)(p-1)\}} k_E[u]/u^{e'p}$ and has all nonzero terms of degree congruent to $l-k$ modulo $p-1$ (*cf.* the discussion in Section 7 of [Sav04]); denote this Breuil module by $\mathcal{P}(x, y, \nu)$. Note that c is fixed while x determines k, since we require $\omega^{k+x} \cdot \mathrm{ur}_{c^{-1}} = \overline{\chi}_1$; similarly d is fixed and y determines l. So this notation is reasonable.

We would like to compare the generic fibres of extensions of different choices of $\mathcal{M}$ and $\mathcal{N}$. To this end, we have the following result. Write $\overline{\chi}_1|_{I_K} = \omega^\alpha$, $\overline{\chi}_2|_{I_K} = \omega^\beta$.

Proposition 5.2.2. *The Breuil module* $\mathcal{P}(x, y, \nu)$ *has the same generic fibre as the Breuil module* $\mathcal{P}'$*, where*

- $\mathcal{P}' = (k_E[u]/u^{e'p}) \cdot v' + (k_E[u]/u^{e'p}) \cdot w'$,
- $\mathrm{Fil}^1\mathcal{P}' = (k_E[u]/u^{e'p}) \cdot u^{e(p-1)}v' + (k_E[u]/u^{e'p}) \cdot (w' + u^{p(e-x)+y}\nu v')$,
- $\phi_1(u^{e(p-1)}v') = cv'$, $\phi_1(w' + u^{p(e-x)+y}\nu v') = dw'$,
- $\widehat{g}(v') = \omega^{\alpha-e}(g)v'$ *and* $\widehat{g}(w') = \omega^\beta(g)w'$ *for all* $g \in \mathrm{Gal}(K_1/K)$*.*

Proof. Consider the Breuil module $\mathcal{P}''$ defined by

- $\mathcal{P}'' = (k_E[u]/u^{e'p}) \cdot v'' + (k_E[u]/u^{e'p}) \cdot w''$,
- $\mathrm{Fil}^1\mathcal{P}'' = (k_E[u]/u^{e'p}) \cdot u^{e(p-1)}v'' + (k_E[u]/u^{e'p}) \cdot (u^{y(p-1)}w'' + u^{p(e-x)}\nu v'')$,
- $\phi_1(u^{e(p-1)}v'') = cv''$, $\phi_1(u^{y(p-1)}w'' + u^{p(e-x)}\nu v'') = dw''$,
- $\widehat{g}(v'') = \omega^{k+x-e}(g)v''$ and $\widehat{g}(w'') = \omega^l(g)w''$ for all $g \in \mathrm{Gal}(K_1/K)$.

(One checks without difficulty that this *is* a Breuil module. For instance the condition on the minimum degree of terms appearing in ν guarantees that $\mathrm{Fil}^1\mathcal{P}''$ contains $u^{e'}\mathcal{P}''$.) Note that $k + x \equiv \alpha \pmod{p-1}$, $l + y \equiv \beta \pmod{p-1}$. We claim that $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{P}''$ all have the same generic fibre. To see this, one can check directly that there is a morphism $\mathcal{P} \to \mathcal{P}''$ given by

$$v \mapsto u^{p(e-x)}v'',\ w \mapsto w'',$$

and a morphism $\mathcal{P}' \to \mathcal{P}''$ given by

$$v' \mapsto v'',\ w' \mapsto u^{py}w''.$$

By Proposition 8.3 of [Sav04], it is enough to check that the kernels of these maps do not contain any free $k_E[u]/(u^{e'p})$-submodules, which is an immediate consequence of the inequalities $p(e-x), py < e'p$. □

Remark 5.2.3. We note for future reference that while the classes in $H^1(G_K, \overline{\chi}_1\overline{\chi}_2^{-1})$ realised by $\mathcal{P}(x,y,\nu)$ and $\mathcal{P}'$ may not coincide, they differ at most by multiplication by a k_E-scalar. To see this, observe that the maps $\mathcal{P} \to \mathcal{P}''$ and $\mathcal{P}' \to \mathcal{P}''$ induce k_E-isomorphisms on the one-dimensional sub- and quotient characters.

We review the constraints on the integers x, y: they must lie between 0 and e, and if we let k, l be the residues of $\alpha - x, \beta - y \pmod{p-1}$ in the interval $[0, p-1)$ then we must have $\{\omega^k, \omega^l\} = \{\omega^{a_1}, \omega^{a_2}\}$. Call such a pair x, y *valid.*

Corollary 5.2.4. *Let x', y' be another valid pair. Suppose that $x' + y' \leq e$ and $p(x'-x) + (y-y') \geq 0$. Then $\mathcal{P}(x,y,\nu)$ has the same generic fibre as $\mathcal{P}(x',y',\nu')$, where $\nu' = u^{p(x'-x)+(y-y')}\nu$.*

Proof. The Breuil module $\mathcal{P}(x',y',\nu')$ is well-defined: one checks, e.g. from the relation $l - k \equiv \beta - \alpha + x - y \pmod{p-1}$, that the congruence condition on the degrees of the nonzero terms in ν' is satisfied, while since $x' + y' \leq e$ there is no condition on the lowest degrees appearing in ν'. Now the result is immediate from Proposition 5.2.2, since $u^{p(e-x)+y}\nu = u^{p(e-x')+y'}\nu'$. □

Recall that $x + y \equiv e \pmod{p-1}$, so that x and $e - y$ have the same residue modulo $p-1$. It follows that if x, y is a valid pair of parameters, then so is $e-y, y$; and similarly for $x, e-x$. Let X be the largest value of x over all valid pairs x, y, and similarly Y the smallest value of y. Then on the one hand $X \geq e - Y$ by definition of X, while on the other hand $e - X \geq Y$ by definition of Y. It follows that $X + Y = e$.

Corollary 5.2.5. *The module $\mathcal{P}(x,y,\nu)$ has the same generic fibre as $\mathcal{P}(X,Y,\mu)$ where $\mu \in k_E[u]/u^{e'p}$ has all nonzero terms of degree congruent to $\beta - \alpha + X - Y$ modulo $p-1$, and has all terms of degree less than $X(p-1)$, unless $\overline{\chi}_1 = \overline{\chi}_2$, in which case it may additionally have a term of degree $pX - Y$.*

Proof. Since $X + Y = e$ and $p(X-x) + (y-Y) \geq 0$ from the choice of X, Y, Corollary 5.2.4 shows that $\mathcal{P}(x,y,\nu)$ has the same generic fibre as some $\mathcal{P}(X,Y,\nu')$; by Proposition 5.2.1 there exists μ as in the statement such that $\mathcal{P}(x,y,\mu)$ has the same generic fibre as $\mathcal{P}(X,Y,\nu')$. (Note that if $\overline{\chi}_1 = \overline{\chi}_2$ then automatically $X \geq Y$, because in this case if (x,y) is a valid pair then so is (y,x).) □

Proposition 5.2.6. *Let X be as above, i.e., X is the maximal integer such that*

- *$0 \leq X \leq e$, and*
- *either $\overline{\chi}_1|_{I_K} = \omega^{a_1+X}$ or $\overline{\chi}_1|_{I_K} = \omega^{a_2+X}$.*

Then L_{flat} is an $\overline{\mathbb{F}}_p$-vector space of dimension at most X, unless $\overline{\chi}_1 = \overline{\chi}_2$, in which case it has dimension at most $X+1$.

Proof. Let $L_{\mathrm{flat},k_E} \subset L_{\mathrm{flat}}$ consist of the classes η such that the containment $\eta \in L_{\mathrm{flat}}$ is witnessed by a k_E-vector space scheme with generic fibre descent data. By Corollary 5.2.5 and Remark 5.2.3 these are exactly the classes arising from the Breuil modules $\mathcal{P}(X, Y, \mu)$ with k_E-coefficients as in Corollary 5.2.5. These classes form a k_E-vector space (since they are *all* the extension classes arising from extensions of $\mathcal{N}(Y)$ by $\mathcal{M}(X)$), and by counting the (finite) number of possibilities for μ we see that $\dim_{k_E} L_{\mathrm{flat},k_E}$ is at most X (resp. $X+1$ when $\overline{\chi}_1 = \overline{\chi}_2$).

Since $L_{\mathrm{flat},k_E} \subset L_{\mathrm{flat},k'_E}$ if $k_E \subset k'_E$ it follows easily that $L_{\mathrm{flat}} = \cup_{k_E} L_{\mathrm{flat},k_E}$ is an $\overline{\mathbb{F}}_p$-vector space of dimension at most X (resp. $X+1$). □

We can now prove our main local result, the promised relation between L_{flat} and L_{crys}.

Theorem 5.2.7. *Provided that either $a_1 - a_2 \neq p-1$ or $\overline{\chi}_1\overline{\chi}_2^{-1} \neq \overline{\epsilon}$, we have $L_{\mathrm{flat}} = L_{\mathrm{crys}}$.*

Proof. Before we begin the proof, we remind the reader that the spaces L_{crys} and L_{flat} depend on the fixed Serre weight a and the fixed representation $\overline{\rho}^{\mathrm{ss}}$, and that we are free to vary J and δ in our arguments. By Proposition 4.2.5, we know that $L_{\mathrm{crys}} \subset L_{\mathrm{flat}}$, so by Proposition 5.2.6 it suffices to show that L_{crys} contains an $\overline{\mathbb{F}}_p$-subspace of dimension X (respectively $X+1$ if $\overline{\chi}_1 = \overline{\chi}_2$). Since L_{crys} is the union of the spaces L_{χ_1,χ_2}, it suffices to show that one of these spaces has the required dimension. Let X be as in the statement of Proposition 5.2.6, so that X is maximal in $[0, e]$ with the property that either $\overline{\chi}_1|_{I_K} = \omega^{a_1+X}$ or $\overline{\chi}_1|_{I_K} = \omega^{a_2+X}$. Note that by the assumption that there is a decomposition $\mathrm{Hom}(\mathbb{F}_p, \overline{\mathbb{F}}_p) = J \coprod J^c$, and an integer $0 \le \delta \le e-1$ such that

$$\overline{\rho}|_{I_K} \cong \begin{pmatrix} \omega^{\delta} \prod_{\sigma\in J} \omega_\sigma^{a_1+1} \prod_{\sigma \in J^c} \omega_\sigma^{a_2} & * \\ 0 & \omega^{e-1-\delta} \prod_{\sigma \in J^c} \omega_\sigma^{a_1+1} \prod_{\sigma\in J} \omega_\sigma^{a_2} \end{pmatrix},$$

we see that if $X = 0$ then $\overline{\chi}_1|_{I_K} = \omega^{a_2}$.

If $\overline{\chi}_1|_{I_K} = \omega^{a_2+X}$ then we can take J to be empty and we take $\delta = X$; otherwise $X > 0$ and $\overline{\chi}_1|_{I_K} = \omega^{a_1+X}$, and we can take J^c to be empty and $\delta = X-1$. In either case, we may define characters χ_1 and χ_2 as in Section 4.2, and we see from Lemma 4.2.2 that $\dim_{\overline{\mathbb{F}}_p} L_{\chi_1,\chi_2} = X$ unless $\overline{\chi}_1 = \overline{\chi}_2$, in which case it is $X+1$. The result follows. □

As a consequence of this result, we can also address the question of the relationship between the different spaces L_{χ_1,χ_2} for a fixed Serre weight $a \in W^?(\overline{\rho})$. If e is large, then these spaces do not necessarily have the same dimension, so they cannot always be equal. However, it is usually the case that the spaces of maximal dimension coincide, as we can now see.

Corollary 5.2.8. *If either $a_1 - a_2 \neq p-1$ or $\overline{\chi}_1\overline{\chi}_2^{-1} \neq \overline{\epsilon}$, then the spaces L_{χ_1,χ_2} of maximal dimension are all equal.*

Proof. In this case $\dim_{\overline{\mathbb{F}}_p} L_{\chi_1,\chi_2} = \dim_{\overline{\mathbb{F}}_p} L_{\mathrm{crys}}$ by the proof of Theorem 5.2.7, so we must have $L_{\chi_1,\chi_2} = L_{\mathrm{crys}}$. □

Finally, we determine L_{crys} in the one remaining case, where the spaces L_{χ_1,χ_2} of maximal dimension no longer coincide.

Proposition 5.2.9. *Suppose that $a_1 - a_2 = p-1$ and that $\overline{\chi}_1\overline{\chi}_2^{-1} = \overline{\epsilon}$. Then $L_{\mathrm{crys}} = H^1(G_K, \overline{\epsilon})$.*

Proof. We prove this in a similar fashion to the proof of Lemma 6.1.6 of [BLGG10]. By twisting we can reduce to the case $(a_1, a_2) = (p-1, 0)$. Let L be a given line in $H^1(G_K, \overline{\epsilon})$, and choose an unramified character ψ with trivial reduction. Let χ be some fixed crystalline character of G_K with Hodge-Tate weights $p, 1, \dots, 1$ such that $\overline{\chi} = \overline{\epsilon}$. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$, uniformiser ϖ and residue field $\mathbb{F}$, such that ψ and χ are defined over E and L is defined over $\mathbb{F}$ (that is, there is a basis for L which corresponds to an extension defined over $\mathbb{F}$). Since any extension of 1 by $\chi\psi$ is automatically crystalline, it suffices to show that we can choose ψ so that L lifts to $H^1(G_K, \mathcal{O}(\psi\chi))$.

Let H be the hyperplane in $H^1(G_K, \mathbb{F})$ which annihilates L under the Tate pairing. Let $\delta_1 : H^1(G_K, \mathbb{F}(\overline{\epsilon})) \to H^2(G_K, \mathcal{O}(\psi\chi))$ be the map coming from the exact sequence $0 \to \mathcal{O}(\psi\chi) \stackrel{\varpi}{\to} \mathcal{O}(\psi\chi) \to \mathbb{F}(\overline{\epsilon}) \to 0$ of G_K-modules. We need to show that $\delta_1(L) = 0$ for some choice of ψ.

Let δ_0 be the map $H^0(G_K, (E/\mathcal{O})(\psi^{-1}\chi^{-1}\epsilon)) \to H^1(G_K, \mathbb{F})$ coming from the exact sequence $0 \to \mathbb{F} \to (E/\mathcal{O})(\psi^{-1}\chi^{-1}\epsilon) \stackrel{\varpi}{\to} (E/\mathcal{O})(\psi^{-1}\chi^{-1}\epsilon) \to 0$ of G_K-modules. By Tate local duality, the condition that L vanishes under the map δ_1 is equivalent to the condition that the image of the map δ_0 is contained in H. Let $n \geq 1$ be the largest integer with the property that $\psi^{-1}\chi^{-1}\epsilon \equiv 1 \pmod{\varpi^n}$. Then we can write $\psi^{-1}\chi^{-1}\epsilon(x) = 1 + \varpi^n \alpha_\psi(x)$ for some function $\alpha_\psi : G_K \to \mathcal{O}$. Let $\overline{\alpha}_\psi$ denote $\alpha_\psi \pmod{\varpi} : G_K \to \mathbb{F}$. Then $\overline{\alpha}_\psi$ is a group homomorphism (i.e. a 1-cocycle), and the choice of n ensures that it is non-trivial. It is straightforward to check that the image of the map δ_0 is the line spanned by $\overline{\alpha}_\psi$. If $\overline{\alpha}_\psi$ is in H for some ψ, we are done. Suppose this is not the case. We break the rest of the proof into two cases.

Case 1: L is très ramifié: To begin, we observe that it is possible to have chosen ψ so that $\overline{\alpha}_\psi$ is ramified. To see this, let m be the largest integer with the property that $(\psi^{-1}\chi^{-1}\epsilon)|_{I_K} \equiv 1 \pmod{\varpi^m}$. Note that m exists since the Hodge-Tate weights of $\psi^{-1}\chi^{-1}\epsilon$ are not all 0. If $m = n$ then we are done, so assume instead that $m > n$. Let $g \in G_K$ be a fixed lift of Frob_K. We claim that $\psi^{-1}\chi^{-1}\epsilon(g) = 1 + \varpi^n \alpha_\psi(g)$ such that $\alpha_\psi(g) \not\equiv 0 \pmod{\varpi}$. In fact, if $\alpha_\psi(g) \equiv 0 \pmod{\varpi}$ then $\psi^{-1}\chi^{-1}\epsilon(g) \in 1 + \varpi^{n+1}\mathcal{O}_K$. Since $m > n$ we see that $\psi^{-1}\chi^{-1}\epsilon(G_K) \subset 1 + \varpi^{n+1}\mathcal{O}_K$ and this contradicts the selection of n. Now let ψ' be the unramified character sending our fixed g to $1 + \varpi^n \alpha_\psi(g)$. Then ψ' has trivial reduction, and after replacing ψ by $\psi\psi'$ we see that n has increased but m has not changed. After finitely many iterations of this procedure we have $m = n$, completing the claim.

Suppose, then, that $\overline{\alpha}_\psi$ is ramified. The fact that L is très ramifié implies that H does not contain the unramified line in $H^1(G_K, \mathbb{F})$. Thus there is a unique $\overline{x} \in \mathbb{F}^\times$ such that $\overline{\alpha}_\psi + u_{\overline{x}} \in H$ where $u_{\overline{x}} : G_K \to \mathbb{F}$ is the unramified homomorphism sending Frob_K to $\overline{x}$. Replacing ψ with ψ times the unramified character sending Frob_K to $(1 + \varpi^n x)^{-1}$, for x a lift of $\overline{x}$, we are done.

Case 2: L is peu ramifié: Making a ramified extension of $\mathcal{O}$ if necessary, we can and do assume that $n \geq 2$ (for example, replacing E by $E(\varpi^{1/2})$ has the effect of replacing n by $2n$). The fact that L is peu ramifié implies that H contains the unramified line. It follows that if we replace ψ with ψ times the unramified character sending Frob_K to $1 + \varpi$, then we are done (as the new $\overline{\alpha}_\psi$ will be unramified). $\square$

6. Global consequences

6.1. We now deduce our main global results, using the main theorems of [BLGG11] together with our local results to precisely determine the set of Serre weights for a global representation in the totally ramified case.

Theorem 6.1.1. *Suppose that Hypothesis 2.2.5 holds. Let $a \in (\mathbb{Z}^2_+)^S_0$ be a Serre weight such that $\bar{r}$ is modular of weight a. Let w be a place of F dividing p, write $a_w = (a_1, a_2)$, and write ω for the unique fundamental character of I_{F_w} of niveau one.*

Then $a_w \in W^?(\bar{r}|_{G_{F_w}})$.

Proof. Let e be the ramification degree of F_w. Suppose first that $\bar{r}|_{G_{F_w}}$ is irreducible. Then the proof of Lemma 5.5 of [GS11a] goes through unchanged, and gives the required result. So we may suppose that $\bar{r}|_{G_{F_w}}$ is reducible. In this case the proof of Lemma 5.4 of [GS11a] goes through unchanged, and shows that we have

$$\bar{r}|_{G_{F_w}} \cong \begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix}$$

where $(\overline{\chi}_1\overline{\chi}_2)|_{I_K} = \omega^{a_1+a_2+e}$, and either $\overline{\chi}_1|_{I_K} = \omega^{a_1+z}$ or $\overline{\chi}_1|_{I_K} = \omega^{a_2+e-z}$ for some $1 \le z \le e$, so we are in the situation of Section 4.2. Consider the extension class in $H^1(G_{F_w}, \overline{\chi}_1\overline{\chi}_2^{-1})$ corresponding to $\bar{r}|_{G_{F_w}}$. By Proposition 2.2.9, either $a_1 - a_2 = p-1$ and $\overline{\chi}_1\overline{\chi}_2^{-1} = \bar{\epsilon}$, or this extension class is in L_{flat}. In either case, by Theorem 5.2.7 and Proposition 5.2.9, the extension class is in L_{crys}, so that $a_w \in W^?(\bar{r}|_{G_{F_w}})$, as required. □

We remark that we have stated Theorem 6.1.1 only when $F_w/\mathbb{Q}_p$ is totally ramified for all places $w|p$ of F in order to avoid recalling the definition of Serre weights in any greater generality; however, the above argument would prove essentially the same result at any totally ramified place $w|p$ of F, even if not all places $w|p$ are totally ramified (just modify Proposition 2.2.9 suitably).

Combining Theorem 6.1.1 with Theorem 5.1.3 of [BLGG11], we obtain our main global result.

Theorem 6.1.2. *Suppose that Hypothesis 2.2.5 holds. Suppose further that $\zeta_p \notin F$ and $\bar{r}(G_{F(\zeta_p)})$ is adequate. Let $a \in (\mathbb{Z}^2_+)^S_0$ be a Serre weight. Then $a_w \in W^?(\bar{r}|_{G_{F_w}})$ for all places $w|p$ of F if and only if $\bar{r}$ is modular of weight a.*

Finally, we may apply our local results to the case of inner forms of GL_2, as considered in [GS11a]. Here is an example of the kind of theorem that one can prove. We refer the reader to [GS11a] for the notion of $\overline{\rho}$ as below being modular (of some weight).

Theorem 6.1.3. *Let F be a totally real field, let $p \ge 7$ be prime, and suppose that p is totally ramified in F, and that $[F(\zeta_p) : F] > 4$. Let $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous modular representation, and suppose that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. Let $a \in \mathbb{Z}^2$ be a Serre weight. Let v be the unique place of F lying over p, and assume that $\overline{\rho}|^{\mathrm{ss}}_{G_{F_v}} \not\cong \bar{\epsilon}\omega^{a_1} \oplus \omega^{a_2}, \bar{\epsilon}\omega^{a_2} \oplus \omega^{a_1}$. Then $\overline{\rho}$ is modular of weight a if and only if $a \in W^?(\overline{\rho}|_{G_{F_v}})$, where v is the unique place of F lying over p.*

Proof. This follows easily from Theorem 5.2.7 together with (the proof of) Corollary 7.3 of [GS11a], replacing the use of Theorem 7.1 of [GS11a] with an appeal to Theorem 6.1.9 of [BLGG10]. □

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