Floer homology and non-fibered knot detection

Steven Sivek
(joint work with John A. Baldwin)

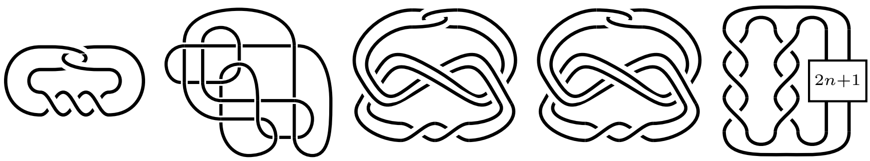
Knot Floer homology assigns to any knot $K \subset S^3$ a bigraded abelian group

$$\widehat{HFK}(K) = \bigoplus_{a,m\in\mathbb{Z}} \widehat{HFK}_m(K,a),$$

and the Seifert genus $g(K)$ is the maximal a such that $\widehat{HFK}_*(K,a)$ is nonzero [OS04a]. Moreover, K is a fibered knot if and only if $\widehat{HFK}(K,g(K))$ has rank 1 [Ghi08, Ni07]. These facts imply that $\widehat{HFK}$ detects the unknot, meaning that $\widehat{HFK}(K) \cong \widehat{HFK}(U)$ as bigraded groups if and only if $K = U$, and likewise the trefoils and figure eight, because these are the only fibered knots of genus ≤ 1. It is also known to detect the cinquefoils [FRW22], which are fibered of genus 2.

This talk focused on recent work with John Baldwin [BS22a], where we proved for the first time that $\widehat{HFK}$ can detect knots which are *not* fibered. The main result is a classification of the "nearly fibered" knots of genus 1.

Theorem 1. *Let $K \subset S^3$ be a knot of Seifert genus 1. Then $\dim \widehat{HFK}(K,1;\mathbb{Q}) = 2$ if and only if K or its mirror is one of the following:*



5_2 $15n_{43522}$ $\mathrm{Wh}^-(T_{2,3},2)$ $\mathrm{Wh}^+(T_{2,3},2)$ $P(-3,3,2n+1)$

Among these knots, we note that $\widehat{HFK}$ uniquely detects 5_2 and $\mathrm{Wh}^+(T_{2,3},2)$; it cannot distinguish $15n_{43522}$ from $\mathrm{Wh}^-(T_{2,3},2)$, or any of the pretzel knots $P(-3,3,2n+1)$ from each other. With a little extra work, we can then use other knot homologies to tell the pretzels apart:

Theorem 2. *Reduced Khovanov homology detects 5_2, and reduced HOMFLY homology detects each of the pretzel knots $P(-3,3,2n+1)$.*

Remark 3. We expect that reduced Khovanov homology should be enough to detect each of the pretzels $P(-3,3,2n+1)$, but we were unable to prove it.

Theorem 1 also lets us draw some purely topological conclusions. We say $r \in \mathbb{Q}$ is a *characterizing slope* for $K \subset S^3$ if $S^3_r(K) \cong S^3_r(J)$ implies that $K = J$.

Theorem 4 ([BS22b, BS22c])**.** *Every $r \in \mathbb{Q} \setminus \mathbb{Z}_{>0}$ is characterizing for 5_2. If K is any of the knots of Theorem 1, then 0 is characterizing for K.*

The first step in the proof of Theorem 1 is to classify the possible complements of genus-minimizing Seifert surfaces. If F is a Seifert surface for K, then the sutured Floer homology of

$$S^3(F) = (S^3 \setminus N(F), \lambda_K)$$

can be identified with $\widehat{HFK}(K, g(F))$. When $\dim SFH(S^3(F)) = 1$, properties of SFH tell us that

$$S^3(F) \cong (F \times [-1,1], \partial F \times \{0\}),$$

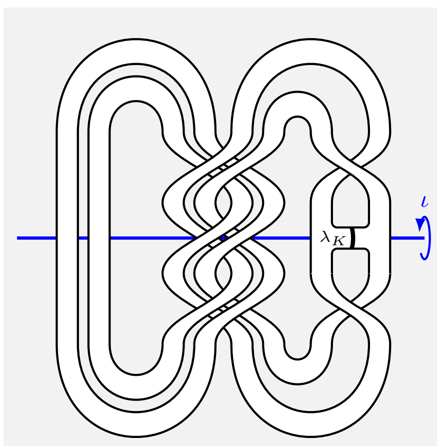
so this recovers the fact that K must be fibered. We are instead concerned with the 2-dimensional case, so $S^3(F)$ is no longer a product sutured manifold; however, work of Juhász [Juh10] tells us that since $\dim SFH(S^3(F))$ is sufficiently small, there must be an essential product annulus in $S^3(F)$. We decompose $S^3(F)$ along this annulus and repeat, and eventually we have simplified the topology enough that only two possibilities remain:

Proposition 5. *Let F be a genus-1 Seifert surface for K, and suppose that* $\dim SFH(S^3(F)) = 2$. *Then $S^3(F)$ is the complement of the union of*

- *the $(2,4)$-cable of either the unknot or the right-handed trefoil, and*
- *a properly embedded, non-separating arc in the cabling annulus,*

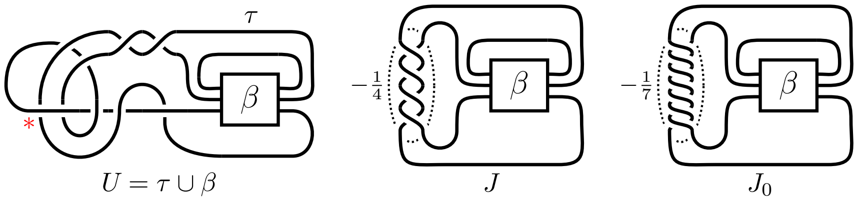
up to orientation reversal. Its suture is a meridian of that arc.

Once we know $S^3(F)$, viewed as the complement of a product $F \times [-1,1]$, it remains to be seen how we can glue $F \times \{1\}$ to $F \times \{-1\}$ to recover the complement of K. The key observation is that in either case, $S^3(F)$ admits an involution ι which restricts to $F \times \{\pm 1\}$ as a hyperelliptic involution. Since $g(F) = 1$, this involution is central in the mapping class group of F, and this allows us to extend ι across $F \times [-1,1]$ to the whole of S^3. Here we illustrate $(S^3(F), \iota)$ in case where $S^3(F)$ is built from a cable of a trefoil:



Taking the quotient by ι, we realize $S^3(F)$ as the branched double cover of a fixed tangle τ in the 3-ball, and $F \times [-1,1]$ as the branched double cover of some 3-braid β in $D^2 \times [-1,1]$. Then $\tau \cup \beta$ must be unknotted, since its branched cover is S^3, so it remains to determine all such β and produce the corresponding K.

We can only give a hint here of how to enumerate the possible braids β in the trefoil case. After some simplification, we are led to the unknot diagram at left:



Changing the indicated crossing turns U into a knot of the form $T_{-2,3} \# J$. The Montesinos trick tells us that its branched double cover $L(3,2) \# \Sigma_2(J)$ arises as some $\frac{2n+1}{2}$-surgery on a knot c in $\Sigma_2(U) \cong S^3$. But half-integer surgeries must be irreducible [GL87], so $L(3,2) \# \Sigma_2(J) \cong L(3,2)$, and then c and J are unknotted and $\frac{2n+1}{2} = \frac{3}{2}$. Now we instead take the 0-resolution of that crossing of U to get J_0; its branched double cover is $S^3_n(c) \cong S^3_1(U) \cong S^3$, so J_0 is unknotted as well.

Both J and J_0 are unknots differing in a single rational tangle, so we can replace it with another rational tangle of slope $\frac{p}{q}$ to get a 2-bridge link with fraction $\frac{p}{q}$. In the cases 0 ( ) or ∞ ( ) we see that the braid closure $\hat{\beta}$ is a 2-component unlink, and that a certain 2-bridge plat closure involving β is unknotted. The 3-braids with $\hat{\beta} = U \sqcup U$ are known up to conjugacy, and from there we can pin down the actual braids β, which end up giving rise to $K = \mathrm{Wh}^{\pm}(T_{2,3}, 2)$.

The remaining knots in Theorem 1 arise when $S^3(F)$ comes from a $(2,4)$-cable of the unknot, and that case is harder but based on similar ideas. These arguments could plausibly generalize to knots K for which $S^3(F)$ comes from a $(2,2n)$-cable of the unknot or of $T_{2,3}$, at least for small values of n, and this would be useful in enumerating genus-1 knots with $\dim \widehat{HFK}(K,1) = n > 2$. The problem is that at present we do not know how to prove the analogue of Proposition 5 that would classify all possible $S^3(F)$, even for $n = 3$.

References

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