MA2AA1 (ODE’s): Lecture Notes

Sebastian van Strien (Imperial College)

March 2013

Contents

0 Introduction ........................................... i
  0.1 Practical Arrangement ............................... i
  0.2 Relevant material ................................ ii
  0.3 Examples of differential equations ........... iii
  0.4 Issues which will be addressed in the course are iv

1 Existence and Uniqueness: Picard Theorem ...... 1
  1.1 Banach spaces ..................................... 1
  1.2 Metric spaces ..................................... 2
  1.3 Metric space versus Banach space ............. 2
  1.4 Examples .......................................... 3
  1.5 Banach Fixed Point Theorem ...................... 4
  1.6 Lipschitz functions ................................ 5
  1.7 The Picard Theorem for ODE’s (for functions which are globally Lipschitz) ............... 6
  1.8 Application to linear differential equations ... 7
  1.9 The Picard Theorem for functions which are locally Lipschitz .............................. 8
  1.10 Some comments on the assumptions in Picard’s Theorem .................................. 10
  1.11 Some implications of uniqueness in Picard’s Theorem .................................. 11
0 Introduction

0.1 Practical Arrangement

- The lectures for this module will take place Tuesday 2-3, Thursday 9-11 in Clore.

- Each week I will hand out a sheet with problems. It is very important you go through these thoroughly, as these will give the required training for the exam and class tests.

- Support classes: Thursday 11-12, from January 23.

- The support classes will be run rather differently from previous years. The objective is to make sure that you will get a lot out of these support classes.

- The main way to revise for the tests and the exam is by doing the exercises.

- There will be two class tests. These will take place in week 4 and 8. Each of these count for 5% .

- Questions are most welcome, during or after lectures and during office hour.

- My office hour is to be agreed with students reps. Office hour will in my office 6M36 Huxley Building.
0.2 Relevant material

- There are many books which can be used in conjunction to the module, but none are required.

- The lecture notes displayed during the lectures will be posted on my webpage: http://www2.imperial.ac.uk/~svanstri/ Click on Teaching in the left column. The notes will be updated during the term.

- The lectures will also be recorded. See my webpage.

- There is no need to consult any book. However, recommended books are

  - Simmons + Krantz, *Differential Equations: Theory, Technique, and Practice*, about 40 pounds. This book covers a significant amount of the material we cover. Some students will love this text, others will find it a bit longwinded.

  - Agarwal + O’Regan, *An introduction to ordinary differential equations*.

  - Teschl, *Ordinary Differential Equations and Dynamical Systems*. These notes can be downloaded for free from the authors webpage.

  - Hirsch + Smale (or in more recent editions): Hirsch + Smale + Devaney, *Differential equations, dynamical systems, and an introduction to chaos*.

  - Arnold, *Ordinary differential equations*. This book is an absolute jewel and written by one of the masters of the subject. It is a bit more advanced than this course, but if you consider doing a PhD, then get this one. You will enjoy it.

Quite a few additional exercises and lecture notes can be freely downloaded from the internet.
### 0.3 Examples of differential equations

- An example of a differential equation is the law of Newton: \( m\ddot{x}(t) = F(x(t)) \quad \forall t \). Here \( F \) is the gravitational force. Using the gravitational force in the vicinity of the earth, we approximate this by

\[
m\ddot{x}_1 = 0, \quad m\ddot{x}_2 = 0, \quad m\ddot{x}_3 = -g.
\]

This has solution

\[
x(t) = x(0) + v(0)t - \frac{g}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t^2.
\]

- According to Newton’s law, the gravitational pull between two particles of mass \( m \) and \( M \) is

\[
F(x) = \gamma m M x / |x|^3.
\]

This gives

\[
m\dddot{x}_i = -\frac{\gamma m M x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}\text{ for } i = 1, 2, 3
\]

Now it is no longer possible to explicitly solve this equation. One needs some theory to be sure there are solutions and that they are unique.

- In **ODE’s** the independent variable is one-dimensional. In a **Partial Differential Equation (PDE)** such as

\[
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0
\]

the unknown function \( u \) is differentiated w.r.t. several variables.
• The typical form for the ODE is the following initial value problem:

\[
\frac{dx}{dt} = f(t, x) \text{ and } x(0) = x_0
\]

where \( f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \). The aim is to find some curve \( t \mapsto x(t) \in \mathbb{R}^n \) so that the initial value problem holds. When does this have solutions? Are these solutions unique?

• An example of an ODE related to vibrations of bridges (or springs) is the following

\[
M x'' + c x' + k x = F_0 \cos(\omega t).
\]

One reason you should want to learn about ODE’s is:


– [http://www.youtube.com/watch?v=3mclp9QmCGs](http://www.youtube.com/watch?v=3mclp9QmCGs)

– [http://www.youtube.com/watch?v=gQK21572oSU](http://www.youtube.com/watch?v=gQK21572oSU)

### 0.4 Issues which will be addressed in the course are

• do solutions of ODE’s exist?

• are they unique?

• most solutions cannot be found explicitly. Can one say something about the behaviour of solutions anyway?
1 Existence and Uniqueness: Picard Theorem

Before we go into differential equations, we need some background.

1.1 Banach spaces

- **A vector space** $X$ is a space so that if $v_1, v_2 \in X$ then $c_1v_1 + c_2v_2 \in X$ for each $c_1, c_2 \in \mathbb{R}$ (or, more usually, for each $c_1, c_2 \in \mathbb{C}$).

- **A norm** on $X$ is a map $\| \cdot \| : X \to [0, \infty)$ so that
  1. $\|0\| = 0, \|x\| > 0 \forall x \in X \setminus \{0\}$.
  2. $\|cx\| = |c|\|x\| \forall c \in \mathbb{R}$ and $x \in X$
  3. $\|x+y\| \leq \|x\| + \|y\| \forall x, y \in X$ (triangle inequality).

- **A Cauchy sequence** in a vector space with a norm is a sequence $(x_n)_{n \geq 0} \in X$ so that for each $\epsilon > 0$ there exists $N$ so that $\|x_n - x_m\| \leq \epsilon$ whenever $n, m \geq N$.

- A vector space with a norm is **complete** if each Cauchy sequence $(x_n)_{n \geq 0}$ converges, i.e. there exists $x \in X$ so that $\|x_n - x\| \to 0$ as $n \to \infty$.

- **X** is a **Banach space** if it is a vector space with a norm which is complete.
1.2 Metric spaces

- A metric space $X$ is a space with together with a function $d : X \times X \to \mathbb{R}^+$ (called metric) so that
  1. $d(x, x) = 0$ and $d(x, y) = 0$ implies $x = y$.
  2. $d(x, y) = d(y, x)$
  3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

- A sequence $(x_n)_{n \geq 0} \in X$ is called Cauchy if for each $\epsilon > 0$ there exists $N$ so that $d(x_n, x_m) \leq \epsilon$ whenever $n, m \geq N$.

- The metric space is complete if each Cauchy sequence $(x_n)_{n \geq 0}$ converges, i.e. there exists $x \in X$ so that $d(x_n, x) \to 0$ as $n \to \infty$.

1.3 Metric space versus Banach space

- Given a norm $\| \cdot \|$ on a vector space $X$ one can also define the metric $d(x, y) = \|y - x\|$ on $X$. So a Banach space is automatically a metric space. A metric space is not necessarily a Banach space.
1.4 Examples

Example 1. $\mathbb{R}^n$ is a Banach space.

Example 2. One can define several norms on the space of $n \times n$ matrices. One, which is often used, is the matrix norm $||A|| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|Ax|}{|x|}$ when $A$ is a $n \times n$ matrix.

Example 3. If $I$ is a compact interval, then $C(I, \mathbb{R})$ endowed with the supremum norm $||x||_\infty = \sup_{t \in I} |x(t)|$ is a Banach space. Remark: this statement is proved in the metric spaces course. That $|| \cdot ||_\infty$ is a norm is easy to check. That $C(I, \mathbb{R}^n)$ is complete when endowed with this norm $|| \cdot ||_\infty$ is beyond the scope of this module. For a proof, see the metric spaces course.

Example 4. The space $C([0,1], \mathbb{R})$ endowed with the $L^1$ norm $||x||_1 = \int_0^1 |x(s)| \, ds$ is not complete. Hint: the sequence of functions $x_n(s) = \min(\sqrt{n}, 1/\sqrt{s})$ for $s > 0$ and $x_n(0) = \sqrt{n}$ is a Cauchy sequence which does not converge: there exists no continuous function $x : [0, 1] \to \mathbb{R}$ so that $||x - x_n||_1$ converges to zero.

Remark: The previous two examples show that the same set can be complete w.r.t. one metric and incomplete w.r.t. to another metric.

Remark: Without saying this explicit everywhere, we will always endow $\mathbb{R}^n$ with the Euclidean metric.
1.5 Banach Fixed Point Theorem

Theorem 1 (Banach Fixed Point Theorem). Let $X$ be a complete metric space and consider $F: X \to X$ so that there exists $\lambda \in (0, 1)$ so that

$$d(F(x), F(y)) \leq \lambda d(x, y) \text{ for all } x, y \in X$$

Then $F$ has a unique fixed point $p$:

$$F(p) = p.$$ 

Proof. (Existence) Take $x_0 \in X$ and define $(x_n)_{n \geq 0}$ by $x_{n+1} = F(x_n)$. This is a Cauchy sequence:

$$d(x_{n+1}, x_n) = d(F(x_n), F(x_{n-1})) \leq \lambda d(x_n, x_{n-1}).$$

Hence for each $n \geq 0$, $d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0)$. Therefore when $n \geq m$, $d(x_n, x_m) \leq d(x_n, x_{n-1}) + \ldots + d(x_{m+1}, x_m) \leq (\lambda^{n-1} + \ldots + \lambda^m)d(x_1, x_0) \leq \lambda^m/(1-\lambda)d(x_1, x_0)$. So $(x_n)_{n \geq 0}$ is a Cauchy sequence and has a limit $p$. As $x_n \to p$ one has $F(p) = p$.

(Uniqueness) If $F(p) = p$ and $F(q) = q$ then $||p - q|| = d(F(p), F(q)) \leq \lambda d(p, q)$. Since $\lambda \in (0, 1), \ p = q$. \qed

Remark: Since a Banach space is also a complete metric space, the previous theorem also holds for a Banach space.

Example 5. Let $g: [0, \infty) \to [0, \infty)$ be defined by $g(x) = (1/2)e^{-x}$. Then $g'(x) = (1/2)e^{-x} \leq 1/2$ for all $x \geq 0$ and so there exists a unique $p \in \mathbb{R}$ so that $g(p) = p$. 

4
1.6 Lipschitz functions

Let $X$ be a Banach space. Then we say that a function $f : X \to X$ is **Lipschitz** if there exists $K > 0$ so that

$$||f(x) - f(y)|| < K||x - y||.$$ 

**Example 6.** Let $A$ be a $n \times n$ matrix. Then $\mathbb{R}^n \ni x \mapsto Ax \in \mathbb{R}^n$ is Lipschitz. Indeed, $|Ax - Ay| \leq K|x - y|$ where $K$ is the **matrix norm** of $A$ defined by $||A|| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|Ax|}{|x|}$.

**Example 7.** The function $\mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$ is not Lipschitz: there exists no constant $K$ so that $|x^2 - y^2| \leq K|x - y|$ for all $x, y \in \mathbb{R}$.

**Example 8.** On the other hand, the function $[0, 1] \ni x \mapsto x^2 \in [0, 1]$ is Lipschitz.

**Example 9.** The function $[0, 1] \ni x \mapsto \sqrt{x} \in [0, 1]$ is not Lipschitz.

**Example 10.** Let $U$ be an open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be continuously differentiable. Then $f : C \to \mathbb{R}$ is Lipschitz for any compact set $C \subset U$. When $n = 1$ this follows from the mean value theorem, and for $n > 1$ this will be proved in Appendix B.
1.7 The Picard Theorem for ODE’s (for functions which are globally Lipschitz)

Theorem 2. Picard Theorem (global version). Consider
\[ f : \mathbb{R}^n \to \mathbb{R}^n \] which satisfies the Lipschitz inequality
\[ |f(s,u) - f(s,v)| \leq K|u - v| \] for all \( s \in \mathbb{R}, \, u, v \in \mathbb{R}^n \).
Let \( h = \frac{1}{2K} \). Then there exists a unique \( x : (-h, h) \to \mathbb{R}^n \)
satisfying the initial value problem
\[ \frac{dx}{dt} = f(t, x) \text{ and } x(0) = x_0. \]  

Proof. The initial value problem is equivalent to finding a fixed point of the operator \( P : B \to B \) defined by
\[ P(x)(t) := x_0 + \int_0^t f(s, x(s)) \, ds \]
on the Banach space \( B := C([-h, h], \mathbb{R}^n) \) with norm \( ||x|| = \max_{t \in [-h, h]} |x(t)| \). Take \( x, y \in [-h, h] \to \mathbb{R}^n \). Then for all \( t \in [-h, h] \) one has
\[ |P(x)(t) - P(y)(t)| = |\int_0^t (f(s, x(s)) - f(s, y(s))) \, ds| \leq \]
\[ \int_0^t |f(s, x(s)) - f(s, y(s))| \, ds \leq K \int_0^t |x(s) - y(s)| \, ds \]
\[ \leq (hK)||x - y|| \leq (1/2)||x - y||. \]
So
\[ ||P(x) - P(y)|| = \sup_{t \in [-h, h]} |P(x)(t) - P(y)(t)| \]
\[ \leq (1/2)||x - y|| \]
and so \( P \) is a contraction on the Banach space \( B \). Hence it has a unique fixed point. \( \square \)
1.8 Application to linear differential equations

Consider
\[ x' = Ax \text{ with } x(0) = x_0 \quad (2) \]
where \( A \) is a \( n \times n \) matrix and \( x \in \mathbb{R}^n \).

- Note that \(|Ax - Ay| \leq K|x - y|\) where \( K \) is the matrix norm of \( A \) defined by \( ||A|| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|Ax|}{|x|} \). So the Picard Theorem implies that the initial value problem (2) has a unique solution \( t \mapsto x(t) \) for \( |t| \) small.

- For each choice of \( x_0 \in \mathbb{R} \) there exists a unique solution \( x(t) \) (for \( |t| \) small). Let \( u_i(t) \) be the solution corresponding to the unit vector \( e_i \), \( i = 1, \ldots, n \). Since linear combinations of solutions of \( x' = Ax \) are also solutions,
\[
c_1 u_1(t) + \cdots + c_n u_n(t)
\]
is the general solution of \( x' = Ax \). (That each solution is of this form follows from the uniqueness part of Picard’s theorem.)

- What form do the solutions take? Apply Picard iteration, taking \( x_0(t) \equiv x_0 \). Then \( x_1(t) = x_0 + \int_0^t Ax_0(s) \, ds = x_0 + tAx_0 \). \( x_2(t) = x_0 + \int_0^t Ax_1(s) \, ds = x_0 + tAx_0 + \frac{t^2}{2} A^2 x_0 \). By induction \( x_n(t) = x_0 + tAx_0 + \frac{t^2}{2} A^2 x_0 + \cdots + \frac{t^n}{n!} A^n x_0 = \sum_{k=0}^{n} \frac{t^k A^k}{k!} x_0 \). So the solution of (2) is
\[
x(t) = e^{At} x_0 \text{ where we write } e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}.
\]

The proof of the Picard Theorem shows that this infinite sum exists (i.e. converges) when \( |t| \) is small. Later on we shall show that it exists for all \( t \).
1.9 The Picard Theorem for functions which are locally Lipschitz

Theorem 3. Picard Theorem (local version). Let $U$ be an open subset of $\mathbb{R} \times \mathbb{R}^n$ containing $(0, x_0)$ and assume that

- $f : U \rightarrow \mathbb{R}^n$ is continuous,
- $|f| \leq M$
- $|f(t, x) - f(t, y)| \leq K|x - y|$ for all $(t, x), (t, y) \in U$
- $h \in (0, \frac{1}{2K})$ is chosen so that $[-h, h] \times \{y; |y - x_0| \leq hM\} \subset U$ (such a choice for $h$ is possible since $U$ open).

Then there exists a unique $(-h, h) \ni t \mapsto x(t)$ so that

$$\frac{dx}{dt} = f(t, x) \text{ and } x(0) = x_0.$$  \hfill (3)

Proof. Fix $h > 0$ as in the theorem, write $I = [-h, h]$, and let $B := \{y \in \mathbb{R}^n; |y - x_0| \leq hM\}$. Next define $C(I, B)$ as the space of continuous functions $x : I \rightarrow B \subset \mathbb{R}^n$ and

$$P : C(I, B) \rightarrow C(I, B) \text{ by } P(x)(t) = x_0 + \int_0^t f(s, x(s)) \, ds$$

Then the initial value problem (3) is equivalent to the fixed point problem

$$x = P(x).$$

To see that $P$ is well-defined, note that $h > 0$ is chosen so that when $B := \{y; |y - x_0| \leq hM\}$ then $[-h, h] \times B \subset U$. So

- when $x \in C(I, B)$ then $f(t, x(t))$ is well-defined for all $t \in [-h, h]$;
- $|P(x)(t) - x_0| \leq \int_0^h |f(s, x(s))| \, ds \leq hM$. So $P(x)(t) \in B$ for all $t \in [-h, h]$ and therefore $P(x) \in C(I, B)$. 

8
Remember:

\[ P: C(I, B) \to C(I, B) \] by \[ P(x)(t) = x_0 + \int_0^t f(s, x(s)) \, ds \]

**Let us now show that** \( P \) **a contraction**: for each \( t \in [-h, h] \),

\[
|P(x)(t) - P(y)(t)| = \left| \int_0^t (f(s, x(s)) - f(s, y(s))) \, ds \right|
\leq \int_0^t |f(s, x(s)) - f(s, y(s))| \, ds
\leq K \int_0^t |x(s) - y(s)| \, ds \quad \text{(Lipschitz)}
\leq Kt \max_{|s| \leq t} |x(s) - y(s)|
\leq Kh ||x - y||
\leq ||x - y||/2 \quad \text{(since } h \in (0, \frac{1}{2K})\text{)}

So the integral equation, and therefore the ODE, has a unique solution. \( \square \)
1.10 Some comments on the assumptions in Picard’s Theorem

- To obtain existence in Theorem 3 it is enough to find some open set \( U \ni (0, x_0) \).

- Often one can apply Theorem 3 but not Theorem 2. Take for example \( x' = (1 + x^2) \). Then the r.h.s. is not Lipschitz on all of \( \mathbb{R} \). It is locally Lipschitz though.

- It is not necessary to take the initial time to be \( t = 0 \). The Picard Theorem also gives that there exists \( h > 0 \) so that the initial value problem

\[
x' = f(t, x), x(t_0) = x_0
\]

has a solution \( (t_0 - h, t_0 + h) \ni t \mapsto x(t) \in \mathbb{R}^n \).

- Let \( V \subset \mathbb{R} \times \mathbb{R}^n \) and assume that the Jacobian matrix \( \frac{\partial f}{\partial x}(t, x) \) exists for \( (x, t) \in V \) and \( (t, x) \ni V \mapsto \frac{\partial f}{\partial x}(t, x) \) is continuous. Then for each convex, compact subset \( C \subset V \) there exists \( K \in \mathbb{R} \) so that

\[
|f(t, x) - f(t, y)| \leq K|x - y|.
\]

This follows from the Mean Value Theorem in \( \mathbb{R}^n \), see Appendix B. (So one can apply the previous theorem for each open set \( U \subset C \).

- If \( (t, x) \mapsto f(t, x) \) has additional smoothness, the solutions will be more smooth. For example, suppose that \( f(t, x) \) is real analytic (i.e. \( f(t, x) \) can be written as a convergent power series), then the solution \( t \mapsto x(t) \) is also real analytic.
1.11 Some implications of uniqueness in Picard’s Theorem

- If the assumptions of the previous theorem hold and

\[ x_1: I_1 \to \mathbb{R}^n, x_2: I_2 \to \mathbb{R}^n \]

are both solutions of the initial value problem. Then

\[ x_1(t) = x_2(t) \text{ for all } t \in I_1 \cap I_2. \]

(See exercises.)

- \( f: U \to \mathbb{R}^n \) does not depend on \( t \) (in this case we could take \( U = \mathbb{R} \times V \)). This case is called autonomous, and so we can write \( x' = f(x), x(0) = x_0 \). In this setting solutions cannot cross:

More precisely, if \( x_1, x_2 \) are solutions with \( x_1(t_1) = x_2(t_2) = p \in V \) then

\[ x_3(t) = x_1(t + t_1) \text{ and } x_4(t) = x_2(t + t_2) \]

are both solutions to \( x' = f(x) \) with \( x(0) = p \). So

\[ x_3 \equiv x_4. \]
1.12 Higher order differential equations

Consider a higher order differential equation of the form

\[ y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = b(t) \]  

(4)

where \( y^{(i)} \) stands for the \( i \)-th derivative of \( y \) w.r.t. \( t \).

- One can rewrite (4) as a first order ODE, by defining

  \[ z_1 = y, z_2 = y^{(1)}, \ldots, z_n = y^{(n-1)}. \]

The higher order differential equation (4) is equivalent to

\[
\frac{d}{dt} \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix} = \begin{pmatrix} z_2 \\ \vdots \\ z_n \\ b(t) - [a_{n-1}(t)z_n + \cdots + a_0(t)z_1] \end{pmatrix}
\]

- Picard’s theorem implies \( \exists! \) solution of this ODE which satisfies \( (z_1(0), \ldots, z_n(0)) = (y(0), \ldots, y^{(n-1)}(0)) \).

- One can rewrite the vectorial equation as

\[
\frac{d}{dt} \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix} = A(t) \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix}
\]

where \( A(t) \) is matrix with coefficients depending on \( t \). Therefore, as in subsection 1.8 the general solution of the non-homogeneous ODE is of the form \( c_1y_1 + \cdots + c_ny_n + p \) where \( p \) is a particular solution. There are at most \( n \) degrees of freedom.
1.13 Continuous dependence on initial conditions

Theorem 4. Continuous dependence on initial conditions
Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be open, $f, g: U \to \mathbb{R}^n$ be continuous and assume that

$$K = \sup_{(t,x),(t,y) \in U} \frac{|f(t,x) - f(t,y)|}{|x - y|}, \quad M = \sup_{(t,x) \in U} |f(t,x) - g(t,x)|$$

are finite. If $x(t)$ and $y(t)$ are respective solutions of the IVP’s

\[
\begin{align*}
&\begin{cases}
x' = f(t,x) \\
x(0) = x_0
\end{cases} \quad \text{and} \quad \begin{cases}
y' = g(t,y) \\
y(0) = y_0
\end{cases}
\end{align*}
\]

Then

$$|x(t) - y(t)| \leq |x_0 - y_0|e^{K|t|} + \frac{M}{K}(e^{K|t|} - 1).$$

1.14 Gronwall Inequality

Proof:

$$|x(t) - y(t)| \leq |x_0 - y_0| + \int_0^t |f(s, x(s)) - g(s, y(s))| \, ds.$$

Moreover,

$$|f(s, x(s)) - g(s, y(s))| \leq |f(s, x(s)) - f(s, y(s))| + |f(s, y(s)) - g(s, y(s))| \leq K|x(s) - y(s)| + M.$$

Hence, writing $u(t) := |x(t) - y(t)|$ we have

$$u(t) \leq |x_0 - y_0| + \int_0^t (K|u(s)| + M)$$

and therefore the required inequality follows from the following lemma.
Lemma 1. Gronwall Inequality

\[ u(t) \leq C_0 + \int_0^t (Ku(s) + M) \, ds \text{ for all } t \in [0, h] \implies \]

\[ u(t) \leq C_0 e^{Kt} + \frac{M}{K} (e^{Kt} - 1) \text{ for all } t \in [0, h]. \]

**Proof.** Let’s only prove this only when \( M = 0 \). Define

\[ U(t) = C_0 + \int_0^t (Ku(s)) \, ds. \]

Then \( u(t) \leq U(t) \). Differentiating, we obtain

\[ U'(t) = Ku(t). \]

Hence

\[ U'(t)/U(t) = Ku(t)/U(t) \leq K \]

and therefore

\[ \frac{d}{dt} \log(U(t)) \leq K. \]

Since \( U(0) = C_0 \) this gives

\[ u(t) \leq U(t) \leq C_0 e^{Kt}. \]

\[ \square \]
1.15 Consequences: the Butterfly effect

- Let us interpret the previous result for $f = g$. Then $M = 0$ and

\[
\begin{align*}
\begin{cases}
x' &= f(t, x) \\
x(0) &= x_0
\end{cases}
\text{ and }
\begin{cases}
y' &= f(t, y) \\
y(0) &= y_0
\end{cases}
\end{align*}
\]

implies

\[|x(t) - y(t)| \leq |x_0 - y_0| e^{K|t|}.\]

In particular, uniqueness follows.

- The previous inequality states:

\[|x(t) - y(t)| \leq |x_0 - y_0| e^{K|t|} + 0.\]

So orbits can separate exponentially fast.

- If solutions indeed separate exponentially fast, the the differential equation is said to have sensitive dependence on initial conditions. (The flapping of a butterfly in the Amazon can cause a hurricane over the Atlantic.)
2 Linear systems in $\mathbb{R}^n$

In this section we consider

$$x' = Ax \text{ with } x(0) = x_0$$

(5)

where $A$ is a $n \times n$ matrix and $\mathbb{R} \ni t \mapsto x(t) \in \mathbb{R}^n$.

In Example 1.8 we saw that

$$e^{tA} = \sum_{k \geq 0} \frac{1}{k!} (At)^k$$

is defined for $|t|$ small and that $x(t) = e^{tA}x_0$ is a solution of (5) for $|t|$ small. In this section we will show that $e^{tA}$ is well-defined for all $t \in \mathbb{R}$ and show how to compute this matrix.

Example 11. Let $A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. Then one has inductively

$$(tA)^k = \begin{pmatrix} (t\lambda)^k & 0 \\ 0 & (t\mu)^k \end{pmatrix}.$$ So $e^{tA} = \begin{pmatrix} e^{t\lambda} & 0 \\ 0 & e^{t\mu} \end{pmatrix}$.

Example 12. Let $A = \begin{pmatrix} \lambda & \epsilon \\ 0 & \lambda \end{pmatrix}$. Then one has inductively

$$(tA)^k = \begin{pmatrix} (t\lambda)^k & \epsilon t^{k-1} \lambda^{k-1} \\ 0 & (t\lambda)^k \end{pmatrix}.$$ So $e^{tA} = \begin{pmatrix} e^{t\lambda} & \epsilon t e^{t\lambda} \\ 0 & e^{t\lambda} \end{pmatrix}$.

Lemma 2. $e^A$ is well-defined for any matrix $A = (a_{ij})$.

Proof. Let $a_{ij}(k)$ be the matrix coefficients of $A^k$ and define $a := ||A||_\infty := \max |a_{ij}|$. Then

$$|a_{ij}(2)| = \sum_{k=1}^n |a_{ik}a_{kj}| \leq na^2$$
$$|a_{ij}(3)| = \sum_{k,l} |a_{ik}a_{kl}a_{lj}| \leq n^2a^3$$
$$\vdots$$
$$|a_{ij}(k)| = \sum_{k,l} |a_{ik}a_{kj}a_{lj}| \leq n^{k-1}a^k$$

16
So \( \sum_{k=0}^{\infty} \left| a_{ij}(k) \right| k! \leq \sum_{k=0}^{\infty} \frac{n^{k-1} a_k}{k!} \leq \exp(na) \) which means that the series \( \sum_{k=0}^{\infty} \frac{a_{ij}(k)}{k!} \) converges absolutely by the ratio test. So \( e^{A} \) is well-defined. \( \square \)

2.1 Some properties of \( \exp(A) \)

**Lemma 3.** Let \( A, B, T \) be \( n \times n \) matrices and \( T \) invertible. Then

1. If \( B = T^{-1} A T \) then \( \exp(B) = T^{-1} \exp(A)T \);
2. If \( A B = B A \) then \( \exp(A + B) = \exp(A) \exp(B) \)
3. \( \exp(-A) = (\exp(A))^{-1} \)

**Proof.** (1) \( T^{-1}(A+B) = T^{-1}AT + T^{-1}BT \) and \( (T^{-1}AT)^k = T^{-1}A^kT \). Therefore

\[
T^{-1} \left( \sum_{k=0}^{n} \frac{A^k}{k!} \right) T = \sum_{k=0}^{n} \frac{(T^{-1}AT)^k}{k!}.
\]

(2) follows from the next lemma and (3) follows from (2) taking \( B = -A \). \( \square \)

For general matrices \( \exp(A + B) \neq \exp(A) \exp(B) \).

Note that if \( AB = BA \) then \( (A + B)^n = n! \sum_{j+k=n} \frac{A^j B^k}{j! k!} \).

So (2) in the previous lemma follows from:

**Lemma 4.**

\[
\sum_{n=0}^{\infty} \sum_{j+k=n} \frac{A^j B^k}{j! k!} = \sum_{j=0}^{\infty} \frac{A^j}{j!} \sum_{k=0}^{\infty} \frac{B^k}{k!}.
\]
Proof: A computation shows
\[
\sum_{n=0}^{2m} \sum_{j+k=n} A^j B^k \frac{j!}{j!} \frac{k!}{k!} - \sum_{j=0}^{m} A^j \sum_{k=0}^{m} B^k \frac{j!}{j!} \frac{k!}{k!} = \sum A^j B^k \frac{j!}{j!} \frac{k!}{k!} + \sum A^j B^k \frac{j!}{j!} \frac{k!}{k!}
\]
where \(\sum'\) respectively \(\sum''\) denote the sum over terms
\[
\begin{align*}
j + k &\leq 2m, 0 \leq j \leq m, m + 1 \leq k \leq 2m, \\
j + k &\leq 2m, m + 1 \leq j \leq 2m, 0 \leq k \leq m.
\end{align*}
\]
So the absolutely values of the coefficients in \(\sum A^j B^k \frac{j!}{j!} \frac{k!}{k!}\) are bounded by \(\sum_{j=0}^{m} ||A^j||_{\infty} \sum_{k=m+1}^{2m} ||B^k||_{\infty} \frac{j!}{j!} \frac{k!}{k!}\). As in the proof Lemma 2 the latter term goes to zero as \(m \to \infty\).

Similarly \(\sum'' A^j B^k \frac{j!}{j!} \frac{k!}{k!}\) goes to zero as \(m \to \infty\). This completes the proof of Lemma 4.

Example 13.
\[
\exp \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) = \left( \begin{array}{cc} e^{at} \cos(bt) & e^{at} \sin(bt) \\ -e^{at} \sin(bt) & e^{at} \cos(bt) \end{array} \right).
\]
This is proved in the last assignment of week 2.

Each coefficient of \(e^{tA}\) depends on \(t\). So define \(\frac{d}{dt} e^{tA}\) to be the matrix obtained by differentiating each coefficient.

Lemma 5. \(\frac{d}{dt} \exp(tA) = A \exp(tA) = \exp(tA)A\).

Proof.
\[
\frac{d}{dt} \exp(tA) = \lim_{h \to 0} \frac{\exp((t+h)A) - \exp(tA)}{h} = \]

Figure 3.1 Saddle phase portrait for \( x' = -x, y' = y \).

\[
\lim_{h \to 0} \frac{\exp(tA) \exp(hA) - \exp(tA)}{h} = \exp(tA) \lim_{h \to 0} \frac{\exp(hA) - I}{h} = \exp(tA)A.
\]

Here the last equality follows from the definition of \( \exp(hA) = I + hA + \frac{h^2}{2!}A^2 + \ldots \).

### 2.2 Solutions of 2 \times 2 systems

\( x(t) = e^{tA}x_0 \) is the solution of \( \dot{x} = Ax, x(0) = x_0 \) because \( \dot{x} = Ae^{tA}x_0 = Ax(t) \) and \( x(0) = e^{0A}x_0 = x_0 \).

**Example 14.** Take \( A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \). So \( x(t) = e^{tA} = \begin{pmatrix} e^{t\lambda} & 0 \\ 0 & e^{t\mu} \end{pmatrix} x_0 \) is a solution of the differential equation.

(Case a) \( \lambda, \mu < 0 \) (sink). Then \( x(t) \to 0 \) as \( t \to \infty \).

(Case b) If \( \lambda, \mu > 0 \) (source). Then \( x(t) = e^{tA}x_0 \to \infty \) as \( t \to \infty \) for any \( x_0 \neq 0 \).

(Case c) \( \lambda < 0 < \mu \) (saddle). Then \( x(t) = e^{tA}x_0 \to \infty \) as \( t \to \infty \) if the 2nd component of \( x_0 \) is non-zero, and \( x(t) \to 0 \) otherwise.

**Example 15.** Take \( A = \begin{pmatrix} \lambda & \epsilon \\ 0 & \lambda \end{pmatrix} \) and let us compute \( e^{tA} \) again. \( tA = t\Lambda + tN \) where \( \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \) and \( N = \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix} \). Note that \( \Lambda N = \lambda N = N \Lambda \) and that \( N^2 = 0 \).
So
\[ e^{tN} = I + tN = \begin{pmatrix} 1 & t\epsilon \\ 0 & 1 \end{pmatrix} , \quad e^{t\Lambda} = \begin{pmatrix} e^{t\lambda} & 0 \\ 0 & e^{t\lambda} \end{pmatrix} \]

and
\[ e^{tA} = e^{t\Lambda} e^{tN} = \begin{pmatrix} e^{t\lambda} & e^{t\epsilon t\lambda} \\ 0 & e^{t\lambda} \end{pmatrix} . \]

In general it is not so easy to compute \( e^{tA} \) directly from the definition. For this reason we will discuss

- eigenvalues and eigenvectors;
- using eigenvectors to put a matrix in a new form;
- using eigenvectors and eigenvalues to obtain solutions directly.

### 2.3 \( n \) linearly independent eigenvectors

Given a concrete \( n \times n \) matrix \( A \), one usually solves the solutions of \( \dot{x} = Ax \) using eigenvalues and eigenvectors.

**Reminder:** A vector \( v \neq 0 \) is an **eigenvector** if \( Av = \rho v \) for some \( \rho \in \mathbb{C} \) where \( \rho \) is called the corresponding **eigenvalue.** So, \( (A - \rho I)v = 0 \) and \( \det(A - \rho I) = 0 \). The equation \( \det(A - \rho I) = 0 \) is a polynomial of degree in \( \rho \).

**Example 16.** Take \( A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix} \). Consider

\[
\det \begin{pmatrix} 1 - \rho & 2 & -1 \\ 0 & 3 - \rho & -2 \\ 0 & 2 & -2 - \rho \end{pmatrix} = -(-1 + \rho)(-2 - \rho + \rho^2).
\]

So \( A \) has eigenvalues 2, 1, -1. Eigenvector w.r.t. 2:
\[
\begin{pmatrix}
-1 & 2 & -1 \\
0 & 1 & -2 \\
0 & 2 & -4
\end{pmatrix} v = 0
\]

which gives \( v = (3, 2, 1) \) (or multiples). \( A \) has eigenvalues \( 2, 1, -1 \) with eigenvectors \((3, 2, 1), (1, 0, 0), (0, 1, 2)\).

- **Case 1: \( n \) linearly independent eigenvectors.** Suppose that \( v_1, \ldots, v_n \) are eigenvectors of \( A \) with eigenvectors \( \rho_1, \ldots, \rho_n \) and assume that these eigenvectors are linearly independent.

**Lemma from Linear Algebra:** if all \( \rho_i \) are distinct then the eigenvectors \( v_1, \ldots, v_n \) are linearly independent and span \( \mathbb{R}^n \).

Then \( x_i(t) = e^{\rho_i t} v_i \) is a solution because

\[
\dot{x}_i = \rho_i e^{\rho_i t} v_i = e^{\rho_i t} A v_i = A x_i(t_i)
\]

Hence

\[
x(t) = c_1 e^{\rho_1 t} v_1 + \cdots + c_n e^{\rho_n t} v_n
\]

is the general solution of the differential equation.

To determine the solution with \( x(0) = x_0 \) one needs to solve \( c_i \) so that \( c_1 v_1 + \cdots + c_n v_n = x_0 \) (which can be done since \( v_1, \ldots, v_n \) are linearly independent and span \( \mathbb{R}^n \)).

### 2.4 complex eigenvectors

- **Case 2: Complex eigenvectors.** If \( v_1 \) is non-real (which implies since \( A \) is real that \( \rho_1 \) is also non-real), then there exists another eigenvector, say \( v_2 \) with \( \bar{v}_2 = v_1, \bar{\rho}_2 = \rho_1 \).
So write $v_1 = \zeta_1 + i\zeta_2$, $v_2 = \zeta_1 - i\zeta_2$, $\rho_1 = a_1 + ib_1$ and $\rho_2 = a_1 - ib_1$ with $\zeta$, $a_1, b_1$ are real. This gives

$$c_1 e^{\rho_1 t} v_1 + c_2 e^{\rho_2 t} v_2 = c_1 e^{a_1 t} \left( (\cos(bt) + i\sin(bt)) (\zeta_1 + i\zeta_2) + c_2 (\cos(bt) - i\sin(bt)) (\zeta_1 - i\zeta_2) \right)$$

By taking suitable choices of $c_1, c_2 \in \mathbb{C}$ one can rewrite this as

$$d_1 e^{at} (\cos(bt) \zeta_1 - \sin(bt) \zeta_2) + d_2 e^{at} (\sin(bt) \zeta_1 + \cos(bt) \zeta_2)$$

where $d_1, d_1 \in \mathbb{R}$.

An alternative way of seeing this goes as follows: $A(\zeta_1 + i\zeta_2) = (a + bi)(\zeta_1 + i\zeta_2) = (a\zeta_1 - b\zeta_2) + i(a\zeta_2 + b\zeta_1)$. So $A(\zeta_1) = a\zeta_1 - b\zeta_2$ and $A(\zeta_2) = (a\zeta_2 + b\zeta_1)$. It follows that if $T$ is the matrix consisting of columns $\zeta_1, \zeta_2$ then

$$T^{-1} A T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$ 

Indeed, $AT(e_1) = A(\zeta_1) = a\zeta_1 - b\zeta_2 = aT(e_1) - bT(e_2)$ and so $T^{-1} A T(e_1) = a e_1 - b e_2$. Similarly $T^{-1} A T(e_2) = b e_1 + a e_2$. So

$$\exp(A) = T \exp(T^{-1} A T) T^{-1} = T \begin{pmatrix} e^{at} \cos(bt) & e^{at} \sin(bt) \\ -e^{at} \sin(bt) & e^{at} \cos(bt) \end{pmatrix} T^{-1}.$$ 

Here we use Example[13] Now check that

$$T \begin{pmatrix} e^{at} \cos(bt) & e^{at} \sin(bt) \\ -e^{at} \sin(bt) & e^{at} \cos(bt) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = d_1 e^{at} (\cos(bt) \zeta_1 - \sin(bt) \zeta_2) + d_2 e^{at} (\sin(bt) \zeta_1 + \cos(bt) \zeta_2).$$

22
2.5 eigenvalues with higher multiplicity

- **Case 1: Repeated eigenvalues** If $\rho = \rho_1 = \cdots = \rho_k$ then we proceed as follows. Let us assume $\rho_1 = \rho_2$ and $v_1$ is an eigenvector w.r.t. $\rho$ but there is not 2nd eigenvector. Then there exists a vector $v_2$ so that $(A - \rho I)v_2 = v_1$. (If $\rho$ appears with multiplicity $k$, the space $\{(A - \rho I)^k v = 0\}$ is $k$ dimensional - we won’t prove this here, but refer to the proof of the Jordan normal form in Linear Algebra.)

So

$$x_1(t) = e^{\rho t}v_1 \text{ and } x_2(t) = t e^{\rho t}v_1 + e^{\rho t}v_2$$

is a solution: indeed

$$\dot{x}_2 = e^{\rho t}v_1 + t \rho e^{\rho t}v_1 + \rho e^{\rho t}v_2 = \rho te^{\rho t}v_1 + (e^{\rho t}v_1 + \rho e^{\rho t}v_2) = A (te^{\rho t}v_1 + e^{\rho t}v_2)$$

where we use that $Av_2 = \rho I v_2 + v_1$.

2.6 A worked example: 1

Computing solutions in several ways

**Example 17.** The matrix $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}$ has eigenvalues 2, 1, −1 with eigenvectors $(3, 2, 1), (1, 0, 0), (0, 1, 2)$. Set

$$T = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \text{ we get } T^{-1}AT = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  

Indeed, $T^{-1}A e_i = T^{-1} A v_i = \rho_i T^{-1} v_i = \rho_i e_i$ where $e_i$ is the $i$-th unit vector. Hence

$$\exp(tA) = \exp \left( tT^{-1}ATT^{-1} \right) = T \exp \left( tT^{-1}AT \right) T^{-1}$$
\begin{align*}
&= T \begin{pmatrix}
e^{2t} & 0 & 0 \\
0 & e^t & 0 \\
0 & 0 & e^{-t}
\end{pmatrix} T^{-1}.
\end{align*}

For each vector \( c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \) there exists \( x_0 \in \mathbb{R}^3 \) so that
\( c = T^{-1}x_0 \). Hence
\[
\exp(tA)x_0 = T \begin{pmatrix}
e^{2t} & 0 & 0 \\
0 & e^t & 0 \\
0 & 0 & e^{-t}
\end{pmatrix} T^{-1}x_0 = T \begin{pmatrix}
e^{2t} & 0 & 0 \\
0 & e^t & 0 \\
0 & 0 & e^{-t}
\end{pmatrix} c = \]
\[
T \begin{pmatrix}
c_1 e^{2t} \\ c_2 e^t \\ c_3 e^{-t}
\end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.
\]
Notice that this agrees with the method suggested in the previous slides!!

The previous example is an instance of the \textbf{diagonal Jordan Normal Form}:

\textbf{Theorem:} If an \( n \times n \) matrix \( A \) has \( n \) distinct eigenvalues \( \lambda_1, \ldots, \lambda_n \) with eigenvectors \( v_i \) then

- The eigenvectors \( v_1, \ldots, v_n \) are linearly independent and span \( \mathbb{R}^n \);
- If we take \( T \) the matrix with columns \( v_1, \ldots, v_n \) then
  \[ T^{-1}AT = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & \lambda_n
\end{pmatrix}.\]
- \( e^{tA} = T \begin{pmatrix}
e^{t\lambda_1} & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & e^{t\lambda_n}
\end{pmatrix} T^{-1}. \]
2.7 A second worked example

Example 18. Take $A = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix}$ and compute the solution of $x' = Ax$ with $x_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. $\det(A - \rho I) = (1 - \rho) \begin{pmatrix} 9 & -5 \\ -1 & -\rho \end{pmatrix} = (\rho + 2)^2$ so the eigenvalue $-2$ appears with double multiplicity. $(A - \rho I)v = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} v = 0$ implies $v$ is a multiple of $v_1 := \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Next $(A - \rho I)v_2 = v_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ gives $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as a solution. So $(\star) \quad x(t) = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} t \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}) e^{-2t}$

is the general solution. $c_1 = 1$, $c_2 = -2$ solves the initial value problem.

$v_1, v_2$ allow us to transform this matrix into what is called a Jordan normal form:

$Av_1 = \rho v_1, Av_2 = \rho v_2 + v_1$.

Take $T$ the matrix with columns $v_1, v_2$, i.e. $Te_i = v_i$.

Then

$T^{-1}AT(e_1) = T^{-1}Av_1 = T^{-1}\rho v_1 = \rho e_1$

$T^{-1}AT(e_2) = T^{-1}Av_2 = T^{-1}(\rho v_2 + v_1) =

= \rho T^{-1}(v_2) + T^{-1}(v_1) = \rho e_2 + e_1$.

This means that

$T^{-1}AT = N := \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$
Hence \( e^{tA} = Te^{tN}T^{-1} = T \left( \begin{array}{cc} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{array} \right) T^{-1}. \)

The previous example is an instance of the Jordan Normal Form Theorem:

If an \( n \times n \) matrix \( A \) has only one eigenvector \( v \) (which implies that its eigenvalue \( \lambda \) appears with multiplicity \( n \)) then

- one can define inductively \( v_1 = v \) and \( (A - \lambda I)v_{i+1} = v_i \).
- \( v_1, \ldots, v_n \) are linearly independent and span \( \mathbb{R}^n \).
- If we take \( T \) the matrix with columns \( v_1, \ldots, v_n \) then \( T^{-1}AT \) takes the Jordan Normal Form:

\[
T^{-1}AT = \begin{pmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda \\
0 & 0 & 0
\end{pmatrix}
\]

- \( e^{tA} = e^{t\lambda}(I + tN + \cdots + \frac{t^{n-1}}{(n-1)!}N^{n-1}) \) where \( N = \begin{pmatrix}
0 & 1 & 0 \\
\vdots & \ddots & \ddots \\
0 & 0 & 1
\end{pmatrix} \).

2.8 Complex Jordan Normal Form (General Case)

**Theorem 5.** For each \( n \times n \) matrix \( A \) there exists a (possibly complex) matrix \( T \) so that \( T^{-1}AT \) takes the Jordan Normal Form:

\[
T^{-1}AT = \begin{pmatrix}
J_1 & \cdots & \\
& \ddots & \\
& & J_p
\end{pmatrix}
\]

where \( J_j = \begin{pmatrix}
\rho_j & 1 & 0 \\
0 & \rho_j & 1 \\
0 & 0 & \rho_j
\end{pmatrix} \)

and where \( \rho_j \) is an eigenvalue of \( A \) appearing with multiplicity equal to the dimension of \( J_j \).
In the computations above, we showed how to determine $T$ so this holds.

## 2.9 Real Jordan Normal Form

Splitting real and complex parts we obtain:

For each real $n \times n$ matrix $A$ there exists a real $n \times n$ matrix $T$ so that $T^{-1}AT$ takes the real Jordan Normal Form:

$$T^{-1}AT = \begin{pmatrix}
J_1 & & \\
& \ddots & \\
& & J_p
\end{pmatrix}$$

where $J_j$ is either as in the complex Jordan Normal form when $\rho_j$ real or if it is complex equal to

$$J_j = \begin{pmatrix}
C_j & I & 0 & 0 \\
0 & C_j & I & 0 \\
& & \ddots & \\
0 & 0 & C_j & I \\
0 & 0 & 0 & C_j
\end{pmatrix}$$

where $C_j = \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix}$

and

$$\rho_j = a_j + ib_j \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
3 Power Series Solutions

Theorem 6. If \( f \) is real analytic near \((x_0, 0)\), then \( x' = f(x, t) \), \( x(0) = x_0 \) has a real analytic solution, i.e. the solution is a power series in \( t \) which converges for \(|t| < h|\).

To prove theorem one considers in the differential equation \( x' = f(x, t) \) the time \( t \) be complex! We will not pursue this here.

In this chapter we will consider some examples. Typically, one the coefficients appearing in the power series expansions of the solutions can be found inductively as in the next examples.

Example 19. \( y' = y \). Then substitute \( y = \sum_{i \geq 0} a_i x^i \) and \( y' = \sum_{j \geq 1} j a_j x^{j-1} = \sum_{i \geq 0} (i + 1) a_{i+1} x^i \). Comparing powers gives \( \sum_{i \geq 0} (a_i x^i - (i + 1) a_{i+1} x^i) = 0 \) and so \( a_{i+1} = a_i / (i + 1) \). So \( a_n = C / n! \) which gives \( y(x) = C \sum_{n \geq 0} x^n / n! = C \exp(x) \).

3.1 Legendre equation

Example 20. Consider the Legendre equation at \( x = 0 \):

\[(1 - x^2)y'' - 2xy' + p(p + 1)y = 0.\]

Write \( y = \sum_{i \geq 0} a_i x^i \),

\[y' = \sum_{j \geq 1} j a_j x^{j-1} = \sum_{i \geq 0} (i + 1) a_{i+1} x^i.\]

\[y'' = \sum_{j \geq 2} j(j - 1) a_j x^{j-2} = \sum_{i \geq 0} (i + 2)(i + 1) a_{i+2} x^i.\]

We determine \( a_i \) as follows.

\[y'' - x^2 y'' - 2xy' + p(p + 1)y = \]
\[
\sum_{i \geq 0} (i+2)(i+1)a_{i+2}x^i - \sum_{i \geq 2} i(i-1)a_ix^i - 2 \sum_{i \geq 1} i a_ix^i + p(p+1) \sum_{i \geq 0} a_ix^i
\]
\[
= \sum_{i \geq 2} [(i + 2)(i + 1)a_{i+2} - i(i - 1)a_i - 2ia_i + p(p+1)a_i]x^i +
\]
\[
+ (2a_2 + 6xa_3) - 2a_1x + p(p+1)(a_0 + a_1)x
\]

\[
\sum_{i \geq 0} (i+2)(i+1)a_{i+2}x^i - \sum_{i \geq 2} i(i-1)a_ix^i - 2 \sum_{i \geq 1} i a_ix^i + p(p+1) \sum_{i \geq 0} a_ix^i
\]
\[
= \sum_{i \geq 2} [(i + 2)(i + 1)a_{i+2} - i(i - 1)a_i - 2ia_i + p(p+1)a_i]x^i +
\]
\[
+ (2a_2 + 6xa_3) - 2a_1x + p(p+1)(a_0 + a_1)x
\]

So collecting terms with the same power of \(x\) together gives
\[
a_2 = - \frac{p(p+1)}{2} a_0 \quad \text{and} \quad a_3 = \frac{(2-p(p+1))}{6} a_1
\]
\[
a_{i+2} = \frac{[i(i-1) + 2i - p(p+1)]a_i}{(i+1)(i+2)} = - \frac{(p-i)(p+i+1)}{(i+2)(i+1)} a_i.
\]

If \(p\) is an integer, \(a_{p+2j} = 0\) for \(j \geq 0\). If \(p \notin \mathbb{N}\) then \(y\) is given by an infinite power series. Convergence for \(|x| < 1\) follows from the ratio test.

### 3.2 Second order equations with singular points

Sometimes one encounters a differential equation where the solutions are not analytic because the equation has a pole. For example

\[
y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0.
\]

Or more generally if the equation can be written in the form

\[
y'' + p(x)y' - q(x) = 0
\]
where \( p \) has a pole of order 1 and \( q \) a pole of order 2. These systems are said to have a \textbf{regular singular points}.

Even though the existence and uniqueness theorem from Chapter 2 no longer guarantees the existence of solutions, it turns out that a solution of the form \( y = x^m \sum_{i \geq 0} a_i x^i \) exists. Here \( m \in \mathbb{R} \) and \( \sum a_i x^i \) converges near 0). For simplicity we always assume \( a_0 \neq 0 \).

\textbf{Example 21.}

\[ 2x^2 y'' + x(2x + 1)y' - y = 0. \]

Substitute \( y = \sum_{i \geq 0} a_i x^{m+i}, \ y' = \sum_{i \geq 0} (m+i)a_i x^{m+i-1} \) and \( y'' = \sum_{i \geq 0} (m+i)(m+i-1)a_i x^{m+i-2} \). Note that \( m \) may not be an integer so we always start with \( i = 0 \). Plugging this in gives

\[
2 \sum_{i \geq 0} (m+i)(m+i-1)a_i x^{m+i} + 2 \sum_{i \geq 0} (m+i)a_i x^{m+i+1} + \\
+ \sum_{i \geq 0} (m+i)a_i x^{m+i} - \sum_{i \geq 0} a_i x^{m+i}.
\]

Collecting the coefficient in front of \( x^m \) gives

\[(2m(m-1) + m - 1)a_0 = 0.\]

Since we assume \( a_0 \neq 0 \) we get the equation \( 2m(m-1) + m - 1 = 0 \) which gives \( m = -1/2, 1 \). The coefficient in front of all the terms with \( x^{m+i} \) gives

\[ 2(m+i)(m+i-1)a_i + 2(m+i-1)a_{i-1} + (m+i)a_i - a_i = 0, \ \text{i.e.} \]

\[ [2(m+i)((m+i-1) + (m+i) - 1)] a_i = -2(m+i-1)a_{i-1}.\]
If $m = -1/2$ this gives $a_j = \frac{3 - 2j}{-3j + 2j^2}a_{j-1}$.

If $m = 1$ then this gives $a_j = \frac{-2j}{3j + 2j^2}a_{j-1}$.

So $y = Ax^{-1/2} \left(1 - x + (1/2)x^2 + \ldots\right) + Bx \left(1 - (2/5)x + \ldots\right)$.

The ratio test gives that $(1 - x + (1/2)x^2 + \ldots)$ and $(1 - (2/5)x + \ldots)$ converge for all $x \in \mathbb{R}$.

**Remark:** The equation required to have the lowest order term vanish is called the **indicial equation** which has two roots $m_1, m_2$ (possibly of double multiplicity).

**Theorem 7.**

- If $m_1 - m_2$ is not an integer then we obtain two independent solutions of the form $y_1(x) = x^{m_1} \sum_{i \geq 0} a_i x^i$ and $y_2(x) = x^{m_2} \sum_{i \geq 0} a_i x^i$.

- If $m_1 - m_2$ is an integer then one can find a 2nd solution of the form $\log(x)y_1(x)$ where $y_1(x)$ is the first solution.

Certain families of this kind of differential equation with regular singular points, appear frequently in mathematical physics.

- **Legendre equation**
  
  $$y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0$$

- **Bessel equation**
  
  $$x^2y'' + xy' + (x^2 - p^2)y = 0$$

- **Gauss’ Hypergeometric equation**
  
  $$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

For suitable choices of $a, b$ solutions of this are the sine, cosine, arctan and log functions.
3.3 Computing invariant sets by power series

One can often obtain curves through certain points as convergent power series.

**Example 22.** Let \( x' = x + y^2, y' = -y + x^2 \).

- This is a autonomous differential equation in the plane.
- Solutions are unique (r.h.s. is locally Lipschitz).
- So solutions are of the form \( t \mapsto \phi_t(x,y) \) where \( \phi_t(x,y) \) has the flow property \( \phi_{t+s}(x,y) = \phi_t \phi_s(x,y) \).
- Of course at \( \phi_t(0,0) = (0,0) \) for all \( t \), since the r.h.s. of the differential equation is zero (the speed is zero there).
- Nevertheless we can find a curve \( \gamma \) of the form \( y' = \psi(x) \) with the property that if \( (x,y) \in \gamma \) then \( \phi_t(x,y) \in \gamma \) for all \( t \) (for which \( \phi_t(x,y) \) exists).
- Later on we shall see that \( x' = x + y^2, y' = -y + x^2 \) locally behaves very much like the equation in which the higher order terms are removed: \( x' = x, y' = -y \). For this linear equation the \( x \)-axis (\( y = 0 \)) is invariant: on that line we have \( x' = -x \) and so orbits go to 0 in the linear case.
- What about the non-linear case? Since, \( x' = x + y^2, y' = -y + x^2 \) we have
  \[
  y'(x) = \frac{dy}{dt}/\frac{dx}{dt} = \frac{-y + x^2}{x + y^2}.
  \]

Let us assume \( 0 \in \gamma \) and write \( y = a_1 x + a_2 x^2 + a_3 x^3 + \ldots \). Comparing terms gives

\[
a_1 + 2a_2 x + 3a_3 x^2 + \ldots = \frac{-[a_1 x + a_2 x^2 + a_3 x^3 + \ldots] + x^2}{x + [a_1 x + a_2 x^2 + a_3 x^3 + \ldots]^2}.
\]
Comparing terms of the same power, shows that $a_1 = 0$, $2a_2 = (1 - a_2 - a_1^2)$ and so on. This gives a curve which is tangent to the $x$-axis so that orbits remain in this curve.
### 4 Boundary Value Problems

5. Boundary Value Problems Instead of initial conditions, in this chapter we will consider boundary values. Examples:

- $y'' + y = 0, y(0) = 0, y(\pi) = 0$. This has infinitely many solutions: $y(x) = c \sin(x)$.
- $y'' + y = 0, y(0) = 0, y(\pi) = \epsilon \neq 0$ has no solutions: $y(x) = a \cos(x) + b \sin(x)$ and $y(0) = 0$ implies $a = 0$ and $y(\pi) = 0$ has no solutions.
- Clearly boundary problems are more subtle.
- We will concentrate on equations of the form $u'' + \lambda u = 0$ with boundary conditions, where $\lambda$ is a free parameter.
- This class of problems is relevant for a large class of physical problems: heat, wave and Schroedinger equations.
- This generalizes Fourier expansions.

#### 4.1 Motivation: wave equation

The foundational example: consider the **wave equation**:

\[
\frac{\partial^2}{\partial t^2} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t)
\]

where $x \in [0, \pi]$ and the end points are fixed:

- $u(0, t) = 0, u(\pi, t) = 0$ for all $t$,
- $u(x, 0) = f(x), \frac{\partial}{\partial t} u(x, t)|_{t=0} = 0$. 

---

34
• This is a model for a string of length $\pi$ on a musical instrument such as a guitar; before the string is released the shape of the string is $f(x)$.

• As usual one solves this by writing $u(x, t) = w(x) \cdot v(t)$ and then obtaining $w''(x)/w(x) = v''(t)/v(t)$. Since the left hand does not depend on $t$ and the right hand side not on $x$ this expression is equal to some constant $\lambda$ and we get

$$w'' = \lambda w \text{ and } v'' = \lambda v.$$ 

• We need to set $w(0) = w(\pi) = 0$ to satisfy the boundary conditions that $u(0, t) = u(0, t) = 0$ for all $t$.

Write $\lambda = -\mu^2$ where $\mu$ is not necessarily real.

• When $\lambda \neq 0$, $v'' = \lambda v$ has solution

$$v(t) = c_1 \cos(\mu t) + c_2 \sin(\mu t).$$

• Consider $w'' - \lambda w = 0$ and $w(0) = w(\pi) = 0$.

- $\lambda = 0$ implies $w(x) = c_3 + c_4 x$ and because of the boundary condition $c_3 = c_4 = 0$. So can assume $\lambda \neq 0$.
- If $\lambda \neq 0$, solution is $w(x) = c_3 \cos(\mu x) + c_4 \sin(\mu x)$. $w(0) = 0 \implies c_3 = 0$ therefore $w(\pi) = c_4 \sin(\mu \pi) = 0$ implies $\mu = n \in \mathbb{N}$ [check: $\mu$ is non-real $\implies \sin(\mu \pi) \neq 0$]. So

$$w(x) = c_4 \sin(n x) \text{ and } \lambda = -n^2 \text{ and } n \in \mathbb{N} \setminus \{0\}.$$
So for any \( n \in \mathbb{N} \) we obtain solution

\[
    u(x, t) = w(x)v(t) = (c_1 \cos(nt) + c_2 \sin(nt)) \sin(nx).
\]

The string can only vibrate with frequencies which are a multiple of \( \mathbb{N} \).

So \( u(x, t) = \sum_{n \geq 1} (c_{1,n} \cos(nt) + c_{2,n} \sin(nt)) \sin(nx) \) is solution provided the sum makes sense and is twice differentiable.

**Lemma 6.** \( \sum n^2 |c_{1,n}| < \infty \) and \( \sum n^2 |c_{2,n}| < \infty \) \( \Rightarrow \) \( u(x, t) = \sum_{n \geq 1} (c_{1,n} \cos(nt) + c_{2,n} \sin(nt)) \sin(nx) \) is \( C^2 \).

**Proof.**

- That \( \sum_{n=1}^{N} (c_{1,n} \cos(nt) + c_{2,n} \sin(nt)) \sin(nx) \) converges follows from

  **Weierstrass test:** if \( M_n \geq 0, \sum M_n < \infty \) and \( u_n : [a, b] \to \mathbb{R} \) is continuous with \( |u_n| \leq M_n \) then \( \sum u_n \) converges uniformly on \( [a, b] \) (and so the limit is continuous too!).

- Since the \( d/dx \) derivative of

  \[
  \sum_{n=1}^{N} (c_{1,n} \cos(nt) + c_{2,n} \sin(nt)) \sin(nx)
  \]

  is equal to \( \sum_{n=1}^{N} (c_{1,n} \cos(nt) + c_{2,n} \sin(nt)) n \cos(nx) \), and the latter converges, hence \( u(x, t) \) is differentiable w.r.t. \( x \).

- and so on...

We need to find \( c_{1,n}, c_{2,n} \) so that

- \( \frac{\partial}{\partial t} u(x, t)|_{t=0} = 0 \) for all \( x \in [0, \pi] \) \( \Rightarrow \) \( \sum c_{2,n} n \sin(nx) \equiv 0 \) \( \Rightarrow \) \( c_{2,n} = 0 \) for all \( n \geq 0 \).
• \( u(x, 0) = \sum c_{1,n} \sin(nx) = f(x) \) for all \( x \in [0, \pi] \). The next Theorem tells us that we can take \( c_{1,n} \) so that this is true, provided \( f \) is \( C^2 \):

**Theorem 8. Fourier Theorem.** Assume \( f : [0, \pi] \to \mathbb{R} \) is \( C^2 \) (twice continuously differentiable) and \( f(0) = f(\pi) = 0 \) then one can find \( c_{1,n} \) so that

\[
\sum_{n=1}^{N} c_{1,n} \sin(nx) \text{ converges uniformly to } f(x) \text{ as } N \to \infty.
\]

• \( u(x, t) = \sum_{n=1}^{\infty} c_{1,n} \cos(nt) \sin(nx) \) is \( C^2 \) and solves wave equation.

• \( c_{1,n} \) can be found by methods you have seen before.

• Since \( w'' = \lambda w \), one calls \( \lambda \) an *eigenvalue* and \( w \) an *eigenfunction*.

### 4.2 A typical Sturm-Liouville Problems

Let us consider another example:

\[
y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(1) = 0.
\]

• If \( \lambda = 0 \) then \( y(x) = c_1 + c_2 x \) and the boundary conditions give \( y(x) = 1 - x \).

• If \( \lambda \neq 0 \) we write again \( \lambda = \mu^2 \). The equation \( y'' + \lambda y = 0 \) gives \( y(x) = c_1 e^{i\mu x} + c_2 e^{-i\mu x} \).

• Plugging in \( y(0) + y'(0) = 0, y(1) = 0 \) gives
  
  \[
  (c_1 + c_2) + i\mu (c_1 - c_2) = 0 \text{ and } c_1 e^{i\mu} + c_2 e^{-i\mu} = 0.
  \]

• So \( c_2 = -c_1 e^{2i\mu} \) and \( (1 + i\mu)e^{-i\mu} - (1 - i\mu)e^{i\mu} = 0 \).
• Hence \( \tan \mu = \mu \). This has infinitely many solutions \( \mu_n \in [0, \infty), \ n = 0, 1, \ldots \) with \( \mu_n \to \infty \) and \( \mu_n \approx (2n + 1)\pi/2 \).

• Eigenvalues: \( \lambda_n = \mu_n^2 \approx (2n + 1)^2(\pi/2)^2, \ n = 0, \ldots \); eigenvectors: \( y_0(x) = 1 - x \) and \( y_n(x) = \sin(\sqrt{\lambda_n}(1 - x)), \ n \geq 1 \).

These are special cases of following type of problem: functions \( p, q, r : [a, b] \to \mathbb{R} \) find \( y : [a, b] \to \mathbb{R} \) so that

\[
(p(x)y')' + q(x)y + \lambda r(x)y = 0. \quad (7)
\]

**Theorem 9. Sturm-Louiville Theorem** Assume that \( p, r > 0 \) are continuous and \( p \) is \( C^1 \) on \([a, b] \). Then \( (7) \) with the boundary conditions \( (8) \) from the next page has infinitely many solutions:

• The eigenvalues \( \lambda_n \) are real, distinct and of single multiplicity;

• The eigenvalues \( \lambda_n \) tend to infinity, so \( \lambda_1 < \lambda_2 < \ldots \) and \( \lambda_n \to \infty \).

• If \( n \neq m \) then corresponding eigenfunctions \( y_n, y_m \) are orthogonal in the sense that

\[
\int_a^b y_m(x)y_n(x)r(x) \, dx = 0
\]

• Each continuous function can be expanded in terms of the eigenfunctions, as in the Fourier case!!

• Suitable **boundary conditions** are:

\[
\alpha_0 y(a) + \alpha_1 y'(a) = 0, \beta_0 y(b) + \beta_1 y'(b) = 0. \quad (8)
\]
• Note that if $y_n, y_m$ are solutions and we set

$$W(y_m, y_n)(x) := \det\begin{pmatrix} y_m(x) & y'_m(x) \\ y_n(x) & y'_n(x) \end{pmatrix} = y_m(x)y'_n(x) - y_n(x)y'_m(x)$$

then $W(a) = 0$ and $W(b) = 0$.

• How to find $a_n$ so that $f(x) = \sum_{n \geq 0} a_n y_n(x)$? Just take

$$(f, ry_k) = \left( \sum_{n \geq 0} a_n y_n(x), ry_k \right) = \sum_{n \geq 0} a_n (y_n, ry_k) = a_k (y_k, ry_k).$$

Here we used in the last equality that $(y_n, ry_k) \neq 0$ implies $n = k$. Hence

$$a_k := \frac{(f, ry_k)}{(y_k, ry_k)}$$

where $(v, w)$ is the inner product: $(v, w) = \int_a^b v(t)\overline{w}(t) \, dt$.

4.3 A glimpse into symmetric operators

• Sturm-Liouville problems are solved using some operator theory: the 2nd order differential equation is equivalent to

$$Ly(x) = \lambda r(x)y(x) \quad \text{where} \quad L = \left( -\frac{d}{dx} p(x) \frac{d}{dx} - q(x) \right).$$

This turns out to be a symmetric operator in the sense that $(Lv, w) = (v, Lw)$ where $(v, w) = \int_a^b v(x)\overline{w}(x) \, dx$ is as defined above.

• The situation for analogous to the finite dimensional case:

• $L$ is a symmetric (and satisfies some additional properties) $\implies$ its eigenvalues are real, and its eigenvectors form a basis.
**L is symmetric (self-adjoint).** Let \( Lu = -(pu')' - qu \) and \( Lv = -(pv')' - qv \).

\[
\int_{a}^{b} L(u)\overline{v} \, dx = \int_{a}^{b} [-(pu')' - qu] \, dx. \quad \int_{a}^{b} uL(v) \, dx = \int_{a}^{b} [-u(pv')' - qv] \, dx.
\]

\[
\int_{a}^{b} - (pu')' \overline{v} \, dx = -pu'|_{a}^{b} + \int_{a}^{b} pu' \overline{v} \, dx
\]

\[
= -pu'|_{a}^{b} + pu'\overline{v}|_{a}^{b} - \int u(p\overline{v})' \, dx
\]

So

\[
\int_{a}^{b} [L(u)\overline{v} - u\overline{L(v)}] \, dx = -p(x)[u' \overline{v} - u\overline{v}']|_{a}^{b}
\]

\[
= p(b)W(u, v)(b) - p(a)W(u, v)(a)
\]

So if \( u, v \) satisfy the boundary conditions, \( L \) is self-adjoint:

\[
\int_{a}^{b} [L(u)\overline{v} - u\overline{L(v)}] \, dx = 0, \ i.e. \ (Lu, v) = (u, Lv).
\]

**Proof that eigenvalues are real and orthogonality of eigenfunctions:** Define \((u, v) = \int_{a}^{b} u(x)v(x) \, dx\). Then the previous slide showed \((Lu, v) = (u, Lv)\).

- Suppose that \( Ly = r\lambda y \). Then the eigenvalue \( \lambda \) is real:

  Indeed,
  \[
  \lambda(ry, y) = (\lambda ry, y) = (Ly, y) = (y, Ly) = \overline{\lambda}(y, ry) = \overline{\lambda}(ry, y)
  \]
  since \( r \) is real. Since \((ry, y) > 0\) it follows that \( \lambda = \overline{\lambda} \).

- Suppose that \( Ly = r\lambda y \) and \( Lz = r\mu z \).

  \( \lambda \neq \mu \implies \int_{a}^{b} r(x)yz(x) \, dx = (ry, z) = 0. \)

  So the eigenvectors \( y, z \) are orthogonal. Indeed,
  \[
  \lambda(ry, z) = (\lambda ry, z) = (Ly, z) = (y, Lz) = (y, \mu rz) = \overline{\mu}(y, rz) = \mu(ry, z) = \mu(ry, z).
  \]
where we have used that $r$ and $\mu$ are real. Since $\lambda \neq \mu$ it follows that $(ry, z) = 0$.

The slide is not examinable  A full discussion of the proof of the above theorem would require:

- to define Hilbert space $H$: this is a Banach space with an inner product for which $(v, w) = \overline{(w, v)}$ where $\overline{z}$ is complex conjugation;

- to define the norm $||v|| = \sqrt{(v, \overline{v})}$ (generalizing $||z|| = \sqrt{z\overline{z}}$ on $\mathbb{C}$); Note $||v|| = \sqrt{\int_a^b |v(x)|^2 \, dx}$, the so-called $L^2$ norm.

- to associate to a linear $A: H \to H$ the operator norm $||A|| = \sup_{f \in H, ||f|| = 1} ||Af||$;

- to call a linear operator $A$ is compact if for each sequence $||f_n|| \leq 1$, there exists a convergent subsequence of $Af_n$.

- to show that if $A: H \to H$ is compact, then there exists a sequence of eigenvalues $\alpha_n \to 0$ and eigenfunctions $u_n$. These eigenvalues are all real and the eigenfunctions are orthogonal. If the closure of $A(H)$ is equal to $H$, then for each $f \in H$ then one can write $f = \sum_{j=0}^{\infty} (u_j, f)u_j$.

The slide is not examinable

- The operator $L$ in Sturm-Liouville problems is not compact, and that is why one considers some related operator (the resolvent).

- This related operator is compact.

- The above theorem then follows.
The Sturm-Liouville Theorem is fundamental in
- quantum mechanics;
- in large range of boundary value problems;
- and related to geometric problems describing properties of geodesics.

### 4.4 Oscillatory equations

Consider \((py')' + ry = 0\) where \(p > 0\) and \(C^1\) as before.

**Theorem 10.** Let \(y_1, y_2\) be solutions. Then the Wronskian \(x \mapsto W(y_1, y_2)(x) := y_1(x)y_2'(x) - y_2(x)y_1'(x)\) has constant sign.

**Proof.** \((py_1')' + ry_1 = 0\) and \((py_2')' + ry_2 = 0\). Multiplying the first equation by \(y_2\) and the second one by \(y_1\) and subtract:

\[
0 = y_2(py_1')' - y_1(py_2')' = y_2p'y_1 + y_2py''_1 - y_1p'y_2 - y_1py''_2.
\]

Differentiating \(W\) and substituting the last equation in

\[
pW' = p[y_1'y_2' + y_1y_2'' - y_2'y_1' - y_2y_1''] = py_1y_2'' - py_2y_1''
\]
gives

\[
pW' = -p'W.
\]

This implies that if \(W(x) = 0\) for some \(x \in [a, b]\) then \(W(x) = 0\) for all \(x \in [a, b]\). \(\square\)

**Lemma 7.** \(W(y_1, y_2) \equiv 0 \implies \exists c \in \mathbb{R} \) with \(y_1 = cy_2\) (or \(y_2 = 0\)).

**Proof.** Since \(W(y_1, y_2) = 0, y_2 \neq 0\) implies \((y_1, y_1')\) is a multiple of \((y_2, y_2')\). Can this multiple depend on \(x\)? No: if \(y_1(x) = c(x)y_2(x)\) and \(y_1'(x) = c(x)y_2'(x)\) \(\forall x\)

\[
\implies c(x)y_2(x) = y_1'(x) = c'(x)y_2(x) + c(x)y_2'(x) \forall x.
\]

Hence \(c' \equiv 0\). \(\square\)
**Theorem 11. Sturm Separation Theorem** Let $y_1, y_2$ be two solutions which are independent (one is not a constant multiple of the other). Then zeros are interlaced: between consecutive zeros of $y_1$ there is a zero of $y_2$ and vice versa.

**Proof.** Assume $y_1(a) = y_1(b) = 0$. $y_1'(a) \neq 0$ (otherwise $y_1 \equiv 0$) and $y_2'(b) \neq 0$. We may choose $a, b$ so that $y_1(x) > 0$ for $x \in (a, b)$. Then $y_1'(a)y_1'(b) < 0$. (Draw a picture.) Also,

$$W(y_1, y_2)(a) = -y_2(a)y_1'(a) \quad \text{and} \quad W(y_1, y_2)(b) = -y_2(b)y_1'(b).$$

Since $y_1'(a)y_1'(b) < 0$ and $W(a)W(b) > 0$ ($W$ does not change sign), we get $y_2(a)y_2(b) < 0$, which implies that $y_2$ has a zero between $a$ and $b$. \qed


5 Calculus of Variations

Many problems result in differential equations. In this chapter we will consider the situation where these arise from a minimisation (variational) problem. Specifically, the problems we will consider are of the type

- Minimize

\[ I[y] = \int_0^1 f(x, y(x), y'(x)) \, dx \]  

(9)

where \( f \) is some function and \( y \) is an unknown function.

- Minimize \((9)\) where \( f \) is some function and \( y \) is an unknown function conditional to some restriction of the type \( J[y] = \int_0^1 f(x, y(x), y'(x)) \, dx = 1. \)

5.1 Examples

Example 23. Let \( A = (0, 0) \) and \( B = (l, -b) \) with \( l, b > 0 \) and consider a path of the form \((x, y(x))\), \( x \in [0, l] \), connecting \( A \) and \( B \). Take a ball starting at \( A \) and rolling along this path under the influence of gravity to \( B \). Let \( T \) be the time this ball will take. Which function \( x \mapsto y(x) \) which will minimise \( T \)?

The sum of kinetic and potential energy is constant

\[ \frac{1}{2}mv^2 + mgh = \text{const.} \]

Since the ball rolls along \((x, y(x))\) we have \( v(x) = \sqrt{-2gy(x)} \).

Let \( s(t) \) be the length travelled at time \( t \). Then \( v = ds/dt \).

Hence \( dt = ds/v \) or

\[ T[y] := \int_0^l \frac{\sqrt{1 + y'(x)^2}}{\sqrt{-2gy(x)}} \, dx. \]
**Task:** minimise $T[y]$ within the space of functions $x \mapsto y(x)$ for which $y$ and $y'$ continuous and $y(0) = 0$ and $y(l) = -b$. This is called the **Brachisotochrome**, go back to Bernouilli in 1696.

**Example 24.** Let $=(0, 0)$ and $B = (1, 0)$ with $l, b > 0$ and consider a path of the form $[0, 1] \ni t \mapsto c(t) = (c_1(t), c_2(t))$, connecting $A$ and $B$. What is the shortest path?

Of course this is a line segment, but how to make this precise?

If we are not in a plane, but in a surface or a higher dimensional manifold, these shortest curves are called **geodesics**, and these are studied extensively in mathematics.

**Task:** Choose $[0, 1] \ni t \mapsto c(t) = (c_1(t), c_2(t))$ with $c(0) = (0, 0)$ and $c(1) = (1, 0)$ which minimises

$$L[c] = \int_0^1 \sqrt{c_1'(t)^2 + c_2'(t)^2} dt.$$  

**Example 25.** Take a closed curve in the plane without self-intersections and of length one. What is the curve $c$ which maximises the area $D$ it encloses? Again, let $[0, 1] \ni t \mapsto c(t) = (c_1(t), c_2(t))$ with $c(0) = c(1)$ and so that $s, t \in [0, 1)$ and $s \neq t$ implies $c(s) \neq c(t)$.

The length of the curve is again $L[c] = \int_0^1 \sqrt{c_1'(t)^2 + c_2'(t)^2} dt$. To compute the area of $D$ we use the Green theorem:

$$\int \int_D Pdx + Qdy = \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

Take $P \equiv 0$ and $Q = x$. Then

$$\int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int \int_D 1 dxdy = \text{area of } D.$$ 

So

$$A[c] = \int \int_D 1 dxdy = \int_c xdy = \int_0^1 c_1(t)c_2'(t) dt.$$
This is the isoperimetric problem: find the supremum of $A[c]$ given $L[c] = 1$.

5.2 Extrema in the finite dimensional case

The one-dimensional case

We say that $f: \mathbb{R} \to \mathbb{R}$ take a local minimum at $\tilde{x} \in \mathbb{R}$ if there exists $\delta > 0$ so that

$$f(x) \geq f(\tilde{x}) \text{ for all } x \text{ with } |x - \tilde{x}| < \delta.$$ 

Theorem 12. Assume that $f$ is differentiable at $a$ and also has a minimum at $\tilde{x}$ then $Df(\tilde{x}) = 0$.

Proof. That $f$ has a minimum means that $f(\tilde{x} + h) - f(\tilde{x}) \geq 0$ for all $h$ near zero. Hence

$$\frac{f(\tilde{x} + h) - f(\tilde{x})}{h} \geq 0 \text{ for } h > 0 \text{ near zero and}$$

$$\frac{f(\tilde{x} + h) - f(\tilde{x})}{h} \leq 0 \text{ for } h < 0 \text{ near zero.}$$

Therefore

$$f'(\tilde{x}) = \lim_{h \to 0} \frac{f(\tilde{x} + h) - f(\tilde{x})}{h} = 0$$

\[ \square \]

The finite dimensional case

We say that $f: \mathbb{R}^n \to \mathbb{R}$ take a local minimum at $\tilde{x}$ if there exists $\delta > 0$ so that

$$f(x) \geq f(\tilde{x}) \text{ for all } x \text{ with } |x - \tilde{x}| < \delta.$$ 

Theorem 13. Assume that $f$ is differentiable at $\tilde{x}$ and also has a minimum at $\tilde{x}$ then $Df(\tilde{x}) = 0$.
Proof. Take a vector $v$ at $\tilde{x}$, define $l(t) = \tilde{x} + t v$ and $g(t) := f \circ l(t)$. By the previous theorem $g'(0) = 0$. By the chain rule this is equal to

$$\frac{\partial f}{\partial x_1}(\tilde{x}) v_1 + \cdots + \frac{\partial f}{\partial x_n}(\tilde{x}) v_n = 0.$$ 

Hence $Df(\tilde{x})v = 0$ where $Df(\tilde{x})$ is the Jacobian matrix at $\tilde{x}$. Since this holds for all $v$, we get $Df(\tilde{x}) = 0$. \hfill \square

This is why $Df(\tilde{x})v$ is the directional derivative of $f$ at $\tilde{x}$ in the direction $v$.

5.3 The Euler-Lagrange equation

The infinite dimensional case: the Euler-Lagrange equation

- In the infinite dimensional case, we need to generalise this.

- Let $C^1[a, b]$ be the space of $C^1$ functions $[a, b] \to \mathbb{R}^n$. This space is an infinite dimensional vector space with norm $|f|_{C^1} = \sup_{x \in [a, b]} (|f(x)|, |Df(x)|)$.

- Choose some function $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Take $(x, y, y') \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^n$ denote by $f_y, f_{y'}$ the corresponding partial derivatives. So $f_y(x, y, y')$ and $f_{y'}(x, y, y')$ vectors. Attention: here $y'$ is just the name of a vector in $\mathbb{R}^n$ (and not - yet - a function).

- Here $f_y$ is the part of the $1 \times (1 + n + n)$ vector $Df$ which concerns the $y$ derivatives.
Let \( f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) with \( f_y, f_{y'} \) continuous. Define \( I : C^1[a, b] \rightarrow \mathbb{R} \) by, for example,

\[
I[y] = \int_a^b f(x, y(x), y'(x)) \, dx.
\]

Given \( \tilde{y} : [a, b] \rightarrow \mathbb{R}^n \), let’s denote

\[
f_y[\tilde{y}](x) = f_y(x, \tilde{y}(x), \tilde{y}'(x)) \quad \text{and} \quad f_{y'}[\tilde{y}](x) = f_{y'}(x, \tilde{y}(x), \tilde{y}'(x))
\]

where \( f_y, f_{y'} \) are the corresponding partial derivatives. Fix \( y_a, y_b \in \mathbb{R}^n \) and define

\[\mathcal{A} = \{ y : [a, b] \rightarrow \mathbb{R}^n \text{ is } C^1 \text{ and } y(a) = y_a, y(b) = y_b \}.\]

**Theorem 14.** If \( \mathcal{A} \ni y \mapsto I[y] \) has a minimum at \( \tilde{y} \) then

1. for every \( v \in C^1[a, b] \) with \( v(a) = v(b) = 0 \) we get
\[
\int_a^b (f_y[\tilde{y}] \cdot v + f_{y'}[\tilde{y}] v') \, dx = 0.
\]

2. \( f_{y'}[\tilde{y}] \) exists, is continuous on \([a, b]\) and
\[
\frac{d}{dx} f_{y'}[\tilde{y}] = f_y[\tilde{y}].
\]
The proof of the Euler-Lagrange equations

- Remember \( A = \{ y; \ y\colon [a, b] \to \mathbb{R}^n \text{ is } C^1 \text{ and } y(a) = y_a, y(b) = y_b \} \). Hence \( v \in C^1[a, b] \) with \( v(a) = v(b) = 0 \), then \( y + hv \in A \) for each \( h \).

- If \( I: C^1[a, b] \to \mathbb{R} \) has a minimum at \( \tilde{y} \) then
  \[
  I[\tilde{y} + hv] \geq I[\tilde{y}] \quad \forall v \in C^1[a, b] \forall h \in \mathbb{R}.
  \]

- So if \( \lim_{h \to 0} \frac{I[\tilde{y} + hv] - I[\tilde{y}]}{h} \) exists then it is equal to zero.
  \[
  \frac{I[\tilde{y} + hv] - I[\tilde{y}]}{h} = \int_a^b \frac{f(x, (\tilde{y} + hv)(x), (\tilde{y} + hv)'(x)) - f(x, \tilde{y}(x), \tilde{y}'(x))}{h} \, dx
  \]

- By the chain rule,
  \[
  \lim_{h \to 0} \frac{f(x, (\tilde{y} + hv)(x), (\tilde{y} + hv)'(x)) - f(x, \tilde{y}(x), \tilde{y}'(x))}{h} = f_y[\tilde{y}]v + f_y'[\tilde{y}]v'.
  \]

- More precisely,
  \[
  \frac{f(x, (\tilde{y} + hv)(x), (\tilde{y} + hv)'(x)) - f(x, \tilde{y}(x), \tilde{y}'(x))}{h} = f_y[\tilde{y}]v + f_y'[\tilde{y}]v' + O(h).
  \]

- So a necessary condition for \( \tilde{y} \) to be a minimum of \( I \) is
  \[
  \int_a^b [f_y[\tilde{y}]v + f_y'[\tilde{y}]v'] \, dx = 0
  \]
  for each \( v \in C^1[a, b] \).

- Partial integration gives
  \[
  \int_a^b f_y'[\tilde{y}]v' \, dx = (f_y'[\tilde{y}]v)|_a^b - \int_a^b \frac{d}{dx} f_y'[\tilde{y}]v \, dx.
  \]
• Remember \( v(a) = v(b) = 0 \), so \( (f_{y'}[\tilde{y}]v)^b_a = 0 \).

• So a necessary condition for \( \tilde{y} \) to be a minimum of \( I \) is:

\[
\forall v \in C^1[a, b] \text{ with } v(a) = v(b) = 0:
\int_a^b \left[f_{y'}[\tilde{y}] - \frac{d}{dx}f_{y'}[\tilde{y}] \right] v \, dx = 0.
\]

• This prove first assertion of Theorem and also the 2nd assertion because of the following lemma:

**Lemma 8.** If \( G : [a, b] \to \mathbb{R} \) is continuous and \( \int_a^b Gv \, dx = 0 \) for each \( v \in C^1[a, b] \) with \( v(a) = v(b) = 0 \), then \( G \equiv 0 \).

**Proof.** If \( G(x_0) > 0 \) then \( \exists \delta > 0 \) so that \( G(x) > 0, \forall x \) with \( |x - x_0| < \delta \). Choose \( v \in C^1[a, b] \) with \( v(a) = v(b) = 0 \), so that \( v > 0 \) on \( x \in (x_0 - \delta, x_0 + \delta) \cap (a, b) \) and zero outside. Then \( \int_a^b G(x)v(x) \, dx > 0 \). 

Quite often \( x \) does not appear in \( f \). Then it is usually more convenient to rewrite the Euler-Lagrange equation:

**Lemma 9.** If \( x \) does not appear explicitly in \( f \), then \( \frac{d}{dx}f_{y'}[\tilde{y}] = f_{y'}[\tilde{y}] \) implies \( f_{y'}[\tilde{y}]\tilde{y}' - f = C \).

**Proof.**

\[
\frac{d}{dx}(f_{y'}[\tilde{y}]\tilde{y}' - f[\tilde{y}]) = (\frac{d}{dx}f_{y'}[\tilde{y}])\tilde{y}' + f_{y'}\tilde{y}''
\]

\[
= (f_x[\tilde{y}] + f_y[\tilde{y}]\tilde{y}' + f_{y'}[\tilde{y}]\tilde{y}'')
\]

\[
= y' \left\{ \frac{d}{dx}f_{y'}[\tilde{y}] - f_y[\tilde{y}] \right\} - f_x[\tilde{y}].
\]

Since \( f_x = 0 \), and by the E-L equation, the term \{·\} = 0 this gives the required result. 

\[\square\]
Example 26. Shortest curve connecting two points $(0, 0)$ and $(1, 0)$. Let us consider curves of the form $x \mapsto (x, y(x))$ and minimise the length: $I[c] = \int_a^b \sqrt{1 + y'(x)^2} \, dx$. The Euler-Lagrange equation is 

$$\frac{d}{dx} f_y[y] = f_y[y] = 0.$$ 

Note $f_y' = \frac{y'}{\sqrt{1 + (y')^2}}$, so 

$$\frac{d}{dx} \frac{y'}{\sqrt{1 + (y')^2}} = 0.$$ 

So 

$$\frac{y'}{\sqrt{1 + (y')^2}} = C.$$ 

This means that $y' = C_1$. Hence $y(x) = C_1 x + C_2$. With the boundary conditions this gives $y(x) = 0$. 

5.4 The brachistochrone problem

Example 27. (See Example 23) The curve \( x \to (x, y(x)) \) connecting \((0, 0)\) to \((l, -b)\) with the shortest travel time \( \text{brachistochrone} \). Then \( f(x, y, y') = \frac{\sqrt{1 + (y')^2}}{\sqrt{-2gy}} \). Since \( y < 0 \), we orient the vertical axis downwards, that is we write \( z = -y \) and \( z' = -y' \), i.e. take \( f(x, z, z') = \frac{\sqrt{1 + (z')^2}}{\sqrt{2gz}} \). Note that

\[
f_{z'} = (1/2) \frac{1}{\sqrt{1 + (z')^2}} \frac{2z'}{\sqrt{2gz}}.
\]

The previous lemma gives \( f_{z'} z' - f = \text{const} \), i.e.

\[
\frac{(z')^2}{\sqrt{1 + (z')^2} \sqrt{z}} - \frac{\sqrt{1 + (z')^2}}{\sqrt{z}} = \text{const}.
\]

Rewriting this gives

\[
z[1 + (z')^2] = \text{const}.
\]

Rewriting this again gives the differential equation

\[
\frac{dz}{dx} = \sqrt{\frac{C - z}{z}} \quad \text{or} \quad \frac{dx}{dz} = \sqrt{\frac{z}{C - z}}
\]

with \( C > 0 \). As usual we solve this by writing \( dx = \sqrt{\frac{z}{C - z}} dz \) and so

\[
x = \int \sqrt{\frac{z}{C - z}} \, dz.
\]

Substituting \( z = C \sin^2(s) \), where \( s \in [0, \pi] \), gives

\[
x = \int \sqrt{\frac{\sin^2(s)}{1 - \sin^2(s)}} (2C) \sin(s) \cos(s) \, ds =
\]
\[ x = \int \sqrt{\frac{\sin^2(s)}{1 - \sin^2(s)}} (2C) \sin(s) \cos(s) \, ds = \]

\[ 2C \int \sin^2(s) \, dt = C \int (1 - \cos(2s)) \, dt = (C/2)(2s - \sin(2s)) + A \]

Since the curve starts at \((0, 0)\) we have \(A = 0\).

So we get

\[
\begin{align*}
x(s) &= \frac{C}{2}(2s - \sin(2s)), \\
z(s) &= C \sin^2(s) = \frac{C}{2}(1 - \cos(2s)).
\end{align*}
\] (10)

Here we choose \(C\) so that \(z = b\) when \(x = L\). This is called a cycloid, an evolute of the circle. This is the path of a fixed point on a bicycle wheel, as the bicycle is moving forward.

Substituting \(2s\) to \(\phi\) and taking \(a = C/2\) we get

\[
\begin{align*}
x(\phi) &= a(\phi - \sin(\phi)), \\
z(\phi) &= a(1 - \cos(\phi)).
\end{align*}
\] (11)

What is \(a\)? Given \(L = x_0\) and \(b = y_0\) we need to choose \(a, \phi\) so that \(x(\phi) = L\) and \(z(\phi) = b\). This amounts two equations and two unknowns.

Two special cases:

- The right endpoint is \((L, 0)\): Then take \(L = 2\pi a\) and \(\phi = 2\pi\). In this case the ball goes all the way back up.

- The right endpoint is \((L, 2a)\). Then this is solved by \(L = \pi\) and \(\phi = \pi a\). 

![Diagram of a cycloid and evolute with labeled points (0,0) and (L,2a)](attachment)
A remarkable property of the brachistochrone: Take an initial point \((\hat{x}, \hat{y})\) on this curve, and release it from rest. Then the time to hit the lower point of the curve is independent of the choice of the initial point!!!

**Theorem 15.** For any initial point \((\hat{x}, \hat{y})\) (i.e. for any initial \(\hat{\phi}\))

\[
T = \int_{\hat{x}}^{L} \sqrt{\frac{1 + (z')^2}{2g(z - z_0)}} \, dx = \sqrt{\frac{a}{g}} \int_{\hat{\phi}}^{\pi} \sqrt{\frac{1 - \cos(\phi)}{\cos(\hat{\phi}) - \cos(\phi)}} \, d\phi
\]

is equal to = \(\pi \sqrt{a/g}\). Wow!

**Proof.** Not examinable. Let us first show the integrals are equal:

\[
x(\phi) = a(\phi - \sin(\phi)), z(\phi) = a(1 - \cos(\phi)) \implies
\]

\[
z' = \frac{dz}{dx} = \frac{dz}{dt} \frac{dt}{dx} = \frac{a \sin(\phi)}{a(1 - \cos(\phi))} \implies
\]

\[
\sqrt{1 + (z')^2} = \sqrt{\frac{(1 - \cos(\phi))^2 + \sin^2(\phi)}{(1 - \cos(\phi))^2}} = \sqrt{\frac{2(1 - \cos(\phi))}{(1 - \cos(\phi))^2}}.
\]

\[
\]

\[
x(\phi) = a(\phi - \sin(\phi)), z(\phi) = a(1 - \cos(\phi)) \implies
\]

\[
z' = \frac{dz}{dx} = \frac{dz}{dt} \frac{dt}{dx} = \frac{a \sin(\phi)}{a(1 - \cos(\phi))} \implies
\]

\[
\sqrt{1 + (z')^2} = \sqrt{\frac{(1 - \cos(\phi))^2 + \sin^2(\phi)}{(1 - \cos(\phi))^2}} = \sqrt{\frac{2(1 - \cos(\phi))}{(1 - \cos(\phi))^2}}.
\]
Since \( dx = a(1 - \cos(\phi)) \, d\phi \) this gives

\[
\sqrt{\frac{1 + (z')^2}{2g(z - z_0)}} \, dx = \sqrt{a} \sqrt{\frac{1 - \cos(\phi)}{\cos(\hat{\phi}) - \cos(\phi)}} \, d\phi.
\]

Showing the two integrals the same.

Claim: the following integral does not depend on \( \hat{\phi} \):

\[
\int_{\phi=\hat{\phi}}^{\pi} \sqrt{\frac{1 - \cos(\phi)}{\cos(\hat{\phi}) - \cos(\phi)}} \, d\phi
\]

Substitute \( \sin(\phi/2) = \sqrt{1 - \cos(\phi)/2} \) and \( \cos(\phi) = 2\cos^2(\phi/2) - 1 \) gives:

\[
\sqrt{\frac{1 - \cos(\phi)}{\cos(\hat{\phi}) - \cos(\phi)}} = \sqrt{2} \frac{\sin(\phi/2)}{\sqrt{2[\cos^2(\hat{\phi}/2) - \cos^2(\phi/2)]}}
\]

Substitute \( u = \cos(\phi/2)/\cos(\hat{\phi}/2) \), then as \( \phi \) varies between \([\hat{\phi}, \pi]\) then \( u \) varies from 1 to 0.

\[
\int_{\hat{\phi}}^{\pi} \frac{\sin(\phi/2)}{\sqrt{\cos^2(\hat{\phi}/2) - \cos^2(\phi/2)}} \, d\phi
\]

Substitute \( u = \cos(\phi/2)/\cos(\hat{\phi}/2) \) gives

\[
\frac{\sin(\phi/2)}{\sqrt{\cos^2(\hat{\phi}/2) - \cos^2(\phi/2)}} = \frac{\sin(\phi/2)}{\cos(\hat{\phi}/2)\sqrt{1 - u^2}}.
\]

Since \( du = -(1/2) \frac{\sin(\phi/2)}{\cos(\hat{\phi}/2)} \, d\phi \) and since \( u \) varies from 1 to 0 the integral is equal to

\[
\int_{0}^{1} \frac{2}{\sqrt{1 - u^2}} \, du = 2 \arcsin(u) \bigg|_{0}^{1} = \pi
\]

So the time to decent from any point is \( \pi \sqrt{a/g} \).

For history and some movies about this problem:
5.5 Are the critical points of the functional $I$ minima?

Are the critical points of $I$ minima?

- In general we cannot guarantee that the solutions of the Euler-Lagrange equation gives a minimum.

- This is of course is not surprising: a minimum $\tilde{x}$ of $f: \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies $Dg(\tilde{x}) = 0$, but the latter condition is not enough to guarantee that $\tilde{x}$ is a minimum.

- It is also not always the case that a functional of the form $I[y] = \int_a^b f(x, y(x), y'(x)) \, dx$ over the set $A = \{y; y: [a, b] \rightarrow \mathbb{R}^n \text{ is } C^1 \text{ and } y(a) = y_a, y(b) = y_b\}$ does have a minimum.

- Additional considerations are often required.
5.6 Constrained minima in \( \mathbb{R}^n \)

Suppose \( \tilde{x} \) is minimum of \( f: M \to \mathbb{R} \) where \( M = \{ x \in \mathbb{R}^n; g(x) = 0 \} \) and \( g: \mathbb{R}^n \to \mathbb{R} \). What does this imply? Write \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_{n-1}, \tilde{x}_n) \).

**Theorem 16. (Lagrange multiplier)** If \( Dg(\tilde{x}) \neq 0 \) and \( \tilde{x} \) is minimum of \( f: M \to \mathbb{R} \), then \( \exists \lambda \in \mathbb{R} \) with \( Df(\tilde{x}) = \lambda Dg(\tilde{x}) \).

**Proof.** Since \( Dg(\tilde{x}) \neq 0 \), we get that \( \frac{\partial g}{\partial x_i}(\tilde{x}) \neq 0 \) for some \( i = 1, \ldots, n \). In order to be definite assume \( \frac{\partial g}{\partial x_n}(\tilde{x}) \neq 0 \) and write \( \tilde{w} = (\tilde{x}_1, \ldots, \tilde{x}_{n-1}) \). By the Implicit Function Theorem, locally near \( \tilde{w} \) there exits \( h \) so that \( g(x) = 0 \iff x_n = h(x_1, \ldots, x_{n-1}) \). So \( \tilde{w} \) is minimum of \( (x_1, \ldots, x_{n-1}) \mapsto f \circ (x_1, x_2, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})) \). This means for all \( i = 1, \ldots, n-1 \):

\[
\frac{\partial f}{\partial x_i}(\tilde{x}) + \frac{\partial f}{\partial x_n}(\tilde{x}) \frac{\partial h}{\partial x_i}(\tilde{w}) = 0.
\]

This means all partial derivatives need to be zero:

\[
\frac{\partial f}{\partial x_i}(\tilde{x}) + \frac{\partial f}{\partial x_n}(\tilde{x}) \frac{\partial h}{\partial x_i}(\tilde{w}) = 0 \; \forall i = 1, \ldots, n - 1.
\]

Since \( g(x_1, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1}) = 0 \) we also get

\[
\frac{\partial g}{\partial x_i}(\tilde{x}) + \frac{\partial g}{\partial x_n}(\tilde{x}) \frac{\partial h}{\partial x_i}(\tilde{w}) = 0 \; \forall i = 1, \ldots, n - 1.
\]

Substituting this into the previous equation and writing

\[
\lambda = \frac{\frac{\partial f}{\partial x_n}(\tilde{x})}{\frac{\partial g}{\partial x_n}(\tilde{x})}
\]

57
gives
\[ \frac{\partial f}{\partial x_i}(\tilde{x}) - \lambda \frac{\partial g}{\partial x_i}(\tilde{x}) = 0 \quad \forall i = 1, \ldots, n - 1. \]
(For \( i = n \) the last equation also holds, by definition.)

5.7 Curves, surfaces and manifolds

- If \( M = \{ x \in \mathbb{R}^n; g(x) = 0 \} \) where \( g: \mathbb{R}^n \to \mathbb{R} \) and \( Dg(\hat{x}) \neq 0 \) then \( M \) is near \( \hat{x} \) a manifold. If \( n = 2 \) then \( M \) is nearby a curve, \( n = 3 \) a piece of a surface, and so on.
- Examples: \( x^2 + 2y^2 = 1, x^2 + y^4 + z^6 = 1. \)
- If \( \gamma: [0, 1] \to M \subset \mathbb{R}^n \) is a \( C^1 \) curve with \( \gamma(0) = \hat{x} \), then \( g \circ \gamma(t) = 0 \) for all \( t \) and so
  \[ \frac{\partial g}{\partial x_1}(\hat{x})\gamma'_1(0) + \cdots + \frac{\partial g}{\partial x_n}(\hat{x})\gamma'_n(0) = 0 \]
  or in other words,
  \[ Dg(\hat{x})\gamma'(0) = 0 \]
  that is \( \nabla g(\hat{x}) \cdot \gamma'(0) = 0 \)
  where \( \cdot \) is the usual dot product in \( \mathbb{R}^n \). So the vector \( \nabla g(\hat{x}) \) is orthogonal to \( \gamma'(0) \). The hyperplane which goes through \( \hat{x} \) and which is orthogonal to \( \hat{x} \) is called the tangent plane.

5.8 Constrained Euler-Lagrange Equations

Let \( I[y] = \int_a^b f(x, y(x), y'(x)) \, dx \) and \( J[y] = \int_a^b g(x, y(x), y'(x)) \, dx \) be functionals on \( \mathcal{A} = \{ y; y: [a, b] \to \mathbb{R}^n \text{ is } C^1 \text{ and } y(a) = y_a, y(b) = y_b \} \).
as before. Define

\[ M = \{ y; \ y \in \mathcal{A} \text{ with } J[y] = 0 \}. \]

**Theorem 17.** If \( M \ni y \mapsto I[y] \) has a minimum at \( \bar{y} \) then there exists \( \lambda \in \mathbb{R} \) so that the E-L condition hold for \( F = f - \lambda g \).

That is,

\[
\frac{d}{dx} F_y[y] = F_y[\bar{y}].
\]

The idea of the proof combines the Lagrange multiplier approach with the proof of the previous Euler Lagrange theorem.

**Example 28.** Maximize the area bounded between the graph of \( y \) and the line segment \([-1, 1] \times \{0\} \), conditional on the length of the arc being \( L \). (This is a special case of Dido’s problem.)

Let \( \mathcal{A} \) be the set of \( C^1 \) functions \( y: [-1, 1] \to \mathbb{R} \) with \( y(-1) = y(1) = 0 \). Fix \( L > 0 \) and let

\[
I[y] = \int_{-1}^{1} y(x) \, dx \quad \text{and} \quad J[y] = \int_{-1}^{1} \sqrt{1 + (y')^2} \, dx - L = 0.
\]

Write

\[
f = y, \ g = \sqrt{1 + (y')^2}, \ F = f - \lambda g = y - \lambda \sqrt{1 + (y')^2}.
\]

The Euler Lagrange equation in the version of Lemma 9 gives \( F_y y' - F = C \) which amounts to

\[
\frac{-\lambda (y')^2}{\sqrt{1 + (y')^2}} - [y - \lambda \sqrt{1 + (y')^2}] = C.
\]

Rewriting this gives

\[
1 = \frac{(y + C)^2}{\lambda^2} (1 + (y')^2).
\]
Substituting $y + C = \lambda \cos \theta$ gives $y' = -\lambda \sin \theta \frac{d\theta}{dx}$. Substituting this in the previous equation gives

$$1 = \cos^2 \theta \left( 1 + \lambda^2 \sin^2 \theta \left( \frac{d\theta}{dx} \right)^2 \right).$$

Since $\cos^2 \theta + \sin^2 \theta = 1$, this implies

$$\lambda \cos \theta \frac{d\theta}{dx} = \pm 1, \text{ i.e. } \frac{dx}{d\theta} = \pm \lambda \cos \theta$$

which means $x = \pm \lambda \sin \theta$ and $y + C = \lambda \cos \theta$: a circle segment!
6 Nonlinear Theory

In the remainder of this course we will study initial value problems associated to autonomous differential equations

\[ x' = f(x), \quad x(0) = x_0 \]  

(12)

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is \( C^\infty \). We saw:

- There exists \( \delta(x) > 0 \) so that this has a unique solution \( x : (-\delta, \delta) \to \mathbb{R}^n \).

- There exists a unique maximal domain of existence \( I(x_0) = (\alpha(x_0), \beta(x_0)) \) and a unique maximal solution \( x : I(x_0) \to \mathbb{R}^n \).

- If \( \beta(x_0) < \infty \) then \( |x(t)| \to \infty \) when \( t \uparrow \beta(x_0) \).

- If \( \alpha(x_0) > -\infty \) then \( |x(t)| \to \infty \) when \( t \downarrow \alpha(x_0) \).

- The solution is often denoted by \( \phi_t(x_0) \).

- One has the flow property: \( \phi_{t+s}(x_0) = \phi_t \phi_s(x_0) \), \( \phi_0(x_0) = x_0 \).

6.1 The orbits of a flow

The orbits of a flow Rather than studying each initial value problem separately, it makes sense to study the flow \( \phi_t \) associated to \( x' = f(x), \quad x(0) = x_0 \). The curves \( t \mapsto \phi_t(x) \) are called the orbits. For example we will show that the flow of

\[
\begin{align*}
\dot{x} &= Ax - Bxy \\
\dot{y} &= Cy + Dxy
\end{align*}
\]

is equal to
6.2 Critical points

Critical Points Consider \( x' = f(x), x(0) = x_0 \).

If \( f(x_0) = 0 \) then \( x(t) \equiv x_0 \) is a solution, and by uniqueness the solution. So \( \phi_t(x_0) = x_0 \) for all \( t \in \mathbb{R} \).

This is so important that many different names are used for this: rest point, fixed point, singular point or critical point.

Near such points usually a linear analysis suffices.

By Taylor’s Theorem we can write

\[
 f(x) = f(x_0) + A(x-x_0) + g(x-x_0) = A(x-x_0) + g(x-x_0)
\]

where \( |g(x-x_0)| \leq O(|x-x_0|^2) \).

\( A = Df(x_0) \) is called the linear part of \( f \) at \( x_0 \).

6.3 Linearisation

6.4 Stable and Unstable Manifold Theorem

Linearisation

\( A \) is called hyperbolic if its eigenvalues \( \lambda_1, \ldots, \lambda_n \) satisfy

\( \Re(\lambda_i) \neq 0, \ i = 1, \ldots, n \). Order the eigenvalues so that

\( \Re(\lambda_i) < 0 \) for \( i = 1, \ldots, s \) and \( \Re(\lambda_i) > 0 \) for \( i = s+1, \ldots, n \).

Let \( E^s \) (resp. \( E^u \)) be the eigenspace associated to the eigenvalues \( \lambda_1, \ldots, \lambda_s \) (resp. \( \lambda_{s+1}, \ldots, \lambda_n \)).
Figure 1: In this situation there are several singularities: with a sink, source and saddle.

**Theorem 18. Stable and Unstable Manifold Theorem**  
Let $x_0$ be a singularity of $f$ and assume $Df(x_0)$ is hyperbolic. Then there exist a manifold $W^s(x_0)$ of dimension $s$ and a manifold $W^u(x_0)$ of dimension $n - s$ both containing $x_0$ so that

\[
x \in W^s(x_0) \implies \phi_t(x) \to x_0 \text{ as } t \to \infty,
\]

\[
x \in W^u(x_0) \implies \phi_t(x) \to x_0 \text{ as } t \to -\infty.
\]

$W^s(x_0), W^u(x_0)$ are tangent to $x_0 + E^s$ resp. $x_0 + E^u$ at $x_0$.

- We will not introduce the notion of a manifold here. But think of a curve if its dimension one and a surface if its dimension two.
- If $s = n$ then the singularity is called a sink.
- If $1 \leq s < n$ then it is called a saddle.
- $s = 0$ then it called a source.
- $W^s(x_0)$ is called the stable manifold.
- $W^u(x_0)$ is called the unstable manifold.

**Example 29.** Take $x' = x + y^2, y' = -y + x^2$. By Theorem 18 there is supposed to an invariant manifold $W^u(0)$ (a curve) which is tangent to the $x$-axis. How to find the power series expansion of $W^u(0)$? Write

\[
y'(x) = \left(\frac{dy}{dt}\right)/\left(\frac{dx}{dt}\right) = \frac{-y + x^2}{x + y^2}.
\]  

(13)
• Since $0 \in W^u(0)$ and $W^u(0)$ is tangent to the horizontal axis, we can describe this curve by $y(x) = a_2 x^2 + a_3 x^3 + \ldots$

• That this power series converges follows from the stable and unstable manifold theorem.

• Substituting this in (13) gives
  
  $$2a_2 x + 3a_3 x^2 + \cdots = \frac{[a_2 x^2 + a_3 x^3 + \ldots]}{x + [a_2 x^2 + a_3 x^3 + \ldots]^2}.$$

• Comparing terms of the same power, shows that $2a_2 = (1 - a_2)$ and so on.

• Thus we determine the power series expansion of $y(x)$.

**Proof of Theorem 18**

• We will only prove this theorem in the case that $s = n$.

• To simplify the argument, we assume that $A$ has $n$ real eigenvalues $\lambda_i < 0$ and $n$ real eigenvectors $v_1, \ldots, v_n$. Let $T$ be the matrix consisting of the vectors $v_1, \ldots, v_n$ (that is $Te_j = v_j$). Then $T^{-1}AT$ is diagonal.

• Define the inner product $(x, y) = [T^{-1}x] \cdot [T^{-1}y]$. Then
  
  $$(v_j, Av_k) = [T^{-1}v_j] \cdot [T^{-1}Av_k] = [e_j] \cdot [T^{-1}\lambda_k v_k] = [e_j] \cdot [\lambda_k e_k] = \lambda_k \delta_{j,k} \text{ (the Kronecker symbol)}.$$  

  Hence if we write $x = \sum x_j v_j$ we get
  
  $$(x, Ax) = (\sum x_j v_j, \sum x_k Av_k) = \sum j, k x_j x_k \lambda_k \delta_{j,k} = \sum \lambda_k x_k^2 \leq -\rho |x|^2, \text{ where } \rho = \min_{i=1, \ldots, n} |\lambda_i|.$$  

  Here $|x|^2 = (x, x) = [T^{-1}x] \cdot [T^{-1}x]$ (note $[T^{-1}x]$).

  $$[T^{-1}x] = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1^2 + \cdots + x_n^2).$$
So \((x, Ax) \leq -\rho|x|^2\) for all \(x\).

Since \(A = Df(0)\) we have \(\lim_{x \to 0} \frac{|f(x) - Ax|}{|x|} = 0\).

\[\implies (\text{by Cauchy-Schwarz's inequality), } \forall \epsilon > 0 \exists \delta > 0 \text{ so that } (x, f(x) - Ax) \leq |x||f(x) - Ax| \leq \epsilon|x|^2 \quad \forall |x| \leq \delta \]

Hence \((x, f(x)) = (x, f(x) - A(x)) + (x, Ax) \leq \epsilon|x|^2 - \rho|x|^2\).

Hence \(\exists \delta > 0 \text{ so that } (x, f(x)) \leq -(\rho/2)|x|^2\) for all \(x\) with \(|x| \leq \delta\).

\[\frac{d}{dt}(x(t), x(t)) = 2(x(t), x'(t)) = 2(x(t), f(x(t))) \leq -2(\rho/2)(x, x).\]

Hence if we write \(U(x) = (x, x)\) we get \(\frac{d}{dt}U \leq -\rho U\). It follows that \(U(x(t)) \leq e^{-t\rho}\) and \(|x(t)| \to 0\).

This completes the proof in this case.

**Example 30.** Let \(A = \begin{pmatrix} -1 & b \\ -b & -1 \end{pmatrix}\) has eigenvalues \(-1 \pm bi\) and consider \(x' = Ax\). Define \(U(t) = |x(t)|^2\). Then

\[
\frac{dU}{dt} = 2x \cdot \dot{x} = 2x \cdot Ax = 2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} -x_1 + bx_2 \\ -bx_1 - x_2 \end{pmatrix} = -2 \left( [x_1(t)]^2 + [x_2(t)]^2 \right) = -2U(t).
\]

Hence \(U(t) = e^{-2t}\) and so \(x(t) \to 0\) as \(t \to \infty\).

When \(x' = f(x)\) with \(f(0) = 0\) and \(A = Df(0)\) we can argue analogously: by Taylor’s theorem \(\forall \epsilon > 0 \exists \delta > 0 \text{ so that } |x| < \delta \implies |f(x) - Ax| \leq \epsilon|x|\). So, by Cauchy-Schwarz,

\[x \cdot [f(x) - Ax] \leq |x||f(x) - Ax| \leq \epsilon|x|^2 = \epsilon U(t). \quad (14)\]
• denote soln $x' = f(x)$ by $x(t)$ and $U(t) = |x(t)|^2$.

• $\dot{U} = 2x \cdot \dot{x} = 2x \cdot f(x) = 2x(Ax + [f(x) - Ax]) = 2x \cdot Ax + 2x(f(x) - Ax) \leq -2U(t) + 2\epsilon |x(t)|^2 \leq -U(t)$.

  Here we take $\epsilon < 1/2$ in (14) and assume $|x| < \delta$.

• Hence $U'(t) \leq -U(t)$ when $x(t)$ is close to 0.

• So $\phi_t(x) \to 0$ as $t \to \infty$ provided $x$ is close to 0.

Example 31. If $A = \begin{pmatrix} -1 & Z \\ 0 & -1 \end{pmatrix}$ where $Z \in \mathbb{R}$. This has eigenvalues $-1$ (with double multiplicity). Take $U(x, y) = ax^2 + bxy + cy^2$. Then

$$
\dot{U} = 2ax\dot{x} + b\dot{xy} + bx\dot{y} + 2cy\dot{y} = 2ax(-x + Zy) + b(-x + Zy)y + bx(-y) + 2cy(-y) = -2ax^2 + (2Za - b - b)xy + (Zb - 2c)y^2.
$$

If $Z$ is large and $a = 1, b = 0, c = 1$ then we definitely don’t get $\dot{U} \leq 0$.

• How to choose $a, b, c$ so that $U \geq 0$ and $\dot{U} \leq 0$?

• If $Z \approx 0$, then we can take $a = 1, b = 0, c = 1$ because then $\dot{U} = -2a^2 + (2Z)xy - 2y^2 \leq 0$ (since $Z \approx 0$).

• What do to when $|Z|$ is large?

• In this case we can set $b = 0$, and write

$$
\dot{U} = -2ax^2 + (2Za)xy - 2cy^2 = -2a[x - (Z/2)y]^2 + (aZ^2/2 - 2c)y^2 = -2[x - (Z/2)y]^2 - y^2.
$$

where in the last line we substitutes $a = 1$ and $c = 1/2 + Z^2/4$. Thus $U = c$ corresponds to a ‘flat’ ellipse when $Z$ is large.
• This is rather ad hoc. How to do this in general? Go back to Jordan normal form: first eigenvector $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (i.e. $(A+I)v_1 = 0$) and choose 2nd vector $v_2$ so that $(A+I)v_2 = \epsilon v_1$ where $\epsilon > 0$ is small. So $v_2 = \begin{pmatrix} 0 \\ \epsilon/Z \end{pmatrix}$.

Taking $T = (v_1 v_2)$ gives $T^{-1}AT = \begin{pmatrix} -1 & \epsilon \\ 0 & -1 \end{pmatrix}$.

• In this new coordinates we are in the same position as if $Z \approx 0$. So define inner product $(x,y) = [T^{-1}x] \cdot [T^{-1}y]$, $U(x) = (x,x)$ and argue as before.

### 6.5 Hartman-Grobman

**Theorem 19. Hartman-Grobman** Let $x_0$ be a singularity and that $A = Df(x_0)$ is a hyperbolic matrix. Then there exists a continuous bijection (a homeomorphism) $h: \mathbb{R}^n \to \mathbb{R}^n$ so that $h(x_0) = 0$ and so that near $x_0$,

$$h \text{ sends orbits of } x' = f(x) \text{ to orbits of } x' = Ax.$$

In other words, there exists an open set $U \ni x_0$ so that

$$h \circ \phi_t(x) = \phi_t^A \circ h(x)$$

for each $x, t$ so that

$$\bigcup_{0 \leq s \leq t} \phi_s(x) \subset U.$$

Here $\phi_t^A$ is the flow associated to $x' = Ax$ and $\phi_t$ the flow for $x' = f(x)$.

**Remark.** In general, a homeomorphism is a continuous bijection whose inverse is also continuous. In Euclidean space (and ‘manifolds’), this is the same as saying that it is continuous bijection.
6.6 Lyapounov functions

Lyapounov functions
Sometimes Theorem 18 does not apply, but its proof will:

**Definition:** Let \( W \subseteq \mathbb{R}^n \) be an open set containing \( x_0 \). \( V : W \rightarrow \mathbb{R} \) is a Lyapounov function for \( x_0 \) if it is \( C^1 \) and

- \( V(x_0) = 0, V(x) > 0 \) for \( x \in W \setminus \{x_0\} \);
- \( \dot{V} \leq 0 \) for \( x \in W \).

Here \( \dot{V} := \frac{dV(x(t))}{dt} = DV_{x(t)} \frac{dx}{dt} = DV_{x(t)} f(x(t)) \).

- So \( V \) measures the distance to \( x_0 \).
- \( \dot{V} \leq 0 \) means that this ‘distance’ is non-increasing.

- \( x_0 \) is called **asymptotically stable** if, for each \( x \) near \( x_0 \), one has \( \phi_t(x) \rightarrow x_0 \).

- \( x_0 \) is called **stable** if for each \( \epsilon > 0 \) there exists \( \delta > 0 \) so that if \( x \in B_\delta(x_0) \) implies \( \phi_t(x) \in B_\epsilon(x_0) \) for all \( t \geq 0 \). (So you nearby points don’t go far.)

**Lemma 10. Lyapounov functions**

1. If \( \dot{V} \leq 0 \) then \( x_0 \) is stable. Moreover, \( \phi_t(x) \) exists for all \( t \geq 0 \) provided \( d(x, x_0) \) is small.

2. If \( \dot{V} < 0 \) for \( x \in W \setminus \{x_0\} \) then \( \forall x \) is close to \( x_0 \) one gets \( \phi_t(x) \rightarrow x_0 \) as \( t \rightarrow \infty \), i.e. \( x_0 \) is asymptotically stable.

**Proof.** (1) Take \( \epsilon > 0 \) so that \( B_{2\epsilon}(x_0) \subset W \). Let \( \delta := \inf_{y \in \partial B_{\epsilon}(x_0)} V(y) \). Since \( V > 0 \) except at \( x_0 \) we get \( \delta > 0 \). It follows that \( V^{-1}(0, \delta) \cap \partial B_{\epsilon}(x_0) = \emptyset \). Hence, since \( t \rightarrow V(\phi_t(x)) \) is non-increasing, \( x \in V^{-1}(0, \delta) \cap B_{\epsilon}(x_0) \) implies \( \phi_t(x) \in V^{-1}[0, \delta] \).
Since \( t \to \phi_t(x) \) is continuous curve, \( \phi_0(x) = x \in B_\epsilon(x_0) \) and \( V^{-1}[0, \delta] \cap \partial B_\epsilon(x_0) = \emptyset \), it follows that \( \phi_t(x) \in B_\epsilon(x_0) \). Hence \( \phi_t(x) \in V^{-1}[0, \delta] \cap B_\epsilon(x_0) \) for all \( t \geq 0 \). In particular \( \phi_t(x) \) remains bounded, and so \( \phi_t(x) \) exists \( \forall t \).

(2) \( \dot{V} < 0 \implies t \to V(\phi_t(x)) \) is decreasing. Take \( x \in V^{-1}[0, \delta] \cap B_\epsilon(x_0) \) and suppose by contradiction that \( V(\phi_t(x)) \) does not tend to 0. Then \( \exists \rho > 0 \) with \( \phi_t(x) \notin B_\rho(x_0) \forall t \geq 0 \). Hence \( \phi_t(x) \) is always in the compact set \( B_\epsilon(x_0) \setminus B_\rho(x_0) \) and therefore \( \exists \kappa > 0 \) so that \( \dot{V} < -\kappa, \forall t \geq 0 \) and \( V(\phi_t(x)) - V(x) \leq -\kappa t \to -\infty \) as \( t \to \infty \), contradicting \( V \geq 0 \).

Example 32.

\[
\begin{align*}
x' &= 2y(z-1) \\
y' &= -x(z-1) \\
z' &= xy
\end{align*}
\]

Its linearisation is \( \mathbf{A} := \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \). Note \( \mathbf{A} \) has eigenvalues \( \pm \sqrt{2}i \) and 0. So \( \mathbf{A} \) is not hyperbolic, and theorem[18] does not apply.

Take \( V(x, y, z) = ax^2 + by^2 + cz^2 \). Then

\[
\dot{V} = 2(ax \dot{x} + by \dot{y} + c \dot{z}) = 4axy(z-1) - 2bxy(z-1) + 2cxyz.
\]

We want \( V \geq 0 \) and \( \dot{V} \leq 0 \). We can achieve this by setting \( c = 0, 2a = b \). This makes \( \dot{V} = 0 \). It follows that solutions stay on level sets of the function \( V = x^2 + 2y^2 \). \( x_0 = (0, 0, 0) \) is not asymptotically stable. \( V \) is not a Lyapounov function because \( V(0, 0, z) = 0 \): more work needed to check if \( x_0 \) is stable.

### 6.7 The pendulum

Pendulum Consider a pendulum moving along a circle of radius \( l \), with a mass \( m \) and friction \( k \). Let \( \theta(t) \) be the angle
from the vertical at time \( t \). The force tangential to the circle is
\[-(kl\frac{d\theta}{dt} + mg\sin(\theta))\). So Newton’s law gives
\[ml\theta'' = -kl\theta' - mg\sin(\theta) \quad \text{i.e.} \quad \theta'' = -(k/m)\theta' - (g/l)\sin(\theta).

Taking \( \omega = \theta' \) gives
\[
\begin{align*}
\theta' &= \omega \\
\omega' &= -\frac{g}{l}\sin(\theta) - \frac{k}{m}\omega.
\end{align*}
\]

Singularities are \((n\pi, 0)\) which corresponds to the pendulum being in vertical position (pointing up or down). Linearizing this at \((0, 0)\) gives
\[
\begin{pmatrix}
0 & 1 \\
-g/l & -k/m
\end{pmatrix}
\]
which gives eigenvalues \((-k/m \pm \sqrt{(k/m)^2 - 4g/l})/2\).

Note that, as \( l > 0 \), the real part of \((-k/m \pm \sqrt{(k/m)^2 - 4g/l})/2\) is negative. (If \((k/m)^2 - 4g/l < 0\) then both e.v. are complex and if \((k/m)^2 - 4g/l > 0\) then both e.v. are real and negative.)

Let us construct a Lyapounov function for this:

\[
E = \text{kinetic energy} + \text{potential energy}
= (1/2)mv^2 + mg(l - l\cos(\theta))
= (1/2)ml^2\omega^2 + mgl(1 - \cos(\theta)).
\]

Then \( E \geq 0 \) and \( E = 0 \) if and only if \( \omega = 0 \) and \( \theta = n\pi \).

Moreover,
\[
\dot{E} = ml(l\omega' + g\theta'\sin(\theta))
= ml(l\omega(-\frac{g}{l}\sin(\theta) - \frac{k}{m}\omega) + g\omega\sin(\theta))
= -kl^2\omega^2.
\]

If the friction \( k > 0 \) then \( \dot{E} < 0 \) except when \( \omega = 0 \). If the friction \( k = 0 \) then \( \dot{E} = 0 \) and so solutions stay on level sets of \( E \).
6.8 Hamiltonian systems

Hamiltonian systems When the friction $k = 0$ we obtain an example of a Hamiltonian system.

This is a system for which there exists a function $H : \mathbb{R}^2 \to \mathbb{R}$ so that the equation of motion (i.e. the differential equation):

$$
\dot{x} = \frac{\partial H}{\partial y}(x, y) \\
\dot{y} = -\frac{\partial H}{\partial x}(x, y)
$$

For such systems

$$
\dot{H} = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial y} \dot{y} \\
= \frac{\partial H}{\partial x} \frac{\partial H}{\partial y} + \frac{\partial H}{\partial y} \left( -\frac{\partial H}{\partial x} \right) \\
= 0.
$$

6.9 Van der Pol’s equation

Van der Pol’s equation In electrical engineering the following
The equation often arises
\[ \dot{x} = y - x^3 + x, \quad \dot{y} = -x. \]
This system has a singularity at \((x, y) = (0, 0)\). Linear part at \((0, 0)\) is \[
\begin{pmatrix}
1 & 1 \\
-1 & 0
\end{pmatrix}
\]
This has eigenvalues \((1 \pm \sqrt{3}i)/2\) and therefore \((0, 0)\) is a source. What happens with other orbits?

**Theorem 20.** There is one periodic solution of this system and every non-equilibrium solution tends to this periodic solution.

\[ \dot{x} = y - x^3 + x, \quad \dot{y} = -x. \]

Define \(v^\pm = \{(x, y); \pm y > 0, x = 0\}\) and \(g^\pm = \{(x, y); \pm x > 0, y = x^3 - x\}\).

This splits up \(\mathbb{R}^2\) in regions \(A, B, C, D\) where horizontal and vertical speed is positive/negative.

\[ \dot{x} = y - x^3 + x, \quad \dot{y} = -x. \]
Lemma 11. For any \( p \in v^+ \), \( \exists t > 0 \) with \( \phi_t(p) \in g^+ \).

Proof. Define \((x_t,y_t) = \phi_t(p)\).

- Since \( x'(0) > 0 \), \( \phi_t(p) \in A \) for \( t > 0 \) small.
- \( x' > 0 \), \( y' < 0 \) in \( A \). So the only way the curve \( \phi_t(p) \) can leave the region \( A \cap \{(x,y); y < y_0\} \) is via \( g^+ \).
- So \( \phi_t(p) \) cannot go to infinity before hitting \( g^+ \).
- Hence \( T = \inf \{ t > 0; \phi_t(p) \in g^+ \} \) is well-defined.
- We need to show \( T < \infty \).
- Choose \( t_0 \in (0,T) \) and let \( a = x_{t_0} \). Then \( a > 0 \) and \( x_t \geq a \) for \( t \in [t_0,T] \).
- Hence \( \dot{y} \leq -a \) for \( t \in [t_0,T] \) and therefore \( y(t) - y(t_0) \leq -a(t - t_0) \) for \( t \in [t_0,T] \).
- \( T = \infty \implies \lim_{t \to \infty} y(t) \to -\infty \) which gives a contradiction since \( (x(t),y(t)) \in A \) for \( t \in (0,T) \).

\[ \dot{x} = y - x^3 + x \]
\[ \dot{y} = -x. \]

Similarly

Lemma 12. For any \( p \in g^+ \), \( \exists t > 0 \) with \( \phi_t(p) \in v^- \).

For each \( y > 0 \) define \( F(y) = \phi_t(0,y) \) where \( t > 0 \) is minimal so that \( \phi_t(0,y) \in v^- \). Similarly, define for \( y < 0 \) define \( F(y) = \phi_t(0,y) \) where \( t > 0 \) is minimal so that \( \phi_t(0,y) \in v^+ \). By symmetry \( F(-y) = -F(y) \).

Define the Poincaré first return map to \( v^+ \) as

\[ P: v^+ \to v^+ \text{ by } (0,y) \mapsto (0,F^2(y)). \]

\( P(p) = \phi_t(p) \) where \( t > 0 \) is minimal so that \( \phi_t(p) \in v^+ \).
Lemma 13.  1. $P: v^+ \rightarrow v^+$ is increasing (here we order $v^+$ as $(0, y_1) < (0, y_2)$ when $y_1 < y_2$);  
2. $P(p) > p$ when $p \approx 0$;  
3. $P(p) < p$ when $p$ is large;  
4. $P: v^+ \rightarrow v^+$ has a unique attracting fixed point.

Remarks:  
• Uniqueness of solns $\implies$ orbits don’t cross $\implies$ $P$ is increasing.  
• In fact we will show  
  
  $$p \mapsto \delta(p) := |F(p)|^2 - |p|^2$$  
  is strictly decreasing  
  $$\delta(p) > 0 \text{ for } p > 0 \text{ small and } \delta(p) \rightarrow -\infty \text{ as } p \rightarrow \infty$$  

  (15)  
• Hence $y \mapsto F(y)$ has a unique attracting fixed point.  
• $\implies$ above lemma and Theorem 20 (see lecture).  
• Define $p^* = (0, y^*) \in v^+$ so that $\phi_t(p^*) = (1, 0)$ for some $t$ and $\phi_s(p^*) \in A$ for $0 < s < t$.  
• Define $U(x, y) = x^2 + y^2$. Pick $p \in v^+$ and let $\tau > 0$ be minimal so that $\phi_\tau(p) \in v^-$. (So $\phi_\tau(p) = F(p)$.)

• Hence  
  $$\delta(p) := |F(p)|^2 - |p|^2 = U(\phi_\tau(p)) - U(\phi_0(p)) = \int_0^\tau \dot{U}(\phi_t(p)) \, dt.$$  
• $\dot{U} = 2x \dot{x} + 2y \dot{y} = 2x(y - x^3 + x) + 2y(-x) = -2x(x^3 - x) = 2x^2(1 - x^2)$.
Hence
\[ \delta(p) = 2 \int_0^\tau [x(t)]^2(1 - [x(t)]^2)dt = 2 \int_\gamma x^2(1 - x^2) dt. \]

Here \( \gamma \) is the curve \([0, \tau] \ni t \to \phi_t(p) \).

- If \( p < p^* \) then \( \delta(p) > 0 \) because then \( 1 - [x(t)]^2 \geq 0 \) for all \( t \in [0, \tau] \).

Now consider \( \delta(p) = 2 \int_\gamma x^2(1 - x^2) ds \) when \( p > p^* \).

- \( \gamma \) meets the line \( x = 1 \) twice; the piece of \( \gamma \) with both endpoint on this line we call \( \gamma_2 \);

- \( \gamma_1 \) is the curve which connects \( p \in v^+ \) to the line \( x = 1 \).

- \( \gamma_3 \) is the curve which connects \( F(p) \in v^- \) to the line \( x = 1 \).

- Now consider \( \delta_i(p) := 2 \int_{\gamma_i} x^2(1 - x^2) ds \) for \( i = 1, 2, 3 \).

- \( \gamma_1 \) is a curve which can be regarded as function of \( x \).

- Hence we can write
\[ \int_{\gamma_1} x^2(1 - x^2) dt = \int_{\gamma_1} \frac{x^2(1 - x^2)}{dx/dt} dx = \int_{\gamma_1} \frac{x^2(1 - x^2)}{y - (x^3 - x)} dx. \]

- As \( p \) moves up, the curve \( \gamma_1 \) (connecting \( p \in v^+ \) to a point on the line \( x = 1 \)) moves up and so \( y - (x^3 - x) \) (along this curve) increases.

- Hence \( p \to \delta_1(p) = 2 \int_{\gamma_1} x^2(1 - x^2) dt \) decreases as \( p \) increases.
• Along $\gamma_2$, $x(t)$ is a function of $y \in [y_1, y_2]$ (where $(1, y_1)$, $y_1 > 0$ and $(1, y_2)$, $y_2 < 0$) are the intersections points of $\gamma$ with the line $x = 1$.

• Since $-x = dy/dt$ we get
$$\int_{\gamma_2} x^2(1 - x^2) \, dt = \int_{y_2}^{y_1} -x(y)(1 - [x(y)]^2) \, dy$$
$$= \int_{y_1}^{y_2} x(y)(1 - [x(y)]^2) \, dy$$

• Since $x(y) \geq 1$ along $\gamma_2$, this integral is negative.

• As $p$ increases, the interval $[y_1, y_2]$ gets larger, and the curve $\gamma_2$ moves to the right and so $x(y)(1 - [x(y)]^2)$ decreases. It follows that $\delta_2(p)$ decreases as $p$ increases.

• It is not hard to show that $\delta_2(p) \to -\infty$ as $p \to \infty$, see lecture.

Exactly as for $\delta_1(p)$, one also gets that $\delta_3(p)$ decreases as $p$ increases.

This completes the proof of the equation (15), Lemma 13 and Theorem 20.

6.10 Population dynamics

Population dynamics A common predator-prey model is the equation
\[
\begin{align*}
\dot{x} &= (A - By)x \\
\dot{y} &= (Cx - Dy)y.
\end{align*}
\]
where $A, B, C, D > 0$

Here $x$ are the number of rabbits and $y$ the number of foxes.

• For example, $x' = Ax - Bxy$ expresses that rabbits grow with speed $A$ but that the proportion that get eaten is a multiple of the number of foxes.
• Singularities are \((x, y) = (0, 0)\) and \((x, y) = (D/C, A/B)\).

• If \(p\) is on the axis, then \(\phi_t(x)\) is on this axis for all \(t \in \mathbb{R}\).

• At \((0, 0)\) the linearisation is \(\begin{pmatrix} A & 0 \\ 0 & -D \end{pmatrix}\), so eigenvalues are \(A, -D\) and \((0, 0)\) is a saddle point.

• At \((x, y) = (D/C, A/B)\) the linearisation is

\[
\begin{pmatrix} A - By & -Bx \\ Cy & Cx - D \end{pmatrix}
\]

which has eigenvalues \(\pm ADi\) (purely imaginary).

\[
\begin{align*}
\dot{x} &= (A - By)x \\
\dot{y} &= (Cx - D)y.
\end{align*}
\]

where \(A, B, C, D > 0\)

• Analysing the direction field, suggests that orbits cycle around \((D/C, A/B)\) (see lecture).

• Try to find Lyapounov of the form \(H(x, y) = F(x) + G(y)\).

\[
\dot{H} = F'(x)\dot{x} + G'(y)\dot{y} = xF'(x)(A - By) + yG'(y)(Cx - D).
\]

• If we set (that is, insist on) \(\dot{H} = 0\) then we obtain

\[
xF' = \frac{yxG'}{Cx - D} = \frac{yG'}{By - A} \quad (16)
\]

• LHS of (16) only depends on \(x\) and RHS only on \(y\). So expression in (16) = const.

• We may as well set \(const = 1\). This gives \(F' = C - D/x\) and \(G' = B - A/y\).
• So \( F(x) = Cx - D \log x, G(y) = By - A \log y \) and \( H(x, y) = Cx - D \log x + By - A \log y \).

Summarising:

**Theorem 21.** Take \((x, y) \neq (D/C, A/B)\) with \(x, y > 0\) and consider its orbits under
\[
\begin{align*}
\dot{x} &= (A - By)x \\
\dot{y} &= (Cx - D)y.
\end{align*}
\]
where \(A, B, C, D > 0\).

Then \(t \mapsto \phi_t(x, y)\) is periodic (i.e. is a closed curve).

**Proof.** Take \(H_0 = H(x, y)\) and let \(\Sigma = \{(u, v); H(u, v) = H_0\}\).

• The orbit \(\phi_t(x, y)\) stays on the level set \(\Sigma\) of \(H\).

• It moves with positive speed.

• So it returns in finite time.

• Orbits exist for all time, because it remains on \(\Sigma\) (and therefore cannot go to infinity).

\(\square\)


7 Dynamical Systems

So far we saw:

- Most differential equations cannot be solved explicitly.
- Nevertheless in many instances one can still prove many properties of its solutions.
- The point of view taken in the field dynamical systems is to concentrate on
  - attractors and limit sets: what happens eventually;
  - statistical properties of orbits.

In this chapter we will discuss a result which describes the planar case (i.e. the two-dimensional case).

Through the remainder of this notes, we will tacitly assume the solution $\phi_t(x)$ through $x$ exists for all $t \geq 0$.

7.1 Limit Sets

Let $\phi_t$ be the flow of a dynamical system and take a point $x$. Then the $\omega$-limit set of $x$, denoted by $\omega(x)$, is the set of limit points of the curve $[0, \infty) \ni t \mapsto \phi_t(x)$. More specifically, $y \in \omega(x)$ if and only if there exists a sequence $t_n \to \infty$ so that $\phi_{t_n}(x) \to y$.

So $\omega(x)$ describes where the point $x$ eventually goes. It is easy to prove that $\omega(x)$ is a closed set. (But, possibly, $\omega(x) = \emptyset$.)

We say that $x$ lies on a periodic orbit if $\phi_T(x) = x$ for some $T > 0$. The smallest such $T > 0$ is called the period of $x$. Note that then
• \( \gamma = \cup_{t \in [0,T)} \phi_t(x) \) is closed curve without self-intersections, and

• \( \omega(x) = \gamma \).

7.2 Local sections

• Take \( p \in \mathbb{R}^n \) and let \( g: \mathbb{R}^n \to \mathbb{R} \) be smooth and \( Dg(p) \neq 0 \). Hence \( S = \{ x \in \mathbb{R}^n; g(x) = 0 \} \) is a manifold near \( p \).

• Let \( f: \mathbb{R}^n \to \mathbb{R}^n \) and consider \( x' = f(x) \).

• Assume that \( f(p) \neq 0 \) and that \( Dg(p)f(p) \neq 0 \). This means that \( f(p) \) does not lie in the tangent space of \( S \) at \( p \). Indeed, \( Dg(p)f(p) \neq 0 \) implies that \( f(p) \) is not perpendicular to \( \nabla g(p) \) (the gradient of \( g \) at \( p \)).

In this setting we that \( S \) is a **local section at** \( p \). \( S \) is a **local section** if \( S \) is a local section at \( x \) for each \( x \in S \).

**Theorem 22. Flow Box Theorem** Assume \( S \) is a local section at \( p \) and assume \( q \) is that \( \phi_{t_0}(q) = p \) for some \( t_0 > 0 \). Then

• there exists a neighbourhood \( U \) of \( q \)

• a smooth function \( \tau: U \to \mathbb{R} \) so that \( \tau(q) = t_0 \) so that

• for each \( x \in U \), \( \phi_{\tau(x)}(x) \in S \).

If \( t_0 > 0 \) is the minimal time so that \( \phi_{t_0}(q) \in S \) then we will also have that \( \tau(x) > 0 \) is minimal so that \( \phi_{\tau(x)}(x) \in S \). \( \tau(x) \) is then called the **first arrival time** and the map \( P(x) = \phi_{\tau(x)}(x) \) the **Poincaré entry map** to \( S \).
Proof. Define \( G(x, t) = g(\phi_t(x)) \). Then \( G(q, t_0) = g(\phi_{t_0}(q)) = g(p) = 0 \). Moreover,

\[
\frac{\partial G}{\partial t}(q, t_0) = \frac{\partial g}{\partial \phi_t}(q) \frac{\partial \phi_t}{\partial t}(q) \bigg|_{t=t_0} = Dg(p)f(\phi_{t_0}(q))
\]

\[
= Dg(p)f(p) \neq 0 \text{ (because } S \text{ is a section at } p).\]

Hence by the implicit function theorem there exists \( x \mapsto \tau(x) \) so that \( G(x, \tau(x)) = 0 \) for \( x \) near \( q \). Hence \( \phi_{\tau(x)} \in S \) for \( x \) near \( q \). \( \Box \)

- If \( S \) is a section at \( p \) and \( \phi_{t_n}(x) \to p \) for some \( t_n \to \infty \) then there exists \( t'_n \to \infty \) so that \( \phi_{t'_n}(x) \to p \) and \( \phi_{t'_n}(x) \in S \).

- If \( f(p) \neq 0 \) then one can find a local section at \( p \): just take \( g: \mathbb{R}^n \to \mathbb{R} \) affine (of the form \( x \mapsto A(x - p) \)) where \( A \) is a \( 1 \times n \) matrix with \( Af(p) \neq 0 \). Then \( S = \{x; g(x) = 0\} \) is a codimension-one hyperplane with the required properties.

- If \( p \) lies on a periodic orbit and \( S \) a local section at \( p \), then \( \phi_T(p) = p \) and then there exists a neighbourhood \( U \) of \( p \) and a map \( P: S \cap U \to S \) so that \( P(p) = p \). This is called the Poincaré return map.

- As in the example of the van der Pol equation, one can use this map to check whether the periodic orbit is attracting.
7.3 Planar Systems

Theorem 23. Let $S$ be a local section for a planar differential equation, so $S$ is an arc $c$. Let $\gamma = \cup_{t \geq 0} \phi_t(x)$ and let $y_0, y_1, y_2 \in S \cap \gamma$. Then $y_0, y_1, y_2$ lie ordered on $\gamma$ if and only if they lie ordered on $S$.

Proof. Take $y_0, y_1, y_2 \in \gamma \cap c$. Assume that $y_0, y_1, y_2$ are consecutive points on $\gamma$, i.e. assume $y_2 = \phi_{t_2}(y_0), y_1 = \phi_{t_1}(y_0)$ with $t_2 > t_1 > 0$. Let $\gamma' = \cup_{0 \leq s \leq t_1} \phi_s(y_0)$ and consider the arc $c'$ in $c$ between $y_0, y_1$. Then

- $c' \cup \gamma'$ is a closed curve which bounds a compact set $D$ (here we use a special case of a deep result namely the Jordan theorem);

- Either all orbits enter $D$ along $c'$ or they all leave $D$ along $c'$.

- Either way, since the orbit through $y$ does not have self-intersections and because of the orientation of $x' = f(x)$ along $c$, $\phi_{t_2}(y_0)$ cannot intersect $c'$, see figure.

\[\square\]
Lemma 14. If \( y \in \omega(x) \). Then the orbit through \( y \) intersects any local section at most once.

Proof. 1. Assume by contradiction that \( y_1 = \phi_u(y) \) and \( y_2 = \phi_v(y) \) (where \( v > u \)) are contained on a local section \( S \).

2. Since \( y \in \omega(x) \) where exists \( t_n \to \infty \) so that \( \phi_{t_n}(x) \to y \). Hence \( \phi_{t_n+u}(x) \to y_1 \) and \( \phi_{t_n+v}(x) \to y_2 \).

3. Since \( y_1, y_2 \in S \), (2) implies that for \( n \) large there exists \( u_n, v_n \to 0 \) so that \( \phi_{t_n+u_n}(x) \in S \), \( \phi_{t_n+u_n}(x) \to y_1 \) and \( \phi_{t_n+v_n}(x) \in S \), \( \phi_{t_n+v_n}(x) \to y_2 \).

4. Take \( n' > n \) so that

\[
  t_n + u + u_n < t_n + v + v_n < t_{n'} + u + v_n. \tag{17}
\]

Then

\[
\phi_{t_n+u_n}(x), \phi_{t_n+v}(x), \phi_{t_{n'}+u+n'}(x)
\]

do not lie ordered on \( S \): the first and last one are close to \( y_1 \) and the middle one close to \( y_2 \). This and (17) contradict the previous theorem.

\[\square\]
### 7.4 Poincaré Bendixson

**Theorem 24.** Consider a planar differential equation, take \( x \in \mathbb{R}^2 \) and assume that \( \omega := \omega(x) \) is non-empty, bounded and does not contain a singular point of the differential equation. Then \( \omega \) is a periodic orbit.

**Proof**

- Assume that \( \omega \) does not contain a singular point.

- Take \( y \in \omega \). Then there exists \( s_m \to \infty \) so that \( \phi_{s_m}(x) \to y \). Hence for each fixed \( t > 0 \), \( \phi_{s_m+t}(x) \to \phi_t(y) \) as \( m \to \infty \). It follows that the forward orbit \( \gamma = \bigcup_{t \geq 0} \phi_t(y) \) is contained in \( \omega \). Since \( \omega \) is compact, any sequence \( \phi_{t_n}(y) \) has a convergent subsequence. Hence \( \omega(y) \neq \emptyset \) and \( \omega(y) \subset \omega \).

- Take \( z \in \omega(y) \). Since \( z \) is not a singular point, there exists a local section \( S \) containing \( z \). Since \( z \in \omega(y) \), there exists \( t_n \to \infty \) so that \( \phi_{t_n}(y) \to z \) and \( \phi_{t_n}(y) \in S \).

- By the previous lemma, \( \phi_{t_n}(y) = \phi_{t_{n'}}(y) \) for all \( n, n' \). So \( \exists T > 0 \) so that \( \phi_T(y) = y \) and \( y \) lies on a periodic orbit.

- We will skip the proof that \( \omega \) is equal to the orbit through \( y \) (but see lecture).
7.5 Further Outlook

- The Poincaré Bendixson theorem implies that planar differential equations cannot have ‘chaotic’ behaviour.

- Differential equations in dimension $\geq 3$ certainly can have chaotic behaviour, see the 3rd year course *dynamical systems* (M3PA23) and for example [http://www.youtube.com/watch?v=ByH8_nKD-ZM](http://www.youtube.com/watch?v=ByH8_nKD-ZM).

- To describe their statistical behaviour one uses probabilistic arguments; this area of mathematics is called *ergodic theory*. This is a 4th year course (M5A36).

- The geometry of attractors is often fractal like, see the 3rd year course *chaos and fractals* (M3PA46).

- Instead of differential equations one also studies discrete dynamical systems, $x_{n+1} = f(x_n)$. When $f: \mathbb{C} \to \mathbb{C}$ is a polynomial this leads to the study of *Julia sets* using tools from complex analysis. For more information, see [http://en.wikipedia.org/wiki/Julia_set](http://en.wikipedia.org/wiki/Julia_set).
Appendix A  Explicit methods for solving ODE’s

This Appendix summarises explicit methods for solving ODE’s. Since most of the material is already covered in first year material, it will not be covered in the lectures.

A.1 State independent

- This section summarises techniques for solving ODE’s.
- The first subsections are about finding $x: \mathbb{R} \to \mathbb{R}$ so that $x' = f(x, t)$ and $x(0) = x_0$ where $f: \mathbb{R}^2 \to \mathbb{R}$.
- So the issue is to find curves with prescribed tangents.
- Let us first review methods for explicitly solving such equations (in part reviewing what you already know).

A.2 State independent $\dot{x} = f(t)$.

In this case, each solution is of the form $x(t) = \int_0^t f(s) \, ds + x(0)$.

Example 33. Assume the graph $t \mapsto (t, x(t))$ has tangent vector $(1, \sin(t))$ at $t$. Then $x'(t) = \sin(t)$ and so $x(t) = -\cos(t) + c$. So the solution of the ODE $x'(t) = \sin(t)$ finds a curve which is tangent to the arrows of the vector field.

A.3 Separation of variables

Separation of variables: $\dot{x} = f(t)g(x)$. Then one can find solutions as follows.
\[
\int_{x(0)}^{x(T)} \frac{dy}{g(y)} = \int_0^T \frac{1}{g(x(t))} \, dx \, dt = \int_0^T f(t) \, dt.
\]

Here the first equality follows from the substitution rule (taking \( y = x(t) \)) and 2nd from \( \frac{1}{g(x(t))} \, dx = f(t) \).

**Example 34.** \( \frac{dx}{dt} = ax + b, x(t) = x_0 \). Then \( \frac{dx}{ax+b} = dt, x(0) = x_0 \) which gives, when \( a \neq 0 \),

\[
\frac{1}{a} \log(ax + b) = T,
\]

and therefore

\[
x(T) = x_0 e^{at} + \frac{e^{at} - 1}{a} b \quad \text{for} \quad T \in (-\infty, \infty)
\]

**Example 35.** \( \frac{dx}{dt} = x^2, x(0) = x_0 \). Then \( \frac{dx}{x^2} = dt, x(0) = x_0 \).

Hence \( [-1/x]_{x_0}^{x(t)} = t \) and so \( x(t) = \frac{1}{-1/x_0 - t} \). Note that \( x(t) \) is well-defined for \( t \in (-\infty, 1/x_0) \) but that \( x(t) \to \infty \) as \( t \uparrow 1/x_0 \). The solution goes to infinity in finite time.

**Example 36.** \( \frac{dx}{dt} = \sqrt{|x|}, x(0) = x_0 \). If \( x_0 > 0 \) and \( x(t) > 0 \) then we obtain \( \frac{dx}{\sqrt{x}} = dt, x(0) = x_0 \) and so \( 2\sqrt{x(t)} - 2\sqrt{x_0} = t \). Thus \( x(t) = (\sqrt{x_0} + t/2)^2 \) for \( t \in (-2\sqrt{x_0}, \infty) \). When \( t = -2\sqrt{x_0} \) then \( x(t) = 0 \), so need to analyse this directly.

When \( x_0 = 0 \) then there are many solutions (non-uniqueness). For any \( -\infty \leq t_0 \leq 0 \leq t_1 \leq \infty \)

\[
x(t) = \begin{cases} 
-(t - t_0)^2/4 & \text{for} \ t \in (-\infty, t_0) \\
0 & \text{for} \ t \in [t_0, t_1] \\
(t - t_1)^2/4 & \text{for} \ t \in (t_1, \infty)
\end{cases}
\]

is a solution.

So, without imposing some assumptions, solutions need not be unique.
A.4 Linear equations \( x' + a(t)x = b(t). \)

To solve this, first consider the **homogeneous case** \( x' + a(t)x = 0 \). This can be solved by separation of variables: \( \frac{dx}{x} = -a(t)dt \) and so \( x(t) = x_0 \exp\left(-\int_0^t a(s) ds\right) \).

To find the solution of the ODE, apply the **variation of variables ‘trick’**: substitute \( x(t) = c(t) \exp\left(-\int_0^t a(s) ds\right) \) in the equation and obtain an equation for \( c(t) \).

**Example 37.** \( x' + 2tx = t \). The homogeneous equation \( x' + 2tx = 0 \) has solution \( x(t) = ce^{-t^2} \).

Substituting \( x(t) = c(t)e^{-t^2} \) into \( x' + 2tx = t \) gives \( c'(t)e^{-t^2} + c(t)(-2te^{-t^2} + 2tc(t)e^{-2t^2}) = t \), i.e. \( c'(t) = te^{t^2} \). Hence \( c(t) = c_0 + (1/2)e^{t^2} \) and therefore \( x(t) = c_0e^{-t^2} + (1/2) \). That the equation is of the form

\[
c_0 \cdot \text{solution of hom.eq} + \text{special solution}
\]

is due to the fact that the space of solutions \( x' + 2tx = 0 \) is linear (linear combination of solutions are again solutions).

A.5 **Exact equations** \( M(x, y)dx + N(x, y)dy = 0 \) when \( \partial M/\partial y = \partial N/\partial x \).

Suppose \( f(x, y) \equiv c \) is a solution. Then \( df = (\partial f/\partial x)dx + (\partial f/\partial y)dy = 0 \) and this corresponds to the ODE if \( \partial f/\partial x = M \) and \( \partial f/\partial y = N \). But if \( f \) is twice differentiable we have

\[
\partial M/\partial y = \partial^2 f/\partial x \partial y = \partial^2 f/\partial y \partial x = \partial N/\partial x.
\]

It turns out that this necessary condition for ‘exactness’ is also sufficient if the domain we consider has no holes (is simply connected).

**Example 38.** \( (y - x^3)dx + (x + y^2)dy = 0 \). The exactness condition is satisfied (check!). How to find \( f \) with \( \partial f/\partial x = \)
$y - x^3$ and $\partial f / \partial y = x + y^2$? The first equation gives $f(x, y) = yx - (1/4)x^4 + c(y)$. The second equation then gives $x + c'(y) = \partial f / \partial y = x + y^2$. Hence $c(y) = y^3/3 + c_0$ and $f(x) = yx - (1/4)x^4 + y^3/3 + c_0$ is a solution.

Sometimes you can rewrite the equation to make it exact.

Example 39. $ydx + (x^2y - x)dy = 0$. This equation is not exact (indeed, $\frac{\partial y}{\partial y} \neq \frac{\partial (x^2y - x)}{\partial x}$). If we rewrite the equation as $y/x^2dx + (y - 1/x)dy = 0$ then it becomes exact.

Clearly this was a lucky guess. Sometimes one can guess that by multiplying by a function of (for example) $x$ the ODE becomes exact.

Example 40. The equation $(xy - 1)dx + (x^2 - xy)dy = 0$ is not exact. Let us consider the equation $\mu(x)(xy - 1)dx + \mu(x)(x^2 - xy)dy = 0$. The exactness condition is $\mu x = \mu'(x^2 - xy) + \mu(2x-y)$. Rewriting this gives $\mu'(x)(x-y)+\mu(x)(x-y) = 0$, and so $x\mu' + \mu = 0$ implies the exactness condition. So we can take $\mu(x) = 1/x$. So instead of the original ODE we solve $(y - 1/x)dx + (x - y)dy = 0$ as in the previous example.

A.6 Substitutions

• Sometimes one can simplify the ODE by a substitution.

• One instance of this method, is when the ODE is of the form $M(x,y)dx + N(x,y)dy = 0$ where $M, N$ are homogeneous polynomials of the same degree.

In this case we can simplify by substituting $z = y/x$.

Example 41. $(x^2 - 2y^2)dx + xydy = 0$. Rewrite this as $\frac{dy}{dx} = \frac{-x^2 + 2y^2}{xy}$. Substituting $z = y/x$, i.e. $y(x) = z(x)x$ gives

$$x \frac{dz}{dx} + z = \frac{dy}{dx} = \frac{-1 + 2z^2}{z}.$$
Hence
\[ \frac{dz}{dx} = -\frac{1}{z} + z. \]
This can be solved by separation of variables.

### A.7 Higher order linear ODE’s with constant coefficients

Note that each \( y_1 \) and \( y_2 \) are solutions of
\[ y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0 \]  \hspace{1cm} (18)
then linear combinations of \( y_1 \) and \( y_2 \) are also solutions.

Substituting \( y(x) = e^{rx} \) in this equation gives:
\[ e^{rn} \left( r^n + a_{n-1}r^{n-1} + \cdots + a_0 \right) = 0. \]
Of course the polynomial equation \( r^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0 \) has \( n \) solutions \( r_1, \ldots, r_n \in \mathbb{C} \).

**Case 1:** If these \( r_i \)'s are all different (i.e. occur with single multiplicity), then we obtain as a solution:
\[ y(x) = c_1 e^{r_1 x} + \cdots + c_n e^{r_n x}. \]

**Case 2:** What if, say, \( r_1 \) is complex? Then \( \bar{r}_1 \) is also a root, so we may (by renumbering) assume \( r_2 = \bar{r}_1 \) and write \( r_1 = \alpha + \beta i \) and \( r_2 = \alpha - \beta i \) with \( \alpha, \beta \in \mathbb{R} \). So
\[ e^{r_1 x} = e^{\alpha x} (\cos(\beta x) + i \sin(\beta x)), \ e^{r_2 x} = e^{\alpha x} (\cos(\beta x) - i \sin(\beta x)), \]
and \( c_1 e^{r_1 x} + c_2 e^{r_2 x} = (c_1 + c_2) e^{\alpha x} \cos(\beta x) + (c_1 - c_2) i e^{\alpha x} \sin(\beta x) \).
Taking \( c_1 = c_2 = A/2 \in \mathbb{R} \implies c_1 e^{r_1 x} + c_2 e^{r_2 x} = Ae^{\alpha x} \cos(\beta x) \). On the other hand, taking \( c_1 = -(B/2)i = -c_2 \implies c_1 e^{r_1 x} + c_2 e^{r_2 x} = Be^{\alpha x} \sin(\beta x) \) (nothing prevents us choosing \( c_i \) non-real!!).
So if \( r_1 = r_2 \) is non-real, we obtain as a general solution
\[
y(x) = Ae^{\alpha x} \cos(\beta x) + Be^{\alpha x} \sin(\beta x) + c_3e^{r_3x} + \cdots + c_n e^{r_nx}.
\]

**Case 3: Repeated roots:** If \( r_1 = r_2 = \cdots = r_k \) then one can check that \( c_1e^{r_1x} + c_2xe^{r_2x} + \cdots + c_kx^k e^{r_1x} \) is a solution.

**Case 4: Repeated complex roots:** If \( r_1 = r_2 = \cdots = r_k = \alpha + \beta i \) are non-real, then we have corresponding roots \( r_{k+1} = r_{k+2} = \cdots = r_{2k} = \alpha - \beta i \) and we obtain as solution
\[
c_1e^{\alpha x} \cos(\beta x) + \cdots + c_kx^k e^{\alpha x} \cos(\beta x) + \\
+ c_{k+1}e^{\alpha x} \sin(\beta x) + \cdots + c_{2k}x^k e^{\alpha x} \sin(\beta x).
\]

**Example 42. Vibrations and oscillations of a spring**
One can model an object attached to a spring by \( Mx'' = F_s + F_d \) where \( F_d \) is a damping force and \( F_s \) a spring force. Usually one assumes \( F_d = -cx' \) and \( F_s = -kx \). So
\[
Mx'' + cx' + kx = 0 \text{ or } x'' + 2bx' + a^2x = 0
\]
where \( a = \sqrt{k/M} > 0 \) and \( b = c/(2M) > 0 \).

Using the previous approach we solve \( r^2 + 2br + a^2 \), i.e. \( r_1, r_2 = \frac{-2b \pm \sqrt{4b^2 - 4a^2}}{2} = -b \pm \sqrt{b^2 - a^2} \).

**Case 1:** If \( b^2 - a^2 > 0 \) then both roots are real and negative. So \( x(t) = x_0(e^{r_1t} + e^{r_2t}) \) is a solution and as \( t \to \infty \) we get \( x(t) \to 0 \).

**Case 2:** If \( b^2 - a^2 = 0 \) then we obtain \( r_1 = r_2 = -a \) and \( x(t) = Ae^{-at} + Be^{-at} \). So \( x(t) \) still goes to zero as \( t \to \infty \), but when \( B \) is large, \( x(t) \) can still grow for \( t \) not too large.

**Case 3:** If \( b^2 - a^2 < 0 \). Then \( x(t) = e^{-bt}(A \cos(\alpha t) + B \sin(\alpha t)) \) is a solution. Solutions go to zero as \( t \to \infty \) but oscillate.
Example 43. Vibrations and oscillations of a spring with forcing

Suppose one has external forcing

\[ Mx'' + cx' + kx = F_0 \cos(\omega t). \]

If \( b^2 - a^2 < 0 \) (using the notation of the previous example) then

\[ e^{-bt}(A \cos(\alpha t) + B \sin(\alpha t)) \]

is still the solution of the homogeneous part and one can check

\[ \frac{F_0}{\sqrt{(k - \omega^2 M)^2 + \omega^2 c^2}}(\omega c \sin(\omega t) + (k - \omega^2 M) \cos(\omega t)) = \]

\[ \frac{F_0}{\sqrt{(k - \omega^2 M)^2 + \omega^2 c^2}} \cos(\omega t - \phi) \]

is a particular solution where \( \omega = \arctan(\omega c/(k - \omega^2 M)) \).

\[ \frac{F_0}{\sqrt{(k - \omega^2 M)^2 + \omega^2 c^2}} \cos(\omega t - \phi) \]

is a particular solution where \( \omega = \arctan(\omega c/(k - \omega^2 M)) \).

Here \( c \) is the damping, \( M \) is the mass and \( k \) is the spring constant.

- If damping \( c \approx 0 \) and \( \omega \approx k/M \) then the denominator is large, and the oscillation has large amplitude.

- \((k - \omega^2 M)^2 + \omega^2 c^2\) is minimal for \( \omega = \sqrt{\frac{k}{M} - \frac{c^2}{2M^2}} \)

and so this is the natural frequency (or eigen-frequency).

- This is important for bridge designs (etc), see

  - [http://www.ketchum.org/bridgecollapse.html](http://www.ketchum.org/bridgecollapse.html)
  - [http://www.youtube.com/watch?v=3mclp9QmCGs](http://www.youtube.com/watch?v=3mclp9QmCGs)
  - [http://www.youtube.com/watch?v=gQK21572oSU](http://www.youtube.com/watch?v=gQK21572oSU)
**A.8 Solving ODE’s with maple**

**Example 44.**

```maple
> ode1 := diff(x(t), t) = x(t)^2;
\[ \frac{d}{dt} x(t) = x(t)^2 \]

> dsolve(ode1);
\[ x(t) = \frac{1}{-t + _C1} \]

> dsolve({ode1, x(0) = 1});
\[ x(t) = -\frac{1}{t - 1} \]
```

**Example 45.** Example: \( y'' + 1 = 0 \).

```maple
> ode5 := diff(y(x), x, x)+1 = 0;
\[ \frac{d^2}{dx^2} y(x) + 1 = 0 \]

> dsolve(ode5);
\[ y(x) = -\frac{1}{2} x + _C1 x + _C2 \]
```

Figure 5: The vector field \((1, \sin(t))\) drawn with the Maple command: `with(plots):fieldplot([1, sin(t)], t = -1 .. 1, x = -1 .. 1, grid = [20, 20], color = red, arrows = SLIM);`
A.9 Solvable ODE’s are rare

It is not that often that one can solve an ODE explicitly. What then?

- Use approximation methods.
- Use topological and qualitative methods.
- Use numerical methods.

This module will explore all of these methods.

In fact, we need to investigate whether we can even speak about solutions. Do solutions exist? Are they unique? Did we find all solutions in the previous subsections?

A.10 Chaotic ODE’s

Very simple differential equations can have complicated dynamics (and clearly cannot be solved analytically). For example the famous Lorenz differential equation

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xy - bz
\end{align*}
\]

(19)

with \(\sigma = 10, r = 28, b = 8/3.\)

has solutions which are chaotic and have sensitive dependence (the butterfly effect).

http://www.youtube.com/watch?v=ByH8_nKD-ZM
Appendix B  Multivariable calculus

Some of you did not do multivariable calculus. This note provides a crash course on this topic. This note also includes some important theorems which are not covered in 2nd year courses.

These notes include examples that are taken from the internet.

B.1 Jacobian

Suppose that $F: U \to V$ where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^p$. We say that $F$ is differentiable at $x \in U$ if there exists a linear map $A: \mathbb{R}^n \to \mathbb{R}^m$ (i.e. a $m \times n$ matrix $A$)

$$\frac{|(F(x + u) - F(x)) - Au|}{|u|} \to 0$$

as $u \to 0$. In this case we define $DF_x = A$.

- In other words $F(x + u) = F(x) + Au + o(|u|)$. ($A$ is the linear part of the Taylor expansion of $F$).
- How to compute $DF_x$? This is just the Jacobian matrix, see below.
- If $f: \mathbb{R}^n \to \mathbb{R}$ then $Df_x$ is a $1 \times n$ matrix which is also called $	ext{grad}(f)$ or $\nabla f(x)$.

Example 46. Let $F(x, y) = \left( \begin{array}{c} x^2 + yx \\ xy - y \end{array} \right)$ then

$$DF_{x,y} = \left( \begin{array}{cc} 2x + y & x \\ y & x - 1 \end{array} \right).$$

Usually one denotes by $(Df_x)u$ is the directional derivative of $f$ (in the direction $u$) at the point $x$. 
Example 47. If $F(x, y) = \left( \begin{array}{c} x^2 + y \vspace{1ex} \vspace{1ex} x \\
\vspace{1ex} \vspace{1ex} y \\
\vspace{1ex} \vspace{1ex} \vspace{1ex} y 
\end{array} \right)$ and $e_1 = \left( \begin{array}{c} 1 \\
0 
\end{array} \right)$ then $(DF_{x,y})e_1 = \left( \begin{array}{c} 2x + y \\
\vspace{1ex} \vspace{1ex} y \\
\vspace{1ex} \vspace{1ex} x - 1 
\end{array} \right)e_1 = \left( \begin{array}{c} 2x + y \\
\vspace{1ex} \vspace{1ex} y 
\end{array} \right)$. This is what you get when you fix $y$ and differentiate w.r.t. $x$ in $F(x, y)$.

For each fixed $y$ one has a curve $x \mapsto F(x, y) = \left( \begin{array}{c} x^2 + yx \\
\vspace{1ex} \vspace{1ex} xy - y 
\end{array} \right)$ and $(DF_{x,y})e_1 = \left( \begin{array}{c} 2x + y \\
\vspace{1ex} \vspace{1ex} y 
\end{array} \right)$ gives its speed vector.

**Remark:** Sometimes one writes $DF(x, y)u$ instead of $DF_{x,y}u$.

If $u$ is the $i$-th unit vector $e_i$ then one often writes $D_i F_{x,y}$ and if $i = 1$ something like $D_x F(x, y)$.

**Theorem 25** (Multivariable Mean Value Theorem). If $f : \mathbb{R} \to \mathbb{R}^m$ is continuously differentiable then $\forall x, y \in \mathbb{R}$ there exists $\xi \in [x, y]$ so that $|f(x) - f(y)| \leq |Df_{\xi}| |x - y|$.

**Proof:** By the Main Theorem of integration, $f(y) - f(x) = \int_x^y \! Df_s \, ds$ (where $Df_s$ is the $n \times 1$ matrix (i.e. vertical vector) of derivatives of each component of $f$). So

\[
|f(x) - f(y)| = |\int_x^y \! Df_s \, ds| \leq \int_x^y |Df_s| \, ds \leq \max_{s \in [x, y]} |Df_s| |x - y| \leq |Df_{\xi}| |x - y|
\]

for some $\xi \in [x, y]$. \qed

**Corollary:** If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable then for each $x, y \in \mathbb{R}^n$ there exists $\xi$ in the arc $[x, y]$ connecting $x$ and $y$ so that $|f(x) - f(y)| \leq |Df_{\xi}(u)| |x - y|$ where $u = (x - y)/|x - y|$. **Proof:** just consider $f$ restricted to the line connecting $x, y$ and apply the previous theorem.
B.2 The statement of the inverse function theorem

**Theorem 26 (The Inverse Function Theorem).** Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $F: U \to \mathbb{R}^n$ be continuously differentiable and suppose that the matrix $DF_p$ is invertible. Then there exist open sets $W \subset U$ and $V \subset \mathbb{R}^n$ with $p \in W$ and $F(p) \in V$, so that $F: W \to V$ is a bijection and so that its inverse $G: V \to W$ is also differentiable.

**Definition** A map $F: U \to V$ which has a differentiable inverse is called a **diffeomorphism**.

**Proof:** Without loss of generality we can assume that $p = 0 = F(p)$ (just apply a translation). By composing with a linear transformation we can even also assume $DF_0 = I$. Since we assume that $x \mapsto DF_x$ is continuous, there exists $\delta > 0$ so that

$$||I - DF_x|| \leq 1/2 \text{ for all } x \in \mathbb{R} \text{ with } |x| \leq 2\delta. \quad (20)$$

Here, as usual, we define the norm of a matrix $A$ to be

$$||A|| = \sup\{|Ax|; |x| = 1\}.$$ 

Given $y$ with $|y| \leq \delta/2$ define the transformation

$$T_y(x) = y + x - F(x).$$

Note that

$$T_y(x) = x \iff F(x) = y.$$ 

So finding a fixed point of $T_y$ gives us a point $x$ for which $G(y) = x$.

We will find $x$ using the Banach Contraction Mapping Theorem.

**(Step 1)** By (20) we had $||I - DF_x|| \leq 1/2$ when $|x| \leq 2\delta$. Therefore, the Mean Value Theorem applied to $x \mapsto x - F(x)$ gives
\[ |x - F(x) - (0 - F(0))| \leq \frac{1}{2} |x - 0| \text{ for } |x| \leq 2\delta \]

Therefore if \(|x| \leq \delta\) then

\[ |T_y(x)| \leq |y| + |x - F(x)| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta. \]

So \(T_y\) maps the closed ball \(B := B_\delta(0)\) into itself.

(Step 2) \(T_y : B \to B\) is a contraction since if \(x, z \in B_\delta(0)\) then \(|x - z| \leq 2\delta\) and so we obtain by the Mean Value Theorem again

\[ |T_y(x) - T_y(z)| = |x - F(x) - (z - F(z))| \leq \frac{1}{2} |x - z|. \quad (21) \]

(Step 3) Since \(B_\delta(0)\) is a Banach space, there exists a unique \(x \in B_\delta(0)\) with \(T_y(x) = x\) i.e. so that \(F(x) = y\).

(Step 4) The upshot is that for each \(y \in B_{\delta/2}(0)\) there is precisely one solution \(x \in B_\delta(0)\) of the equation \(F(x) = y\). Hence there exists \(W \subset B_\delta(0)\) so that the map

\[ F : W \to V := B_{\delta/2}(0) \]

is a bijection. So \(F : W \to V\) has an inverse, denoted by \(G\).

(Step 5) \(G\) is continuous: Set \(u = F(x)\) and \(v = F(z)\). Applying the triangle and the 2nd inequality in equation (21),

\[ |x - z| = |(x - z) - (F(x) - F(z)) + (F(x) - F(z))| \leq |(x - z) - (F(x) - F(z))| + |F(x) - F(z)| \leq \frac{1}{2} |x - z| + |F(x) - F(z)|. \]

So \(|G(u) - G(v)| = |x - z| \leq 2|F(x) - F(z)| = 2|u - v|\).

(Step 6) \(G\) is differentiable:

\[ |(G(u) - G(v)) - (DF_z)^{-1}(u - v)| = |x - z - (DF_z)^{-1}(F(x) - F(z))| \leq \]
\[(|DF_z|^{-1}||DF_z(x-z)-(F(x)-F(z))|=o(|x-z|)=2o(|u-v|)).\]
as \[(|DF_z|^{-1}||\text{is bounded, using the definition and the last inequality in step 5. Hence}\]
\[|G(u)-G(v)-(DF_z)^{-1}(u-v)|=o(|u-v|)\]
proving that \(G\) is differentiable and that \(DG_v=(DF_z)^{-1}\).

**Example 48.** Consider the set of equations
\[
x^2 + y^2 = u, \sin(x) + \cos(y) = v.
\]
Given \((u, v)\) near \((u_0, v_0) = (2, \cos(1) + \sin(1))\) is it possible to find a unique \((x, y)\) near to \((x_0, y_0) = (1, 1)\) satisfying this set of equations? To check this, we define
\[
F(x, y) = \left( \begin{array}{c}
x^2 + y^2 \\
\sin(x) + \cos(y)
\end{array} \right).
\]
The Jacobian matrix is
\[
\left( \begin{array}{cc}
x^2 - y^2 \\
\cos(x)
\end{array} \right).
\]
The determinant of this is \(u^2 - x^2 \sin(y) - \frac{2u}{x} \cos(x)\) which is non-zero near \((1, 1)\). So for every \((u, v)\) sufficiently close to \((u_0, v_0)\) one can find a unique solution near to \((x_0, y_0)\) to this set of equations. Near \((\pi/2, \pi/2)\) probably not.

**B.3 The Implicit Function Theorem**

**Theorem 27** (Implicit Function Theorem). Let \(F: \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^n\) be differentiable and assume that \(F(0, 0) = 0\). Moreover, assume that \(n \times n\) matrix obtained by deleting the first \(p\) columns of the matrix \(DF_{0,0}\) is invertible. Then there exists a function \(G: \mathbb{R}^p \to \mathbb{R}^n\) so that for all \((x, y)\) near \((0, 0)\)
\[y = G(x) \iff F(x, y) = 0.\]
The proof is a fairly simple application of the inverse function theorem, and won’t be given here. The $\mathbb{R}^p$ part in $\mathbb{R}^p \times \mathbb{R}^n$ can be thought as parameters.

**Example 49.** Let $f(x, y) = x^2 + y^2 - 1$. Then one can consider this as locally as a function $y(x)$ when $\partial f/\partial y = 2y \neq 0$.

**Example 50.** What can you say about solving the equations

\[
\begin{align*}
x^2 - y^2 - u^3 + v^2 + 4 &= 0 \\
2xy + y^2 - 2u^2 + 3v^4 + 8 &= 0
\end{align*}
\]

for $u, v$ in terms of $x, y$ in a neighbourhood of the solution $(x, y, y, v) = (2, -1, 2, 1)$. Define

\[F(x, y, y, v) = (x^2 - y^2 - u^3 + v^2 + 4, 2xy + y^2 - 2u^2 + 3v^4 + 8)\]

We have to consider the part of the Jacobian matrix which concerns the derivatives w.r.t. $u, v$ at this point. That is

\[
\left. \begin{pmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{pmatrix} \right|_{(2, -1, 2, 1)} = \begin{pmatrix} -12 & 2 \\ -8 & 12 \end{pmatrix}
\]

which is an invertible matrix.

So locally, near $(2, -1, 2, 1)$ one can write

\[(u, v) = G(x, y)\] that is $F(x, y, G_1(x, y), G_2(x, y)) = 0$.

To determine $\partial G_1/\partial x$ (i.e. $\partial u/\partial x$) we differentiate this. Indeed, writing $u = G(x, y)$ and $v = G(x, y)$ and differentiating

\[
\begin{align*}
x^2 - y^2 - u^3 + v^2 + 4 &= 0, \\
2xy + y^2 - 2u^2 + 3v^4 + 8 &= 0,
\end{align*}
\]

w.r.t. $x$ one has

\[
\begin{align*}
2x - 3u^2 \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} &= 0, \\
2y - 4u \frac{\partial u}{\partial x} + 12v^3 \frac{\partial v}{\partial x} &= 0.
\end{align*}
\]
So
\[
\left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial x} \right) = \begin{pmatrix} 3u^2 & -2v \\ 4u & -12v^3 \end{pmatrix} \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \frac{1}{8uv - 36u^2v^2} \begin{pmatrix} -12v^3 & 2v \\ -4u & 3u^2 \end{pmatrix} \begin{pmatrix} 2x \\ 2y \end{pmatrix}
\]

Hence
\[
\frac{\partial u}{\partial x} = \frac{(-24xv^3 + 4vy)}{8uv - 36u^2v^2}.
\]
Appendix C  Prerequisites

C.1  Function spaces

1. Let $f : [0, 1] \to \mathbb{R}$ be a function and $f_n : [0, 1] \to \mathbb{R}$ be a sequence functions. Define what it means to say that $f_n \to f$ uniformly.

   Answer: for all $\epsilon > 0$ there exists $n_0$ so that for all $n \geq n_0$ and all $x \in [0, 1]$ one has $|f_n(x) - f(x)| < \epsilon$.

   Answer 2: $||f_n - f||_\infty \to 0$ as $n \to \infty$ where $||f_n - f||_\infty = \sup_{x \in [0,1]} |f_n(x) - f(x)|$.

2. Let $f : [0, 1] \to \mathbb{R}$ be a function and $f_n : [0, 1] \to \mathbb{R}$ be a sequence functions. Define what it means to say that $f_n \to f$ point wise.

   Answer: for all $\epsilon > 0$ and all $x \in [0, 1]$ there exists $n_0$ so that for all $n \geq n_0$ one has $|f_n(x) - f(x)| < \epsilon$.

3. Let $f : [0, 1] \to \mathbb{R}$ be a function and $f_n : [0, 1] \to \mathbb{R}$ be a sequence functions. Assume that $f_n \to f$ uniformly and that $f_n$ is continuous. Show that $f$ is continuous.

   Answer: Take $\epsilon > 0$, $x \in [0, 1]$. Choose $n_0$ so that $|f_n - f|_\infty < \epsilon/3$ for $n \geq n_0$ and pick $\delta > 0$ so that $|f_{n_0}(x) - f_{n_0}(y)| < \epsilon/3$ for all $y$ with $|y - x| < \delta$. Then for all $y$ with $|y - x| < \delta$, $|f(x) - f(y)| < |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(y)| + |f_{n_0}(y) - f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$.

4. Let $f : [0, 1] \to \mathbb{R}$ be a function and $f_n : [0, 1] \to \mathbb{R}$ be a sequence functions. Assume that $f_n \to f$ point wise and that $f_n$ is continuous. Show that $f$ is not necessarily continuous.

   Answer: Take $f_n(x) = (1 - nx)$ for $x \in [0, 1/n]$ and $f_n(x) = 0$ elsewhere. Then $f_n \to f$ point wise, where $f(0) = 1$ and $f(x) = 0$ for $x \in (0, 1]$. 

102