Quasisymmetric rigidity of real maps

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The maps that we are iterating.

$f : [0,1] \to [0,1]$ or $S^1 \to S^1$ that are C^3 and satisfy some extra conditions.

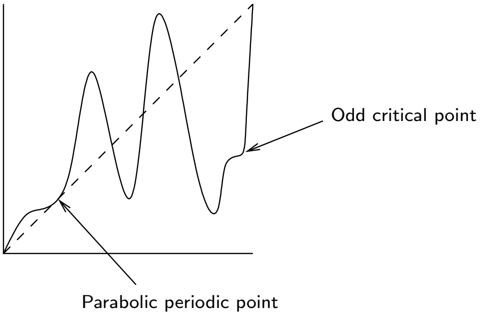


Figure : the type of map we will consider

Aim: Complete Sullivan's quasi-symmetric rigidity programme

A homeomorphism $h\colon [0,1] \to [0,1]$ is called **quasi-symmetric** (often abbreviated as *qs*) if there exists $K < \infty$ so that

$$\frac{1}{K} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq K$$

for all $x - t, x, x + t \in [0,1]$. ($\implies$ Hölder; has h qc extension to $\mathbb{C}$).

Sullivan's programme: prove that f is **quasi-symmetrically rigid**, i.e.

$$f, \tilde{f} \text{ is topologically conjugate } \implies$$

$$\tilde{f}, f \text{ are quasi-symmetrically conjugate.}$$

That is, homeomorphism h with $h \circ f = g \circ h$ is **'necessarily' qs**.

Remark about Sullivan's aim:

- Quasi-symmetric maps have a quasiconformal extension to $\mathbb{C}$.
- **Sullivan's aim:** $\mathcal{C}$ should be **infinite dimensional Teichmuller space** with metric $d(f,\tilde{f}) = \inf K_h$ where K_h is the dilatation of qc extension $H\colon \mathbb{C} \to \mathbb{C}$ of qs conjugacy $h\colon N \to N$ between f and $\tilde{f}$.
- Define $f \sim \tilde{f}$ when f are smooth conjugate. Is d a metric on $\mathcal{C}/\sim$?
- Yes (in the unimodal case, and probably also in the multimodal case). Indeed:
  $d(f,\tilde{f}) = 0 \implies$
  multipliers at corresponding periodic point of $f,\tilde{f}$ are equal $\implies$
  if $f,\tilde{f}$ unimodal, they are C^3 conjugate (by result of Li-Shen).
- Current project: endow $\mathcal{C}/\sim$ with manifold structure.

Completing Sullivan's qs-rigidity programme

Theorem (Clark-vS)

Let $N = [0,1]$ or $N = S^1$. Suppose $f, \tilde{f} : N \to N$ are **topologically conjugate** *and are in $\mathcal{C}$ with* **at least one critical point**. *Moreover, assume that the topological conjugacy is a bijection between*

- *the sets of critical points and the orders of corresponding critical points are the same, and*
- *the set of parabolic periodic points.*

Then f and $\tilde{f}$ are **quasisymmetrically conjugate**.

- This completes a programme initiated in the 80's by
  - **Sullivan** for interval maps: in his work on renormalisation;
  - **Herman** for circle homeo's: to use quasiconformal surgery.
- The result is optimal, in the sense that no condition can be dropped.
- When $N = S^1$, the assumption $\exists$ critical point implies $\exists$ periodic point.

Real versus complex methods

The space $\mathcal{C}$ consists of real interval maps, and includes

- all real analytic maps;
- all C^∞ maps with finitely many critical points of integer order;
- all C^3 maps with finitely many critical points of integer order and without parabolic cycles.

This is a totally real setting, but

- in the proof we shall use **complex methods**
- **having qs-conjugacies** makes it possible to **apply powerful complex tools** such as measurable Riemann mapping etc.

Assume C^3 because then f extends to a C^3 map $F\colon U \to \mathbb{C}$ with U neighbourhood of I in $\mathbb{C}$, so that F is **asymptotically holomorphic** of order 3 on I; that is,

$$\frac{\partial}{\partial \bar{z}} f(x,0) = 0, \text{ and } \frac{\frac{\partial}{\partial \bar{z}} f(x,y)}{|y|^2} \to 0$$

uniformly as $(x,y) \to I$ for $(x,y) \in U \setminus I$.

Class of maps, $\mathcal{C}$

- $\exists$ finitely many critical points $c_1, \ldots, c_b$,
- $x \mapsto f(x)$ is C^3 when $x \neq c_1, \ldots, c_b$
- near each critical point $c_i, 1 \leq i \leq b$, we can express

  $$f(x) = \pm|\phi(x)|^{d_i} + f(c_i),$$

  where ϕ is C^3 and d_i is an integer ≥ 2.
- **extra regularity** near **parabolic** periodic points.
  - Let $\lambda \in \{-1, 1\}$ be multiplier and s the period of p, then $\exists n$ with

    $$f^s(x) = p + \lambda(x-p) + a(x-p)^{n+1} + R(|x-p|), R(|x-p|) = o(|x-p|^{n+1})$$

  - $f \in C^{n+2}$ near p,

The extra regularity makes it possible to use the Taylor series of f to study the local dynamics near the parabolic periodic points.

Why is qs-rigidity useful?

QS (QC) rigidity plays a crucial role in the following results:

- Density of hyperbolic maps (maps where each critical point converges to an attracting periodic point) (Lyubich, Graczyk-Świątek, Kozlovski, Shen, Kozlovski-Shen-vS).
- Density of hyperbolicity of transcendental maps: Rempe-vS (e.g. maps from Arnol'd family).
- Topological conjugacy classes of certain maps are connected and analytic (infinite dimensional) manifolds.
  This is a crucial fact in the proof that in a non-trivial family of analytic unimodal maps almost every map is **regular or Collet-Eckmann** (Lyubich, Avila-Lyubich-de Melo, Avila-Moreira, Avila-Lyubich-Shen, Clark).
- Hyperbolicity of renormalization (Lyubich, Avila-Lyubich). (**Multimodal Palis conjecture**).
- Monotonicity of entropy for real polynomial multimodal maps (Bruin-vS) and trigonometric families (Rempe-vS).

Previous results

- **Polynomial case:** Real polynomials *with only real critical points all of even order* (Kozlovski-Shen-vS, 2007).
- **Semi-local results:** conjugacy is qs restricted to the post-critical set (renorm. of bdd type Sullivan 1990s', critical covering maps Levin-vS 2000, persistently recurrent + extra condition Shen 2003).
- **Critical covering maps of the circle:** *Real analytic maps with one critical point and no parabolic points* (Levin-vS 2000).
- **Critical circle homeomorphisms:** *One critical point* (Herman-Świątek, 1988).
- **Smooth maps:** for maps for particular combinatorics and fast decaying geometry (Jakobson-Świątek, Lyubich, early 1990's).

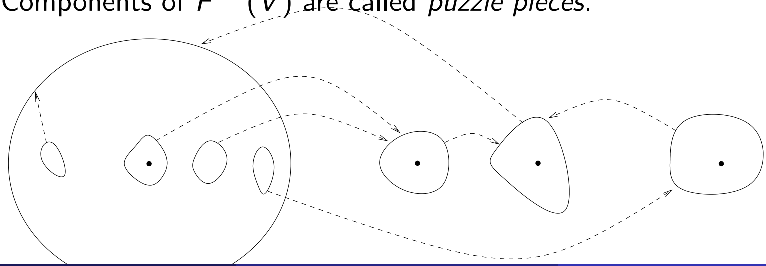
Some issues to overcome: make qs global; not polynomial, not even real analytic; match critical points with different behaviours; parabolic periodic points; odd critical points.

Real versus non-real maps

- For **complex** (non-real) polynomials there are partial results (**qc-rigidity**), due to Kozlovski-vS, Lyubich-Kahn, Levin, Cheraghi, Cheraghi-Shishikura. However, in general **wide open** (**related to local connectivity of Mandelbrot set** and **Fatou conjecture**).
- So methods require a mixture of real and complex tools.
- One of the main ingredients, **complex bounds**, fails for general complex maps.

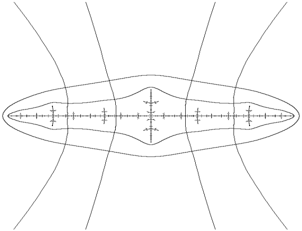
Go to complex plane: complex box mappings

- qs-rigidity, requires control of high iterates ('**compactness**').
- Turns out to be useful to **construct** an extension to $\mathbb{C}$: when f, g are real analytic, use holomorphic extension of f, g to small neighbourhoods of $[0, 1]$ in $\mathbb{C}$.
- Prove that first return maps of f, g to small intervals, extend to a **'complex box mapping'** $F \colon U \to V$, see figure.
- Each component of U is mapped as a branched covering onto a component of V, and components of U are either compactly contained or equal to a component of V.
- Components of $F^{-n}(V)$ are called *puzzle pieces*.



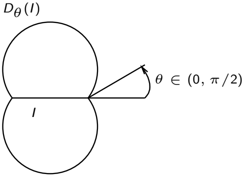
How to construct a complex box mapping?

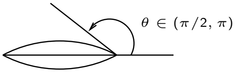
- In the polynomial case one ucan se the Yoccoz puzzle partition (using rays and equipotentials).
- In the real analytic case or smooth case one has to do this by hand: not obvious at all that pullbacks of V is contained in V.
- In the non-renormalizable case one can repeatedly take first return maps to central domains.
- In the infinitely renormalizable case one has to start from scratch again and again (note: there is no straightening theorem when f is not holomorphic).



Poincaré disks and their diffeomorphic pullbacks

Let d be Poincaré metric on $\mathbb{C}_I := (\mathbb{C} - \mathbb{R}) \cup I$. A Poincaré disk is a set of the form $\{z; d(z, I) \leq d_0\}$ and is bounded by the union of two circle segments. These are used to construct a "Yoccoz puzzle" by hand.





Poincaré lens domain

- If f is polynomial with only real critical points and $f : J \to I$ a diffeomorphism: no loss of angle when pulling back $D_\theta(I)$ (by the Schwarz inclusion lemma).
- If f is real analytic or only C^3 one looses angle, whose amount depends on the size of $|I|^2$. One therefore needs to control this term along a pullback.

Poincaré disks and their pullbacks through critical points

- If $f : J \to I$ has a unique critical point then one looses more angle:

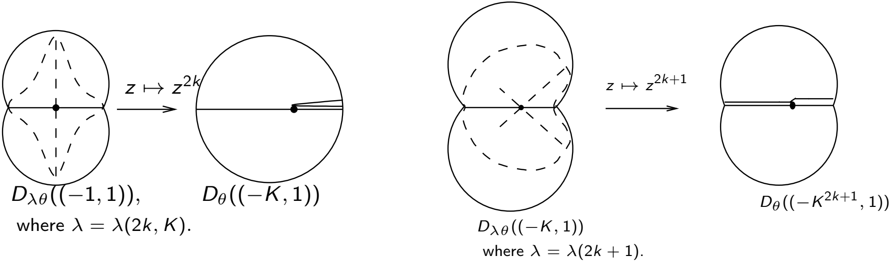


Figure : Inverses of Poincaré disks through a critical point

Control of high iterates: complex bounds

In addition we need **complex bounds,** i.e. **universal control** on **shape** and **position** of components of U inside components of V.

Clark-Trejo-vS:

Theorem (Complex box mappings with complex bounds)

One can construct complex box mappings with complex bounds on arbitrarily small scales.

- Previous similar partial results by Sullivan, Levin-vS, Lyubich-Yampolsky and Graczyk-Świątek, Smania, Shen.
- **Key ingredient** in e.g. **renormalisation**, e.g. Avila-Lyubich.
- Complex bounds give better control than real bounds.
- Clark-Trejo-vS: something similar even for C^3 maps, but then F is only asymptotically holomorphic.

Proving complex bounds

- From the **enhanced nest** construction (see next •) and a **remarkable result due to Kahn-Lyubich**, given a non-renormalizable complex box mapping at one level, one can obtain **complex box mappings with complex bounds at arbitrary deep levels**.
- The **enhanced nest** is a sophisticated choice of a sequence of puzzle pieces $U_{n(i)}$, so that
  1. $\exists k(i)$ for which $F^{k(i)}\colon U_{n(i+1)} \to U_{n(i)}$ is a branched covering map with degree bounded by some universal number N.
  2. its inverse transfers geometric information efficiently from scale $U_{n(i)}$ to the smallest possible scale $U_{n(i+1)}$.
- Other choices will not give complex bounds, in general.
- In the **renormalizable case** and also in the C^3 case, the construction of complex box mappings and the proof of complex bounds is significantly *more involved*.
- Critical points of odd order require quite a bit of additional work.

Touching box mappings

If $f, \tilde{f} \in \mathcal{C}$ are topologically conjugate, the role of the Böttcher coordinate for polynomials is played by the construction of an "external conjugacy" between touching box mappings F_T and $\tilde{F}_T$.

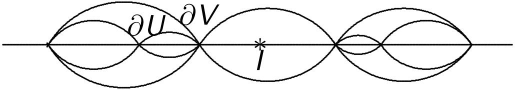


Figure : A touching box mapping $F_T \colon U \to V$: at first the domain U does not contain critical points of f (marked with the symbol $*$), but V covers the whole interval.

- No loss of angle at periodic boundary points (these will include all parabolic points).
- Real trace of the range contains a neighbourhood of the set of critical points, immediate basins of attracting cycles, and covers the interval.
- Used to pullback qc-conjugacies through branches that avoid $\mathrm{Crit}(f)$.

Idea for proving quasi-symmetric rigidity

- Using the **complex bounds** and a *methodology for constructing quasi-conformal homeomorphisms* (building on papers of Kozlovski-Shen-vS and Levin-vS), we construct quasi-conformal pseudo conjugacies on **small scale**.
  Here we use our so-called QC-criterion (related to result of Heinonen-Koskela; something similar obtained by Smania).
- Eventually we will need take **infinitely** many lens-shaped domains in some components of V.
- Then develop a technology to glue the local information together. Requires additional care when there are several critical points.
- Need to consider regions whose boundaries are no longer quasi-circles.

Remarks:

- In the C^3 case f, g have *asymptotically holomorphic extensions* near $[0, 1]$. Issue to deal with: arbitrary high iterates of f and g are not necessarily close to holomorphic.

A remark about qs-rigidity of critical circle homeomorphisms

The following result follows from work of de Faria-de Melo.

Theorem (follows from: de Faria-de Melo who use a result of Yoccoz)

Suppose that $f, \tilde{f} : S^1 \to S^1$ are critical circle homeomorphisms with irrational rotation number and one critical point. If $h : S^1 \to S^1$ is a homeomorphism such that $h \circ f = \tilde{f} \circ h$, then h is quasisymmetric.

Observation: No need to assume h maps the critical point of f to the critical point of $\tilde{f}$: it turns out the dynamical partition generated by any point (not just the critical point) the lengths of adjacent intervals are comparable.

We have more:

Theorem (Clark-vS)

Suppose that $f, \tilde{f} \in \mathcal{C}$ are topologically conjugate critical circle homeomorphisms, then f and $\tilde{f}$ are quasisymmetrically conjugate.

So far I discussed what is **qs-rigidity**, and **why it holds**. Next: Why is **qs-rigidity useful**?

Roughly, because it provides a comprehensive understanding of the dynamics, which opens up a pretty full understanding.

I will discuss **two applications**. Both are based on **tools from complex analysis** that become available because of **quasi-symmetric rigidity**.

A third application will *hopefully* be a resolution of the *1-dimensional Palis conjecture* in full generality.

Application 1: Hyperbolic maps

A smooth map $f : \mathbb{R} \to \mathbb{R}$ is **hyperbolic** if

- Lebesgue a.e. point is attracted to some periodic orbit with multiplier λ so that $|\lambda| < 1$, or *equivalently*
- each critical point of f is attracted to a periodic orbit *and* each periodic orbit is hyperbolic (i.e. with multiplier $\lambda \neq \pm 1$).

Martens-de Melo-vS: the **period** of periodic **attractors** is **bounded** $\implies$ hyperbolic maps have at most finitely many periodic attractors.

The notion of hyperbolicity was introduced by Smale and others because these maps are well-understood and:

- **Every hyperbolic map** satisfying an additional transversality condition, that no critical point is eventually mapped onto another critical point, is **structurally stable**. (A nearby map is *topologically conjugate*, i.e. same up to topological coordinate change.)

Hyperbolic one-dimensional maps are dense

- Fatou (20's) conjectured most rational maps on the Riemann sphere are hyperbolic.
- Smale (60's) conjectured that in higher dimensions, hyperbolic maps are dense. This turned out to be **false**.

Kozlovski-Shen-vS:

Theorem (Density of hyperbolicity for real polynomials)

Any real polynomial can be approximated by a hyperbolic real polynomials of the same degree.

and

Theorem (Density of hyperbolicity for smooth one-dimensional maps)

Hyperbolic 1-d maps are C^k dense, $k = 1, 2, \dots, \infty$.

This solves one of Smale's problems for the 21st century.

Density of hyperbolicity for real transcendental maps

Rempe-vS:

Theorem (Density of hyperbolicity for transcendental maps)

Density of hyperbolicity holds within the following spaces:

1. *real transcendental entire functions, bounded on the real line, whose singular set is finite and real;*
2. *transcendental functions* $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ *that preserve the circle and whose singular set (apart from* $0, \infty$*) is contained in the circle.*

Remarks:

- Hence, density of hyperbolicity within the famous **Arnol'd family** and within space of trigonometric polynomials.
- Result implies conjectures posed by de Melo-Salomão-Vargas.

Hyperbolicity is dense within generic families

Theorem (vS: Hyperbolicity is dense within generic families)

For any (Baire) C^∞ generic family $\{g_t\}_{t\in[0,1]}$ of smooth maps:

- *the number of critical points of each of the maps g_t is bounded;*
- *the set of t's for which g_t is hyperbolic, is open and dense.*

and

Theorem (vS: $\exists$ family of cubic maps with robust chaos)

There exists a real analytic one-parameter family $\{f_t\}$ of interval maps (consisting of cubic polynomials) so that

- *there exists no $t \in [0,1]$ with f_t is hyperbolic;*
- *f_0 and f_1 are not topologically conjugate.*

Question: What if f_0 and f_1 are '**totally different**'?

Density of hyperbolicity on $\mathbb{C}$?

Density of hyperbolicity for rational maps (Fatou's conjecture) is wide open. By Mañé-Sad-Sullivan it follows from:

Conjecture

If a rational map carries a measurable invariant line field on its Julia set, then it is a Lattès map.

Eremenko-vS:

Theorem

Any rational map on the Riemann sphere such that the multiplier of each periodic orbit is real, either is

- *an interval or circle map (Julia set is 1d), or*
- *a Lattès map.*

In the first case, the Julia set of course does not carry measurable invariant line field.

Strategy of the proof: local versus global perturbations

One approach: take g to be a **local perturbation** of f, i.e. find a 'bump' function h which is small in the C^k sense so that $g = f + h$ becomes hyperbolic.

- Difficulty with this approach: orbits pass many times through the support of the bump function.
- Jakobson (1971, in dimension one) and Pugh (1967, in higher dimensions but for diffeo's) used this approach to prove a C^1 **closing lemma**.
- In the C^2 **category** this approach has proved to be **unsuccessful** (but Blokh-Misiurewicz have partial results). Shen (2004) showed C^2 density using qs-rigidity results.

Proving density of hyperbolicity for $z^2 + c$

Density of hyperbolicity with family $z^2 + c$, $c \in \mathbb{R}$ holds if there exists no interval of parameters c of non-hyperbolic maps.

Sullivan showed that this follows *from* **quasi-symmetric rigidity** *of any* **non-hyperbolic map** f_c (by an open-closed argument):

- Measurable Riemann Mapping Theorem $\implies$
  $$I(f_c) = \{\tilde{c} \in \mathbb{R} \text{ s.t. } f_{\tilde{c}} \text{ topologically conjugate to } f_c\}$$
  is either *open or a single point*.
- Basic kneading theory $\implies$ $I(f_c)$ is *closed set*.

$\emptyset \subsetneq I(f_c) \subsetneq \mathbb{R}$ gives a contradiction unless $I(f_c)$ is a single point.

Using a slightly more sophisticated argument, Kozlovski-Shen-vS also obtain that quasi-symmetric rigidity implies density of hyperbolicity when there are more critical points.

Application 2: monotonicity of entropy

In the early 90's, Milnor posed the

Monotonicity Conjecture. The set of parameters within a family of real polynomial interval maps, for which the topological entropy is constant, is connected.

- A version of this conjecture was proved in the 1980's for the quadratic case.
- Milnor-Tresser (2000) proved conjecture for cubics using
  - planar topology (in the cubic case the parameter space is two-dimensional) and
  - density of hyperbolicity for real quadratic maps.
- Bruin-vS: the set of parameters corresponding to polynomials of degree $d \geq 5$ with constant entropy is in general NOT locally connected.

Monotonicity of entropy: the multimodal case

Given $d \geq 1$ and $\epsilon \in \{-1, 1\}$, let P_ϵ^d space of

1. real polynomials $f\colon [0,1] \to [0,1]$ of degree $= d$;
2. all critical points in $(0,1)$;
3. $\mathrm{sign}(f'(0)) = \epsilon$.

Bruin-vS show:

Theorem (Monotonicity of Entropy)

For each integer $d \geq 1$, each $\epsilon \in \{-1, 1\}$ and each $c \geq 0$,

$$\{f \in P_\epsilon^d ; h_{top}(f) = c\}$$

is connected.

- Main ingredient: is quasi-symmetric rigidity.
- Hope to remove assumption (2): (currently $d = 4$ with Cheraghi).
- Rempe-vS $\implies$ top. entropy of $x \mapsto a \sin(x)$ monontone in a.