Math 273 Lecture 23

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We have now shown that every open book decomposition of a closed 3-manifold Y supports a unique contact structure up to isotopy, and also that every contact structure ξ on Y is supported by an open book. However, there can be many open books supporting ξ: for example, we have seen that (S^3, ξ_{st}) is supported by every Milnor open book $(B = f^{-1}(0), \pi = \arg f)$, where $f : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial with $f(0,0) = 0$ and an isolated singularity at the origin. The final step in proving the Giroux correspondence is to understand the relationship between all of the open books supporting a given (Y, ξ); for this we need the notion of positive stabilization.

We can define an open book abstractly, without reference to an ambient manifold, as a pair (Σ, h) consisting of a surface Σ with boundary and a monodromy map $h : \Sigma \to \Sigma$ which restricts to the identity on a neighborhood of $\partial\Sigma$. From this description we can construct the manifold Y by taking the mapping torus

$$M_h = \frac{\Sigma \times I}{(h(x), 0) \sim (x, 1)}$$

and for each component $\gamma \subset \partial\Sigma$ we glue an $S^1 \times D^2$ to the torus $\gamma \times S^1 \subset \partial M_h$, identifying curves of the form $S^1 \times \{*\}$ in $S^1 \times D^2$ with $\gamma \times \{*\}$ so that each circle $\{x\} \times S^1$ on ∂M_h, $x \in \partial\Sigma$, bounds a disk. It is clear that this description in terms of (Σ, h) is equivalent to the description in terms of (B, π), and that if $f : \Sigma \to \Sigma$ is a diffeomorphism then (Σ, h) and $(\Sigma, f \circ h \circ f^{-1})$ produce the same Y.

Definition 1. Let (Σ, h) be an abstract open book. A *positive stabilization* of (Σ, h) is an open book of the form $(\Sigma', h \circ \tau_c)$, where Σ' is obtained by attaching a 1-handle H to Σ along its boundary and τ_c is a right-handed Dehn twist along a curve $c \subset \Sigma'$ which intersects the cocore of H in a single point.

Positive stabilization is actually a special case of a more general operation.

Definition 2. The *Murasugi sum* of two open books $\mathcal{B}_1 = (\Sigma_1, h_1)$ and $\mathcal{B}_2 = (\Sigma_2, h_2)$, denoted $\mathcal{B}_1 * \mathcal{B}_2$, is constructed as follows: Choose two properly embedded arcs $a_i \subset \Sigma_i$ with product neighborhoods $R_i = a_i \times I$. Then $\mathcal{B}_1 * \mathcal{B}_2$ is the open book with pages

$$\Sigma = \Sigma_1 \cup_{R_1 = R_2} \Sigma_2,$$

glued together via the identification sending $\partial a_1 \times I \mapsto a_2 \times \partial I$ and $a_1 \times \partial I \mapsto \partial a_2 \times I$, and monodromy map $h = h_1 \circ h_2$.

Proposition 3. *Let (Y_i, ξ_i) be contact structures supported by open books $\mathcal{B}_i$ for $i = 1, 2$. Then $(Y_1 \# Y_2, \xi_1 \# \xi_2)$ is supported by the Murasugi sum $\mathcal{B}_1 * \mathcal{B}_2$.*

Proof. Topologically we can remove a ball $R_1 \times [\frac{1}{2}, 1]$ from $\Sigma_1 \times I$ and another ball $R_2 \times [0, \frac{1}{2}]$ from $\Sigma_2 \times I$ and then form $\Sigma \times I$ by gluing them together by the identification $R_1 \xrightarrow{\sim} R_2$. Gluing $\Sigma \times \{1\}$ to $\Sigma \times \{0\}$ by the map $h_1 \circ h_2$ then produces the connected sum $Y_1 \# Y_2$.

To see that this open book supports $\xi_1 \# \xi_2$, suppose that each ξ_i has been isotoped so that the Reeb vector field is transverse to the pages of $\mathcal{B}_i$. If we force each ball $\xi_i|_{R_i \times I}$ to have the same Reeb vector field on a neighborhood of the boundary, then when we glue $Y_1 \backslash B^3$ and $Y_2 \backslash B^3$ we can glue the Reeb vector fields together as well to ensure that the result always remains tangent to the new binding and transverse to the new pages. Therefore the supported contact structure is exactly $\xi_1 \# \xi_2$. □

Proposition 4. *Let $\mathcal{B}'$ be a positive stabilization of $\mathcal{B}$. Then both $\mathcal{B}$ and $\mathcal{B}'$ support the same contact structure.*

Proof. Positive stabilization is the same as taking a Murasugi sum with the open book (A, τ) where A is an annulus and τ a right-handed Dehn twist around its core. This open book supports the standard contact structure (S^3, ξ_{st}): one can see this using the Milnor fibration of $S^3 \subset \mathbb{C}^2$ induced by the positive Hopf link H,

$$\pi(z_1, z_2) = \frac{z_1 z_2}{|z_1 z_2|},$$

whose pages $\Sigma_\phi = \pi^{-1}(\phi)$ are parametrized by the map

$$f(t, \theta) = (\sqrt{t}, \theta, \sqrt{1-t}, \phi - \theta)$$

in polar coordinates $(r_1, \theta_1, r_2, \theta_2)$, for $t \in [0, 1]$ and $\phi \in S^1$. Indeed, if we cut $S^3 \backslash H$ open along the page $\Sigma_{2\pi} = \Sigma_0$ and identify the result as $\Sigma_0 \times I$, then we must apply a right-handed Dehn twist to $\Sigma_{2\pi} = \Sigma_0 \times \{1\}$ to glue it back to $\Sigma_0 = \Sigma_0 \times \{0\}$.

We conclude that if $\mathcal{B}$ supports the contact structure (Y, ξ), then $\mathcal{B}'$ supports $(Y, \xi) \# (S^3, \xi_{\mathrm{st}})$, which is contact isotopic to (Y, ξ) as desired. □

It is also possible to see that positive stabilization preserves the underlying contact structure by realizing the 1-handle attachment and subsequent Dehn twist as a cancelling pair of Weinstein handles; this argument can be generalized to open books in higher dimensions. See [2] for details.

Corollary 5. *Every contact structure has a supporting open book with connected binding.*

Proof. Take an arbitrary supporting open book (B, π), possibly with $|B| > 1$. If we positively stabilize by attaching a handle along two points on different components of B, then the resulting open book (B', π') has $|B'| = |B| - 1$ and we can repeat until we get $|B| = 1$. $\square$

Lemma 6. *Any open book (Σ, h) for (Y, ξ) can be positively stabilized several times so that it is the open book constructed from a contact cell decomposition of (Y, ξ).*

Proof. We can find an embedded graph $G \subset \Sigma$ onto which Σ retracts. We would like to apply the Legendrian realization principle to G, but Σ is not convex: it is transverse to a contact vector field, namely the Reeb vector field (after an isotopy of ξ), but its boundary is transverse rather than Legendrian. However, the double $F = \Sigma \cup_{\partial\Sigma} (-\Sigma')$ is convex, where Σ' is another page such that the union is smooth, and its dividing set is $B = \partial\Sigma$. Since G is nonisolating in this closed surface (every component of $\Sigma \backslash G$ intersects $\Gamma_F = \partial\Sigma$) we can find a singular foliation which contains G as a union of leaves, is divided by B, and agrees with F_ξ near $B \cup (-\Sigma')$. Then as in the proof of LeRP we have a C^∞-small perturbation of F which is fixed except on the interior of Σ and realizes G as a Legendrian curve; after this perturbation Σ is the ribbon of the Legendrian graph G.

Let $\{\alpha_i\}$ be a union of disjoint, properly embedded arcs in Σ such that $\Sigma \backslash \bigcup \alpha_i$ is a disk, and take a neighborhood N of Σ for which $\partial\Sigma \subset \partial N$. Cut Y open along Σ, so that we can identify $Y \backslash \Sigma$ as $\Sigma \times I$, and consider the disks $A_i = (\alpha_i \times I) \cap \overline{Y \backslash N}$. Each A_i intersects $\partial\Sigma$ at the two points $(\partial\alpha_i \times I) \cap \partial\Sigma$, so if we enlarge A_i inside N so that its boundary lies on G then we have $|tw(\partial A_i, \Sigma)| \leq 1$. This follows from the fact that there is no twisting along G, which does not intersect the dividing set Γ_F, and along either $\partial\alpha_i \times I$ component of ∂A_i the point of intersection with $\partial\Sigma$ contributes $\pm\frac{1}{2}$ to the twisting number.

At this point, the disks A_i and the surface Σ cut Y into a union of 3-cells, each of which is tight because $Y \backslash \Sigma$ is tight (embed Σ in the convex surface F, parametrize $\Sigma \times I$ by the Reeb flow so that its contact structure is clearly I-invariant, and apply Giroux's criterion to F), and this gives a CW-decomposition of Y whose 1-skeleton $\partial A_i \cup G$ is Legendrian. We have almost built a contact cell decomposition, but we may have $tw_\xi(\partial D, D) = tw_{T\Sigma}(\partial D, D) > -1$ for each 2-cell D; if this happens, we must positively stabilize the open book near ∂D and push ∂D onto the new handle to lower the twisting number as needed. $\square$

Theorem 7 (Giroux)**.** *Any two open books which support (Y, ξ) are related by a sequence of positive stabilizations.*

Proof. Stabilize these open books until they both correspond to contact cell decompositions. Then there is a common refinement of these decompositions which can be obtained from either one by a sequence of the following moves:

1. Add a Legendrian arc c to a 2-cell D so that $|c \cap \Gamma_D| = 1$, dividing D into a pair of new 2-cells.

2. Add a Legendrian arc c and a 2-cell D so that $c \subset \partial D$ and $\partial D \backslash c$ is already in the 1-skeleton, and $tw(\partial D, D) = 1$.

3. Add a 2-cell D along a closed curve in the 1-skeleton so that $tw(\partial D, D) = -1$.

One can show that the third move does not change the associated open book: the 1-skeleton G is unchanged, so that the ribbon of G is also unchanged and hence the pages are the same, and then the fibration is unchanged as well because we can order the 2-cells in the construction of the fibration so that we cut the complement of a page along all the original disks before cutting along the new 2-cell, so that once we glue back together along the new 2-cell we end up repeating the original construction.

The first and second moves each correspond to a positive stabilization. In either case we are adding a Legendrian arc c to the 1-skeleton, which adds a 1-handle to the ribbon. These moves create a new 2-cell D with Γ_D a single arc intersecting c in a point, and since $tw(\partial D, D) = -1$ the monodromy of the new ribbon is a positive Dehn twist along ∂D composed with the original monodromy map. This is a positive stabilization of the original open book, as desired. □

References

[1] Vincent Colin, *Chirurgies d'indice un et isotopies de sphères dans les variétés de contact tendues,* C. R. Acad. Sci. Paris Sér. I Math. 324 (1997), no. 6, 659–663.

[2] Otto van Koert, *Lecture notes on stabilization of contact open books,* *http*://arxiv.org/abs/1012.4359.