Math 273 Lecture 7

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Last time, when we wanted to destabilize a Legendrian unknot with $tb < -1$, we found a disk with corners Δ' that we could cut out of a Seifert surface to realize this destabilization. This disk turns out to be a fundamental object in convex surface theory called a *bypass*. We will examine the effect of gluing them to convex surfaces to see how the dividing sets can change, but first we need to figure out how dividing sets on a surface with corners change when we round those corners.

Lemma 1. *Let Σ and Σ' be convex surfaces in (M, ξ) which intersect transversely along the Legendrian knot K. Then points of Γ_Σ and $\Gamma_{\Sigma'}$ alternate along K.*

Proof. We can assume that $tw(K, \Sigma) = tw(K, \Sigma') = -n$ with $n > 0$, or else neither dividing set intersects K. Then K has a neighborhood contactomorphic to a neighborhood of $\{(0, 0)\} \times S^1$ in $\mathbb{R}^2 \times S^1$, with contact form

$$\xi = \ker(\cos(n\theta)dx - \sin(n\theta)dy).$$

Furthermore, we can choose coordinates near K so that Σ is the half-plane $\{x \geq 0, y = 0\}$ and Σ' is the half-plane $\{x = 0, y \geq 0\}$; since ∂_x and ∂_y are contact vector fields, both Σ and Σ' are indeed convex.

On Σ the dividing curves are the points where $\partial_x \in \xi|_\Sigma$, i.e. where $\cos(n\theta) = 0$, so $K \cap \Gamma_\Sigma$ is the set of points $(0, 0, \frac{(2k+1)\pi}{2n})$; likewise, the dividing set $\Gamma_{\Sigma'}$ is the set where $\partial_y \in \xi|_{\Sigma'}$, or equivalently $\sin(n\theta) = 0$, so $K \cap \Gamma_{\Sigma'} = (0, 0, \frac{k\pi}{n})$. These sets of points alternate along K, as desired. □

Proposition 2 (Edge rounding)**.** *Let Σ and Σ' be convex surfaces intersecting transversely at a right angle along the Legendrian knot $K \subset \partial\Sigma \cap \partial\Sigma'$. Then we can smooth $\Sigma \cup \Sigma'$ in a neighborhood of K so that the dividing curves on $\Sigma \cup \Sigma'$ turn right as they approach K, as seen from "outside" the right angle.*

Proof. Fix the model neighborhood of the previous lemma and a suitably small $r > 0$, and replace $\Sigma \cup \Sigma'$ within the cylinder around K of radius r by the quarter-cylinder

$$C = \{(x, y, \theta) \mid (x - r)^2 + (y - r)^2 = r^2\}.$$

Along points (x, y, θ) of C, the unit vectors pointing toward (r, r, θ) define a contact vector field v which interpolates between ∂_x along Σ and ∂_y along Σ', and v is transverse to C, so the dividing set Γ is defined by the condition $v \in \xi$, which along C is equivalent to

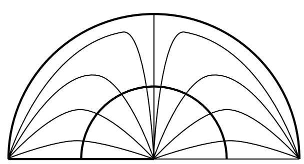
$$v = \sin(n\theta)\partial_x + \cos(n\theta)\partial_y.$$

We can see that these curves connect each arc of Γ_Σ to the next arc of $\Gamma_{\Sigma'}$ to its right along K (viewed along an arc of Γ_Σ as it approaches K), and vice versa.

We remark that the rounded surface and associated contact vector field in this proof are not smooth, but they are C^1 and we can perturb them slightly to make them smooth. Since the dividing curves are determined by C^1 data, namely the singularities and flow lines of the characteristic foliation, this does not change the result. □

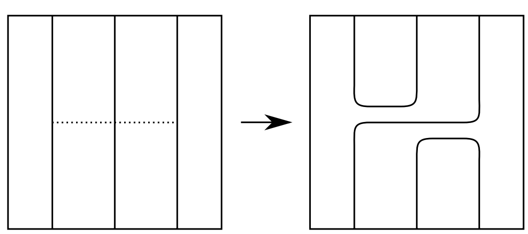
Definition 3. Let $\Sigma \subset (M, \xi)$ be a convex surface, and let $a \subset \Sigma$ be a Legendrian arc which intersects Γ_Σ in three points, including both points of ∂a. A *bypass* for Σ along a is a convex half-disk D with Legendrian boundary, intersecting Σ transversely along a and satisfying $tb(\partial D) = -1$.

It follows that Γ_D is a single arc with both endpoints on the interior of a. By Giroux flexibility we can fix the characteristic foliation to have elliptic singularities at all three points of $a \cap \Gamma_\Sigma$ and a hyperbolic singularity along the interior of $\partial D \backslash a$:



Note that this is exactly the form of the disk Δ' we used above to destabilize a $tb < -1$ unknot. If D is oriented, then the *sign* of the bypass is the sign of the singularity along $\text{int}(a)$.

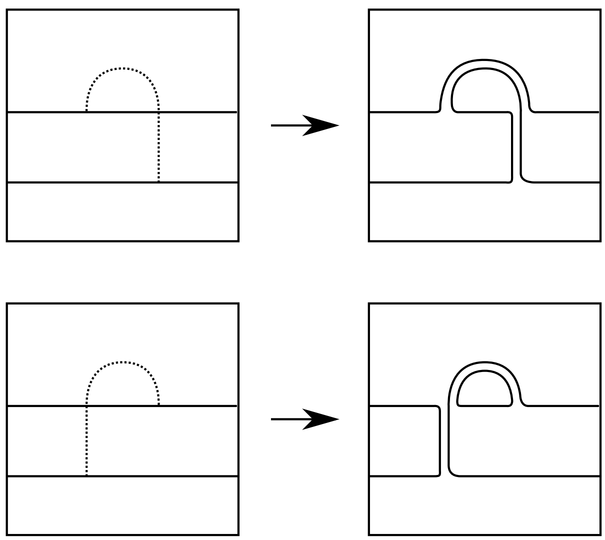
Theorem 4. *Let Σ be a convex surface and attach a bypass D along some arc $a \subset \Sigma$. Isotoping Σ across D has the following effect on the dividing curves Γ_Σ:*



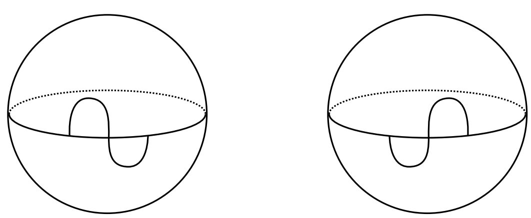
Proof. Extend a to a simple closed curve γ in a neighborhood of a which is transverse to Γ_Σ and use the Legendrian realization principle to make γ Legendrian;

in fact, we can foliate a neighborhood $N(\gamma) = \gamma \times [-\epsilon, \epsilon] \subset \Sigma$ by Legendrian curves parallel to γ. Glue a vertically invariant neighborhood $N(\gamma) \times [0, 1]$ onto Σ along $N(\gamma) \times \{0\}$, and then glue $D \times [-\epsilon, \epsilon]$ to $N(\gamma) \times \{1\}$ along $a \times [-\epsilon, \epsilon]$. The theorem now follows from several applications of the edge rounding lemma, though we must be careful about rounding the corners $\partial a \times \{\pm\epsilon\}$. $\square$

For example, let Σ be a convex surface in a tight contact manifold. There are two ways in which the attaching arc of a bypass can intersect the same dividing curve at two consecutive points:



The first of these does not change the isotopy class of Γ_Σ, so it is called a *trivial* bypass, while the second cannot exist by Giroux's criterion. If Σ is a sphere, then Γ_Σ is connected and so there are only two possible attaching arcs up to isotopy:



The first of these is a trivial bypass, and the second one is forbidden, so every attaching arc of a bypass on a convex sphere in a tight contact structure looks like the picture on the left. Information like this about the existence of bypasses will be extremely useful in studying tight contact structures.

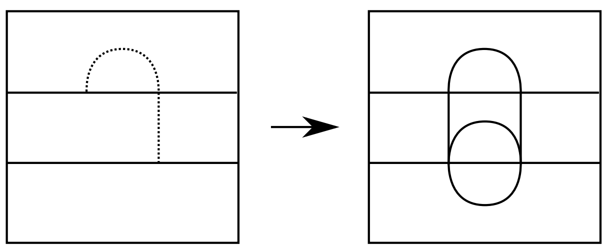
In general bypasses are hard to find, but there are some situations where this is not the case. For example, the idea of the proof of the following proposition was already used once, when we showed that Legendrian unknots with $tb < -1$ can be destabilized.

Proposition 5 (Imbalance Principle)**.** *Let $\Sigma \subset (M, \xi)$ be a convex annulus with Legendrian boundary $K_0 \sqcup K_1$, and suppose that $|\Gamma_\Sigma \cap K_0| < |\Gamma_\Sigma \cap K_1|$ and Γ_Σ is disconnected. Then Σ contains a bypass along K_1.*

Proof. There must be two points of $\Gamma_\Sigma \cap K_1$ which are connected to each other by an arc γ of Γ_Σ, so that γ cuts off a disk D from Σ. Passing to an outermost arc, we may assume that there are no dividing curves on the interior of D, and then take D' to be a slightly larger disk. We can Legendrian realize the arc $\overline{\partial D' \backslash \partial\Sigma}$, since it is nonisolating, and then D' is a bypass. □

Proposition 6 (Right-to-Life Principle)**.** *Let (M, ξ) be tight, and suppose that $a \subset \Sigma$ is an arc which corresponds to a trivial bypass move. Then there exists a bypass along a in any vertically invariant neighborhood of Σ.*

Proof. Complete a to a simple closed curve γ_0 which shares an arc δ of $\gamma_0 \backslash \Gamma_\Sigma$ with a smaller simple closed curve γ_1 as shown below:



Since $\gamma_0 \cup \gamma_1$ is non-isolating, we can use the Legendrian Realization Principle. Then $(\gamma_0 \times \{0\}) \cup (\gamma_1 \times \{1\})$ is Legendrian in $\Sigma \times \mathbb{R}$, so we can find an annulus A bounded by these curves and perturb it to be convex. The dividing set on A intersects γ_0 in four points but only intersects γ_1 twice, so by the Imbalance Principle, A contains a bypass D along γ_0.

The product region $\delta \times I$ of A is vertically invariant, and $\delta \times \{t\}$ intersects the dividing set of $\Sigma \times \{t\}$ exactly at the endpoints, so $\Gamma_A \cap (\delta \times I)$ consists of a single vertical arc. Thus the attaching arc α of the bypass D must not intersect the interior of δ. But then there are only two possibilities for α: one corresponds to a forbidden bypass, and the other is the arc a, as desired. □

We will show that trivial bypasses really are trivial: in other words, attaching a trivial bypass to a convex surface does not change the contact structure in a neighborhood of that surface. In order to do that, we first need the following fact about families of convex surfaces.

Proposition 7. *Suppose that $\Sigma \times I$ has two contact structures ξ_0, ξ_1 which induce the same characteristic foliation on $\Sigma \times \partial I$ and for which every surface $\Sigma_t = \Sigma \times \{t\}$ is convex and divided by a multicurve Γ_t. Then ξ_0 and ξ_1 are isotopic rel boundary.*

Proof. Each ξ_i has contact form $\alpha_i = \eta_t^i + f_t^i dt$, where we can take $\eta_0^0 = \eta_0^1$ and $\eta_1^0 = \eta_1^1$, and since $d\alpha_i = d\eta_t^i - \dot{\eta}_t^i \wedge dt + df_t^i \wedge dt$, the contact condtion $\alpha_i \wedge d\alpha_i > 0$ is equivalent to

$$f_t^i d\eta_t^i + \eta_t^i \wedge (df_t^i - \dot{\eta}_t^i) > 0.$$

Since each Σ_t is convex, we can find a smooth family of functions g_t^i on Σ which vanish to first order along Γ_t and which satisfy

$$g_t^i d\eta_t^i - dg_t^i \wedge \eta_t^i > 0,$$

following our proof that surfaces with dividing sets are convex, and we can also take $g_t^0 = g_t^1$ for $t = 0, 1$. At each time t, the functions g_t^i both vanish to first order on Γ_t and have sign ± 1 on $(\Sigma_t)_\pm$, so their quotient is a well-defined positive function on all of Σ. In particular, we can check that

$$g^0 d\left(\frac{g^0}{g^1}\eta\right) - dg^0 \wedge \left(\frac{g^0}{g^1}\eta\right) = \left(\frac{g^0}{g^1}\right)^2 (g^1 d\eta - dg^1 \wedge \eta)$$

at any time t, and so if we replace α_1 with the contact form

$$\alpha_1' = (\eta_t')^1 + (f_t')^1 dt := \frac{g_t^0}{g_t^1}\left(\eta_t^1 + f_t^1 dt\right)$$

for ξ_1 then we can take $g_t^0 = g_t^1$ and the condition $\eta_t^0 = \eta_t^1$ for $t \in \{0, 1\}$ still holds.

Now for each i and a sufficiently large constant λ we have a family of contact forms

$$\alpha_s^i = \eta_t^i + ((1-s)f_t^i + s\lambda g_t)dt,$$

since α_s^i is contact near $s = 0$ and $\alpha_s^i \wedge d\alpha_s^i$ is dominated by $s\lambda(g_t d\eta_t^i - dg_t \wedge \eta_t^i)$ away from $s = 0$. Thus by an isotopy through contact forms we can replace each α_i by the form

$$\alpha_i = \eta_t^i + \lambda g_t dt.$$

Finally, we interpolate between these using the isotopy

$$\alpha_s = ((1-s)\eta_t^0 + s\eta_t^1) + \lambda g_t dt.$$

Again $\alpha_s \wedge d\alpha_s$ is dominated by

$$\lambda\left((1-s)(g_t d\eta_t^0 - dg_t \wedge \eta_t^0) + s(g_t d\eta_t^1 - dg_t \wedge \eta_t^1)\right) \wedge dt,$$

so if $\lambda \gg 0$ then these are all contact forms and now Gray stability gives us the desired contact isotopy. $\square$