Recent developments in interval dynamics

Sebastian van Strien, Imperial College London
Weixiao Shen, National University of Singapore

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Interval dynamics: take a smooth map $f : \mathbb{R} \to \mathbb{R}$ and initial point $x_0 \in \mathbb{R}$. Then study orbits $x_{n+1} = f(x_n)$.

- These orbits can converge to periodic orbit, but also can converge in a complicated (chaotic) way to a large set.

In spite of large non-linearity of high iterates, many (and in a topological sense even most) such systems are well-understood.

The metric structure of orbits can be understood:
- from an ergodic point of view (Second 1/2 of talk);
- smoothness on certain invariant Cantor sets: using renormalisation (e.g. Avila & Lyubich’s recent result);
- global quasi-symmetric rigidity: Sullivan’s programme.

Aim 1/2 talk: what is qs-rigidity, its origins and applications?
What is quasi-symmetric rigidity?

A homeomorphism \( h: [0, 1] \to [0, 1] \) is called **quasi-symmetric** (often abbreviated as \( qs \)) if there exists \( K < \infty \) so that

\[
\frac{1}{K} \leq \frac{h(x + t) - h(x)}{h(x) - h(x - t)} \leq K
\]

for all \( x - t, x, x + t \in [0, 1] \). (\( \Rightarrow \) Hölder).

**Sullivan**'s programme: prove that \( f \) is **quasi-symmetrically rigid**, i.e.

\[
f, \tilde{f} \text{ is topologically conjugate} \implies \tilde{f}, f \text{ are quasi-symmetrically conjugate.}
\]

That is, homeomorphism \( h \) with \( h \circ f = g \circ h \) is ‘**necessarily**’ \( qs \).
Quasi-symmetric rigidity

Clark-vS:

**Theorem (Quasi-symmetric rigidity)**

Let \( N = S^1 \) or \( N = [0, 1] \). Assume \( f, g : N \to N \) are real analytic and topologically conjugate with at least one critical point and the conjugacy is a bijection between:

1. the set of **critical points** and the order of corresponding critical points are the same;
2. the set of **parabolic periodic points**.

Then \( f \) and \( g \) are **quasi-symmetrically** conjugate.

**Remarks:**

- This result is sharp: if one of the assumptions is not satisfied, then the result is false in general.
- Clark-vS also prove qs-rigidity when \( f \) and \( g \) are merely \( C^3 \) maps, under some very weak additional assumptions.
History of qs-rigidity results

- This completes a programme initiated in the 80’s by
  - **Sullivan** for interval maps: in his work on renormalisation;
  - **Herman** for circle homeo’s: to use quasiconformal surgery.


- The **presence of critical points** is necessary for result to hold: for circle diffeomorphisms the analogous statement is false (due to longer and longer saddle-cascades).

- If there are periodic points, conjugacies **cannot be** $C^1$, since then corresponding periodic orbits have the same multiplier.

- For **complex** (non-real) polynomials there are partial results (**qc-rigidity**), due to Kozlovski-vS, Lyubich-Kahn, Levin, Cheraghi, Cheraghi-Shishikura. However, in general **wide open** (related to local connectivity of Mandelbrot set).
qs-rigidity, requires control of high iterates (‘compactness’).

Turns out to be useful to **construct** an extension to $\mathbb{C}$: when $f, g$ are real analytic, use holomorphic extension of $f, g$ to small neighbourhoods of $[0, 1]$ in $\mathbb{C}$.

Prove that first return maps of $f, g$ to small intervals, extend to a ‘**complex box mapping**’ $F: U \to V$, see figure.

Each component of $U$ is mapped as a branched covering onto a component of $V$, and components of $U$ are either compactly contained or equal to a component of $V$.

Components of $F^{-n}(V)$ are called **puzzle pieces**.
In addition we need **complex bounds**, i.e. **universal control** on **shape** and **position** of components of $U$ inside components of $V$.

Clark-Trejo-vS:

### Theorem (Complex box mappings with complex bounds)

One can construct complex box mappings with complex bounds on arbitrarily small scales.

- Previous similar partial results by Sullivan, Levin-vS, Lyubich-Yampolsky and Graczyk-Świątek, Smania, Shen.

- **Key ingredient** in **renormalisation**, e.g. Avila-Lyubich.

- Complex bounds give better control than real bounds.

- Clark-Trejo-vS: something similar even for $C^3$ maps, but then $F$ is only asymptotically holomorphic.
Proving complex bounds

- From the **enhanced nest** construction (see next ●) and a remarkable result due to Kahn-Lyubich, given a non-renormalizable complex box mapping at one level, one can obtain complex box mappings with complex bounds at arbitrary deep levels.

- The **enhanced nest** is a sophisticated choice of a sequence of puzzle pieces $U_{n(i)}$, so that
  1. $\exists k(i)$ for which $F^{k(i)}: U_{n(i+1)} \rightarrow U_{n(i)}$ is a branched covering map with degree bounded by some universal number $N$.
  2. its inverse transfers geometric information efficiently from scale $U_{n(i)}$ to the smallest possible scale $U_{n(i+1)}$.

- Other choices will not give complex bounds, in general.

- In the **renormalizable case**, the construction of complex box mappings and the proof of complex bounds is significantly more involved.
Idea for proving quasi-symmetric rigidity

- Using the **complex bounds** and a *methodology for constructing quasi-conformal homeomorphisms* (building on papers of Kozlovski-Shen-vS and Levin-vS), we construct quasi-conformal conjugacies on **small scale**.
- Then develop a technology to glue the local information together. Requires additional care when there are several critical points.

Remarks:

- In the $C^3$ case $f, g$ have *asymptotically holomorphic extensions* near $[0, 1]$. Issue to deal with: arbitrary high iterates of $f$ and $g$ are not necessarily close to holomorphic.
So far I discussed what is qs-rigidity, and why it holds. Next:

Why is qs-rigidity useful?

Roughly, because it provides a comprehensive understanding of the dynamics, which opens up a pretty full understanding.

I will discuss two applications. Both are based on tools from complex analysis that become available because of quasi-symmetric rigidity.

A third application will hopefully be a resolution of the 1-dimensional Palis conjecture in full generality.
A smooth map $f : \mathbb{R} \to \mathbb{R}$ is **hyperbolic** if

- Lebesgue a.e. point is attracted to some periodic orbit with multiplier $\lambda$ so that $|\lambda| < 1$, or equivalently
- each critical point of $f$ is attracted to a periodic orbit and each periodic orbit is hyperbolic (i.e. with multiplier $\lambda \neq \pm 1$).

Martens-de Melo-vS: the **period** of periodic attractors is **bounded** $\implies$ hyperbolic maps have at most finitely many periodic attractors.

The notion of hyperbolicity was introduced by Smale and others because these maps are well-understood and:

- **Every hyperbolic map** satisfying an additional transversality condition, that no critical point is eventually mapped onto another critical point, is **structurally stable**. (A nearby map is *topologically conjugate*, i.e. same up to topological coordinate change.)
Hyperbolic one-dimensional maps are dense

- Fatou (20’s) conjectured most rational maps on the Riemann sphere are hyperbolic.
- Smale (60’s) conjectured that in higher dimensions, hyperbolic maps are dense. This turned out to be **false**.

Kozlovski-Shen-vS:

**Theorem (Density of hyperbolicity for real polynomials)**

*Any real polynomial can be approximated by a hyperbolic real polynomials of the same degree.*

and

**Theorem (Density of hyperbolicity for smooth one-dimensional maps)**

*Hyperbolic 1-d maps are $C^k$ dense, $k = 1, 2, \ldots, \infty$.***

This solves one of Smale’s problems for the 21st century.
Density of hyperbolicity for real transcendental maps

Rempe-vS:

**Theorem (Density of hyperbolicity for transcendental maps)**

*Density of hyperbolicity holds within the following spaces:*

1. *real transcendental entire functions, bounded on the real line, whose singular set is finite and real;*
2. *transcendental functions* $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ *that preserve the circle and whose singular set (apart from* $0, \infty$ *) is contained in the circle.*

**Remarks:**

- Hence, density of hyperbolicity within the famous **Arnol’d family** and within space of trigonometric polynomials.
- Result implies conjectures posed by de Melo-Salomão-Vargas.
Hyperbolicity is dense within generic families

**Theorem (vS: Hyperbolicity is dense within generic families)**

For any (Baire) $C^\infty$ generic family $\{g_t\}_{t \in [0,1]}$ of smooth maps:

- the number of critical points of each of the maps $g_t$ is bounded;
- the set of $t$'s for which $g_t$ is hyperbolic, is open and dense.

and

**Theorem (vS: $\exists$ family of cubic maps with robust chaos)**

There exists a real analytic one-parameter family $\{f_t\}$ of interval maps (consisting of cubic polynomials) so that

- there exists no $t \in [0,1]$ with $f_t$ is hyperbolic;
- $f_0$ and $f_1$ are not topologically conjugate.

Question: What if $f_0$ and $f_1$ are ‘**totally different**’?
Density of hyperbolicity for rational maps (Fatou’s conjecture) is wide open. By Mañé-Sad-Sullivan it follows from:

**Conjecture**

*If a rational map carries a measurable invariant line field on its Julia set, then it is a Lattès map.*

**Eremenko-vS:**

**Theorem**

*Any rational map on the Riemann sphere such that the multiplier of each periodic orbit is real, either is*

- *an interval or circle map (Julia set is 1d), or*
- *a Lattès map.*

In the first case, the Julia set of course does not carry measurable invariant line field.
Strategy of the proof: local versus global perturbations

One approach: take $g$ to be a local perturbation of $f$, i.e. find a ‘bump’ function $h$ which is small in the $C^k$ sense so that $g = f + h$ becomes hyperbolic.

- Difficulty with this approach: orbits pass many times through the support of the bump function.
- Jakobson (1971, in dimension one) and Pugh (1967, in higher dimensions but for diffeo’s) used this approach to prove a $C^1$ closing lemma.
- In the $C^2$ category this approach has proved to be unsuccessful (but Blokh-Misiurewicz have partial results). Shen (2004) showed $C^2$ density using qs-rigidity results.
Density of hyperbolicity with family $z^2 + c$, $c \in \mathbb{R}$ holds if there exists no interval of parameters $c$ of non-hyperbolic maps.

Sullivan showed that this follows from quasi-symmetric rigidity of any non-hyperbolic map $f_c$ (by an open-closed argument):

- Measurable Riemann Mapping Theorem $\implies$ $I(f_c) = \{ \tilde{c} \in \mathbb{R} \text{ s.t. } f_{\tilde{c}} \text{ topologically conjugate to } f_c \}$ is either open or a single point.

- Basic kneading theory $\implies$ $I(f_c)$ is closed set.

$\emptyset \subsetneq I(f_c) \subsetneq \mathbb{R}$ gives a contradiction unless $I(f_c)$ is a single point.

Using a slightly more sophisticated argument, Kozlovski-Shen-vS also obtain that quasi-symmetric rigidity implies density of hyperbolicity when there are more critical points.
In the early 90’s, Milnor posed the

**Monotonicity Conjecture.** The set of parameters within a family of real polynomial interval maps, for which the topological entropy is constant, is connected.

- A version of this conjecture was proved in the 1980’s for the quadratic case.
- Milnor-Tresser (2000) proved conjecture for cubics using
  - planar topology (in the cubic case the parameter space is two-dimensional) and
  - density of hyperbolicity for real quadratic maps.
- Bruin-vS: the set of parameters corresponding to polynomials of degree $d \geq 5$ with constant entropy is in general NOT locally connected.
Monotonicity of entropy: the multimodal case

Given \( d \geq 1 \) and \( \epsilon \in \{-1, 1\} \), let \( P^d_\epsilon \) space of

1. real polynomials \( f : [0, 1] \rightarrow [0, 1] \) of degree = \( d \);
2. all critical points in (0, 1);
3. \( \text{sign}(f'(0)) = \epsilon \).

Bruin-vS show:

**Theorem (Monotonicity of Entropy)**

For each integer \( d \geq 1 \), each \( \epsilon \in \{-1, 1\} \) and each \( c \geq 0 \),

\[
\{ f \in P^d_\epsilon ; h_{\text{top}}(f) = c \}
\]

is connected.

- Main ingredient: is quasi-symmetric rigidity.
- Hope to remove assumption (2): (currently \( d = 4 \) with Cheraghi).
- Rempe-vS \( \implies \) top. entropy of \( x \mapsto a \sin(x) \) monotone in \( a \).
Let’s now discuss ergodic properties in one-dimensional dynamics.
Let
\[ Q_a(x) = ax(1 - x), \quad 0 < a \leq 4. \]
Let
\[ \mathcal{H} = \{ a \in (0, 4] : Q_a \text{ is hyperbolic} \}. \]
As we have seen above, \( \mathcal{H} \) is an open and dense subset of the parameter space \((0, 4]\).
On the other hand, Jakobson has shown that the set
\[ \mathcal{A} = \{ a \in (0, 4] : Q_a \text{ has an ergodic acip} \} \]
has positive Lebesgue measure.
Here, an \textit{acip} is an invariant Borel probability measure that is absolutely continuous with respect to the Lebesgue measure.
### Theorem (Jakobson 1981)

*a = 4 is a one-sided Lebesgue density point of \( \mathcal{A} \):

\[
\lim_{\varepsilon \to 0} \frac{|\mathcal{A} \cap [4 - \varepsilon, 4]|}{\varepsilon} = 1.
\]

### Theorem (Lyubich 2002)

*For almost every \( a \in (0, 4] \), either \( Q_a \) is hyperbolic or it has an acip. In other words,*

\[
\mathcal{H} \cup \mathcal{A} = (0, 4] \mod 0.
\]
For generic smooth family $f_\lambda$ of interval maps, the following hold: For a.e. parameter $\lambda$,

- $f_\lambda$ has finitely many ‘physical’ measures (acip’s or supported on periodic orbits) whose basins cover the whole phase interval (mod 0).
- the system is ‘stochastically stable’.

Theorem (Avila-de Melo-Lyubich 2003, Avila-Moreira 2005+)

The Palis conjecture holds for unimodal maps with non-degenerate critical point.

Indeed, they strengthened Lyubich’s theorem, showing that for almost every $a \in \mathcal{A}$, $Q_a$ satisfies the Collet-Eckmann condition, and hence is stochastically stable (by a result of Baladi-Viana).

The multimodal case remains a major challenge in real one-dimensional dynamics.

In the remainder of this talk we will discuss partial results towards the Palis conjecture for multimodal maps:

- the existence of absolutely continuous invariant measures;
- expansion properties;
- stochastic stability;
- a generalisation of Jakobson’s theorem.

Here we will push as far as possible what can be obtained using only real methods. Hoping real and complex will meet.
∃ many proofs of Jakobson’s theorem and extensions: Benedicks-Carleson, Tsujii, · · ·. Most proofs use a purely real analytic method, showing certain expanding properties of an interval map persist under small perturbations, starting from an almost hyperbolic map.

To prove the Palis conjecture we need to consider perturbations starting from arbitrary maps.

Lyubich has a proof using a complex analytic method, which makes it possible to start near arbitrary unimodal maps. Using holomorphic motion, he constructs a parameter-phase space relationship which is almost quasiconformal.

For multimodal maps, it is not clear how to obtain regularity of the parameter-phase relationship starting from a general non-hyperbolic map.
What is needed for the existence of an acip?

The problem on existence of acip has a long history:

- Bowen (1970s): postcritically finite maps
- Misiurewicz (1981): Misiurewicz (non-recurrent) maps
- Collet-Eckmann (1983): S-unimodal map with the Collet-Eckmann condition
- Nowicki-van Strien (1991): S-unimodal maps with a summability condition
We say that a smooth map $f : [0, 1] \to [0, 1]$ satisfies the *large derivatives condition* (LD) if for each critical point $c$,

$$|Df^n(f(c))| \to \infty \text{ as } n \to \infty.$$

**Theorem (Bruin, Rivera-Letelier, S, van Strien 2008)**

Let $f$ be a smooth interval map with non-flat critical points and all periodic points hyperbolic repelling. If $f$ satisfies the (LD) condition, then $f$ has an acip.

How to prove existence of acip

- Construct an induced Markov map (‘tower’) $F$.
- Show that the induced map $F$ has an acip.
- This requires distortion estimates on $F$.
- To get invariant measure for the original map $f$ the inducing times need to be summable.
- Here we need to quantify expanding properties of $f$. 

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Lemma (Expansion)

Assume $f$ satisfies (LD) and (for simplicity) all critical points are non-degenerate. Let $V_\varepsilon = \bigcup_{c \in \text{Crit}} (c - \varepsilon, c + \varepsilon)$. For any $x \in V_\varepsilon$, if $n$ is the minimal positive integer such that $f^n(x) \in V_\varepsilon$, then

$$|Df^n(x)| \geq \Lambda(\varepsilon) \frac{|Df(x)|}{\varepsilon} \exp \{\rho(\varepsilon)n\},$$

where $\Lambda(\varepsilon) \to \infty$ and $\log \rho(\varepsilon)/\log \varepsilon \to 0$ as $\varepsilon \to 0$.

The inequality is equivalent to the so-called backward contraction property defined by Rivera-Letelier (2006) $\Rightarrow$ critical pullbacks of $V_\varepsilon$ have length much shorter than $\varepsilon$ $\Rightarrow$ fast decay of branches with long return times (see next slide).

Li-S : BC $\Leftrightarrow$ LD.
An induced Markov map is a map $F : \bigcup_{j=1}^{\infty} J_j \to I$, where

- $I$ is an open interval, and $J_j$’s are open subintervals;
- $|I \setminus (\bigcup J_j)| = 0$;
- $\forall j$, $F|J_j$ is a diffeomorphism onto $I$;
- $\exists C > 0$ such that for each component $U$ of $F^{-n}(I)$,
  \[
  \sup_{x,y \in U} \log \frac{|DF^n(x)|}{|DF^n(y)|} \leq C|F^n(x) - F^n(y)|;
  \]
- $\forall j$, $\exists s_j$ such that $F|J_j = f^{s_j}|J_j$.

**Theorem (Rivera-Letelier+S, in press)**

Suppose $f$ satisfies the (LD) condition and has all periodic points hyperbolic repelling. Then $f$ has an induced Markov map $F : \bigcup_{j=1}^{\infty} J_j \to I$ with super-polynomially small tail, i.e.

\[
\sum_{s_j=s} |J_j| \text{ is super-polynomially small in } s.
\]
A smooth interval map is called \textit{summable (with exponent 1)} if for each critical point \( c \), \( \sum_{n=1}^{\infty} |Df^n(f(c))|^{-1} < \infty \).

An \( \varepsilon \)-\textit{pseudo-orbit} is a sequence \( \{x_j\}_{j=0}^{n} \) such that for each \( 0 \leq j < n \), \( |f(x_j) - x_{j+1}| \leq \varepsilon \).

\textbf{Lemma (Robust expansion, S 2013)}

Assume that \( f \) is summable and all critical points are non-degenerate. Let \( V_\varepsilon = \bigcup_{c \in \text{Crit}} (c - \varepsilon, c + \varepsilon) \). For any \( \varepsilon \)-pseudo-orbit \( \{x_j\}_{j=0}^{\infty} \), if \( x_0 \in V_\varepsilon \) and \( n \) is the minimal positive integer such that \( x_n \in V_\varepsilon \), then

\[
\prod_{j=0}^{n-1} |Df(x_j)| \geq \Lambda(\varepsilon) \frac{|Df(x_0)|}{\varepsilon} \exp \{ \rho(\varepsilon)n \},
\]

where \( \Lambda(\varepsilon) \rightarrow \infty \) and \( \log \rho(\varepsilon)/\log \varepsilon \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \).
Now consider $\varepsilon$-pseudo orbits which are chosen randomly. What happens as $\varepsilon$ goes to zero?

**Theorem (S 2013)**

Let $f : [0, 1] \to [0, 1]$ be a $C^3$ interval map with non-flat critical points and all periodic points repelling. If $f$ is summable and $f$ is ergodic with respect to the Lebesgue measure, then $f$ is stochastically stable with respect to a large class of random perturbations.


**Theorem (S-van Strien 2013)**

The Manneville-Pommeau map $x \mapsto x + x^{1+\alpha} \mod 1$ is stochastically stable.
Recall that a map $f$ satisfies the Collet-Eckmann condition if each critical value has positive lower Lyapunov exponent.

**Theorem (Gao-S 2014)**

Let $f_t : [0, 1] \to [0, 1]$ be a $C^3$ family of interval maps with non-degenerate critical points. Put

$$E = \{ t \in [0, 1] : f_t \text{ satisfies } (CE) \}.$$ 

Assume that $f_0$ is summable and a parameter transversality condition. Then

$$\frac{\text{Leb}([0, \varepsilon] \cap E)}{\varepsilon} \to 1 \text{ as } \varepsilon \to 0.$$

Tsujii (1993) obtains the same conclusion assuming $f_0$ satisfies a slow recurrence condition in addition to the (CE) condition.

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Now restrict to $\mathcal{P}_d$, the space of real polynomials of degree $d$ which map $[0,1]$ into itself.

Levin (2008): the **transversality condition automatically holds** for a generic one-parameter family passing through a summable map.

Thus we have

**Corollary**

*Almost every summable map in $\mathcal{P}_d$ satisfies the Collet-Eckmann condition.*
A version of the Palis conjecture states that Lebesgue almost all non-renormalizable maps satisfy the Collet-Eckmann condition. For the moment, we cannot prove this in the multimodal case. Current aim: prove that Lebesgue almost all non-renormalizable maps which satisfy the large derivative condition satisfy the Collet-Eckmann condition.

**Conjecture.** Let $f_0$ be a map in the interior of $\mathcal{P}_d$. Then $f_0$ is a Lebesgue density point of Collet-Eckmann maps in $\mathcal{P}_d$ if and only if $f_0$ satisfies the large derivatives condition.

We expect that some combination of real and complex methods will allow us to prove this and the full one-dimensional Palis conjecture subsequently.