Representations of some characteristic algebras

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The purpose of this note, which accompanies the paper [6], is to document some representations $\mathcal{C} \to \mathrm{Mat}_2(\mathbb{F})$, where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{C}$ is the characteristic algebra of a Legendrian knot. As noted in [6], the characteristic algebra of any tb-maximizing knot of at most 10 crossings has a representation of dimension 1 or 2 unless possibly if the knot is one of $m(9_{42})$, $m(10_{128})$, $m(10_{132})$, or $m(10_{136})$. We will give representations of characteristic algebras for Legendrian knots in each of these knot types except $m(10_{132})$, and also in the knot types $11n_{19}$, $m(11n_{38})$, and $m(12n_{243})$.

Recall that $\mathrm{Mat}_2(\mathbb{F})$ can be presented as

$$\frac{\mathbb{F}\langle a, b\rangle}{\langle a^2 = b^2 = 0, ab + ba = 1\rangle},$$

where a and b are the matrices $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ respectively. We will specify the image of each generator using this presentation, omitting generators corresponding to right cusps since these do not appear in any differential and thus can all be sent to 0. The differentials in the Chekanov-Eliashberg algebra of each knot can be computed using the program [4].

1 $m(9_{42})$

The $m(9_{42})$ representative with $(tb, r) = (-5, 0)$ in the table of [5] has a plat closure with braid word

$$2, 1, 1, 4, 5, 3, 5, 3, 2, 4, 3, 3, 2, 4.$$

If the crossings are labeled from left to right with generators $x_1, \ldots, x_{14}$ and the right cusps are x_{15}, x_{16}, x_{17} from top to bottom, then it has a representation

$$\begin{aligned} x_1, x_4, x_9, x_{10}, x_{11} &\mapsto 0 \\ x_6 &\mapsto 1 \\ x_3, x_7, x_{12} &\mapsto a \\ x_8 &\mapsto a + 1 \\ x_2, x_5, x_{13}, x_{14} &\mapsto b. \end{aligned}$$

2 $m(10_{128})$

The $m(10_{128})$ representative with $(tb, r) = (-14, 1)$ in [2] has a plat closure with braid word

$$6, 5, 5, 4, 3, 3, 2, 1, 5, 4, 3, 2, 2, 4, 1, 3, 5, 7, 1, 2, 3, 4, 5, 6.$$

If $x_1, \ldots, x_{24}$ are the crossings from left to right and $x_{25}, \ldots, x_{28}$ are the right cusps from top to bottom, then it has a representation

$$\begin{aligned} x_1, x_2, x_3, x_4, x_5, x_7, x_8, x_{11}, x_{12}, x_{18}, x_{23}, x_{24} &\mapsto 0 \\ x_{20} &\mapsto 1 \\ x_6, x_9, x_{13}, x_{14}, x_{19}, x_{22} &\mapsto a \\ x_{10}, x_{15}, x_{16}, x_{17}, x_{21} &\mapsto b. \end{aligned}$$

3 $m(10_{136})$

The $m(10_{136})$ representative with $(tb, r) = (-6, 1)$ in [2] has a plat closure with braid word

$$6, 5, 4, 3, 7, 5, 3, 3, 2, 1, 4, 3, 2, 4, 5, 2, 3, 1, 1, 2, 3, 4, 5, 6.$$

Its generators are the crossings $x_1, \ldots, x_{24}$ and right cusps $x_{25}, \ldots, x_{28}$, and it has a representation

$$\begin{aligned} x_1, x_3, x_4, x_5, x_9, x_{10}, x_{11} x_{12}, x_{13}, x_{22}, x_{23}, x_{24} &\mapsto 0 \\ x_{15}, x_{17}, x_{20}, x_{21} &\mapsto 1 \\ x_6, x_8, x_{18} &\mapsto a \\ x_2, x_7, x_{14}, x_{19} &\mapsto b \\ x_{16} &\mapsto ab. \end{aligned}$$

4 $11n_{19}$

The $11n_{19}$ representative with $(tb, r) = (-8, 1)$ in [2] has a plat closure with braid word

$$2, 1, 4, 5, 3, 1, 2, 5, 2, 1, 3, 2, 4, 3, 3, 2, 4.$$

Its generators are the crossings $x_1, \ldots, x_{17}$ and right cusps x_{18}, x_{19}, x_{20}, and it has a representation

$$\begin{aligned} x_1, x_2, x_3, x_7, x_9, x_{13}, x_{14} &\mapsto 0 \\ x_{10}, x_{17} &\mapsto 1 \\ x_5, x_6 &\mapsto a \\ x_4, x_8, x_{10}, x_{11}, x_{12}, x_{16} &\mapsto b \\ x_{15} &\mapsto ba. \end{aligned}$$

5 $m(11n_{38})$

The $m(11n_{38})$ representative with $(tb, r) = (-4, 1)$ in [2] has a plat closure with braid word

$$6, 7, 8, 9, 5, 7, 6, 9, 4, 3, 2, 1, 5, 4, 3, 2, 2, 4, 6, 1, 3, 5, 7, 1, 2, 3, 4, 5, 6, 7, 8.$$

Its generators are the crossings $x_1, \ldots, x_{30}$ and right cusps $x_{31}, \ldots, x_{36}$, and it has a representation

$$\begin{aligned} x_1, x_3, x_5, x_7, x_9, x_{11}, x_{12}, x_{14}, x_{15}, x_{16}, x_{23}, x_{25}, x_{28}, x_{30} &\mapsto 0 \\ x_2, x_6, x_{17}, x_{18}, x_{19}, x_{24} &\mapsto a \\ x_4, x_{10}, x_{20}, x_{21}, x_{22}, x_{26}, x_{29} &\mapsto b \\ x_8, x_{27} &\mapsto a+1 \\ x_{13} &\mapsto ba. \end{aligned}$$

6 $m(12n_{243})$

A representative of $m(12n_{243})$ with $(tb, r) = (-17, -2)$ was found using Gridlink [3], where it was specified as a grid diagram with coordinates

$$X = [5, 6, 7, 8, 0, 9, 1, 4, 2, 3]; \qquad O = [8, 9, 1, 2, 3, 4, 5, 0, 6, 7].$$

It has a plat closure with braid word

$$2, 4, 3, 2, 1, 5, 4, 4, 3, 2, 5, 4, 3, 3, 6, 5, 4, 3, 2, 7, 6, 6, 5, 4, 7, 7, 6$$

so that its generators are the crossings $x_1, \ldots, x_{27}$ and the right cusps $x_{28}, \ldots, x_{31}$, and it has a representation

$$\begin{aligned} x_2, x_3, x_4, x_7, x_{10}, x_{12}, x_{13}, x_{15}, x_{16}, x_{17}, x_{24}, x_{27} &\mapsto 0 \\ x_{18} &\mapsto 1 \\ x_1, x_6, x_8, x_{14}, x_{20}, x_{21}, x_{22}, x_{25}, x_{26} &\mapsto a \\ x_5, x_9, x_{11}, x_{19}, x_{23} &\mapsto b. \end{aligned}$$

We remark that according to KnotInfo [1], the value of $\overline{tb}(m(12n_{243}))$ is unknown as of the time of this writing; it is either -17 or -16, so either this knot is not tb-maximizing or $\overline{tb}(m(12n_{243})) = -17$.

References

[1] J. C. Cha and C. Livingston, *KnotInfo: Table of Knot Invariants*, `http://www.indiana.edu/~knotinfo`, November 23, 2011.

[2] Wutichai Chongchitmate and Lenhard Ng, *An atlas of Legendrian knots*, arXiv:1010.3997.

[3] Marc Culler, *Gridlink*, http://www.math.uic.edu/~culler/gridlink, 2007.

[4] Paul Melvin et al., Legendrian Invariants.nb, Mathematica program available at http://www.haverford.edu/math/jsabloff/Josh_Sabloff/Research.html.

[5] Paul Melvin and Sumana Shrestha, *The nonuniqueness of Chekanov polynomials of Legendrian knots*, Geom. Topol. **9** (2005), 1221–1252 (electronic).

[6] Steven Sivek, *The contact homology of Legendrian knots with maximal Thurston-Bennequin invariant*, arXiv:1012.5038.