Real one-dimensional dynamics: real and complex methods

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October 1, 2013
Throughout these talks we assume that $N$ is an *interval or a circle* and that $f : N \to N$ is *real analytic*. For example:

$$f(x) = ax(1 - x) \text{ or } f(x) = x^2 + c$$
One of the reasons real one-dimensional dynamics has been such an exciting field is because

- the theory is *far from trivial, yet almost complete*;
- the theory can be considered as a *model for what can happen in higher dimensions*.

- My first talk will be about theorems that can be obtained by real tools.
- The later talks will then discuss why one introduces complex tools.
Notation: $\omega(x)$ is the set of accumulation points of the sequence $x, f(x), f^2(x), \ldots$.

It would be great to describe all orbits of $f$, but it turns out to be much more fruitful to describe attractors and ergodic properties.

We say that a compact forward invariant set is a topological resp. metric attractor if

$$B(X) = \{ x ; \omega(x) \subset X \}$$

is of second Baire category (i.e. countable intersection of open and dense) resp. has positive Lebesgue measure, and if for any $X' \subsetneq X$, $B(X')$ does not have this property.
The nicest maps are those where each attractor is a hyperbolic periodic orbits. These maps are called the \textit{hyperbolic}.

We will sketch a proof that - in some sense - most maps are hyperbolic. This problem goes back to Fatou (1930’s).

\textit{These latter results rely on constructing an extension of $f$ to the complex plane.}

This interplay of real and complex methods in interval dynamics will be one of the main topics of these lectures.
Real Bounds and Ergodic Properties.
Description of attractors

**Theorem**

*Each map has at least one and at most a finite number of attractors. If $X$ is an attractor then one of the following:*

1. $X$ is a periodic attractor;
2. $X = \omega(c)$ where $c$ is a critical point of $f$ so that $\omega(c)$ is a Cantor set which is minimal and has zero Lebesgue measure;
3. $X$ is equal to a finite union of intervals which contains a critical point (or equal to the entire space $\mathbb{N}$)

**Definition:** An invariant set is called *minimal* if each forward orbit is dense in $\omega(c)$.

**Corollary (Denjoy):** An attractor of a circle diffeomorphism is either the whole circle or a periodic orbit. (Iterates are either dense or converge to a periodic orbit.)
Examples of maps $f : [0, 1] \to [0, 1]$:

- $f(x) = 2x(1 - x)$. Then $x = 1/2$ is an attracting fixed point: $f(1/2) = 1/2$ and $f'(1/2) = 0$.

- $f(x) = 3x(1 - x)$. Then $x = 2/3$ is a weakly attracting fixed point: $f(2/3) = 2/3$ and $f'(2/3) = -1$.

- $f(x) = ax(1 - x)$ is an infinitely renormalizable map where the parameter $a \approx 3.569..$ is at the accumulation of period doubling: there exists a sequence of intervals $J_n$ and integers $p(n)$ so that

  \[ J_n, \ldots, f^{p(n)-1}(J_n) \text{ have disjoint interiors} \]

  and

  \[ f^{p(n)}(J_n) \subset J_n. \]

The resulting attractor is a Cantor set. It turns out this NOT the only example of a Cantor attractor.
Examples

- $f(x) = 4x(1 - x)$. In this case the attractor is $[0, 1]$ and the map is conjugate to a tent map with slope $\pm 2$. There are infinitely many periodic orbits (or all periods), but a.e. point $x \in [0, 1]$ has a dense orbit in $[0, 1]$. 
The proof of the previous theorem in the non-invertible case has a long history: [Guc79, dMvS89, BL89, Lyu89, MdMvS92, vSV04]

Note that the objects in the classification in the theorem are the same, regardless whether the attractor is a topological or a metric one.

Milnor posed the question whether a metric attractor is also a topological attractor (and vice versa). The answer turns out to be NO, as we will see.
Nice Intervals and first return maps

For *simple* maps such as \( f(x) = 4x(1 - x) \) one describe the map through a *Bernoulli or Markov setting*:

\[
J_1 \rightarrow J_1 \cup J_2, \quad J_2 \rightarrow J_1 \cup J_2
\]

where \( J_1 = [0, 1/2] \) and \( J_2 = [1/2, 1] \). However, for most maps this is not possible.

Instead: use first return maps to so-called nice intervals.

- Let \( I \) be an interval, and assume that there exists a *(minimal)* \( n > 0 \) so that \( f^n(x) \in I \). Then we denote the component of \( f^{-n}(I) \) containing \( x \) by \( \mathcal{L}_x(I) \).

- An interval is called *nice* if no iterate of \( x \in \partial I \) ever gets mapped into the interior of \( I \).

- If \( I \) is nice then \( \mathcal{L}_x(I) \subset I \) whenever \( x \in I \).

- This makes it useful to work with first return domains.
• If $I$ is nice, then a pullback $\mathcal{L}_x(I)$ is also nice.

• Two pullbacks of $I$ are either disjoint, or one is contained in the other.

• Nice intervals are easy to find.

• Indeed, let's say $f$ is unimodal. Then take a periodic orbit, choose $p$ in the orbit ‘closest to the critical point’. Then $I = [p, p']$ is a nice interval where $p'$ so that $f(p') = p$.

• The first return map $R_I : Dom(I) \to I$ to $I$ has (usually infinitely many) diffeomorphic branches and a folding branch.

One of the main challenges is to control the distortion. If all the branches were linear, then one knows essentially everything.
Notation and terminology

- If $T = [a - x, a + x]$ is an interval in $\mathbb{R}$ and $\tau > 0$ then
  $$\tau \cdot T := [a - \tau x, a + \tau x].$$

- Reminder: $\mathcal{L}_x(I)$ is the component of $f^{-n}(I)$ containing $x$ where $n$ is minimal so that $f^n(x) \in I$.

- $J_1, \ldots, J_k$ have *intersection multiplicity* $m$ if any point $x$ is contained in at most $m$ of the intervals $J_1, \ldots, J_k$.

- If $f^n(x) \in T$ where $n$ is minimal, then the *pullback* of $T$ is $\mathcal{L}_x(T)$ and the pullback *chain* is the collection of intervals
  $$\mathcal{L}_x(T), \mathcal{L}_{f(x)}(T), \ldots, \mathcal{L}_{f^{n-1}(x)}(T), \mathcal{L}_{f^n(x)}(T) = T.$$ 

- Let $f$ have $b$ critical points $c_i$ with critical order $l_i$. Then we say that $f$ has type $b = (l_1, \ldots, l_b)$. 

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Real one-dimensional dynamics
The theorem require estimates on high iterates of a map. Since \( f \) is non-linear, as there are critical points, this is not so easy.

- **Schwarzian derivative**: Define

\[
Sf(x) = \frac{f'''(x)f'(x) - (3/2)f''(x)}{[f'(x)]^2}.
\]

Then \( S(f \circ g) = Sf [g'(x)]^2 + Sg \). Hence

\[
Sf < 0 \implies Sf^n < 0 \text{ for all } n \in \mathbb{N}.
\]

- **Koebe**: Then for \( \delta > 0 \) there exists \( K \) so that the following holds. If \( g : T \to T' := g(T) \) is a diffeomorphism and \( Sg < 0 \) then for each \( x, y \in T \) so that \( g(x) \in (1 - \delta)g(T) \) one has \( |Dg(x)|/|Dg(y)| \leq K \). *See blackboard*
Negative Schwarzian appear naturally:

- Let \( f \) be a polynomial of degree \( \geq 2 \) with real coefficients and assume that all zeros of \( Df \) are real. Then \( S_f < 0 \).

  (Hint: By assumption \( Df(x) = A \prod_{j=1}^{n} (x - a_j) \) where \( a_j \) are real. Then

  \[
  S_f(x) = 2 \sum_{i<j} \frac{1}{(x - a_i)(x - a_j)} - \frac{3}{2} \left[ \sum_{i} \frac{1}{x - a_i} \right]^2.
  \]

  It is not hard to see that this is negative for \( x \) real. (There is a more insightful way of showing this, which we will discuss briefly below.)
Many papers assumes that a map has negative Schwarzian. This simplifies because

- each periodic attractor has a critical point in its immediate basin
- one has Koebe control on diffeomorphic branches.

It turns out that the assumption $Sf < 0$ (with extra work) can always be replaced by assuming:

*all periodic points of $f$ are hyperbolic and repelling.*
Schwarzian derivative is closely related to cross-ratio (there is a formula...)

When $J \subset T \subset \mathbb{R}$ are intervals, then

$$C(T, J) := \frac{|T||J|}{|L||R|}$$

where $L, R$ are the components of $T - J$.

If $f : T \to f(T)$ is a continuous bijection then one can consider the expansion of the cross-ratio:

$$\frac{C(fT, fJ)}{C(T, J)}.$$
Expansion of cross-ratio corresponds to $Sf < 0$.

One can put the Poincaré metric on $\mathbb{C}_T = (\mathbb{C} - \mathbb{R}) \cup T$. Then $C(T, J)$ is equal to the Poincaré metric of $J$.

Another way of showing $Sf < 0$ for certain polynomials:

- Take $f : \mathbb{C} \to \mathbb{C}$ a real polynomial with only real critical points and so that $f|T$ is a diffeomorphism.
- Then define $f^{-1} : \mathbb{C}_{f(T)} \to \mathbb{C}_T$ by analytic continuation.
- Example $f(z) = z^2$, $T = [1, 2]$, $f(T) = [1, 4]$, see blackboard.
- The map $f^{-1} : \mathbb{C}_{f(T)} \to f^{-1}(\mathbb{C}_{f(T)})$ is a conformal bijection and therefore an isomorphism w.r.t. the Poincaré metric on these sets.
- $f^{-1}(\mathbb{C}_{f(T)}) \subset \mathbb{C}_T \implies f^{-1} : \mathbb{C}_{f(T)} \to \mathbb{C}_T$ contracts the Poincaré metric on these sets. This proves $Sf < 0$. 

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Many maps do not have negative Schwarzian. It turns out that one can use distortion of cross-ratios instead, provided one has some disjointness:

**Theorem (Koebe in the case of disjoint intervals)**

Assume that $J \subset T$ and $f^n|_T$ is a diffeomorphism, and the intersection multiplicity of $T, \ldots, f^{n-1}(T)$ is at most $m$.

Then Koebe holds (with bounds depending on $m$).
One of the most basic tools in real one-dimensional dynamics are real bounds.

**Theorem**

[vSV04] Assume that $I$ is a nice interval, $x \in I$ and assume that $R_I(x) \notin \mathcal{L}_x(I)$. Then $(1 + \delta)\mathcal{L}_x^2(I) \subset \mathcal{L}_x(I)$.

Let’s explain some of the ideas behind the proof and why this is helpful.
Let \( J, \ldots, f^n(J) \) be disjoint intervals. Then one of them, \( f^k(J) \), is the smallest. Assume the smallest is \textbf{not} the left or right most interval. Then the smallest \( f^k(J) \) has two larger intervals \( f^l(J) \) and \( f^r(J) \) to its right and its left. Now take

\[
T' = [f^l(J), f^r(J)] \supset f^k(J)
\]

and pullback \( T' \supset f^n(J) \) to \( T \supset J \).

- the resulting chain has multiplicity \( \leq 3 \);
- \( (1 + \delta) T \supset J \).
The spiral structure argument

Let’s take $x \in I$ and assume that $x$ visits $I$ at least $n$ times. Let $J_0, J_1, J_2, \ldots$ be the return domains $x$ visits consecutively.

Fix $\rho > 0$. For each $n$ there exists $i_0 \leq n$ so that

**P1** $(1 + \rho)J_{i_0} \subset I$;

**P2** $J_{i_0}$ has (at least) one $\rho$-small side and $J_{i_0+1}$ is contained in a $\rho$-small side of $J_{i_0}$;

**P3** For all $i < i_0$ properties [P1] and [P2] do not hold (which means that one has a spiral structure up to time $i_0$) and the interval $J_{i_0+1}$ breaks the spiral structure;

**P4** Properties [P1] and [P2] do not hold and the spiral structure holds until time $n$.

In each situation one obtains space, see blackboard.

Combining these ideas one obtains the proof of the real bounds.
Corollary (Absence of wandering intervals)

If $W, f(W), \ldots$ are disjoint, then $f^n(W)$ converges to an attracting periodic orbit.

Sketch of proof of Corollary (proof by contradiction):

- By ‘surgery’ one can always assume that $f$ has periodic orbits (and therefore has nice intervals), see blackboard.
- Let $x$ be an accumulation point of the orbit of $W$. 
Sketch of the proof of absence of wandering intervals

Using (a small part of) the proof of real bounds, one can show that there exists a *nested* sequence of nice intervals $I_n \supset I'_n \ni x$ so that

- $(1 + \rho)I'_n \subset I_n$
- *the first visit of $W$ to $I_n$ is contained in $I'_n$*
- after some further time $W$ enters $I_{n+1}$.
- *Any pullback of $I_{n+1}$ which intersects $I'_n$ is contained in $I'_n$.*

This gives

$$(1 + \rho')\mathcal{L}_W(I'_n) \subset \mathcal{L}_W(I_n) \text{ and } \mathcal{L}_W(I_{n+1}) \subset \mathcal{L}_W(I'_n).$$

Combining this gives

$$(1 + \rho')\mathcal{L}_W(I_{n+1}) \subset \mathcal{L}_W(I_n)$$

and therefore

$$(1 + \rho')^n W \subset (1 + \rho')^n \mathcal{L}_W(I_{n+1}) \subset \mathcal{L}_W(I_0).$$

But since the length of $(1 + \rho')^n W$ tends to infinity, this gives a contradiction.
Theorem [Koebe without disjointness]

Assume all periodic orbits of $f$ are hyperbolic and repelling.

Then Koebe holds on diffeomorphic branches.
Examples of results one can obtain using real methods: Invariant measures

**Definition:** $f$ has an *absolutely continuous invariant probability measure* or an *acip*, if there exists a *probability* measure $\mu$, so that
- $\mu(f^{-1}(B)) = \mu(B)$ for each measurable set $B$ and
- $\mu(B)$ is small when $B$ has small Lebesgue measure.

**Theorem**

_The map $f(x) = 4x(1-x)$ has an acip._

- It is *enough to show* that $\forall \epsilon > 0$ there exists $\delta > 0$ so that for each measurable set $A \subset [0,1]$ of Lebesgue measure $< \delta$ one has $f^{-n}(A)$ has Lebesgue measure $< \epsilon$ for any $n \geq 0$.
- If $A$ is an interval which does not contain 0 or 1, then this immediately follows from Koebe.
- One can reduce the general case to this situation.
There is a long history of results on this (going back to the 50’s), with well-known results by Misiurewicz, Benedicks-Carleson, Collet-Eckmann, Nowicki-vS and others. The sharpest result is:

**Theorem ([BSvS03] and [BRLSvS08])**

Assume that $f$ is real analytic and has no periodic attractors. Then there exists constant $C(b)$ such that if

$$\liminf_n |(f^n)'(f(c))| \geq C$$

for each critical point $c$ then $f$ has an acip.

There is a remarkable sequel to this result, by Rivera-Lettelier and Shen: one has superpolynomial decay of mixing in this case.
Consider a set $A$ of small size

Aim: estimate $f^{-n}(A)$.

Distinguish components of $f^{-n}(A)$.

A ‘good’ component $J$ is one for which exists a neighbourhood $T \supset J$ so that

- $f^n|T$ is a diffeomorphism
- $f^n(T) \supset (1 + \xi)f^n(J)$ where $\xi$ is large

The Lebesgue measure of all good components of $f^{-n}(A)$ is obtained in this way.

Other components: decompose these branches and use an inductive estimate.
Theorem (Existence of Wild attractors, [BKNvS96])

There exist maps of the form $f(z) = z^d + c$ with $c \in \mathbb{R}$ and $d$ even (and large) with an invariant Cantor set which is a metric, but not a topological attractor.

Theorem (Non-existence of Wild attractors in the quadratic case, [Lyu94])

Assume that $f$ is unimodal and has a quadratic critical point then the notions of topological and metric attractor coincide.

Note that wild attractors also exist for certain real polynomials of higher degree with only non-degenerate critical points.
One can decompose the space into disjoint intervals $J_n$.
Each interval $J_n$ maps diffeomorphically onto a countable union of such intervals.
One has sufficient control on non-linearity.
Probabilistic proofs to show what happens with points, see blackboard.
A map with several critical points can have several attractors whose basins are **intermingled**:

**Theorem ([vS96])**

There exists a polynomial \( f : [0, 1] \rightarrow [0, 1] \) with two critical points with two disjoint invariant Cantor sets \( \Lambda_i \) so that the basin of each of these sets is dense and has positive Lebesgue measure.
Some questions about higher dimensional systems

Question (Wild attractors for two-dimensional diffeomorphisms)

Let $M = S^2$. Are there diffeomorphisms $f : M \to M$ which have wild Cantor attractors? (That is, metric but not topological attractor.) It is well-known that Hénon maps can have a Cantor set as an attractor, see [GvST89], [DCLM05]. Do these Cantor sets necessarily have to be of solenoidal type?

Question (Wandering domains for Hénon maps)

Let $H$ be a Hénon map. Is it possible for $H$ to have wandering domains, i.e. is it possible that there exists an open set $U$ so that $U, f(U), \ldots$ are all disjoint and so that $U$ is not contained in the basin of a periodic attractor?
Second Lecture: Density of Hyperbolicity and Complex Methods
Density of Hyperbolicity

The *simplest situation* is when the *attractors of f are all hyperbolic periodic orbits*: such f called *hyperbolic* (also called *Axiom A*). It would be nice if *every map can be approximated by a hyperbolic map*. This problem goes back in some form to

- **Fatou**, who stated this as a conjecture in the 1920’s.
- **Smale** gave this problem ‘naively’ as a thesis problem in the 1960’s, see [Sma80].
- **Jakobson** proved that the set of hyperbolic maps is dense in the $C^1$ topology, see [Jak71];
- The quadratic case $x \mapsto ax(1 - x)$ was proved in a major breakthrough in the the mid 90’s by **Lyubich** [Lyu97] and also **Graczyk and Swiatek** [GŚ97].
- **Blokh and Misiurewicz** [BM00] proved a partial result towards the density of hyperbolic maps in the $C^2$ topology.
- **Shen** [She04] then proved the $C^2$ density of hyperbolic maps.
The general result is:

**Theorem (Density of hyperbolicity for real polynomials, [KSvS07a])**
Any real polynomial can be approximated by hyperbolic real polynomials of the same degree.

The above theorem allows us to solve the 2nd part of Smale’s eleventh problem for the 21st century. [Sma00]:

**Theorem (Density of hyperbolicity for smooth one-dimensional maps, [KSvS07b])**
Hyperbolic maps are dense in the space of $C^k$ maps of the compact interval or the circle, $k = 1, 2, \ldots, \infty, \omega$. 
For quadratic maps $f_a = ax(1 - x)$, the above theorems assert that the periodic windows are dense in the bifurcation diagram.

The quadratic case turns out to be special, because in this case certain return maps become almost linear. This special behaviour does not even hold for maps of the form $x \mapsto x^4 + c$. 
It turns out that it is often *not possible to perturb a map to a hyperbolic map by local methods* (in the $C^k$ topology, $k \geq 2$).

Instead one shows that a non-hyperbolic map is essentially uniquely determined by its conjugacy class: *if $f$ and $g$ are conjugate then show they are quasi-symmetrically conjugate.* This approach goes back to Sullivan.

In [KSvS07a] we showed that this rigidity holds for polynomials with certain additional restrictions (e.g. all critical points real).

In fact, it holds in general:
Theorem (SvS)

Assume that \( f, g : [0, 1] \to [0, 1] \) are real analytic, topologically conjugate and that the topologically conjugacy is a bijection between

- the set of critical points and the order of corresponding critical points is the same;
- the set of parabolic periodic points.

Then the conjugacy between \( f \) and \( g \) is quasi-symmetric.

A homeomorphism \( h : [0, 1] \to [0, 1] \) is called quasi-symmetric if there exists \( K < \infty \) so that

\[
\frac{1}{K} \leq \frac{h(x + t) - h(x)}{h(x) - h(x - t)} \leq K
\]

for all \( x - t, x, x + t \in [0, 1] \).

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Note that $f$ and $g$ can only have finitely many parabolic periodic orbits (see [MdMvS92]).

All conditions are necessary

Previous results:

- *Khanin and Teplinsky* show this for critical circle maps (building on earlier work of de *Faria, de Melo and Yampolsky*).
- *Levin + vS* show that for covering maps with one inflection point $c$, one can obtain a qs conjugacy restricted to $\omega(c)$, provided $\omega(c)$ is either minimal or every periodic orbit in $\omega(c)$ is repelling.
- *Kozlovski-Shen-vS* for real polynomials with only real critical points.

In our proof complex methods are essential.

Interestingly, the proof even goes through to the $C^3$ category, (joint work with *Trevor Clark*).
Quasi-symmetric rigidity $\implies$ density of hyperbolicity?

Let’s explain this for the family $z^2 + c$.

- Assume, by contradiction, that there exists a non-trivial interval of parameters $[c_l, c_r]$ so that the corresponding map are all non-hyperbolic.
- Hence, $\forall c \in [c_l, c_r]$ and for all $n \geq 0$ one has $f_c^n(0) \neq 0$.
- So all maps $f_c$ with $c \in [c_l, c_r]$ are topologically conjugate.
- Assume that $[c_l, c_r]$ is a maximal interval with this property (that this interval is closed when $f$ is non-hyperbolic follows from kneading theory).
- By qs-rigidity Thm, $f_c, f_{c'}$ are qs-conjugate $\forall c, c' \in [c_l, c_r]$.

Now assume that $c_l \neq c_r$. Then we will use quasiconformal maps to obtain an open neighbourhood $O \supset [c_l, c_r]$ so that for all $c, c' \in O$ the maps $f_c, f_{c'}$ are also topologically conjugate.

This contradicts maximality of $[c_l, c_r]$. Hence $c_l = c_r$, and density follows.

So let’s go to the complex plane!
An orientation preserving homeomorphism $h$ is called $K$-quasiconformal if

- there exists a constant $K < \infty$ such that for Lebesgue almost all $x \in \mathbb{C}$

$$\limsup_{r \to 0} \frac{\sup_{|y-x|=r} |h(y) - h(x)|}{\inf_{|y-x|=r} |h(y) - h(x)|} \leq K.$$ 

- If $K = 1$ then $h$ is conformal.
- Such maps are, for example, Hölder and Lebesgue almost everywhere differentiable (as maps from $\mathbb{C} = \mathbb{R}^2$ to $\mathbb{C} = \mathbb{R}^2$).
- (In general, a conjugacy cannot be $C^1$, because then multipliers at periodic points would be the same.)
In this case $qs$-rigidity $\implies$ qc-rigidity.

Assume first that $f(z) = z^2 + c_l$ and $\tilde{f}(z) = z^2 + c_r$ are $qs$-conjugate.

- **Fact:** Any $qs$-homeomorphism $h$ on $\mathbb{R}$ can be extended to a $K$-quasiconformal-homeomorphism $H$ on $\mathbb{C}$.
- Hence $\exists$ a qc map $H$ so that $H \circ f = \tilde{f} \circ H$ on $\mathbb{R}$ and near $\infty$.
- Now define a sequence of lifts $H_n$ inductively by $H_0 = H$ and $\tilde{f} \circ H_{n+1} = H_n \circ f$. This can be done, see blackboard.
- $H_n$ is again $K$-qc for any $n$ with the same $K$.
- $H_{n+1} = H_n$ on ever larger sets.
- **Fact:** The space of $K$-quasiconformal maps is compact.
- Hence $H_n$ converges to some $K$-qc homeomorphism $H$.
- Therefore $\tilde{f} \circ H = H \circ f$. 
\( \tilde{f} = H \circ f \circ H^{-1} \) for some \( qc \)-homeo \( H \). So what?

Now we use the **Measurable Riemann Mapping Theorem**:

- \( DH(z) \) exists for a.e. \( z \).
- So \( DH(z) \) sends ellipse based at \( z \) to circle based at \( H(z) \).
- One can associate to this ellipse some number \( \mu(z) \in \mathbb{D} = \{ w; |w| < 1 \} \) where \( |\mu(z)| \) is the eccentricity of the ellipse.
- By this theorem, associated to \( t\mu(z) \) there is another \( qc \) map \( H_t \) with the same long and short axis and eccentricity \( t|\mu(z)| \).
- Normalize so that \( H_t(0) = 0 \) and \( H_t(x)/x \to 1 \) as \( x \to \infty \).
- \( Since \ \tilde{f} = H \circ f \circ H^{-1} \) is holomorphic, the map \( f_t = H_t \circ f \circ H_t^{-1} \) is again conformal, see blackboard.
- \( \tilde{f}_t \) has a unique critical point, is holomorphic and the normalisation implies that \( f_t(z) = z^2 + c(t) \).
What’s useful about $f_t = H_t \circ f \circ H_t^{-1}$?

Reminder: $f_t(z) = H_t \circ f \circ H_t^{-1}(z) = z^2 + c(t)$

- $H_0 = id \implies f_0 = H_0 \circ f \circ H_0^{-1} = f = z^2 + c_l \implies c(0) = c_l$;
- $f_1 = H \circ f \circ H^{-1} = \tilde{f} = z^2 + c_r \implies c(1) = c_r$.

By the Measurable Riemann Mapping Theorem, $t \mapsto f_t(0)$ is holomorphic. Hence $t \mapsto c(t)$ is holomorphic.

- By construction, $t \mapsto c(t)$ is real and has no critical points.
- Hence for $t > 1$, $t \approx 1$ one has $c(t) > c_r$ and the map $f_t$ is still conjugate to $f$.
- $\implies$ open neighbourhood of $[c_l, c_r]$ of conjugate maps.

Together this shows $qs$-rigidity $\implies$ density of hyperbolicity.
qs-rigidity \implies \text{density of hyperbolicity for real polynomials with real critical points}

If the two qs-conjugate polynomials \textbf{only have real critical points} then one can generalise this argument:

- use an inductive dimension reduction:
- restrict to algebraic varieties of the form \( \{ f; f^n(c_1) = c_2 \} \) of lower and lower dimension.
qs-rigidity $\iff$ density of hyperbolicity for real polynomials

- If the two qs-conjugate polynomials $f, \tilde{f}$ have **non-real critical points** then $f, \tilde{f}$ qs-conjugate $\implies f, \tilde{f}$ are qc-conjugate.
- Lifting $\tilde{f} \circ H_{n+1} = H_n \circ f$ not possible: one has no information about the orbits of the complex critical points.
- Want all critical points to be **captured** (in hyperbolic basin). Let's capture more and more critical points:
  - **Step 1:** Take any one-parameter families $f_t, t \in [-1, 1]$ of *regular maps*: each neutral periodic orbit of $f_t$ has a critical point in its basin. Assume this family so that $f_1$ has more captured points than $f_0$ and so that captured critical points for $f_0$ remain captured for $f_t \forall t \in [0, 1]$.
  - **Step 2:** *Thm:* $\exists t \approx 0$ so that $f_t$ has new captured critical points. Here use *holomorphic motion* and *geometric control* for certain complex box mappings. (Not soft...)
How to construct regular families?

- **Step 3:** Approximate \( f \) by a polynomial \( \tilde{f} \) of the same degree without neutral periodic orbits and *same captured critical pts*.
- **Step 4:** All maps \( C^3 \) near \( \tilde{f} \) are *regular*.
- **Step 5:** Locally perturb \( \tilde{f} \) to a \( C^3 \) hyperbolic map \( g \) (here use ‘complex bounds’)!!! Note \( \tilde{f} \) and \( g \) are not \( C^\infty \) close at all.
- **Step 6:** Approximate the smooth map \( g \) by a *polynomial map* \( G \) of much higher degree.
- **Step 7:** Consider the family \( f_t = \tilde{f} + tG \). By Step 4 this a *regular family*.
- **Step 8:** Using Step 2: \( \exists t \approx 0 \) so that \( f_t \) has more captured critical points. However, \( f_t \) has *much higher degree*.
- **Step 9:** \( f_t, \tilde{f} \) and \( f \) are all \( C^0 \) close on a large disc \( \mathbb{D}_R \). Hence, using the so-called *Straightening Theorem*, \( \exists \) a real pol. \( \hat{f} \)
  - of the same degree as \( f \)
  - conjugate to \( f_t \) on \( \mathbb{D}_{R/2} \)
  - still close to \( \tilde{f} \).
Theorem (Hyperbolicity for entire maps (with Lasse Rempe))

Let $f$ be an entire function with a finite number of critical values and either

- $f$ is bounded is on the real axis.
- some sector condition is satisfied.

Then there exist orientation preserving homeomorphisms $\phi, \psi$ arbitrarily close to the identity such that $g := \psi \circ f \circ \phi^{-1}$ is entire and hyperbolic.
Consider generalized trigonometric polynomial $F_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$:

$$F_\mu(t) = D \cdot t + \mu_1 + \mu_{2m} \sin(2\pi m t) + \sum_{j=1}^{m-1} (\mu_{2j} \sin(2\pi j t) + \mu_{2j+1} \cos(2\pi j t)).$$

Note that if $\mu, \mu' \in \mathbb{R}^{2m}$ with $\mu_1 - \mu'_1 \in \mathbb{Z}$, then $f_\mu = f_{\mu'}$. So choose $\mu = (\mu_1, \ldots, \mu_{2m}) \in \Delta$, where

$$\Delta := \{\mu \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2m-1} : \mu_{2m} > 0 \text{ and } f_\mu \text{ is } 2m - \text{multimodal} \}.$$

For example: the *Arnol’d family* $x \mapsto x + \alpha + \beta \sin(2\pi x)$. In this case we have the following theorem:
Hyperbolic parameters in $\Delta$ for which $f_{\mu}$ are dense. Furthermore,

1. Consider the set $[\mu_0]$ of parameters $\mu$ for which $f_{\mu}$ is topologically conjugate to $f_{\mu_0}$ by an order-preserving homeomorphism of the circle. Then $[\mu_0]$ has at most $m$ components.

2. If $f_{\mu_0}$ has no periodic attractors on the circle, then each component of $[\mu_0]$ is equal to a point.

This answers the conjectures posed by de Melo, Salomão and Vargas.
Here

- we need to pay attention to points that go repeatedly to infinity and back again and show absence of line fields on this set.
- We also need to show that $f, \tilde{f}$ are qs conjugate on the real line.
- In the polynomial case this was not fully needed, but now we do not have a straightening theorem.
Summary:

real method $\implies$ real bounds $\implies$ \{ Koebe complex bounds \}

complex method $\implies$ \{ quasiconformal maps, Measurable Riemann Mapping Theorem, Holomorphic Motion \}
How to construct qs-symmetries?

One approach is to use Carleson box construction.

We shall use complex methods, namely a complex analogue of the nice interval (puzzle pieces) and then to use our

**QC-Criterion:** For any $\epsilon > 0$ there exists a constant $K$ with the following property.

Let $\phi: \Omega \rightarrow \tilde{\Omega}$ be a qc homeomorphism between two Jordan domains. Let $X \subset \Omega$ consist of pairwise disjoint topological discs (possibly infinitely many).

Assume that the following hold

- the components of $X$ are topological discs with $\epsilon$-bounded geometry each of which $\epsilon$-well-inside $\Omega$ (and the same holds for $\phi(X)$).

- $\phi$ is 1-qc on $\Omega - X$.

Then there exists a new $K$-qc homeo $\tilde{\psi}: \Omega \rightarrow \tilde{\Omega}$ which agrees with $\phi$ on $\partial\Omega$. 
Third Lecture: Complex Box Mappings and Complex Bounds $\implies$ qc-Rigidity
Given a polynomial $f$, there exists a way of constructing nice sets, i.e. sets $P_n$ so that no point on the boundary is ever mapped into the interior of $P_n$.

This construction uses external rays and equipotentials landing on periodic orbits, see Misha’s lectures and blackboard. These curves come from the Böttcher coordinates near $\infty$. The partition elements are called Yoccoz puzzle pieces.
Existence of good rays

More precisely. One can prove

\[ f \text{ has } \text{no neutral periodic orbits and } J(f) \text{ is connected} \implies \exists \text{ Jordan domain } P \text{ which is strictly nice:} \]

\[ f^n(x) \notin \overline{P} \text{ for all } x \in \partial P \text{ and all } n \geq 0. \]
What we will get:

This is, an object analogous to a polynomial-like map $F: U \to V$, except that

- there are several components of $V$,
- some components of $V$ are also components of $U$,
- there may be infinitely many components of $U$ intersecting the orbit of the critical points.
Definition (Complex Box Mappings)

\( F : U \to V \) between open sets in \( \mathbb{C} \) is a Complex Box Mapping if it is holomorphic and the following hold:

- \( V \) is a union of finitely many pairwise disjoint Jordan discs;
- every connected component \( V' \) of \( V \) is either a connected component of \( U \) or the intersection of \( V' \) and \( U \) is a union of Jordan discs with pairwise disjoint closures which are compactly contained in \( V' \),
- for each component \( U' \) of \( U \), \( F(U') \) is a component of \( V \).
- \( F \) has finitely many critical points; in our setting each component of \( V \) contains precisely one critical point.

Even in the case of non-renormalizable polynomials, we obtain a complex box mapping with each component of \( V \) containing only one critical point, only after obtaining complex bounds.
We will say that this comes with \textit{complex bounds} if

- \textit{bounds on the moduli} of the relevant annuli
- \textit{bounded geometry}

That is, the assumptions of the \textit{QC-criterion} hold.
Its *Julia set* is \( J(F) = \{ x; F^n(x) \in U \text{ for all } n \geq 0 \} \).

The first return map to a component of \( U \) is again a complex box mapping. This is called a *renormalisation* of \( F: U \to V \).

A component of \( F^{-n}(V) \) is called a *puzzle piece* of \( F: U \to V \).

We fix a *parametrisation* of \( \partial V \) (say, coming from Böttcher coordinates). This gives a *boundary marking* of the boundary of the boundary of each puzzle piece.

\( F \) is *renormalizable* at \( c \iff \exists s \text{ with } F^{sn}(c) \in \mathcal{L}_c V \text{ for all } n \geq 0. \)

If no first return map to any puzzle piece containing a critical point is renormalizable, then we say that \( F \) (and the original polynomial) is *non-renormalizable*.

In the real case, renormalizable \( \iff \exists \text{ periodic interval.} \)
QC-rigidity of *non-renormalizable polynomials*

A generalisation of the theorem of Yoccoz that Misha talked about:

**Theorem (QC-rigidity of non-renormalizable complex box mappings without neutral points [KvS09])**

Two such $F: U \to V$ and $\tilde{F}: \tilde{U} \to \tilde{V}$ are topol. conjugate, then $F, \tilde{F}$ qc-conjugate.

**Theorem (QC-rigidity of non-renormalizable polynomials without neutral points [KvS09])**

$f, \tilde{f}$ topol. conjugate $\implies f, \tilde{f}$ qc-conjugate.

The proof of Yoccoz for $z \to z^2 + c$ does not work:

- He uses that $2 \times 1/2 = 1$; here 2 is because of combinatorics and $1/2$ because of the degree is $z \mapsto z^2 + c$.
- $z \mapsto z^4 + c$ gives $2 \times 1/4 < 1$ which leads to failure.
Ingredients of the proof (for *non-renormalizable polynomials*):

1. **The spreading principle**: allows one to extend these qc-maps globally
2. **QC-criterion**: geometric control gives qc maps defined on small puzzle pieces
3. **The enhanced nest from [KSvS07a]**: a way to choose suitable puzzle pieces
4. **A lemma of Kahn-Lyubich** for estimating the moduli of pullbacks of annuli.
Spreading Principle:

Assume

- Take sets $W, \tilde{W}$ containing all critical points consisting of a union of puzzle pieces.
- Assume that one has $K$-qc map $h: W \to \tilde{W}$ which respects the boundary marking.

Then

- $\exists K$-qc map $H: V \to \tilde{V}$
- $\tilde{F} \circ H = H \circ F$ outside $W$.

So it is clear what to aim for:

*Find a sequence of puzzle pieces $W_n, \tilde{W}_n$ with the components of $W_n, \tilde{W}_n$ shrinking to points and $K$-qc maps $h_n: W_n \to \tilde{W}_n$. Then $\exists K$-qc maps $H_n: V \to \tilde{V}$ with $\tilde{F} \circ H_n = H_n \circ F$ outside $W_n$. Taking limits gives a $K$-qc conjugacy.*
**QC-Criterion [KSvS07a]**: For any $\epsilon > 0$ there exists a constant $K$ with the following property.

Let $\phi: \Omega \to \tilde{\Omega}$ be a qc homeomorphism between two Jordan domains. Let $X \subset \Omega$ consist of pairwise disjoint topological discs (possibly infinitely many).

Assume

- the components of $X$ are topological discs with
  - $\epsilon$-bounded geometry
  - are all $\epsilon$-well-inside $\Omega$

  (and the same holds for $\phi(X)$).

- $\phi$ is 1-qc on $\Omega - X$.

Then there exists a new $K$-qc homeo $\tilde{\psi}: \Omega \to \tilde{\Omega}$ which agrees with $\phi$ on $\partial \Omega$. 

The QC-criterion
Pick a critical point $c$.

One could choose inductively first return domains $\mathcal{L}_c(V)$, $\mathcal{L}_c^2(V) := \mathcal{L}_c(\mathcal{L}_c(V))$, \ldots and so on. This is called the principal nest.

The reason we don't use this is because the geometry of these domains gets unbounded, see blackboard.
One possibility is that

- there exists \( D \in \mathbb{N} \)
- a puzzle piece \( J \),
- infinitely many puzzle pieces \( J_n \) containing a critical point and \( p_n \) so that \( f^{p_n} : J_n \to J \) are branched covering of degree \( \leq D \).

This is called the \textit{reluctantly recurrent case}.

In this case, one can go \textit{from arbitrarily small scale with ‘bounded distortion’ to large scale}.

Then one can easily use the QC-criterion on these smaller and smaller pieces, and using the spreading principle we are done.
Instead, we will choose inductively a sequence of puzzle pieces $I_n$ inductively with some suitable properties. So choose $I_0 = V$.

- $I_{n+1}$ is a pullback of $I_n$ of bounded order: $\exists$ an integer $p_n$ so that $f^{p_n} : I_{n+1} \rightarrow I_n$ is a branched covering of degree $\leq D$ where $D$ is fixed.

- there exists a dynamically defined annulus $A_n \supset \partial I_n$ which does not intersect $\omega(c)$.

- some combinatorial properties, namely
  
  - the minimal amount of time it takes from $I_{n+1}$ to $I_{n+1}$ is at least $(1/3)p_n$
  - $p_{n+1} \geq 2p_n$.

Doing this carefully one obtains the enhanced nest.
In [KSvS07a] we proved complex bounds for the enhanced nest for real polynomials – by bare hand.

In [KvS09], we use a remarkable lemma due to Kahn & Lyubich, which allows us to prove the previous result for non-renormalizable polynomials (not necessarily real).

The lemma by Kahn-Lyubich is hard to prove and hard to state. Kozlovski and I have a short proof in the real case for an easier to state and somewhat sharper result.

I will state the result for the real setting on the next slide, but won’t show how to apply it (this will use the properties of the enhanced nest from the previous slide).

The statement on the next slide will apply to a branched covering map $G: U' \rightarrow V$ of the form $G = F^n$. Here

- $F: U \rightarrow V$ is a real-symmetric complex box mapping.
- $U'$ is a real-symmetric component of the domain of $G := F^n$. 
Proposition (Small Distortion of Thin Annuli)

∀K ∈ (0, 1) ∃ κ > 0. Let G : U' → V be as before, B a real-symmetric region in V and A a real-symmetric connected component of G⁻¹(B). Write d = \text{deg}(G|_A) = d and D = \text{deg}(G|_{U'}).

\[ \text{mod}(U' - A) \geq \frac{K^D}{2d} \min\{\kappa, \text{mod}(V - B)\}. \]

Corollary: For each D there exists κ > 0 so that if G : A → B is univalent then

\[ \text{mod}(U' - A) \geq (1/4) \min\{\kappa, \text{mod}(V - B)\}. \]

Improves on \text{mod}(U' - A) \geq (1/D) \text{mod}(V - B) when mod(V - B) ≈ 0, see blackboard: \( x_{n+1} \geq \min(\kappa, 2x_n) \).
What about real analytic or $C^3$ maps?

- For *real analytic maps* we do not have rays and equipotentials, let alone for $C^3$ maps. So what then?
- For *infinitely renormalizable* maps we also cannot use the previous method.

So we need to *construct the complex box mappings – together with the complex bounds – by hand.*

Let’s assume, for now, that $f : N \rightarrow N$ is *real analytic.*

Then *exists a neighbourhood $O$ of $N$ on which $f$ extends to a complex analytic map $f : O \rightarrow \mathbb{C}.*

However, there is not relation between $O$ and $f(O).$
Let’s take the *Poincaré metric* defined on the slit region $\mathbb{C}_J$. The *set of points with constant $\leq R$ to $J$* w.r.t. Poincaré distance consists of two circle segments. Let $D_{\theta}(J)$ be the Poincaré disc with angle $\theta$ based on $J$. 
Pullback of Poincaré discs in the univalent case

- If \( \phi \) is a univalent map of \( \mathbb{C}_J \) into \( \mathbb{C}_{\phi(J)} \) then 
  \( \phi(D_{\theta}(J)) \subset D_{\theta}(\phi(J)) \).
- In particular, take a real polynomial \( f \) with only real critical point and so that \( f|T \) is a diffeomorphism. Then define 
  \( \phi: \mathbb{C}_{f(T)} \to \mathbb{C}_T \) to be the analytic continuation of 
  \( f^{-1}: f(T) \to T \).
- Hence the pullback under \( f \) of \( D_{\theta}(f(T)) \) is \( \subset D_{\theta}(T) \).

Take a real analytic map \( f: [0, 1] \to \mathbb{R} \).
- It extends to an analytic map \( f: U \to \mathbb{C} \) without additional critical points.
- Consider a very small interval \( J \subset [0, 1] \).
- Find a much bigger disc \( D \supset J \) with \( D \subset U \).
- Then one can apply almost the same argument, see next slide.
Lemma (Almost Schwarz Inclusion \cite{dFdM99})

There exist $K < \infty$, $a_0 > 0$ and a function $\theta$ with $\theta(a) \to 0$ as $a \to 0$ such that the following holds. Let $F : \mathbb{D} \to \mathbb{C}$ be univalent and real-symmetric, and assume that $I \subset \mathbb{R}$ is an interval containing 0 and $|I| < a_0$. Let $I' = F(I)$. Then

(a) for all $\theta \geq \theta(|I|)$ (small $\theta$),

$$F(D_\theta(I)) \subset D_{(1-K|I|^{1+\delta})\theta}(I'),$$

where $0 < \delta < 1$ is a universal constant;

(b) for all $\theta \in (\pi/2, \pi)$ (lens shaped region),

$$F(D_{\pi-\theta}(I)) \subset D_{\pi-K|I|\theta}(I').$$
Lemma

Let $\ell \geq 2$ be a natural number. Let $P(z) = z^\ell$. Then there exists $\lambda = \lambda(K, \ell) \in (0, 1)$ so that the following holds:

- if $\ell$ is even, then
  \[
P^{-1}(D_\theta(-K, 1)) \subset D_{\lambda\theta}(-1, 1)
  \]
  for all $\theta \in (0, \pi)$;
- if $\ell$ is an odd integer, then
  \[
P^{-1}(D_\theta(-K^\ell, 1)) \subset D_{\lambda\theta}(-K, 1)
  \]
  for all $\theta \in (0, \pi)$. 

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Real one-dimensional dynamics
So we need real bounds

See blackboard.

The enhanced nest will give us \textit{with some considerable care } \textit{free space}.
What about the $C^3$ case: asymptotically conformal extensions

As before: $\frac{\partial}{\partial z} = \frac{1}{2} (\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}).$

**Definition:** Let $K$ be a compact subset of $\mathbb{R}^2$, $U \supset K$ open and $H : U \to \mathbb{C}$ be $C^1$. We say $H$ is *asymptotically holomorphic of order* $t \geq 1$, on $K \subset \mathbb{R}^2$ if for every $(x, y) \in K$

$$\frac{\partial}{\partial \bar{z}} H(x, y) = 0, \text{ and } \frac{\partial}{\partial z} H(x, y) \frac{d((x, y), K)^{t-1}}{d((x, y), K)} \to 0$$

uniformly as $(x, y) \to K$ for $(x, y) \in U - K$.

In our application, $K$ will be an interval contained in the real line.
Proposition (Graczyk-Sands-Swiatek)

Let $h : I \to \mathbb{R}$ be a $C^3$ diffeomorphism. Then there exist

- a $C^3$ extension $H$ of $h$ to a complex neighbourhood of $I$, with $H$ asymptotically holomorphic of order 3 on $I$.
- $K > 0$ and $\delta > 0$ such that if $a, c \in I$ are distinct, $0 < \alpha < \pi$ and $\operatorname{diam}(D_\alpha(a, c)) < \delta$, moreover,

$$H(D_\alpha(a, c)) \subset D_{\tilde{\alpha}}(h(a), h(c)),$$

where $\tilde{\alpha} = \alpha - K|c - a|\operatorname{diam}(D_\alpha(a, c))$ and $\tilde{\alpha} < \pi$.

This implies the previous inclusion lemma.
Summary

Done:
- We constructed complex box mappings for non-renormalizable polynomials without neutral periodic points.
- We showed that because of the spreading principle and QC-criterion it is enough to obtain complex bounds.
- We showed that one can get complex bounds by Kahn-Lyubich’s lemma in this case.

To do:
- We need to show how to construct complex box mappings for $C^3$ and real analytic maps, and even for real infinitely renormalizable polynomials.
- We will rely on real bounds and analysing pullbacks carefully.
- Big bounds will not always be our friends.
- When there are many critical points, we will need to fit all this together in order to obtain a global conjugacy.
Fourth Lecture: Complex Box Mappings and Complex Bounds for real analytic (and $C^3$) maps
We want to show that:

**Real analytic maps which are topologically conjugate are qs-conjugate**

provided critical points and parabolic points are mapped each other; critical points to critical points of the same order.

e.g. \( f(x) = 4x(1 - x) \), \( \tilde{f}(x) = \sin(\pi x) \) are qs-conjugate on \([0, 1]\).

Ingredients of the proof:

1. **The spreading principle**: allows one to extend these qc-maps globally
2. **QC-criterion**: geometric control gives qc maps defined on small puzzle pieces
3. **The enhanced nest from [KSvS07a]**: a way to choose suitable puzzle pieces
4. **Complex bounds**
Assume \( F : U \to V \) is a complex box mapping.

- Take sets \( W, \tilde{W} \) containing all critical points consisting of a union of puzzle pieces.
- need to have a \( K \)-qc external map. \( H : V \to \tilde{V} : H(U) = \tilde{U} \) and \( H \circ F = \tilde{F} \circ H \) on \( \partial U \), (replacing boundary marking in polynomial case).

\[\Rightarrow\]

- \( \exists \) \( K \)-qc map \( H : V \to \tilde{V} \)
- \( \tilde{F} \circ H = H \circ F \) outside \( W \).

So it is clear what to aim for:

Find a sequence of puzzle pieces \( W_n, \tilde{W}_n \) with the components of \( W_n, \tilde{W}_n \) shrinking to points and \( K \)-qc maps \( h_n : W_n \to \tilde{W}_n \).

Then \( \exists \) \( K \)-qc maps \( H_n : V \to \tilde{V} \) with \( \tilde{F} \circ H_n = H_n \circ F \) outside \( W_n \). Taking limits gives a \( K \)-qc conjugacy.
The QC-criterion

Define $H(x) := \liminf_{r \to 0} \frac{\sup_{|y-x|=r} |\phi(y) - \phi(x)|}{\inf_{|y-x|=r} |\phi(y) - \phi(x)|}.$

QC-Criterion [KSvS07a]: $\forall \epsilon, H > 0 \ \exists K < \infty$ with:

Let $\phi: \Omega \to \tilde{\Omega}$ be a qc homeo between two Jordan domains. Let $X \subset \Omega$ consist of pairwise disjoint topological discs (possibly $\infty$ many) and a set $Z \subset \Omega$ of zero Lebesgue measure.

Assume

1. the components of $X$ are topological discs with
   - $\epsilon$-bounded geometry
   - are all $\epsilon$-well-inside $\Omega$

   (and the same holds for $\phi(X)$).

2. $H(x) \leq H$ for $x \in \Omega - (X \cup Z)$.

3. $H(x) < \infty$ for $x \in Z$.

Then $\exists$ new $K$-qc homeo $\tilde{\psi}: \Omega \to \tilde{\Omega}$ which agrees with $\phi$ on $\partial \Omega$. 
Complex box mapping with complex bounds when \( \omega(c) \) is persistently recurrent

**Lemma**

Assume that \( f \) is real and non-renormalizable.

- \( f \) is persistently recurrent at \( c \) \( \implies \) \( \omega(c) \) is minimal
- \( \omega(c) \) minimal \( \implies \) only finitely many domains of the first return map to any nice interval intersect \( \omega(c) \).

We construct a generalised enhanced nest:

- \( I_n \) is non-terminating then \( I_{n+1} = f^{-p_n}(I_n) \) where \( f^{p_n} : I_{n+1} \to I_n \) has degree \( \leq D \).
- If \( I_n \) is terminating (renormalisable), then \( I_{n+1} = \mathcal{L}_c(\mathcal{R}(I)) \):
Real bounds for the enhanced nest – when ω(c) is persistently recurrent.

**Real Bounds (a priori bounds):** there exists ρ > 0 so that, e.g:

- \( I_n \) non-terminating \( \implies \)
  - \( I_{n+1} \) is \( ρ\)-nice: first return domains are \( ρ\)-well-inside \( I_{n+1} \)
  - \( I_{n+1} \) is externally \( ρ\)-free
  - If \( |I_n|/|I_{n+1}| < K \) then \( I_{n+1} \) is \( ρ\)-internally free.

- For any \( C' > 0 \) there exists \( C > 0 \) such that \( I_n \) non-terminating and \( \exists x \in ω(c_0) \cap I_n \) with \( (1 + 2C)\mathcal{L}_x(I_n) \subset I_n \) \( \implies (1 + 2C')I_{n+1} \subset I_n \) : **Big Bounds**

- If \( I_{n-1} \) is terminating and \( I_n \) is non-terminating, then \( \forall C > 0 \) \( \exists \rho > 0 \) so that \( |I_n|/|I_{n+1}| < C \) \( \implies I_{n+1} \) is \( ρ\)-nice and \( ρ\)-free.

- If \( I_{n-1} \) and \( I_n \) are terminating (Feigenbaum case), then ....

Critical points of odd order cause a lots of troubles here.
Using these, and similar real bounds:

**Theorem (Trevor Clark, Sofia Trejo and SvS)**

Assume that $f$ is a real-analytic and $\omega(c)$ is persistently recurrent. Then $\exists \rho > 0$ so that the first return map to

$$\hat{I}_n = \bigcup_{c'} \mathcal{L}_{c'} I_n.$$ 

extends to a complex box mapping $F: U \to V$ with complex (a priori) bounds:

the domains are $\rho$-well-inside and the range is $\rho$-free.

Provided we choose $n$ large enough, $\rho$ does not depend on $f$, only on the number and order of the critical points of $f$. 
First we obtain quasi box mapping.

But there is a procedure to obtain from this a complex box mapping. One can bound the domains by Poincaré domains.

Then obtain

To construct the map $H: V \to \tilde{V}$ so that $H(U) = \tilde{U}$ and $H \circ F = \tilde{F} \circ U$ on $\partial U$ is then easy.
If $\omega(c)$ is non-persisently recurrent case it is possible that $\exists$ 
*infinitely many first return domains* intersecting $\omega(c)$.

Again one there exists a complex box mapping to $I$. But now there may be infinitely many regions and they are lens shaped:

Now to construct the map $H: V \to \tilde{V}$ so that $H(U) = \tilde{U}$ and $H \circ F = \tilde{F} \circ U$ on $\partial U$ is less easy.
To construct a qc map $H: V \to \tilde{V}$ such that $h(U) = \tilde{U}$ and so that $\tilde{F} \circ h = h \circ F$ on $\partial U$.

1. $U$ has infinitely many components.

2. $(V - U) \cap \mathbb{C}^+$ is a quasi-circle: this can be done as for critical covering maps, see Levin and SvS, Inventiones 2000.

3. There exists a qc map from $(V - U) \cap \mathbb{C}^+$ to $(\tilde{V} - \tilde{U}) \cap \mathbb{C}^+$ which respects the dynamics. To do this we show in particular:

4. There exists a qc conjugacy $(V - U) \cap \mathbb{R} \to (\tilde{V} - \tilde{U}) \cap \mathbb{R}$. To do this we will need to use another type of box mapping, the so-called touching complex box mappings $G: B \to A$. 

Touching complex box mappings $G : B \to A$.

Figure: A touching box mapping: the range contains critical points, but not the domain.

- $\partial B$ will contain the fixed points and parabolic periodic points (and a number of pre images of those).
- This is one reason why $B$ and $A$ will have tangencies.
- So $B - A$ is not a quasi-circle.
- For this reason it requires some work to show that $G : B \to A$ and $\tilde{G} : \tilde{B} \to \tilde{A}$ are qc-conjugate.
This is not enough for what we need, because one has to glue together this information around each of the critical points.
Here we use a partial ordering on the set of critical points and versions of the spreading lemma and pullback arguments.
For example, in the minimal case, there are many pre images of the fixed point (or other points which escape). These have to be adjusted. This is again done using the touching box mapping $G : B \to A$. 
An application to local connectivity

Theorem

The Julia set of any real polynomial or rational maps with only real critical points has a Julia set which is locally connected.

There were several previous papers dealing with special cases. For example, in a paper with Levin we deal with covering maps of the circle with one critical points, but those methods break down completely with two odd critical points.
An application to monotonicity of entropy

In the late 70’s, the following question attracted a lot of interest: does the topological entropy of the interval map $x \mapsto ax(1 - x)$ depends monotonically on $a \in [0, 4]$? In the mid 80’s this was question was solved:

Theorem

The topological entropy of the interval map $x \mapsto ax(1 - x)$ depends monotonically on $a \in [0, 4]$. 

There are several proofs for this theorem.

- In the 80’s this was proved using
  - Thurston’s rigidity theorem;
  - by a geometric argument due to Milnor;
  - Douady-Hubbard’s univalent parametrisation of hyperbolic components together with additional ray arguments;
  - by a method due to Sullivan, see in [MT88].

All of these proofs rely on considering the map $x \mapsto ax(1 - x)$ as a polynomial acting on the complex plane.

- A somewhat different real method was used by Tsujii, [Tsu94]. He showed that periodic orbits never get destroyed using a calculation on how the multiplier depends on the parameter. In hindsight this method turned out to be somewhat related to Adam Epstein’s work.
In the early 90’s, Milnor posed in [Mil92] the more general

**Question**

*Monotonicity Conjecture.* The set of parameters within a family of real polynomial interval maps, for which the topological entropy is constant, is connected.

Milnor and Tresser proved this conjecture for cubic polynomials, see [MT00] (see also [DGMT95]). Their ingredients are planar topology (in the cubic case the parameter space is two-dimensional) and density of hyperbolicity for real quadratic maps.
The general solution of this conjecture

Take \( d \geq 1 \), \( \epsilon \in \{-1, 1\} \) and the space \( P^d_\epsilon \) of real polynomials \( f : [0, 1] \to [0, 1] \) of fixed degree \( d \) with

- \( f(\{0, 1\}) \subset \{0, 1\} \),
- all critical points in \((0, 1)\)
- first lap orientation preserving/reversing if \( \epsilon = 1 \) resp. \( \epsilon = -1 \).

Thank you.
Applying the methods from this talk we get

**Theorem**

*The topological entropy of* $[0, 1] \ni x \mapsto a \sin(2\pi x)$ *increases with* $a$.

Even for *polynomials with non-real critical points*, much is unknown, but *real degree four polynomials with one real critical point and two non-real critical points*, we have

**Theorem (Davoud Cheraghi and SvS)**

*The set of maps in this space with constant entropy is connected.*

Remark: we do not have a rigidity result in this space.
This motivates the following question

**Question**

Let $H(x, y) = (1 - ax^2 + by, y)$ be the family of Hénon maps. It follows from [KKY92] and [DGY+92] that if we fix $b$ then the set of parameters $\{a; h_{top}(H_{a,b}) = c\}$ is not connected.

However, is it possible that the set $\{(a, b); h_{top}(H_{a,b}) = c\}$ are connected?
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Wild Cantor attractors exist.

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Nonexistence of wandering intervals and structure of topological attractors of one-dimensional dynamical systems. II. The smooth case.

Alexander Blokh and Michał Misiurewicz.
Typical limit sets of critical points for smooth interval maps.

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A structure theorem in one-dimensional dynamics.

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Generic hyperbolicity in the logistic family.

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Sensitive dependence to initial conditions for one-dimensional maps.
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Density of hyperbolicity in dimension one.

Oleg Kozlovski and Sebastian van Strien.
Local connectivity and quasi-conformal rigidity of non-renormalizable polynomials.

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Nonexistence of wandering intervals and structure of topological attractors of one-dimensional dynamical systems. I. The case of negative Schwarzian derivative.

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M. Martens, W. de Melo, and S. van Strien.
Julia-Fatou-Sullivan theory for real one-dimensional dynamics.

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Remarks on iterated cubic maps.

John Milnor and William Thurston.
On iterated maps of the interval.

John Milnor and Charles Tresser.
On entropy and monotonicity for real cubic maps.
Weixiao Shen.
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A note on Milnor and Thurston’s monotonicity theorem.
Sebastian van Strien.
Transitive maps which are not ergodic with respect to Lebesgue measure.

Sebastian van Strien and Edson Vargas.
Real bounds, ergodicity and negative Schwarzian for multimodal maps.