Math 273 Lecture 22

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Last time we showed that every open book decomposition (B, π) of a closed, oriented 3-manifold Y is supported by a unique contact structure $\xi_{(B,\pi)}$ up to isotopy, so that there is a well-defined map

$$\{\text{open book decompositions of } Y\} \longrightarrow \{\text{contact structures on Y}\} / \text{isotopy}.$$

In this lecture we will construct a map in the opposite direction.

Definition 1. A *contact cell decomposition* of (Y, ξ) is a CW-decomposition of Y such that the 1-skeleton is Legendrian, each 2-cell D satisfies $tw(\partial D, D) = -1$, and the restriction of ξ to any 3-cell is tight.

Proposition 2. *Let (Y, ξ) be a closed contact 3-manifold. Then (Y, ξ) admits a contact cell decomposition.*

Proof. Since Y is compact, it admits a cover by finitely many Darboux balls. Take a triangulation T which is fine enough so that every 3-cell is contained in one of these Darboux balls, and then apply a C^0-small perturbation to the 1-skeleton to make it Legendrian, then each condition is satisfied except $tw(\partial D, D) = -1$. However, the Thurston-Bennequin inequality tells us that $tw(\partial D, D) \leq -1$ for each 2-cell D, so we can perturb it to be convex. If some face satisfies $tw(\partial D, D) < -1$, then since D lives inside a tight Darboux ball its dividing set Γ_D consists of $-tw(\partial D, D) \geq 2$ properly embedded arcs. We can then find a non-isolating multicurve $C \subset D$ so that each component of $D \backslash C$ contains exactly one dividing arc, apply the Legendrian realization principle to C, and add it to the 1-skeleton. This splits D into convex faces D_i which each have one dividing curve and hence satisfy $tw(\partial D, D) = -1$, as desired. ☐

The 1-skeleton of a contact cell decomposition is a Legendrian graph G. Given such a graph, we can find an embedded connected surface $\Sigma \subset Y$ which retracts onto G and satisfies $T_p\Sigma = \xi_p$ if and only if $p \in G$. We call this the *ribbon* of G, and let $B = \partial\Sigma$ be its boundary; then B is a transverse link, as one can see by constructing Σ inside a model neighborhood of G.

Proposition 3. *The boundary B of the ribbon of G is the binding of an open book decomposition of Y with pages diffeomorphic to Σ.*

Proof. Let $X(B) = \overline{Y \backslash N(B)}$ and $\Sigma_X = \Sigma \cap X(B)$. If we parametrize the neighborhood $N(B)$ as a union of components $S^1_\theta \times D^2_{(r,\phi)}$, then we can express a neighborhood $\overline{N(\Sigma_X)}$ as the product $\Sigma_X \times [-\epsilon, \epsilon]$ where each $\Sigma_X \times \{t\}$ has boundary $\{\phi = t\}$ on $\partial N(B)$. It remains to be seen that $X(\Sigma) = \overline{X(B) \backslash N(\Sigma_X)}$ can be viewed as a similar product. Note that the exterior $X(\Sigma)$ has boundary $A \cup (\Sigma_X \times \{\pm\epsilon\})$, where $A \subset \partial N(B)$ is the union of annuli $\phi \notin (-\epsilon, \epsilon)$, fibered by circles $\{\phi = \text{const.}\}$, and there is one annulus for each component of B.

Let $D_1, \ldots, D_k$ be the 2-cells of the contact cell decomposition. Then each D_i has Legendrian boundary with twisting number -1, and since B is locally a pair of push-offs of ∂D_i determined by the contact framing we can take B to intersect each D_i in exactly two points. Each $D'_i = D_i \cap X(\Sigma)$ intersects A in a pair of arcs which connect two components of ∂A, and it intersects each $\Sigma_X \times \{\pm\epsilon\}$ in a single arc: this is easily seen by drawing a picture of $N(\Sigma_X) \cup N(B)$ twisting along ∂D_i. Furthermore, since $N(\Sigma_X) \cup N(B)$ is a neighborhood of a surface which retracts onto the 1-skeleton G, it follows that if we take these neighborhoods sufficiently small then $X(\Sigma) \backslash \bigcup_i D'_i$ is a disjoint union of balls, each contained in a different 3-cell of the contact cell decomposition.

Cut $X(\Sigma)$ open along D'_1. In the boundary of the resulting manifold $X_1 = X(\Sigma) \backslash N(D'_1)$, we have cut the surfaces $\Sigma_X \times \{\pm\epsilon\}$ along a single arc and the annuli A along a pair of arcs; let $\Sigma_1^\pm$ be the complement of the arc in $\Sigma_X \times \{\pm\epsilon\}$ and let A_1 be the complement of those arcs in A together with the two copies of D'_1 in ∂X_1. Since the arcs $D'_i \cap A$ are transverse to the fibration of A by circles, we can extend the fibration of $A \backslash \partial D'_1$ across each copy of D'_1 to fiber each component of A_1 by circles. The result is that $\partial X_1 = \Sigma_1^+ \cup \Sigma_1^- \cup A_1$ and A_1 is fibered by circles which agree with the original fibration of A.

If we repeat this process for each of $D'_2, \ldots, D'_k$, the resulting manifold X_k is a union of balls B^3_i, as argued above, and each ∂B^3_i consists of two disks belonging to $\Sigma_k^\pm$ (which can be viewed as a subsurface of $\Sigma_X \times \{\pm\epsilon\}$) and an annulus A^i_k fibered by circles. We can extend this fibration of A^i_k across all of B^3_i to write $B^3_i = D^2 \times I$, where each circle $\partial D^2 \times \{t\}$ is one of the circle fibers on A^i_k. Thus we can also write $X_k = (\bigcup D^2_i) \times I$ in a way which respects the fibration on A_k.

Now we reverse the cutting process and glue the copies of D'_k back together. Since the fibration of D'_k by arcs is the same on either copy, we can extend the fibration of X_k to a fibration of X_{k-1}, and similarly for each D'_i. The result is a fibration $X(\Sigma) = \Sigma' \times I$ for some Σ', and since $\Sigma' \times \partial I \cong \partial X(\Sigma) = \Sigma_X \times \{\pm\epsilon\}$ we conclude that $\Sigma' \cong \Sigma_X$. Furthermore, our original annuli A are identified with $\partial\Sigma' \times I$ so that each $\partial\Sigma' \times \{t\}$ is a fiber of the fibration of A by circles. This means that $X(B)$ is fibered over the circle by surfaces diffeomorphic to Σ_X, as desired. □

We remark that this construction can be taken as a proof that every closed 3-manifold Y admits an open book decomposition: using contact surgery and the standard contact structure on S^3, we can construct a contact structure ξ on Y and then construct this open book decomposition from a contact cell decomposition. Of course, we would also like to know that the open book we have constructed supports the original contact structure. For this, we recall one characterization of this property:

Lemma 4. *The open book decomposition (B, π) supports the contact structure ξ if and only if ξ admits a contact form (possibly after an isotopy) for which the Reeb vector field is positively tangent to B and positively transverse to the pages of π.*

We will also need the following gluing result.

Lemma 5. *Let (M, Ξ) be a contact manifold, and let D be a convex, properly embedded disk whose dividing set consists of boundary-parallel arcs. If $\Xi|_{M\backslash D}$ is tight, then Ξ is tight on all of M.*

Proof. We use a technique due to Colin called "isotopy discretization" [1]. Suppose that M contains an overtwisted disk Δ which intersects D in some potentially complicated way, and let D' be an embedded disk in M which is isotopic rel boundary to D but disjoint from Δ. We take an isotopy from D to D' and discretize it, finding a sequence of embedded disks

$$D_0 = D, D_1, \ldots, D_m = D'$$

so that each pair D_i, D_{i+1} intersect only along the boundary and cobound a ball. Then the contact structure on $M\backslash D_{i+1}$ is obtained from the contact structure on $M\backslash D_i$ by attaching some bypasses along D_i. If D_i has boundary-parallel dividing set, then all possible bypasses are either trivial or forbidden, so if $M\backslash D_i$ is tight, then these bypasses must all be trivial. This means that $M\backslash D_i$ is actually contact isotopic to $M\backslash D_{i+1}$, and $\Gamma_{D_{i+1}} \cong \Gamma_{D_i}$. But $M\backslash D_0$ is tight by assumption, so we conclude that $M\backslash D_m = M\backslash D'$ is well, contradicting the fact that $\Delta \subset M\backslash D'$. Therefore (M, Ξ) must have actually been tight. $\square$

Theorem 6. *The open book decomposition (B, π) constructed from (Y, ξ) in Proposition 3 supports ξ.*

Proof. Since we already observed that B is a transverse knot, we will take the neighborhood $N(B)$ to be a standard contact neighborhood of radius r_0 which is contactomorphic to $(S^1_\theta \times D^2_{(r,\phi)}, \ker \alpha)$ with $\alpha = d\theta + r^2 d\phi$ near each component. If we consider the vector field

$$R = (1 - a(r))\partial_\theta + b(r)\partial_\phi$$

for some smooth, nondecreasing function $a(r)$ satisfying $a(r) = r^4$ near $r = 0$ and $a(r) = 1$ near $r = r_0$, and

$$b(r) = \int_0^r \frac{a'(t)}{t^2} dt$$

so that $b(0) = 0$ and $a'(r) = r^2 b'(r)$, then we can check that $\mathcal{L}_R \alpha = 0$, so R is a contact vector field and thus it is the Reeb vector field for the contact form

$\frac{1}{\alpha(R)}\alpha$ once we have arranged that $\alpha(R) \neq 0$ everywhere. (This can be done by shrinking $N(B)$ if necessary.) At $r = 0$ we have $R = \partial_\theta$, so R is positively tangent to the binding B, and for $r > 0$ the ∂_ϕ-component of R is positive so it is positively transverse to the pages $\{\phi = \text{const.}\}$ inside $N(B)$. Furthermore, at $r = r_0$ we have $R = b(r_0)\partial_\phi$, and so the flow of R preserves the boundary $\partial N(B)$.

Next, we extend the Reeb vector field across the neighborhood $\overline{N(\Sigma_X)} = \Sigma_X \times [-\epsilon, \epsilon]$. On any page $\Sigma_X \times \{t\}$, we have already arranged $R = b(r_0)\partial_t$ near $\partial\Sigma_X \times \{t\}$, and we can extend it over the interior of the page by arranging the product $\Sigma_X \times [-\epsilon, \epsilon]$ to be an I-invariant neighborhood of $\Sigma_X \times \{0\} = \Sigma \cap X(B)$ and letting $R = b(r_0)\partial_t$. Again, this is a contact vector field and it is positively transverse to the pages, so R is now the Reeb vector field for an appropriate rescaling of the contact form on $N(B) \cup \overline{N(\Sigma_X)}$. In particular, since R never lies in the contact planes it follows that Σ_X is a convex surface with no dividing curves; this is allowed because its boundary is transverse rather than Legendrian.

We now wish to extend the Reeb vector field to the exterior $X(\Sigma)$. In the notation of the proof of Proposition 3, the boundary of $X(\Sigma)$ is the union of $\Sigma_X \times \{\pm\epsilon\}$ and the annuli A, and we know that R points into $X(\Sigma)$ along $\Sigma_X \times \{\epsilon\}$, points out of $X(\Sigma)$ along $\Sigma_X \times \{-\epsilon\}$, and is tangent to the annuli A. We can identify a neighborhood $N(A) \subset X(\Sigma)$ of the form $A \times I$, where $A = A \times \{0\}$ and $A' = A \times \{1\}$ is a parallel set of annuli and the Reeb flow takes $N(A) \cap (\Sigma_X \times \{-\epsilon\})$ to $N(A) \cap (\Sigma_X \times \{\epsilon\})$.

Let $F = \Sigma_X \cup A' \cup -\Sigma_X$ be a surface diffeomorphic to the double of Σ_X. We give F a singular foliation induced by the characteristic foliation on $\Sigma_X \cup -\Sigma_X$ and extend it across $A' \cong \partial\Sigma_X \times I$ so that it connects the leaves of Σ_X to the leaves of $-\Sigma_X$ in a nonsingular fashion; then F is divided by a set Γ which we identify with the cores of the annuli A. This means we can find an $\mathbb{R}$-invariant contact structure ξ_F on $F \times \mathbb{R}$ for which this foliation is the characteristic foliation on any $F \times \{t\}$. Since ∂_t is a contact vector field for ξ_F and is positively transverse to ξ_F along Σ_X, we can take a contact form for ξ_F such that ∂_t is the Reeb vector field on $\Sigma_X \times \mathbb{R}$. We then take a diffeomorphism

$$f : \Sigma_X \times \{\epsilon\} \to \Sigma_X \times \{0\} \subset F \times \mathbb{R}$$

which identifies the characteristic foliation on $\Sigma_X \times \{\epsilon\}$ induced by ξ with $(\Sigma_X)_{F_\xi}$. Using the flows of the Reeb vector fields R and ∂_t, respectively, we extend f to a neighborhood of $\Sigma_X \times \{\epsilon\} \subset X(\Sigma)$. Since $N(A) \subset X(\Sigma)$ is the image of a neighborhood of $\partial\Sigma_X \times \{\epsilon\}$ under the flow of $-R$, we use the flow of $-\partial_t$ to extend f to an embedding of $N(\Sigma_X \times \{\epsilon\}) \cup N(A)$ into $F \times \mathbb{R}$, and then finally we extend f along $\Sigma_X \times \{-\epsilon\}$ by an embedding which misses the image of the neighborhood of $\Sigma_X \times \{\epsilon\}$ but preserves the characteristic foliation. The result is a contact embedding of a neighborhood of $\partial X(R)$ into $F \times \mathbb{R}$, whose image lies in a neighborhood of $\Sigma_X \times \mathbb{R}$.

The embedding f extends topologically over all of $X(\Sigma)$, though maybe not as a contact embedding, and the pulled back contact structure $f^*\xi_F$ is tight because ξ_F is tight by Giroux's criterion. Furthermore, we know that $f^*\xi_F$ is identical to ξ on a neighborhood of $\partial X(\Sigma)$. Thus we would like to show that $\xi|_{X(\Sigma)}$ is tight, and then since $X(\Sigma)$ is a contact handlebody it is not hard to show $\xi|_{X(\Sigma)}$ must be isotopic to $f^*\xi_F$ rel boundary: cut $X(\Sigma)$ into a union of tight 3-balls along convex disks, each of which has a single dividing arc, so there is a unique way to glue them back together.

In order to see that $\xi|_{X(\Sigma)}$ is tight, we recall that $X(\Sigma)$ can be cut open along finitely many disks $D'_1, \dots, D'_k$, each of which is convex with dividing set a single arc, to get a union of 3-balls B^3_i which all lie in Darboux balls and are therefore tight. We conclude by the above gluing lemma that $\xi|_{X(\Sigma)}$ is tight, so we can perform an isotopy to replace $\xi|_{X(\Sigma)}$ with $f^*\xi_F$, arranging for the Reeb vector field to be $f^*\partial_t$, which is positively transverse to the pages of the open book on $X(\Sigma)$. We now have a globally defined Reeb vector field on Y which is positively transverse to the pages everywhere and also positively tangent to the binding. We conclude that (B, π) supports ξ after all. □

Corollary 7. *Given any Legendrian knot $L \subset (Y, \xi)$, there is an open book decomposition (B, π) such that L lies entirely within one of the pages Σ_θ.*

Proof. We form a contact cell decomposition which contains L in its 1-skeleton G, and then L lies in the ribbon of G, which forms a page of the corresponding open book decomposition. □

References

[1] Vincent Colin, *Chirurgies d'indice un et isotopies de sphères dans les variétés de contact tendues,* C. R. Acad. Sci. Paris Sér. I Math. 324 (1997), no. 6, 659–663.