Recent developments in interval dynamics

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Aim:

Interval dynamics: take a smooth map $f\colon \mathbb{R} \to \mathbb{R}$ and initial point $x_0 \in \mathbb{R}$. Then study orbits $x_{n+1} = f(x_n)$.

- These orbits can converge to periodic orbit, but also can
- converge in a complicated (chaotic) way to a large set.

In spite of large **non-linearity** of high iterates, many (and in a topological sense even most) such systems are well-understood.

The **metric structure of orbits can be understood**:

- from an **ergodic** point of view (*Second* 1/2 of talk);
- **smoothness** on certain invariant **Cantor sets**: using *renormalisation* (e.g. Avila & Lyubich's recent result);
- **global quasi-symmetric rigidity**: *Sullivan*'s programme.

Aim 1/2 **talk:** what is **qs-rigidity**, its **origins** and **applications**?

What is quasi-symmetric rigidity?

A homeomorphism $h\colon [0,1] \to [0,1]$ is called **quasi-symmetric** (often abbreviated as *qs*) if there exists $K < \infty$ so that

$$\frac{1}{K} \leq \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leq K$$

for all $x-t, x, x+t \in [0,1]$. $\quad (\implies$ Hölder).

Sullivan's programme: prove that f is **quasi-symmetrically rigid**, i.e.

$$f, \tilde{f} \text{ is topologically conjugate } \implies$$

$$\tilde{f}, f \text{ are quasi-symmetrically conjugate.}$$

That is, homeomorphism h with $h \circ f = g \circ h$ is **'necessarily' qs**.

Quasi-symmetric rigidity

Clark-vS:

Theorem (Quasi-symmetric rigidity)

Let $N = S^1$ or $N = [0, 1]$. Assume $f, g \colon N \to N$ are real analytic and **topologically conjugate** *with at least one critical point and the* **conjugacy** *is a* **bijection** *between*

1. *the set of* **critical points** *and the order of corresponding critical points are the same;*
2. *the set of* **parabolic periodic points**.

Then f and g are **quasi-symmetrically** *conjugate.*

Remarks:

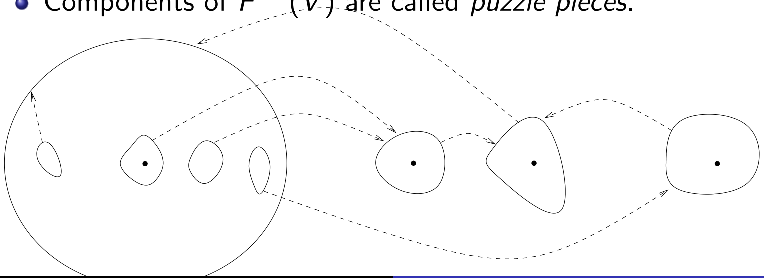
- This result is sharp: if one of the assumptions is not satisfied, then the result is false in general.
- Clark-vS also prove qs-rigidity when f and g are merely C^3 maps, under some very weak additional assumptions.

History of qs-rigidity results

- This completes a programme initiated in the 80's by
  - **Sullivan** for interval maps: in his work on renormalisation;
  - **Herman** for circle homeo's: to use quasiconformal surgery.
- **Partial results** by Herman-Świątek (1988), Sullivan (1988), Lyubich (1998), Graczyk-Świątek (1998), Levin-vS (2000), Shen (2004), Kozlovski-Shen-vS (2007).
- The **presence of critical points** is **necessary** for result to hold: for circle diffeomorphisms the analogous statement is false (due to longer and longer saddle-cascades).
- If there are periodic points, conjugacies **cannot be** C^1, since then corresponding periodic orbits have the same multiplier.
- For **complex** (non-real) polynomials there are partial results (**qc-rigidity**), due to Kozlovski-vS, Lyubich-Kahn, Levin, Cheraghi, Cheraghi-Shishikura. However, in general **wide open** (**related to local connectivity of Mandelbrot set**).

Go to complex plane: complex box mappings

- qs-rigidity, requires control of high iterates ('**compactness**').
- Turns out to be useful to **construct** an extension to $\mathbb{C}$: when f, g are real analytic, use holomorphic extension of f, g to small neighbourhoods of $[0, 1]$ in $\mathbb{C}$.
- Prove that first return maps of f, g to small intervals, extend to a **'complex box mapping'** $F\colon U \to V$, see figure.
- Each component of U is mapped as a branched covering onto a component of V, and components of U are either compactly contained or equal to a component of V.
- Components of $F^{-n}(V)$ are called *puzzle pieces*.



Control of high iterates: complex bounds

In addition we need **complex bounds,** i.e. **universal control** on **shape** and **position** of components of U inside components of V.

Clark-Trejo-vS:

Theorem (Complex box mappings with complex bounds)

One can construct complex box mappings with complex bounds on arbitrarily small scales.

- Previous similar partial results by Sullivan, Levin-vS, Lyubich-Yampolsky and Graczyk-Świątek, Smania, Shen.
- **Key ingredient** in **renormalisation**, e.g. Avila-Lyubich.
- Complex bounds give better control than real bounds.
- Clark-Trejo-vS: something similar even for C^3 maps, but then F is only asymptotically holomorphic.

Proving complex bounds

- From the **enhanced nest** construction (see next •) and a **remarkable result due to Kahn-Lyubich**, given a non-renormalizable complex box mapping at one level, one can obtain **complex box mappings with complex bounds at arbitrary deep levels**.
- The **enhanced nest** is a sophisticated choice of a sequence of puzzle pieces $U_{n(i)}$, so that
  1. $\exists k(i)$ for which $F^{k(i)}\colon U_{n(i+1)} \to U_{n(i)}$ is a branched covering map with degree bounded by some universal number N.
  2. its inverse transfers geometric information efficiently from scale $U_{n(i)}$ to the smallest possible scale $U_{n(i+1)}$.
- Other choices will not give complex bounds, in general.
- In the **renormalizable case**, the construction of complex box mappings and the proof of complex bounds is significantly *more involved*.

Idea for proving quasi-symmetric rigidity

- Using the **complex bounds** and a *methodology for constructing quasi-conformal homeomorphisms* (building on papers of Kozlovski-Shen-vS and Levin-vS), we construct quasi-conformal conjugacies on **small scale**.
- Then develop a technology to glue the local information together. Requires additional care when there are several critical points.

Remarks:

- In the C^3 case f, g have *asymptotically holomorphic extensions* near $[0, 1]$. Issue to deal with: arbitrary high iterates of f and g are not necessarily close to holomorphic.

So far I discussed what is **qs-rigidity**, and **why it holds**. Next:

Why is **qs-rigidity useful**?

Roughly, because it provides a comprehensive understanding of the dynamics, which opens up a pretty full understanding.

I will discuss **two applications**. Both are based on **tools from complex analysis** that become available because of **quasi-symmetric rigidity**.

A third application will *hopefully* be a resolution of the *1-dimensional Palis conjecture* in full generality.

Application 1: Hyperbolic maps

A smooth map $f : \mathbb{R} \to \mathbb{R}$ is **hyperbolic** if

- Lebesgue a.e. point is attracted to some periodic orbit with multiplier λ so that $|\lambda| < 1$, or *equivalently*
- each critical point of f is attracted to a periodic orbit *and* each periodic orbit is hyperbolic (i.e. with multiplier $\lambda \neq \pm 1$).

Martens-de Melo-vS: the **period** of periodic **attractors** is **bounded** $\implies$ hyperbolic maps have at most finitely many periodic attractors.

The notion of hyperbolicity was introduced by Smale and others because these maps are well-understood and:

- **Every hyperbolic map** satisfying an additional transversality condition, that no critical point is eventually mapped onto another critical point, is **structurally stable**. (A nearby map is *topologically conjugate*, i.e. same up to topological coordinate change.)

Hyperbolic one-dimensional maps are dense

- Fatou (20's) conjectured most rational maps on the Riemann sphere are hyperbolic.
- Smale (60's) conjectured that in higher dimensions, hyperbolic maps are dense. This turned out to be **false**.

Kozlovski-Shen-vS:

Theorem (Density of hyperbolicity for real polynomials)

Any real polynomial can be approximated by a hyperbolic real polynomials of the same degree.

and

Theorem (Density of hyperbolicity for smooth one-dimensional maps)

Hyperbolic 1-d maps are C^k dense, $k = 1, 2, \ldots, \infty$.

This solves one of Smale's problems for the 21st century.

Density of hyperbolicity for real transcendental maps

Rempe-vS:

Theorem (Density of hyperbolicity for transcendental maps)

Density of hyperbolicity holds within the following spaces:

1. *real transcendental entire functions, bounded on the real line, whose singular set is finite and real;*
2. *transcendental functions* $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ *that preserve the circle and whose singular set (apart from* $0, \infty$*) is contained in the circle.*

Remarks:

- Hence, density of hyperbolicity within the famous **Arnol'd family** and within space of trigonometric polynomials.
- Result implies conjectures posed by de Melo-Salomão-Vargas.

Hyperbolicity is dense within generic families

Theorem (vS: Hyperbolicity is dense within generic families)

For any (Baire) C^∞ generic family $\{g_t\}_{t\in[0,1]}$ of smooth maps:

- *the number of critical points of each of the maps g_t is bounded;*
- *the set of t's for which g_t is hyperbolic, is open and dense.*

and

Theorem (vS: $\exists$ family of cubic maps with robust chaos)

There exists a real analytic one-parameter family $\{f_t\}$ of interval maps (consisting of cubic polynomials) so that

- *there exists no $t \in [0,1]$ with f_t is hyperbolic;*
- *f_0 and f_1 are not topologically conjugate.*

Question: What if f_0 and f_1 are '**totally different**'?

Density of hyperbolicity on $\mathbb{C}$?

Density of hyperbolicity for rational maps (Fatou's conjecture) is wide open. By Mañé-Sad-Sullivan it follows from:

Conjecture

If a rational map carries a measurable invariant line field on its Julia set, then it is a Lattès map.

Eremenko-vS:

Theorem

Any rational map on the Riemann sphere such that the multiplier of each periodic orbit is real, either is

- *an interval or circle map (Julia set is 1d), or*
- *a Lattès map.*

In the first case, the Julia set of course does not carry measurable invariant line field.

Strategy of the proof: local versus global perturbations

One approach: take g to be a **local perturbation** of f, i.e. find a 'bump' function h which is small in the C^k sense so that $g = f + h$ becomes hyperbolic.

- Difficulty with this approach: orbits pass many times through the support of the bump function.
- Jakobson (1971, in dimension one) and Pugh (1967, in higher dimensions but for diffeo's) used this approach to prove a C^1 **closing lemma**.
- In the C^2 **category** this approach has proved to be **unsuccessful** (but Blokh-Misiurewicz have partial results). Shen (2004) showed C^2 density using qs-rigidity results.

Proving density of hyperbolicity for $z^2 + c$

Density of hyperbolicity with family $z^2 + c$, $c \in \mathbb{R}$ holds if there exists no interval of parameters c of non-hyperbolic maps.

Sullivan showed that this follows *from* **quasi-symmetric rigidity** *of any* **non-hyperbolic map** f_c (by an open-closed argument):

- Measurable Riemann Mapping Theorem $\implies$

  $$I(f_c) = \{\tilde{c} \in \mathbb{R} \text{ s.t. } f_{\tilde{c}} \text{ topologically conjugate to } f_c\}$$

  is either *open or a single point*.
- Basic kneading theory $\implies$ $I(f_c)$ is *closed set*.

$\emptyset \subsetneq I(f_c) \subsetneq \mathbb{R}$ gives a contradiction unless $I(f_c)$ is a single point.

Using a slightly more sophisticated argument, Kozlovski-Shen-vS also obtain that quasi-symmetric rigidity implies density of hyperbolicity when there are more critical points.

Application 2: monotonicity of entropy

In the early 90's, Milnor posed the

Monotonicity Conjecture. The set of parameters within a family of real polynomial interval maps, for which the topological entropy is constant, is connected.

- A version of this conjecture was proved in the 1980's for the quadratic case.
- Milnor-Tresser (2000) proved conjecture for cubics using
  - planar topology (in the cubic case the parameter space is two-dimensional) and
  - density of hyperbolicity for real quadratic maps.
- Bruin-vS: the set of parameters corresponding to polynomials of degree $d \geq 5$ with constant entropy is in general NOT locally connected.

Monotonicity of entropy: the multimodal case

Given $d \geq 1$ and $\epsilon \in \{-1, 1\}$, let P_ϵ^d space of

1. real polynomials $f\colon [0,1] \to [0,1]$ of degree $= d$;
2. all critical points in $(0,1)$;
3. $\text{sign}(f'(0)) = \epsilon$.

Bruin-vS show:

Theorem (Monotonicity of Entropy)

For each integer $d \geq 1$, each $\epsilon \in \{-1, 1\}$ and each $c \geq 0$,

$$\{f \in P_\epsilon^d; h_{top}(f) = c\}$$

is connected.

- Main ingredient: is quasi-symmetric rigidity.
- Hope to remove assumption (2): (currently $d = 4$ with Cheraghi).
- Rempe-vS $\implies$ top. entropy of $x \mapsto a\sin(x)$ monontone in a.

Let's now discuss ergodic properties in one-dimensional dynamics.

Typical dynamics in the logistic family

Let

$$Q_a(x) = ax(1-x),\ 0 < a \leq 4.$$

Let

$$\mathcal{H} = \{a \in (0,4] : Q_a \text{ is hyperbolic}\}.$$

As we have seen above, $\mathcal{H}$ is an open and dense subset of the parameter space $(0,4]$.
On the other hand, Jakobson has shown that the set

$$\mathcal{A} = \{a \in (0,4] : Q_a \text{ has an ergodic acip}\}$$

has positive Lebesgue measure.
Here, an *acip* is an invariant Borel probability measure that is absolutely continuous with respect to the Lebesgue measure.

Abundance of non-uniform hyperbolicity

Theorem (Jakobson 1981)

$a = 4$ is a one-sided Lebesgue density point of $\mathcal{A}$:

$$\lim_{\varepsilon \to 0} \frac{|\mathcal{A} \cap [4 - \varepsilon, 4]|}{\varepsilon} = 1.$$

Theorem (Lyubich 2002)

For almost every $a \in (0, 4]$, either Q_a is hyperbolic or it has an acip. In other words,

$$\mathcal{H} \cup \mathcal{A} = (0, 4] \mod 0.$$

Palis' conjecture (1D case)

For generic smooth family f_λ of interval maps, the following hold: For a.e. parameter λ,

- *f_λ has finitely many 'physical' measures (acip's or supported on periodic orbits) whose basins cover the whole phase interval (mod 0).*
- *the system is 'stochastically stable'.*

Theorem (Avila-de Melo-Lyubich 2003, Avila-Moreira 2005+)

The Palis conjecture holds for unimodal maps with non-degenerate critical point.

Indeed, they strengthened Lyubich's theorem, showing that for almost every $a \in \mathcal{A}$, Q_a satisfies the Collet-Eckmann condition, and hence is stochastically stable (by a result of Baladi-Viana).

Unimodal maps with arbitrary criticality: Bruin-S-van Strien 2006, Avila-Kahn-S-Lyubich 2009, Avila-S-Lyubich 2011, Avila-Lyubich 2012.

The multimodal case remains a major challenge in real one-dimensional dynamics.

In the remainder of this talk we will discuss partial results towards the Palis conjecture for multimodal maps:

- the existence of absolutely continuous invariant measures;
- expansion properties;
- stochastic stability;
- a generalisation of Jakobson's theorem.

Here we will push as far as possible what can be obtained using only real methods. Hoping real and complex will meet.

Unimodal versus multimodal

- $\exists$ many proofs of Jakobson's theorem and extensions: Benedicks-Carleson, Tsujii, $\cdots$. Most proofs use a purely real analytic method, showing certain expanding properties of an interval map persist under small perturbations, **starting from an almost hyperbolic map**.
- To prove the Palis conjecture we need to consider **perturbations starting from arbitrary maps**.
- Lyubich has a proof using a complex analytic method, which makes it possible to **start near arbitrary unimodal maps**. Using holomorphic motion, he constructs a parameter-phase space relationship which is almost quasiconformal.
- For **multimodal** maps, it is not clear how to obtain regularity of the parameter-phase relationship starting from a general non-hyperbolic map.

Previous conditions for the existence of an acip

What is needed for the existence of an acip?

The problem on existence of acip has a long history:

- Bowen (1970s): postcritically finite maps
- Misiurewicz (1981): Misiurewicz (non-recurrent) maps
- Collet-Eckmann (1983): S-unimodal map with the Collet-Eckmann condition
- Nowicki-van Strien (1991): S-unimodal maps with a summability condition

Existence of acip without growth conditions

We say that a smooth map $f : [0, 1] \to [0, 1]$ satisfies the *large derivatives condition* (LD) if for each critical point c,

$$|Df^n(f(c))| \to \infty \text{ as } n \to \infty.$$

Theorem (Bruin, Rivera-Letelier, S, van Strien 2008)

Let f be a smooth interval map with non-flat critical points and all periodic points hyperbolic repelling. If f satisfies the (LD) condition, then f has an acip.

The unimodal case: Bruin-S-van Strien (2003).

How to prove existence of acip

- Construct an induced Markov map ('tower') F.
- Show that the induced map F has an acip.
- This requires distortion estimates on F.
- To get invariant measure for the original map f the inducing times need to be summable.
- Here we need to quantify expanding properties of f.

Constructing acip's: quantifying expansion

Lemma (Expansion)

Assume f satisfies (LD) and (for simplicity) all critical points are non-degenerate. Let $V_\varepsilon = \bigcup_{c \in \mathrm{Crit}} (c - \varepsilon, c + \varepsilon)$. For any $x \in V_\varepsilon$, if n is the minimal positive integer such that $f^n(x) \in V_\varepsilon$, then

$$|Df^n(x)| \geq \Lambda(\varepsilon) \frac{|Df(x)|}{\varepsilon} \exp\{\rho(\varepsilon) n\},$$

where $\Lambda(\varepsilon) \to \infty$ and $\log \rho(\varepsilon) / \log \varepsilon \to 0$ as $\varepsilon \to 0$.

The inequality is equivalent to the so-called *backward contraction* property defined by Rivera-Letelier (2006) $\implies$ critical pullbacks of V_ε have length much shorter than ε $\implies$ fast decay of branches with long return times (see next slide).

$$\text{Li-S} : \text{BC} \Leftrightarrow \text{LD}.$$

Constructing acip's: tail estimates

An induced Markov map is a map $F : \bigcup_{j=1}^{\infty} J_j \to I$, where

- I is an open interval, and J_j's are open subintervals;
- $|I \setminus (\bigcup_j J_j)| = 0$;
- $\forall j$, $F|J_j$ is a diffeomorphism onto I;
- $\exists C > 0$ such that for each component U of $F^{-n}(I)$,

$$\sup_{x,y \in U} \log \frac{|DF^n(x)|}{|DF^n(y)|} \leq C|F^n(x) - F^n(y)|;$$

- $\forall j$, $\exists s_j$ such that $F|J_j = f^{s_j}|J_j$.

Theorem (Rivera-Letelier+S, in press)

Suppose f satisfies the (LD) condition and has all periodic points hyperbolic repelling. Then f has an induced Markov map $F : \bigcup_{j=1}^{\infty} J_j \to I$ with super-polynomially small tail, i.e. $\sum_{s_j=s} |J_j|$ is super-polynomially small in s.

Pseudo orbits and robust expansion

A smooth interval map is called *summable (with exponent* 1*)* if for each critical point c, $\sum_{n=1}^{\infty} |Df^n(f(c))|^{-1} < \infty$.

An ε*-pseudo-orbit* is a sequence $\{x_j\}_{j=0}^{n}$ such that for each $0 \le j < n$, $|f(x_j) - x_{j+1}| \le \varepsilon$.

Lemma (Robust expansion, S 2013)

Assume that f is summable and all critical points are non-degenerate. Let $V_\varepsilon = \bigcup_{c \in \mathrm{Crit}} (c - \varepsilon, c + \varepsilon)$. For any ε-pseudo-orbit $\{x_j\}_{j=0}^{\infty}$, if $x_0 \in V_\varepsilon$ and n is the minimal positive integer such that $x_n \in V_\varepsilon$, then

$$\prod_{j=0}^{n-1} |Df(x_j)| \ge \Lambda(\varepsilon) \frac{|Df(x_0)|}{\varepsilon} \exp\{\rho(\varepsilon) n\},$$

where $\Lambda(\varepsilon) \to \infty$ and $\log \rho(\varepsilon) / \log \varepsilon \to 0$ as $\varepsilon \to 0$.

Stochastic stability

Now consider ε-pseudo orbits which are chosen randomly. What happens as ε goes to zero?

Theorem (S 2013)

Let $f : [0,1] \to [0,1]$ be a C^3 interval map with non-flat critical points and all periodic points repelling. If f is summable and f is ergodic with respect to the Lebesgue measure, then f is stochastically stable with respect to a large class of random perturbations.

Builds on earlier work of Baladi+Benedick+Maume-Deschamps, (2001) and Alves-Vilarinho (2013).

Theorem (S-van Strien 2013)

The Manneville-Pommeau map $x \mapsto x + x^{1+\alpha} \mod 1$ is stochastically stable.

Summability implies Collet-Eckmann a.s.

Recall that a map f satisfies the Collet-Eckmann condition if each critical value has positive lower Lyapunov exponent.

Theorem (Gao-S 2014)

Let $f_t : [0,1] \to [0,1]$ be a C^3 family of interval maps with non-degenerate critical points. Put

$$E = \{t \in [0,1] : f_t \textit{ satisfies } (CE)\}.$$

Assume that f_0 is summable and a parameter transversality condition. Then

$$\frac{Leb([0,\varepsilon] \cap E)}{\varepsilon} \to 1 \textit{ as } \varepsilon \to 0.$$

Tsujii (1993) obtains the same conclusion *assuming f_0 satisfies a slow recurrence condition in addition to the (CE) condition.*

Now restrict to $\mathcal{P}_d$, the space of real polynomials of degree d which map $[0, 1]$ into itself.

Levin (2008): the **transversality condition automatically holds** for a generic one-parameter family passing through a summable map.

Thus we have

Corollary

Almost every summable map in $\mathcal{P}_d$ satisfies the Collet-Eckmann condition.

A possible approach to the multimodal Palis conjecture

A version of the Palis conjecture states that Lebesgue almost all non-renormalizable maps satisfy the **Collet-Eckmann condition.**

For the moment, we cannot prove this in the multimodal case.

Current aim: prove that Lebesgue almost all non-renormalizable maps which satisfy the **large derivative condition** satisfy the **Collet-Eckmann condition**.

Conjecture. *Let f_0 be a map in the interior of $\mathcal{P}_d$. Then f_0 is a Lebesgue density point of Collet-Eckmann maps in $\mathcal{P}_d$ if and only if f_0 satisfies the large derivatives condition.*

We expect that some combination of real and complex methods will allow us to prove this and the full one-dimensional Palis conjecture subsequently.