Recent developments in interval dynamics

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Abstract. Dynamics in dimension-one has been an extremely active research area over the last decades. In this note we will describe some of the new developments of the recent years.

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1. Density of hyperbolicity

Interval maps $f\colon [0,1] \to [0,1]$ can have a surprisingly rich and complicated dynamics. In this paper we will describe results which show that in spite of this one can describe the metric orbit structure of 'most' maps extremely well.

The dynamics of *hyperbolic* maps can be described most easily: for these maps, Lebesgue almost every point in the interval is attracted to some hyperbolic periodic orbit (with multiplier between -1 and 1). By a result by Mañé [65] (for a simpler proof see [98]) it is equivalent to say that a map is hyperbolic if (i) each critical point of f is in the basin of a periodic attractor and (ii) each periodic orbit is hyperbolic. Since the period of periodic attractors is bounded, see [66], it follows that hyperbolic maps have at most finitely many periodic attractors.

As mentioned, hyperbolic maps are very well-understood. The following theorem (which was obtained by the authors, jointly with Kozlovski, see [50]) shows that 'most' maps are hyperbolic.

Theorem 1.1 (Density of hyperbolicity for real polynomials). *Any real polynomial can be approximated by hyperbolic real polynomials of the same degree.*

The above theorem allows us to prove the analogue of the Fatou conjecture in the smooth case, see [51], thus solving the 2nd part of Smale's eleventh problem for the 21st century [91]:

Theorem 1.2 (Density of hyperbolicity for smooth one-dimensional maps). *Hyperbolic maps are dense in the space of C^k maps of the compact interval or the circle, $k = 1, 2, \dots, \infty, \omega$.*

For quadratic maps $f_a = ax(1-x)$, the above theorems assert that the periodic windows (corresponding to hyperbolic maps with attracting periodic orbits) are dense in the bifurcation diagram. The quadratic case turns out to be special, because in this case certain return maps become almost linear. This special behaviour does not even hold for maps of the form $x \mapsto x^4 + c$.

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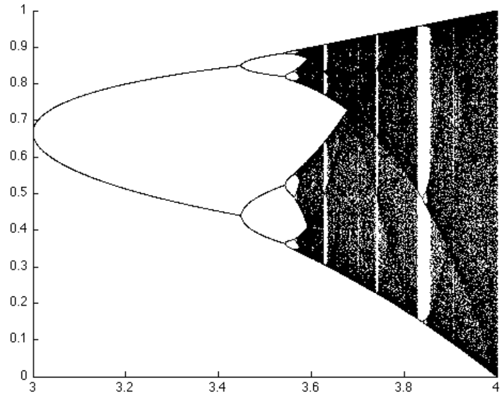

Figure 1.1. The Feigenbaum diagram

The problem of density of hyperbolicity in dimension-one has been considered since the 1920's. Indeed:

- Fatou stated the analogue of this problem in the context of rational maps on the Riemann sphere as a conjecture in the 1920's, see [33, page 73] and also [67, Section 4.1].
- Smale gave this problem 'naively' as a thesis problem in the 1960's, see [90].
- In 1971, Jakobson proved that the set of hyperbolic maps is dense in the C^1 topology, see [43].
- In the mid 1990's, the conjecture was solved in the quadratic case $x \mapsto ax(1-x)$ in a major breakthrough by Lyubich [61] and independently also by Graczyk and Świątek, [36] and [37].
- In 2000, Blokh and Misiurewicz [15] considered the problem of density of hyperbolicity in the C^2 topology, and were able to obtain a partial result.
- A few years later, Shen [87] proved C^2 density of hyperbolic maps.

Note that every hyperbolic map satisfying a mild transversality condition, namely that no critical point is eventually mapped onto another critical point, is *structurally stable*. So density of hyperbolicity implies that structural stable maps are dense.

1.1. Density of hyperbolicity within a large space of real transcendental map. Density of hyperbolicity also holds within classes of much more general maps, for example within the famous Arnol'd family and within the space of trigonometric polynomials. Indeed it was shown by the second author in a joint paper with Rempe, see [79], that

Theorem 1.3. *Density of hyperbolicity holds within the following spaces:*

1. *real transcendental entire functions, bounded on the real line, whose singular set is finite and real;*
2. *transcendental functions $f\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$ that preserve the circle and whose singular set (apart from $0,\infty$) is contained in the circle.*

In [79] also a number of other open problems are solved, including a number of conjectures of behaviour de Melo, Salomão and Vargas [29]. In this paper density of (real) hyperbolicity is also established replacing in assumption (1) the boundedness condition by a sector condition.

1.2. Hyperbolicity is dense within generic one-parameter families of one-dimensional maps.

Theorem 1.4 (Hyperbolicity is dense within generic families). *For any generic family $\{g_t\}_{t\in[0,1]}$ of smooth intervals maps (generic, in the sense of Baire), the following properties hold:*

- *the number of critical points of each of the maps g_t is bounded;*
- *the set of parameters t for which all critical points of g_t are in basins of periodic attractors, is open dense.*

The proof of this result follows easily from the theorems in the previous subsection, see [99]. On the other hand, as is shown in the same paper, it is easy to construct a real analytic one-parameter family f_t, $t \in [0,1]$ of polynomials so that none of the polynomials in this family are hyperbolic:

Theorem 1.5 (A family of cubic maps with robust chaos). *There exists a real analytic one-parameter family $\{f_t\}$ of interval maps (consisting of cubic polynomials) so that f_t has no periodic attractor for any $t \in [0,1]$, and so that not all maps within this family are topologically conjugate.*

1.3. Density of hyperbolicity for more general maps. Density of hyperbolicity is false in dimension ≥ 2. For a list of related interesting questions concerning the higher dimensional case, see [77].

The situation for rational maps on the Riemann sphere may well be more hopeful. In that context one has the following well-known conjecture, going back to Fatou:

Conjecture 1.6 (Density of hyperbolicity for rational maps). *Hyperbolic maps are dense within this space of rational maps of degree d on the Riemann sphere.*

In [64] it was shown that this conjecture follows from

Conjecture 1.7. *If a rational map carries a measurable invariant line field on its Julia set, then it is a Lattès map.*

More about this conjecture and related results can be found in [67]. In [50, 86] and finally [52] it was shown that real polynomials (acting on $\mathbb{C}$) do not carry such invariant line fields. Moreover, real polynomials have Julia sets which are locally connected, see [26, 50, 52, 55]. In [80] it was shown that, under some mild assumptions, real transcendental maps also do not carry invariant line fields.

Interestingly, any rational map on the Riemann sphere such that the multiplier of each periodic orbit is real, either has a Julia set which is contained in a circle (or line) or is a Lattès map, see [32].

1.4. Strategy of the proof: local versus global perturbations. Density of hyperbolicity means that given a map f one can find a map g so that g is hyperbolic and so that $g-$

f is ‘small’ in the C^k topology. It is tempting to consider the setting where g is a local perturbation of f. The purpose would then be to find a small ‘bump’ function h so that $g = f + h$ becomes hyperbolic. The difficulty with this approach is that orbits will pass many times through the support of the bump function. Pugh’s approach in his proof of the C^1 closing lemma, is to find a suitable neighbourhood U of x so that the first return of x to U is not too close to the boundary of x. In this way he is able to construct a function h whose support is in U, which creates a new fixed point of the first return of $g = f + h$ to U, in such a manner that h is C^1 close to zero. A related approach was used successfully in [43] to prove density of hyperbolicity in the C^1 topology, and in [15] for the C^2 topology, but with added assumptions on the dynamics of f. In [87], this approach was used in the case when one has a ‘lot of Koebe space’ while in the ‘essentially bounded geometry’ the proof relied on rigidity (in the sense described below). This rigidity approach also is the key ingredient in the proof of Theorem 1.1. As there is a great deal of evidence that local perturbations cannot be used to prove density of hyperbolicity in general, we discuss rigidity extensively in the next section.

1.5. Strategy of the proof: quasi-symmetric rigidity. Consider the following situation. Take a family of real quadratic maps $f_c(z) = z^2 + c$. To prove density of hyperbolicity we need to prove that there exists no interval of parameters $[c', c'']$ so that each map f_c with $c \in [c', c'']$ is non-hyperbolic. Sullivan showed that this follows from quasi-symmetric rigidity of any non-hyperbolic map f_c. Here f_c is called *quasi-symmetrically rigid* if the following property holds:

> If $f_{\tilde{c}}$, $f_{\hat{c}}$ are topologically conjugate to f_c, then $f_{\tilde{c}}$, $f_{\hat{c}}$ are quasi-symmetrically conjugate.

Here, as usual, a homeomorphism $h\colon [0,1] \to [0,1]$ is called *quasi-symmetric* (often abbreviated as *qs*) if there exists $K < \infty$ so that

$$\frac{1}{K} \le \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \le K$$

for all $x - t, x, x + t \in [0,1]$. By results about quasi-conformal maps (specifically the Measurable Riemann Mapping Theorem) it follows that the set of parameters $\tilde{c}$ so that $f_{\tilde{c}}$ is topologically conjugate to f_c is either a single point or an open interval $I(f_c)$. Since $I(f_c)$ is also a closed set (this follows from some basic kneading theory), the fact that $I(f_c)$ and its complement are both non-empty gives a contradiction unless $I(f_c)$ is a single point.

This argument does not go through directly for real polynomial maps with more than one critical point, but using related arguments, one still obtains that quasi-symmetric rigidity implies density of hyperbolicity, see [50, Section 2]. In the case of real analytic maps the argument to prove density of hyperbolicity is more subtle, see [51].

2. Quasi-symmetric rigidity

As remarked in the previous section, all current proofs of density of hyperbolicity rely on quasi-symmetric rigidity. The most general form can be found in [25], and states:

Theorem 2.1 (Quasi-symmetric rigidity). *Assume that $f, g\colon [0,1] \to [0,1]$ are real analytic and topologically conjugate. Alternatively, assume that $f, g\colon S^1 \to S^1$ are topologically conjugate and that f and g each have at least one critical point or at least one periodic point. Moreover, assume that the topologically conjugacy is a bijection between*

(1) *the set of critical points and the order of corresponding critical points is the same;*

(2) *the set of parabolic periodic points.*

Then the conjugacy between f and g is quasi-symmetric.

The proof of this theorem builds on the machinery developed in [50]. This paper was written jointly by the authors and Kozlovski; it developed many of the key ingredients required to prove density of hyperbolicity, see [51]. Theorem 2.1 is an extension of these results, and was obtained jointly by Clark and the 2nd author, and uses all of the technology from[51], but also extends ideas from [56].

Indeed, when f, g are real analytic, then we will use the fact that these maps have holomorphic extensions to small neighbourhoods of $[0,1]$. Nevertheless, in [25] we prove the analogous result when f and g are merely C^3 maps, under some weak additional assumptions; in this case we will use that f, g have asymptotically holomorphic extensions near $[0,1]$, but will need to deal with the fact that high iterates of f and g are not necessarily close to holomorphic.

It is not hard to see that if conditions (1) or (2) in the previous theorem are not satisfied, then the maps are not even necessarily Hölder conjugate.

Special cases of this theorem we known before: Lyubich [61] and Graczyk & Świątek [37] proved this result for real quadratic maps. As we will see their method of proof in the quadratic case does not work if the degree of the map is > 2. For the case of real polynomials with only real critical points (of even order), this theorem was proved in [50]. For maps which are real analytic, it was shown in [87, Theorem 2, page 345] that there exists a qs-conjugacy *restricted to $\omega(c)$* under the additional assumptions that the maps have no neutral periodic points, only non-degenerate critical points and have *'essentially bounded geometry'*. For covering maps of the circle (of degree ≥ 2) with one-critical point a global qs-conjugacy was constructed under the additional assumption that $\omega(c)$ is non-minimal and have no neutral periodic points, see [56]. When $\omega(c)$ is minimal, a qs-conjugacy restricted to $\omega(c)$ was constructed in [56].

For circle maps without periodic points, it is known that any two analytic critical circle homeomorphisms with one critical point, with the same irrational rotation number and the same order of the critical points are C^1-smoothly conjugate, see [47] (their work builds on earlier work of de Faria, de Melo and Yampolsky on renormalisation and in a recent paper was generalised to the smooth case, [39]). In ongoing work, Clark and the 2nd author are aiming to show that the methods in 2.1 can be extended to the case of circle homeomorphisms with several critical points. Note that the presence of critical points is necessary for circle homeomorphisms, because for circle diffeomorphisms the analogous statement is false. Indeed, otherwise one can construct maps for which some sequence of iterates has *almost a saddle-node fixed point*, resulting in larger and larger passing times near these points. This phenomenon is also referred to as *a sequence of saddle-cascades*. It was used by Arnol'd and Herman to construct examples of diffeomorphisms of the circle which are conjugate to irrational rotations, but where the conjugacy is neither absolutely continuous, nor qs and for which the map has no σ-finite measures, see [40] and also Section I.5 in [30]. In the diffeomorphic case, to get qs or C^1 one needs assumptions on the rotation number (to avoid these sequences of longer and longer saddle-cascades).

In general, one cannot expect C^1, because having a C^1 conjugacy implies that corresponding periodic orbits have the same multiplier.

We should also remark that there are also analogues of these theorems for polynomials in $\mathbb{C}$, but then one must assume that f is only finitely renormalizable, see for example [52], but also see [24].

2.1. Applications of quasi-symmetric rigidity. Quasi-symmetric rigidity is a crucial step towards proving the following types of results:

(1) hyperbolicity is dense, see subsection 1.5.

(2) within certain families of maps, conjugacy classes are connected, see Theorems A and 2.2 in [21].

(3) monotonicity of entropy; for families such as $[0,1] \ni x \mapsto a\sin(\pi x)$, see Section 3.

2.2. Complex box mappings. It turns out to be rather convenient to show quasi-symmetric rigidity by using extensions to the complex plane. This approach is rather natural, as a quasi-symmetric homeomorphism on the real line is always the restriction of a quasi-conformal homeomorphism on the complex plane. More precisely, the idea is to construct an extension of the first return map to some interval, to the complex plane as a 'complex box mapping', see Figure 2.1 in the multimodal case. Roughly speaking, this is a map $F\colon U \to V$ so that each component of U is mapped as a branched covering onto a component of V, and components of U are either compactly contained in a component of V or they are equal to such a component. Components of $F^{-n}(V)$ are called *puzzle pieces*. We also require (roughly speaking) that F is unbranched near the boundary of U (slightly more precisely, that there exists an annulus neighbourhood A of ∂V so that $F\colon F^{-1}(A) \to A$ is an unbranched covering and so that$\mathrm{mod}\,(A)$ is universally bounded from below). If one has such numerical bounds, then F is said to have *complex bounds*. The existence of these complex bounds was first proved by Sullivan for certain unimodal maps. The general unimodal case was dealt with in [55] and somewhat later in [35] and [60]. Later this was extended to the multimodal case for certain maps in [92] and more generally in [87]. The most general result appears in a joint paper of the 2nd author with Clark and Trejo [26]. In that paper complex bounds are associated to any real analytic interval map. In fact, even in the C^3 case complex bounds are constructed in that paper, but in the smooth case the map F is only asymptotically holomorphic.

We should note that in the non-renormalisable real-analytic case one obtains complex bounds at arbitrary deep levels, as soon as one has a complex box mapping. That this is the case follows from the construction of the enhanced nest (discussed in the next subsection) and an interesting lemma due to Kahn and Lyubich, see [45]. This tool is about pulling back a thin annulus, and shows that the modulus of the pullback of this annulus is much better than one might expect. In the real case, one can simplify and strengthen the statement and proof of Kahn and Lyubich's result as follows, see [52, Lemma 9.1]:

Lemma 2.2 (Small Distortion of Thin Annuli). *For every $K \in (0,1)$ there exists $\kappa > 0$ such that if $A \subset U$, $B \subset V$ are simply connected domains symmetric with respect to the real line, $F : U \to V$ is a real holomorphic branched covering map of degree D with all critical points real which can be decomposed as a composition of maps $F = f_1 \circ \cdots \circ f_n$ with all maps f_i real and either real univalent or real branched covering maps with just one critical*

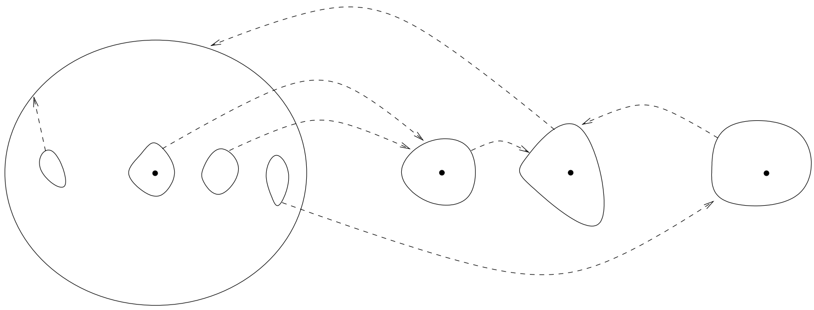

Figure 2.1. A box mapping.

point, the domain A is a connected component of $f^{-1}(B)$ symmetric with respect to the real line and the degree of $F|_A$ is d, then

$$\mathrm{mod}\ (U - A) \geq \frac{K^D}{2d} \min\{\kappa,\ \mathrm{mod}\ (V - B)\}.$$

2.3. How to prove quasi-symmetric rigidity? Consider the complex box mappings associated to two conjugate maps. To show that the conjugacy is quasi-symmetric one proceeds as follows:

(1) Define a sequence of puzzle pieces U_{n_i} called the **enhanced nest**, so that there exists k_i for which $F^{k(i)}\colon U_{n_{i+1}} \to U_{n(i)}$ is a branched covering map with degree bounded by some universal number N. This enhanced nest is chosen so that it transfers geometric information rather efficiently from small scale to large scale, but so that the degree of $F^{k(i)}\colon U_{n_{i+1}} \to U_{n(i)}$ remains universally bounded. This enhanced nest was one of the main new ingredients in [50]. It turns out that the post-critical sets do not come close to the boundary of the puzzle pieces in the enhanced nest, which implies that the puzzle pieces have uniformly bounded shape. Another important property of the enhanced nest is that decaying geometry and bounded geometry alternate quite regularly in the nest, which was used in [58] to study the Hausdorff dimension of Cantor attractors. The enhanced nest construction is also used for example in [26, 52, 75, 78, 94].

(2) In fact, if the interval maps extend to a holomorphic map on a neighbourhood of the real line, then one can partially define a quasi-conformal conjugacy near critical points, and then spread the definition to the whole complex plane fairly easily. This method was called the *spreading principle* in [50].

(3) Because of the spreading principle mentioned above, it then suffices to construct a partial-conjugacy on a puzzle piece in the enhanced nest which is 'natural on the boundary'. Given the above, this can easily be done using the *QC-criterion* from the appendix of [50]. and bounded shape of the puzzle pieces (bounded shape is very easy to derive from complex bounds, see [52, Section 10]). One can also proceed as in [5]. Our QC criterion was a variation of Heinonen-Koskela's theorem [42]. This theorem and its variations were used to prove rigidity result previously in [34, 41, 56, 83**?**], where in the last work, the author explicitly stated that a bounded shape property of puzzle pieces implies rigidity for non-renormalizable unicritical maps.

It is of course conceivable that one can prove quasi-symmetric rigidity using entirely real methods. This hinges on questions of the following type:

Question 2.3. *Consider the space $\mathcal{A}$ of maps of the form $z \mapsto |z|^d + c$ where $d > 1$ is not necessarily an integer and where c is real. Does one have quasi-symmetric rigidity for maps within the space $\mathcal{A}$? Are two topologically conjugate maps in $\mathcal{A}$ without periodic attractors (or both critically finite) necessarily the same?*

One of the difficulties with such a real approach is that it is not so easy to know how to use the information that the exponent d is fixed within the family $\mathcal{A}$: the exponent is not 'visible' in the real line. On the other hand, if d is an even integer, and $z \mapsto z^d + c$, then of course the local degree of the map at 0 is different for different values of d. Without fixing the degree d the answer to the question above is definitely negative. An affirmative answer to the above question would imply density of hyperbolicity and monotonicity of entropy in this family.

3. Monotonicity of entropy

In the late 70's, the following question attracted a lot of interest: does the topological entropy of the interval map $x \mapsto ax(1-x)$ depend monotonically on $a \in [0,4]$? In the mid 80's this question was solved in the affirmative:

Theorem 3.1. *The topological entropy of the interval map $x \mapsto ax(1-x)$ depends monotonically on $a \in [0,4]$.*

In the 80's several proofs of this appeared. One of these uses Thurston's rigidity theorem, see [70]. Another proof relies on Douady-Hubbard's univalent parametrisation of hyperbolic components, see [31], and a third proof is due to Sullivan; for a description of these proofs see [30]. All these proofs consider the map $x \mapsto ax(1-x)$ as a polynomial acting on the complex plane. A rather different method was used by Tsujii, [97]. He showed that periodic orbits bifurcate in the 'right' direction using a calculation on how the multiplier depends on the parameter. Unfortunately, Tsujii's proof also does not work for maps of the form $z \mapsto |z|^a + c$ with a not an integer.

In the early 90's, Milnor (see [69]) posed the more general

Conjecture 3.2 (Monotonicity Conjecture). *The set of parameters within a family of real polynomial interval maps, for which the topological entropy is constant, is connected.*

Milnor and Tresser proved this conjecture for cubic polynomials, see [71] (see also [28]). Their ingredients are planar topology (in the cubic case the parameter space is two-dimensional) and density of hyperbolicity for real quadratic maps.

A few years ago, Bruin and the 2nd author were able to give a proof of the general case of this conjecture. More precisely, given $d \geq 1$ and $\epsilon \in \{-1, 1\}$, consider the space P^d_ϵ of real polynomials $f\colon [0,1] \to [0,1]$ of fixed degree d with $f(\{0,1\}) \subset \{0,1\}$, with all critical points in $(0,1)$ and with the first lap orientation preserving if $\epsilon = 1$ and orientation reversing if $\epsilon = -1$. We call ϵ the *shape* of f. In [21] we proved the general case:

Theorem 3.3 (Monotonicity of Entropy). *For each integer $d \geq 1$, each $\epsilon \in \{-1, 1\}$ and each $c \geq 0$,*

$$\{f \in P^d_\epsilon ; h_{top}(f) = c\}$$

is connected.

The proof in [21] also shows that the set of maps in P^d_ϵ with the same kneading sequence is connected and gives a precise description of the bifurcations that occur when one of the periodic attractors loses hyperbolicity. The main ingredient in the proof is quasi-symmetric rigidity. Recently, Kozlovski announced a simplification of the proof in [21] of this theorem (using semi-conjugacies to maps with constant absolute value of the slopes, rather than stunted sawtooth maps).

3.1. Non-local connectivity of isentropes and non-monotonicity in separate variables. It is possible to parametrize the family P^d by critical values. The following example shows that it is not true that topological entropy depends monotonically on each of these parameters. Define $f_{a,b}(x) = 2ax^3 - 3ax^2 + b$ for $a = b + 0.515$. This cubic map has critical points 0 and 1 and critical values $f(0) = b$, and $f(1) = b - a = 0.515$. It is shown in [21] that there are values of b such that the map $a \mapsto h_{top}(f_{a,b})$ is not monotone.

Related to this, it is shown in [22] that isentropes in P^d, when $d \geq 5$ are not locally connected. It is not known whether isentropes in P^3 or in P^4 are locally connected. For related results and questions, see [100].

4. Measure-theoretical dynamics

We shall now discuss the dynamics of a map $f : N \to N$, where $N = [0, 1]$ or S^1 from measure-theoretical point of view. Recall that a Borel probability measure μ is *invariant* for f if for each Borel set $A \subset [0, 1]$ we have $\mu(f^{-1}A) = \mu(A)$. We say that μ is *ergodic* if a Borel set A with $f^{-1}(A) = A$ satisfies either $\mu(A) = 0$ or $\mu(A) = 1$. The basin $B(\mu)$ of μ is the set of points $x \in [0, 1]$ for which

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} \to \mu \text{ as } n \to \infty, \tag{4.1}$$

where the convergence is with respect to the weak star topology. If $B(\mu)$ has positive Lebesgue measure, then we say that μ is a *physical measure*. Clearly, if O is an attracting periodic orbit, then the averaged Dirac measure $\mu_O = \frac{1}{\#O}\sum_{p\in O}\delta_p$ is a physical measure. An ergodic *acip*, i.e., an invariant probability measure which is absolutely continuous with respect to the Lebesgue measure, is also a physical measure, by Birkhorff's ergodic theorem.

4.1. Typical physical measures. Conjecturally these are the only two types of physical measures for typical interval maps, from measure-theoretical point of view. Indeed, in the major breakthrough [63], Lyubich proved that *within the quadratic family* $f_a(x) = ax(1 - x)$, $1 \leq a \leq 4$, *for almost every* a, *either* f_a *is hyperbolic or* f_a *has an ergodic acip*. In an earlier celebrated work [44], Jakobson showed that the set of a for which f_a has an ergodic acip has positive Lebesgue measure.

An analogue of Lyubich's theorem in the multi-critical case is widely open at the moment, due to the multi-dimensional feature of the corresponding parameter space. However, a generalization to the case of unimodal polynomials of even degree $d \geq 2$ is nearly completed. The work [6] extends the result of [62], showing that for any even integer $d \geq 2$, and almost every $a \in [1, 4]$, $f_a(x) = \frac{a}{4}(1 - (1 - 2x)^d)$ either is hyperbolic, or has an ergodic acip, or is infinitely renormalizable. Moreover, Avila and Lyubich [4] developed a novel way to obtain exponential convergence along hybrid classes for infinitely renormalizable maps. One can expect a complete proof of the generalization of Lyubich's theorem for unimodal maps of a given degree will be available soon. Nevertheless, let us mention in a joint work with Bruin, the authors of this paper proved that for all even integer d, and almost every $1 \le a \le 4$, $\frac{a}{4}(1-(1-2x)^d)$ has a unique physical measure which might be supported on a Cantor set.

4.2. Existence of acip. We shall now discuss some recent advances on existence of acip for smooth interval maps. In order to apply some version of the real Koebe distortion to control distortion, we often assume f lies in the class $\mathcal{A}_3$ defined below. A map $f : [0,1] \to [0,1]$ is in the class $\mathcal{A}_k$ if the following holds: f is C^1 and C^k outside the critical set $\mathrm{Crit}(f) = \{c : f'(c) = 0\}$; moreover, for each $c \in \mathrm{Crit}(f)$, there exists $\ell_c > 1$ and C^k diffeomorphismsms φ_c, ψ_c of $\mathbb{R}$ such that $\varphi_c(c) = \psi_c(f(c)) = 0$ and $|\psi_c(f(x))| = |\varphi_c(x)|^{\ell_c}$ holds in a neighborhood of c.

The following theorem was obtained by the authors in joint with Bruin and Rivera-Letelier.

Theorem 4.1 (Existence of acip [19]). *Let $f \in \mathcal{A}_3$ be an interval map with all periodic points hyperbolic repelling. Assume that the following* large derivatives condition *holds: for each $c \in \mathrm{Crit}(f)$,*

$$|Df^n(f(c))| \to \infty \text{ as } n \to \infty.$$

Then f has an acip μ with density $\frac{d\mu}{dLeb} \in L^p$ for each $p < \ell_{\max}/(\ell_{\max}-1)$ where $\ell_{\max} = \sup_{c\in\mathrm{Crit}(f)} \ell_c$.

The unimodal case was done earlier by the authors in joint with Bruin [20]. The existence of acip for interval maps has been proved previously in more restrictive settings, including

- in [72], for maps satisfying the *Misiurewicz* condition: $\omega(c) \cap \mathrm{Crit}(f) = \emptyset$ for each $c \in \mathrm{Crit}(f)$;
- in [27] for unimodal maps satisfying the *Collet-Eckmann* condition (together with other conditions): for the critical point c, $\liminf_{n\to\infty} \frac{1}{n} \log|Df^n(f(c))| > 0$;
- in [74] for unimodal maps satisfying the following summability condition: if c is the critical point and ℓ is the order, then $\sum_{n=0}^{\infty} |Df^n(f(c))|^{-1/\ell} < \infty$,

among others. All of the following results assume that f has negative Schwarizian outside $\mathrm{Crit}(f)$ in order to apply the real Koebe principle to control distortion, but now we know that the required distortion control is also valid for maps $f \in \mathcal{A}_3$, after [48] and [101, Theorem C].

We should however note that the large derivatives condition is not a necessary condition for the existence of an acip, even though an acip necessarily has positive metric entropy: there exists a unimodal map in the class $\mathcal{A}_3$ with $\liminf|Df^n(f(c))| = 0$ and with an acip [16]. It is also known (not surprisingly) that existence of acip is not a topological (or quasisymmetric) condition [17].

Question 4.2. *Determine topological (or quasisymmetric) conjugacy classes in $\mathcal{A}_3$ such that each map in the class has an acip.*

4.2.1. Ingredients of the proof of Theorem 4.1. An intermediate step of the proof is to show that the large derivatives condition implies *backward contraction* in the sense of Rivera-Letelier [84], which means the following: if $\widetilde{B}_c(\delta)$ denotes the component of $f^{-1}(f(c)-\delta, f(c)+\delta)$ which contains c and

$$\Gamma(\delta) = \inf \left\{ \frac{\delta}{|U|} : \begin{array}{l} U \text{ is a component of } f^{-n}(\widetilde{B}_c(\delta)) \text{ containing } f(c') \\ \text{for some } c, c' \in \mathrm{Crit}(f) \text{ and } n \geq 0 \end{array} \right\}$$

then $\Gamma(\delta) \to \infty$ as $\delta \to 0$. It turns out that the backward contraction property is equivalent to the large derivatives condition [57].

It is well-known that for any Borel probability measure ν, any accumulation point of the following sequence

$$\frac{1}{n}\sum_{i=0}^{n-1}(f^i)_*(\nu)$$

in the weak star topology is an invariant probability measure of f, where $(f^i)_*\nu(A) = \nu(f^{-i}(A))$. Thus it suffices to prove the following statement: for each $0 < \kappa < 1$ there exists $C = C(\kappa)$ such that

$$(f^n)_*(\mathrm{Leb})(A) = |f^{-n}(A)| \leq C|f(A)|^{\kappa/\ell_{\max}},$$

holds for all Borel $A \subset [0,1]$ and all $n \geq 0$. The backward contraction property makes it possible to obtain the estimate when A is an interval close to the critical set. For general A, the paper uses a sliding argument from [74], and Mãné's theorem [65].

4.3. Decay of correlation. A different way to obtain existence of acip is via *inducing*. Let us say a map $F : \mathcal{U} \to \mathcal{V}$, where $\mathcal{U} \subset \mathcal{V}$ are open subsets of $[0,1]$, is a *Markov* map, if for each component U of $\mathcal{U}$, $F|U$ is a C^1 diffeomorphism onto a component of $\mathcal{V}$. A Markov map F is *induced* by a map f if there is a continuous function $s : \mathcal{U} \to \{1, 2, \ldots\}$ such that $F(x) = f^{s(x)}(x)$. (So $s(\cdot)$ takes constant value in each U.) We shall often consider Markov maps with extra properties:

(i) $\mathcal{V}$ is an interval;

(i′) $\mathcal{V}$ consists of finitely many intervals;

(ii) (Bounded distortion) There exist $C > 0$ and $\alpha \in (0,1)$ such that

$$\frac{|DF^n(x)|}{|DF^n(y)|} \leq C|F^n(x) - F^n(y)|^{\alpha},$$

whenever $F^i(x)$ and $F^i(y)$ belong to the same component of $\mathcal{U}$ for each $i = 0, 1, \ldots, n-1$.

A Markov map $F : \mathcal{U} \to \mathcal{V}$ with the properties (i') and (ii) has an absolutelty continuous invariant propbability measure ν such that $d\nu/d\mathrm{Leb}$ is bounded away from 0 and ∞. If we can construct an induced Markov map F for a map f such that (i'), (ii) and the following hold:

$$a_s := |\{s(x) \geq s\}| \to 0 \text{ as } s \to \infty,$$

then the original system f has an acip

$$\mu := \frac{1}{\sum_{s=1}^{\infty} a_s} \sum_{U} \sum_{j=0}^{s|U-1} (f^j)_*(\nu|U),$$

where the sum runs over all components of $\mathcal{U}$. One advantage of inducing is that through estimating the speed of convergence of $a_s \to 0$, one can obtain finer statistical properties of the system.

The following theorem was proved by the 1st author in joint with Rivera-Letelier, improving an earlier result [18] considerably.

Theorem 4.3 (Decay of correlation [85]). *Assume that $f \in \mathcal{A}_3$ is topologically exact and satisfies the large derivatives condition. Then there is an induced Markov map $F : \mathcal{U} \to \mathcal{V}$ such that (i) and (ii) and the following tail estimate hold:*

$$a_s = O(s^{-p}) \text{ for each } p > 0, \text{ as } s \to \infty.$$

In particular, the unique acip μ of f is super-polynomially mixing: for each essentially bounded $\varphi : [0,1] \to \mathbb{R}$ and each Hölder continuous $\psi : [0,1] \to \mathbb{R}$,

$$C_n(\varphi,\psi) := \int_0^1 \varphi \circ f^n \psi d\mu - \int_0^1 \varphi d\mu \int_0^1 \psi d\mu$$

converges to 0 superpolynomially fast as $n \to \infty$.

Here we say that f is topologically exact if for each non-empty open subset U of $[0,1]$, there exists a positive integer n such that $f^n(U) = [0,1]$. This is a necessary condition for f to have a mixing acip. The last statement was deduced from the tail estimate via Young's tower [102]. Note that the tail estimate also implies finer statistical properties of the sequence $\{\psi \circ f^n\}_{n=0}^{\infty}$ (considered as a sequence of random variables with identical distribution), such as the Central Limit Theorem [102], Almost Sure Invariance Principle [68], etc, for ψ Hölder. The paper [85] also dealt with existence and mixing properties of invariant probablity measures with respect to conformal measures (supported on Julia sets) of maximal dimension for a large class of complex rational maps. This paper used the induced Markov map to study the geomtery of the Julia set.

Much recent progress on theomodynamical formalism for one-dimensional maps also used inducing to construct invariant probablity measures with respect to various conformal measures, see for example [23, 76, 82].

For the proof of Theorem 4.3, an adaptation is used of the inducing scheme, called *canonical inducing*, developed in [81, 82]. A crucial new estimate is the following backward shrinking estimate for maps with large derivatives (Theorem B): *there exists $\rho > 0$ such that*

$$\theta_n := \{|J|: J \text{ is an interval such that } |f^n(J)| \leq \rho\}$$

converges to zero super-polynomially fast. Theorem C relates the quantity θ_n to the tail estimate of a suitably constructed induced Markov map, provided the map has *badness exponent* 0 which was the statement of Theorem A.

It is known that $\theta_n \to 0$ exponentially fast (the topological Collet-Eckmann condition, equivalent to the Collet-Eckmann condition in the unimodal case) is equivalent to having an exponentially mixing acip [73, 81]. It would be interesting to know

Question 4.4. *For a topologically exact interval map $f \in \mathcal{A}_3$, is $\theta_n \to 0$ superpolynomially fast equivalent to having a unique acip which is superpolynomially mixing?*

An affirmative solution to Question 2.11 in [85] implies an affirmative answer to the question above.

4.4. Stochastic stability. An interval map with an acip is not hyperbolic and hence not structurally stable. The notion of stochastic stability, posed by Kolmogrov and Sinai, asks for stability of statistical properties under random perturbations. Given a map $f : [0,1] \to [0,1]$, an *ε-random (pseudo) orbit* is by definition a sequence $\{x_n\}_{n=0}^{\infty}$ such that $|f(x_n) - x_{n+1}| \leq \varepsilon$. Roughly speaking, stochastic stability means when $\varepsilon > 0$ is small, for most of the ε-random orbits $\{x_n\}_{n=0}^{\infty}$, the asymptotic distribution, $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \delta_{x_i}$, is close to a physical measure of f. Note that if $f([0,1]) \subset (0,1)$ and $\varepsilon > 0$ small enough, then the space of all ε-random orbits can be identified with $[0,1] \times [-\varepsilon, \varepsilon]^{\mathbb{N}}$ by the following formula:

$$\{x_n\}_{n=0}^{\infty} \mapsto (x_0, x_1 - f(x_0), x_2 - f(x_1), \ldots).$$

So the space of sequences $\{x_n\}_{n=0}^{\infty}$ can be endowed with a probability measure $\mathbb{P}_\varepsilon$ which corresponds to $m \times m_\varepsilon^{\mathbb{N}}$, where m denotes the Lebesgue measure on $[0,1]$ and m_ε denotes the normalised Lebesgue measure on $[-\varepsilon, \varepsilon]$. In the literature, reference measures other than $\mathbb{P}_\varepsilon$ have also been considered on the space of ε-random orbits, corresponding to different types of random perturbations. The measure $\mathbb{P}_\varepsilon$ corresponds to the so-called *additive noise* model.

Recently the 1st author proved the following theorem.

Theorem 4.5 (Stochastic Stability [88]). *Suppose $f \in \mathcal{A}_3$ is ergodic with respect to the Lebsgue measure and that the following summability condition holds: for each $c \in \mathrm{Crit}(f)$,*

$$\sum_{n=0}^{\infty} |Df^n(f(c))|^{-1} < \infty. \tag{4.2}$$

Then the unique acip of f is stochastic stable in the strong sense: For each $\varepsilon > 0$ there exists a unique probability measure μ_ε absolutely continuous with respect to the Lebesgue measure, such that for $\mathbb{P}_\varepsilon$-a.e. ε-random orbits $\{x_n\}_{n=0}^{\infty}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{x_i} \to \mu_\varepsilon$$

as $n \to \infty$ in the weak star topology. Moreover, the density $\frac{d\mu_\varepsilon}{dLeb}$ converges in L^1 to the density of the unique acip of f as $\varepsilon \to 0$.

See the Main Theorem of [88] for a more general statement, which covers a very general type of random perturbation. Previously, stochastic stability was studied for interval maps with a Benedicks-Carleson type condition [12, 13] (or even stronger) which thus has exponential decay of correlation, see [11, 14, 46, 95]. It is surprising that the stochastic stability of the Manneville-Pomeau map $x \mapsto x + x^{1+\alpha} \mod 1$, which is probably the simplest non-uniformly expanding dynamical system, was only established very recently by the authors in [89].

Li and Wang [59] proved stochastic stability for unimodal maps f with a wild attractor where the physical measure is supported on the Cantor attractor. It raises a curious question whether there exists an interval map with a stochastically unstable physical measure.

The crucial step in the proof of Theorem 4.5 was to establish the first estimate on the first return maps to critical neighborhoods: Let $\varepsilon > 0$ be small and let $\widetilde{B}_c(\varepsilon)$ be defined as in § 4.2.1. Then for *all* ε-random orbits $\{x_i\}_{i=0}^n$ with $x_0 \in \widetilde{B}_{c_1}(\varepsilon)$, $x_n \in \widetilde{B}_{c_2}(\varepsilon)$ for some $c_1, c_2 \in \text{Crit}(f)$ and $x_1, x_2, \dots, x_{n-1} \notin \bigcup_{c\in\text{Crit}(f)} \widetilde{B}_c$, we have

$$\prod_{i=1}^{n-1} |Df(x_i)| \geq \frac{\Lambda(\varepsilon)}{\varepsilon^{1-\ell_{c_2}^{-1}}} \exp(\varepsilon^{\alpha(\varepsilon)} n),$$

where $\Lambda(\varepsilon) \to \infty$ and $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$. The measure μ_ε was constructed using a random inducing scheme initiated in [8]. See also [1, 2].

4.5. Jakobson's theorem. The lower bound for derivative plays a crucial role in a generalization of Jakobson's theorem by B. Gao and the 1st author [38]. Among a huge number of works in generalizing Jakobson's theorem, our approach is close to that of [96]. While the paper worked with general one-parameter families, the following is the main result obtained for polynomial maps.

Theorem 4.6 (Summability implies Collet-Eckmann alomost surely [38]). *Fix an integer $n \geq 2$. For each $\boldsymbol{a} = (a_0, a_1, \dots, a_n) \in \mathbb{R}^{n+1}$ write $P_{\boldsymbol{a}}(x) = \sum_{i=0}^n a_i x^i$. Let Λ_n denote the collection of $\boldsymbol{a} \in \mathbb{R}^{n+1} \setminus \{\boldsymbol{0}\}$ for which the following hold: (i) $P_{\boldsymbol{a}}([0,1]) \subset [0,1]$ and (ii) $P_{\boldsymbol{a}} : [0,1] \to [0,1]$ satisfies the summability condition (4.2). Then Λ_n has positive measures and almost every $\boldsymbol{a} \in \Lambda_n$ satisfies the Collet-Eckmann condition, and the following polynomial recurrence conditions: for each $\beta > 1$, and any critical points c, c' of $P_{\boldsymbol{a}}|[0,1]$, we have $|P_{\boldsymbol{a}}^k(c) - c'| \geq k^{-\beta}$ for all k sufficiently large.*

The proof is done by purely real analytic method, except we had to use a recent tranversality result due to Levin [54] which was based on complex methods. For the case $n = 2$, the transversality result was known before in [3, 53]. For the quadratic family, the Collet-Eckmann and polynomial recurrence conditions are satisfied by almost every non-hyperbolic map [7]. It would be interesting to push the real analytic method further, for instance, to see whether the summability condition can be replaced by the large derivatives condition in Theorems 4.5 and 4.6.

Finally we would like to draw the reader's attention to the works [9, 10] where the "modulus of continuity" of $t \mapsto \mu_t$ over "good" non-uniformly expanding maps is studied for families f_t of unimodal maps, where μ_t is the acip for f_t.

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