Math 273 Lecture 9

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Last time we saw that a handful of common 3-manifolds – $S^2 \times [0,1]$, S^3, $S^1 \times S^2$, B^3, and $\mathbb{R}^3$ – have unique tight contact structures up to isotopy (rel boundary, with the boundary assumed convex, where applicable). Contact geometry would probably not be very interesting if this were true in general, so today we'll study the 3-torus $T^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$ and prove that it has infinitely many tight contact structures.

Before considering T^3, we'll need to examine certain tight contact structures on a solid torus $D^2 \times S^1$. The complete classification on $D^2 \times S^1$ was carried out by Honda [3], but we only need some special cases for now.

Proposition 1. *Let ξ_1 and ξ_2 be tight contact structures on $D^2 \times S^1$ with convex boundary, such that the characteristic foliations on $\partial D^2 \times S^1$ agree and $\partial D^2 \times S^1$ has two parallel dividing curves which represent the homology class $m[\partial D^2] + [S^1]$ for some $m \in \mathbb{Z}$. Then ξ_1 is isotopic rel boundary to ξ_2.*

Proof. Pick a curve γ isotopic to $\partial D^2 \times \{*\}$ which intersects each dividing curve once; then γ can be Legendrian realized simultaneously in both ξ_i. Let $\Delta_i \subset D^2 \times S^1$ be convex disks in each ξ_i with boundary γ. Since $|\gamma \cap \Gamma_{\Delta_i}| = |\gamma \cap \Gamma_{\partial D^2 \times S^1}| = 2$ and ξ_i is tight we know that the dividing set on each Δ_i is a single arc, so by Giroux flexibility we can insist that the Δ_i have the same characteristic foliation as well. Thus we can cut $D^2 \times S^1$ along Δ_i to get tight contact structures ξ_1' and ξ_2' on B^3 with identical boundaries. Then the ξ_i' are contact isotopic rel boundary and we can extend this isotopy trivially to the glued-up contact structures ξ_1 and ξ_2 on $D^2 \times S^1$. $\square$

Definition 2. The contact structure $\xi_n = \ker \alpha_n$, $n \geq 1$, on $T^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$ is defined by the 1-form

$$\alpha_n = \cos(nz)dx - \sin(nz)dy.$$

Note that ξ_n is obtained as an n-fold cover of ξ_1 in the z-direction, and ξ_1 is tight because it is the boundary of the unit disk bundle of T^*T^2, hence Stein fillable.

Proposition 3. *Each contact structure ξ_n is tight.*

Proof. We can check that the family of 1-forms

$$\alpha_n^s = (1-s)\alpha_n + sdz$$

consists of contact forms for all $0 \leq s < 1$, and so Gray stability says that ξ_n is contact isotopic to $\ker(\alpha_n^s)$. Now write $T^3 = \partial(T^2 \times D^2)$, where T^2 has coordinates x, y and D^2 has coordinates z, θ, and give $T^2 \times D^2$ the symplectic form $\omega = \omega_{T^2} + \omega_{D^2}$ where each ω_Σ is an area form on Σ. Certainly ω is positive on $\ker(\alpha_n^1) = \ker(dz)$, which is a foliation rather than a contact structure, but this means that ω is also positive on $\ker(\alpha_n^s)$ for some s arbitrarily close to 1. It follows that $\xi_n \cong \ker(\alpha_n^s)$ is weakly symplectically fillable, hence tight. □

Eliashberg [2] distinguished some of these by proving that ξ_1 is the only Stein fillable one, but the complete classification is due to Kanda [5].

Theorem 4. *No two of the ξ_n are contactomorphic.*

Proof. For any Legendrian $K \subset (T^3, \xi_n)$ isotopic to a linear curve, we let $tw(K)$ denote $tw(K, \Sigma)$ for any incompressible torus Σ containing K. This is well-defined if $tw(K) \leq 0$: we can make any two such tori be convex and intersect transversely along K, and then their dividing sets must intersect K in the same number of points, namely $-2tw(K, \Sigma)$. Fix a primitive nonzero homology class $a[S_x^1] + b[S_y^1] + c[S_z^1] \in H_1(T^3)$. We claim that any Legendrian knot K in this class with $tw(K) \leq 0$ satisfies the inequality

$$tw(K) \leq -|c|n,$$

and that there is some representative with $tw(K) = -|c|n$.

Suppose that a knot K with $(a, b, c) = (0, 0, \pm 1)$ violates this inequality, and stabilize if needed so that $tw(K) = -n + 1$. We can pass to a finite cover in the x– and y–directions (i.e. to some $(\mathbb{R}^2/2\pi k\mathbb{Z}^2) \times (\mathbb{R}/2\pi\mathbb{Z})$ so that there are lifts T_1 and T_2 of the xz– and yz–planes which are disjoint from a lift K' of K, and we observe that $tw(K') = tw(K)$. Each T_i is convex, with $2n$ horizontal dividing curves, and if we remove a neighborhood of $T_1 \cup T_2$ then we can check via edge-rounding that the complement is a tight $D^2 \times S^1$ retracting onto K' whose boundary has two parallel dividing curves homologous to $-n[\partial D^2] + [S^1]$. There is a diffeomorphism ϕ carrying this $D^2 \times S^1$ to a neighborhood of the standard Legendrian unknot with $tb = -1$ in $(\mathbb{R}^3, \xi_{\mathrm{st}})$ which sends dividing curves to dividing curves, and since the contact structure on $D^2 \times S^1$ is unique rel boundary we can take ϕ to be a contactomorphism; then since the boundary of the neighborhood of the unknot has dividing curves representing $-[\partial D^2] + [S^1]$ we conclude that $\phi(K')$ is a Legendrian unknot in $(\mathbb{R}^3, \xi_{\mathrm{st}})$ with $tb = 0$. But this violates the Thurston-Bennequin inequality, so we must have $tw(K) \leq -n$. Furthermore, the z-axis is Legendrian with $tw(K) = -n$, achieving equality.

Now suppose that $|c| = 1$ but $(a, b) \in \mathbb{Z}^2$. There is an element ϕ of $SL_3(\mathbb{R})$ which fixes each plane parallel to the xy–plane setwise and takes (a, b, c) to $(0, 0, c)$, and in block form we can write

$$\phi = \begin{pmatrix} A_0 & v_0 \\ 0 & 1 \end{pmatrix}$$

where $A_0 \in SL_2(\mathbb{R})$ and v is a 2×1 vector. Then we can define a 1-parameter family

$$\phi_s = \begin{pmatrix} A_s & v_s \\ 0 & 1 \end{pmatrix}$$

where $A_s \in SL_2(\mathbb{R})$ is a path of matrices from A_0 to I and v_s is a path in $\mathbb{R}^2$ from v_0 to the origin. The family of 1-forms $\phi_s^*(\alpha_n)$ are well-defined on T^3, with $\phi_0^*(\alpha_n) = \phi^*(\alpha_n)$ and $\phi_1^*(\alpha_n) = \alpha_n$, and they are all contact forms, so by Gray stability ξ_n is isotopic to $\phi^*\xi_n$ and thus ϕ can be taken to be a contactomorphism. It now follows from the previous case that $tw(K) \leq -n$ and equality can be achieved.

More generally, suppose that $|c| > 0$. Then the covering map $\phi : T^3 \to T^3$ defined by $\phi(x, y, z) = (x, y, |c|z)$ satisfies $\phi^*\xi_n = \xi_{|c|n}$, and if we lift $K \subset (T^3, \xi_n)$ to $K' \subset (T^3, \xi_{|c|n})$ by this map then $tw(K') \leq -|c|n$ by the previous case. But $tw(K) = tw(K')$, so the inequality is satisfied and again we can achieve equality. Note that in all cases with $c \neq 0$, we have not needed the hypothesis $tw(K) \leq 0$ to conclude that $tw(K) \leq -|c|n$.

Finally, if $c = 0$ then we observe that the torus $\{z = z_0\}$ has nonsingular characteristic foliation by curves of slope $\cot(nz_0)$. If we pick z_0 so that these curves have rational slope and are homologous to K, then any one of these curves is Legendrian and has $tw(K) = 0$.

This completes the proof of the inequality, and it now follows immediately that no two ξ_n are contactomorphic: indeed, given a tight contact structure ξ on T^3 we can define a quantity

$$f(\xi) = \min(-tw(K)),$$

where the minimum is taken over all Legendrian knots representing primitive nonzero homology classes which do not contain $tw = 0$ representatives; if there are no such classes, we can set $f(\xi) = \infty$. Then $f(\xi)$ is clearly invariant under contactomorphisms, and $f(\xi_n) = n$, so the ξ_n are all distinct. $\square$

Theorem 5. *Let ξ be a tight contact structure on T^3. Then ξ is contactomorphic to some ξ_n.*

Proof. Suppose we have two incompressible convex tori T_1 and T_2 which intersect in a Legendrian knot K such that $\#\Gamma_{T_i} = -2tw(K, T_i)$ for each i, and let $n = -tw(K, T_i)$. Since ξ is tight, each dividing curve on each T_i is homotopically (hence also homologically) nontrivial, so the dividing set on either T_i consists of $\#\Gamma_{T_i}$ parallel curves which each represent a primitive element of $H_1(T_i)$. Since $\Gamma_i = -2tw(K, T_i) = |K \cap \Gamma_{T_i}|$, each dividing curve on T_i intersects K once. If h_1 and h_2 are the homology classes of a dividing curve on either T_i, then $h_1, h_2, [K]$ is an integral basis of $H_1(T^3)$ and so there is a diffeomorphism $T^3 \to T^3$ which takes K to the z–axis and T_1 and T_2 to the xz– and yz–planes with $2n$ horizontal dividing curves on each T_i.

The dividing sets induced by ξ on T_1 and T_2 now match those of ξ_n, so by Giroux flexibility we can make their characteristic foliations agree, and then

ξ is identical to ξ_n on a neighborhood of $T_1 \cup T_2$. The complement of this neighborhood is a solid torus $D^2 \times S^1$ with tight contact structure, and by edge-rounding we see that it has two dividing curves on its boundary with slope $\frac{1}{m}$, i.e. representing the homology class $m[\partial D^2]+[S^1]$. We have shown that this contact structure on $D^2 \times S^1$ is unique up to isotopy rel boundary, so ξ and ξ_n are contactomorphic as desired.

It now remains to find the tori T_1 and T_2 and the knot K. Let T be a convex torus and L a Legendrian knot isotopic to a linear curve such that L intersects T transversely in a single point, $\#\Gamma_T$ is as small as possible, and L maximizes $tw(L) \leq 0$ among all such choices of L and T.

Let c be a connected component of Γ_T, and let γ and γ' be linear curves in T for which $\gamma \cdot c = \gamma' \cdot c = 1$ and γ is not homologous to γ'; then $\gamma \cup \gamma'$ is non-isolating, so we can Legendrian realize it with $|\gamma \cap \Gamma_T| = |\gamma' \cap \Gamma_T| = \#\Gamma_T$. We can also insist that γ and γ' intersect at the point $L \cap T$, although they will not be transverse at this point.

We claim that there is a convex torus S containing L for which $S \cap T = \gamma$. Take S to be any incompressible torus containing L which intersects T transversely at γ. In a neighborhood of T we can choose a generic contact vector field v which is transverse to both T and L, and we can perturb S along γ so that v is transverse to S as well in a neighborhood of γ. Now S is convex along a neighborhood of $\gamma \cup L$, and we can find a C^∞–small perturbation away from that neighborhood which makes S convex as desired.

Now we claim that we are done if γ is not homotopic to a dividing curve of S. Indeed, in this case we observe that γ intersects every dividing curve of S, so

$$\#\Gamma_T = |\gamma \cap \Gamma_T| = -2tw(\gamma) = |\gamma \cap \Gamma_S| \geq \#\Gamma_S.$$

Since we assumed $\#\Gamma_T$ was minimal, we must have $\#\Gamma_S = \#\Gamma_T = -2tw(\gamma)$, and so we can take $(T_1, T_2, K) = (S, T, \gamma)$. Thus from now on we will assume that γ is homotopic to a dividing curve of S, and by the same argument as above we will construct a convex torus $S' \supset L$ with $S' \cap T = \gamma'$ and $S \cap S' = L$. Again, if γ' is not homotopic to a dividing curve of S' we are done because we can take $(T_1, T_2, K) = (S', T, \gamma')$.

At this point we choose $(T_1, T_2, K) = (S, S', L)$. Indeed, we have $\#\Gamma_S = \#\Gamma_{S'}$, and we just need to check that both of these equal $-2tw(L)$. The components of Γ_S are homotopic to $\gamma = S \cap T$, so algebraically L intersects each of them once. If L intersects each dividing curve once geometrically then we are done, since this implies that $\#\Gamma_S = |L \cap \Gamma_S| = -2tw(L)$, so suppose instead that L intersects some component $c \subset \Gamma_S$ more than once. Then we can find a disk in S cobounded by arcs of c and L, and an innermost such disk would give rise to a bypass D in $S \backslash L$; let L' be the destabilization of L along D. If D is disjoint from γ (and hence T) then $|L' \cap T| = |L \cap T| = 1$ and $0 \geq tw(L') > tw(L)$, contradicting the assumption that $tw(L)$ was maximal. Therefore $D \subset S$ must intersect $\gamma = S \cap T$.

Push the curve γ off of D in S, and call the resulting curve γ_0. We can Legendrian realize $L' \cup \gamma_0$ so that $tw(\gamma_0) > tw(\gamma)$. If T' is a convex torus isotopic to T and containing $\gamma' \cup \gamma_0$, then $\Gamma_{T'}$ must consist of curves parallel to γ_0. Indeed, if this is not the case then each curve of $\Gamma_{T'}$ intersects γ_0 at least once, and so

$$\#\Gamma_{T'} \leq |\Gamma_{T'} \cap \gamma_0| = -2tw(\gamma_0) < -2tw(\gamma) = \#\Gamma_T$$

which contradicts the minimality of $\#\Gamma_T$. Then each curve in $\Gamma_{T'}$ must intersect γ', so

$$\#\Gamma_T \leq \#\Gamma_{T'} \leq -2tw(\gamma') = \#\Gamma_T$$

and so we must have $\#\Gamma_T = \#\Gamma_{T'}$. But now since $tw(L') > tw(L)$ we should have chosen (T', L') rather than (T, L), which is a contradiction. We conclude that L does intersect each dividing curve in S exactly once, so the choice $(T_1, T_2, K) = (S, S', L)$ suffices and we are done. $\square$

One can ask which closed 3-manifolds other than the 3-torus have infinitely many tight contact structures. This question has been completely answered:

Theorem 6. *An irreducible, closed* 3*-manifold has infinitely many isomorphism classes of tight contact structures if and only if it contains an incompressible torus.*

The proof that an incompressible torus suffices is due to Honda, Kazez, and Matić [4], and involves cutting a tight contact structure open along such a convex torus and splicing in a $T^2 \times I$ which looks like some (T^3, ξ_n) cut open along a convex torus isotopic to $\{z = z_0\}$. The fact that an incompressible torus is necessary was proved by Colin, Giroux, and Honda [1].

References

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