Math 273 Lecture 5

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Now that we know that tight contact structures exist, we can begin to answer some basic questions about them: for example, when does a convex surface have a tight neighborhood? The answer relies in part on an understanding of Legendrian knots in convex surfaces, originally due to Kanda and proved by Honda in full generality.

Theorem 1 (Legendrian Realization Principle)**.** *Let $\Sigma \subset (M, \xi)$ be a convex surface, and let $C \subset \Sigma$ be a multicurve which is transverse to Γ_Σ. Suppose that C is* nonisolating*: in other words, that every component of $\Sigma \backslash C$ intersects Γ_Σ. Then there is an isotopy ϕ_s of Σ through convex surfaces, supported in an arbitrarily small neighborhood of Σ and fixing Γ_Σ, so that $\phi_s(C)$ is always transverse to Γ_Σ and $\phi_1(C)$ is Legendrian.*

Proof. A curve $\gamma : S^1 \to \Sigma$ is Legendrian if and only if $\dot{\gamma}(t) \in \xi_{\gamma(t)} \cap T_{\gamma(t)}\Sigma$, i.e. if $\gamma(S^1)$ is a union of flow lines of the characteristic foliation. Thus our goal will be to construct such a foliation $\mathcal{F}$ of Σ for the multicurve C which is divided by Γ_Σ; having done so, we can then apply the Giroux flexibility theorem.

Consider a component Σ_0 of $\Sigma \backslash (\Gamma_\Sigma \cup C)$ as a surface with corners, and assume without loss of generality that $\Sigma_0 \subset \Sigma_+$. Each smooth component γ of $\partial\Sigma_0$ is a circle or arc. If γ belongs to Γ_Σ then $\mathcal{F}$ must be flowing out of Σ_0 on a neighborhood of γ. If γ belongs to C then it is either a circle, in which case we let it be a repelling closed orbit of $\mathcal{F}$, or an arc. If $\gamma \subset C$ is an arc, we put a positive hyperbolic point on its interior with γ its unstable separatrix, and if its endpoints do not belong to Γ_Σ then we put positive elliptic points on its boundary; otherwise we just let γ be a nonsingular flowline of $\mathcal{F}$.

For $\gamma \subset \partial\Sigma$, let $n = tw(\gamma, \Sigma)$. We assume that Σ has been perturbed into the standard form, so that the contact structure near γ is modeled by $\gamma = S^1 \times \{(0,0)\} \subset S^1_\theta \times \mathbb{R}^2_{x,y}$ with contact form

$$\alpha = \cos(n\theta)dx - \sin(n\theta)dy$$

if $n > 0$, or $\alpha = dy - xd\theta$ if $n = 0$, and $\Sigma = S^1 \times [0,1] \times \{0\}$. Then $\mathcal{F}$ is determined by Σ_ξ near γ. Thus we have defined $\mathcal{F}$ on a neighborhood of $\partial\Sigma_0$.

Write $\partial\Sigma_0 = \gamma_- \cup \gamma_+$, where $\mathcal{F}$ flows out of Σ_0 along γ_- and in along γ_+. We know that γ_- is nonempty because the nonisolating assumption means that $\partial\Sigma_0$ intersects Γ_Σ. If γ_+ is empty, we place a single elliptic source in the interior of Σ_0 and then remove a small neighborhood of that point from Σ_0; the new component of $\partial\Sigma_0$ lies in γ_+. We now extend $\mathcal{F}$ to all of Σ_0 by embedding Σ_0 in $\Sigma_0 \times [-1, 1]$ so that $\gamma_\pm$ is the part of Σ_0 at height ± 1. For generic embeddings, the height function has no extrema along $\text{int}(\Sigma_0)$ and is Morse-Smale, so we define $\mathcal{F}$ to be its downward gradient flow. The singularities added by this process are all hyperbolic, and we can arrange for them to be positive.

Repeating this procedure on each component of $\Sigma_+ \backslash (\Gamma_\Sigma \cup C)$, and the analogous procedure on $\Sigma_- \backslash (\Gamma_\Sigma \cup C)$, yields the desired foliation. □

Proposition 2. *Let C be a closed Legendrian curve transverse to Γ_Σ. Then $tw(C, \Sigma) = -\frac{1}{2}|C \cap \Gamma_\Sigma|$.*

Proof. The characteristic foliation we have just constructed puts a hyperbolic singularity on every arc of $C \backslash \Gamma_\Sigma$, so there are $|C \cap \Gamma_\Sigma|$ singularities of $\mathcal{F}$ along C and these are exactly the points $p \in C$ where $\xi_p = T_p\Sigma$. We claim that at each singularity, ξ and $T\Sigma$ intersect negatively, so that each of the points contributes $-\frac{1}{2}$ to the twisting number.

To see this, consider a standard neighborhood $\Sigma \times \mathbb{R}$ of Σ with contact form $\alpha = \eta + f dt$ and contact vector field $v = \partial_t$, and recall that $\Gamma_\Sigma = f^{-1}(0)$. Let $\{v_1, v_2\}$ be an oriented basis for $T\Sigma$ along C, with v_1 tangent to C, and suppose that $\iota_{v_2}\eta = 1$ near $C \cap \Gamma_\Sigma$. Then $-f v_2 + \partial_t \in \xi$ near these points, so the intersection of ξ with $T\Sigma$ is indeed negative, as desired. □

Using the Legendrian Realization Principle, we can now identify convex surfaces with tight neighborhoods.

Theorem 3 (Giroux's Criterion)**.** *Let $\Sigma \subset (M, \xi)$ be a convex closed surface. Then Σ has a tight neighborhood if and only if $\Sigma = S^2$ and Γ_Σ is connected or $\Sigma \neq S^2$ and Γ_Σ has no contractible components.*

Proof. We deal first with the case $\Sigma = S^2$. If Γ_Σ is connected, then we have already seen that Σ has a tight neighborhood: the unit sphere in $(\mathbb{R}^3, \xi_{\text{st}})$ provided one such example. Otherwise, let $D \subset S^2$ be a disk in a slightly larger neighborhood D' for which $D' \cap \Gamma_\Sigma = \partial D$. Take a simple closed curve $C \subset D' \backslash D$ parallel to ∂D. Then C is nonisolating since $|\Gamma_\Sigma| > 1$, so we can apply the Legendrian Realization Principle to make it Legendrian. Then C is a Legendrian knot which bounds a disk and $tw(C, \Sigma) = 0$, so that disk is overtwisted.

Suppose that $\Sigma \neq S^2$ and some component of Γ_Σ bounds a disk D. If $|\Gamma_\Sigma| = 1$, then we pick a nonseparating curve away from this disk and realize it as a Legendrian curve γ consisting entirely of singularities of Σ_ξ. We can then apply a "folding" operation to create three parallel circles of singularities, the result of which is that we have added two dividing curves parallel to γ in a neighborhood of γ. Thus we have perturbed Σ by an arbitrarily small isotopy (but *not* one through convex surfaces) supported away from D to ensure that $|\Gamma_\Sigma| > 1$. We can now use the Legendrian Realization Principle just as before to find an overtwisted disk in a neighborhood of D.

Finally, consider the case $\Sigma \neq S^2$ and no component of Γ_Σ bounds a disk. Pass to an $\mathbb{R}$-invariant neighborhood $\Sigma \times \mathbb{R}$, and let $\tilde{\Sigma} \cong \mathbb{R}^2$ be the universal cover of Σ. Let $D_1 \subset D_2 \subset \ldots$ be a set of convex disks, not necessarily with Legendrian boundary, whose union is all of $\tilde{\Sigma}$, and perturb them along their boundaries so that $\Gamma_{\tilde{\Sigma}}$ is transverse to each ∂D_i; we remark that $\Gamma_{\tilde{\Sigma}}$ has no closed components by assumption. If $\Sigma \times \mathbb{R}$ is overtwisted, then we can lift an overtwisted disk to $\tilde{\Sigma} \times \mathbb{R}$ and so some $D_i \times \mathbb{R}$ will contain an overtwisted disk.

We will now embed D_i in a convex S^2 with a single dividing curve, proving that D_i is in fact tight. If $\Gamma_\Sigma \cap D_i$ consists of more than one arc, we take two endpoints which are adjacent along ∂D_i and belong to different components; otherwise we take the endpoints of the unique arc of $\Gamma_\Sigma \cap D_i$. In either case Γ_Σ must intersect ∂D_i with opposite signs at these points, since Γ_Σ can be oriented as the boundary of Σ_+ or $-\Sigma_-$ and the arc connecting these points in ∂D_i runs through exactly one of $\Sigma_\pm$. We enlarge D_i by gluing on a small disk along a neighborhood of that arc, with dividing curve a single arc δ connecting the two chosen endpoints of Γ_Σ; this operation lowers $|\Gamma_\Sigma|$ by 1, or if it was already equal to 1 then it makes Γ_Σ into a single closed curve. Once Γ_Σ is a single closed curve, we can cap off D_i with a disk to get the desired convex S^2.

The only thing that remains is to check that this enlargement operation can be done. We first extend the contact form $\eta + f dt$ along a neighborhood N of δ so that $f = 0$ exactly on δ. We now have to extend it to the disk D cobounded by ∂N and ∂D_i, which lies entirely within either Σ_+ or Σ_-, so by rescaling we can assume that $f = 1$ along ∂D (or $f = -1$ if $D \subset \Sigma_+$). The contact form must then be $\beta + dt$ for some 1-form β such that $d\beta$ is an area form on D and $d\beta = d\eta$ along ∂D. Let ω be an area form on D satisfying

$$\int_D \omega = \int_{\partial D} \eta$$

and for which $\omega = d\eta$ on a neighborhood of ∂D. Let $\psi : D \to [0, 1]$ be a cutoff function which is supported in a small neighborhood of ∂D where η is defined, and for which $\psi = 1$ on a smaller neighborhood of ∂D. Then $\omega - d(\psi\eta)$ is a well-defined 2-form which is zero near ∂D, and

$$\int_D \omega - d(\psi\eta) = \int_D \omega - \int_{\partial D} \psi\eta = 0$$

so $[\omega - d(\psi\eta)] = 0$ in $H^2_c(D)$. By the Poincaré lemma it is exact, so we can write $\omega - d(\psi\eta) = d\lambda$ for some λ supported on $\mathrm{int}(D)$ and thus $\omega = d(\psi\eta + \lambda)$. Letting $\beta = \psi\eta + \lambda$ completes the proof. □

We can now use Giroux's criterion to prove the previously mentioned fact that tight contact structures can have only finitely many Euler classes on a given manifold. The following proposition is due to Eliashberg.

Proposition 4. *Let M be a closed, oriented 3-manifold with tight contact structure ξ, and let $\Sigma \subset M$ be a closed embedded surface. Then*

$$|\langle e(\xi), [\Sigma] \rangle| \leq -\chi(\Sigma)$$

if $\Sigma \neq S^2$, and if $\Sigma = S^2$ then $\langle e(\xi), [\Sigma] \rangle = 0$.

Proof. Perturb Σ to make it convex.We have shown that for any convex surface $\Sigma \subset M$,

$$\langle e(\xi), [\Sigma] \rangle = \chi(\Sigma_+) - \chi(\Sigma_-),$$

and if $\Sigma = S^2$ then Giroux's criterion says that Σ_+ and Σ_- are both disks, hence the right side is zero. For $\Sigma \neq S^2$, it is clear that $\chi(\Sigma) = \chi(\Sigma_+) + \chi(\Sigma_-)$, and so

$$\chi(\Sigma) + |\langle e(\xi), [\Sigma] \rangle| = 2 \max(\chi(\Sigma_+), \chi(\Sigma_-)).$$

No component of Σ_+ is closed: if that were the case, then we could rescale the contact form on a neighborhood of that component C to be $\alpha = \eta + dt$, so then $d\alpha = d\eta$ is an area form on C with total area

$$\int_C d\alpha = \int_{\partial C} \alpha = 0,$$

a contradiction. Furthermore, no component of Σ_+ is a disk by Giroux's criterion, so every component must have Euler characteristic at most 0 and thus $\chi(\Sigma_+) \leq 0$. The same is true for Σ_-, so we are done. □

Theorem 5. *Let M be closed and oriented. There are only finitely many classes $e \in H^2(M; \mathbb{Z})$ which are Euler classes of tight contact structures.*

Proof. The universal coefficient theorem says that

$$H^2(M; \mathbb{Z}) \cong \mathrm{Hom}(H_2(M; \mathbb{Z}), \mathbb{Z}) \oplus \text{torsion},$$

so let $\Sigma_1, \dots, \Sigma_r$ be embedded surfaces representing an integral basis of $H_2(M; \mathbb{Z})$, none of which are spheres. For any tight contact structure ξ on M, we can write $e(\xi) = \sum c_i f_i + g$, where the f_i are the classes dual to Σ_i and g is torsion, and then the previous proposition implies $|c_i| \leq -\chi(\Sigma_i)$ for all i, so there are only finitely many possible values for $e(\xi)$. □

Since we have seen that *any* even Euler class supports an overtwisted contact structure, this suggests that tight contact structures are in general much harder to find.