Math 273 Lecture 2

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Last time we proved that every closed, oriented 3-manifold admits a (co-orientable) contact structure, using open book decompositions and an argument of Thurston and Winkelnkemper. Today we will prove that in fact there are lots of contact structures on any such 3-manifold: we can find an overtwisted contact structure in any homotopy class of co-orientable plane field. Recall what it means to be overtwisted:

Definition 1. A contact 3-manifold (M, ξ) is *overtwisted* if it contains an embedded disk D such that $\xi|_{\partial D} = TD|_{\partial D}$. If (M, ξ) does not contain an overtwisted disk, then it is said to be *tight*.

Example 2. The overtwisted structure $(\mathbb{R}^3, \xi_{\text{ot}})$ with contact form $\alpha_{\text{ot}} = \cos(r)dz + r\sin(r)d\theta$ has overtwisted disk $D = \{r = \pi, z = 0\}$ since $\xi|_{r=\pi} = \ker(-dz) = \text{span}\{\partial_x, \partial_y\}$.

Let $K \subset (M, \xi)$ be a positive transverse knot. Then K has a model neighborhood of the form

$$S^1 \times \{0\} \subset S^1_\theta \times D^2_\delta$$

for some $\delta > 0$, where D^2 has polar coordinates (r, ϕ) and rectangular coordinates (x, y) and the contact form on $S^1 \times D^2$ is

$$\alpha = d\theta + r^2 d\phi = d\theta + xdy - ydx.$$

As we saw in Thurston-Winkelnkemper's proof, another contact structure on $S^1 \times D^2$ can be specified by

$$\alpha' = f(r)d\theta + g(r)d\phi$$

as long as $(f(r), g(r)) = (1, r^2)$ near $r = \delta$ and $fg' - gf' > 0$ for all r. We will take functions f, g which satisfy $(f(r), g(r)) = (-1, -r^2)$ near $r = 0$; a parametric graph of (f, g) would start by moving downward from $(-1, 0)$ and traveling counterclockwise around the origin, avoiding the positive y-axis, until it reaches $(1, \delta^2)$ moving upward at $r = \delta$.

Definition 3. Replacing $\xi = \ker(\alpha)$ with $\xi' = \ker(\alpha')$ on a neighborhood of K is called performing a *Lutz twist* along K.

Lemma 4. *Performing a Lutz twist results in an overtwisted contact structure.*

Proof. Let $r_0 \in (0, \delta)$ be a point where $g(r_0) = 0$. At a point $p = (\theta_0, (r_0, \phi_0)) \in S^1 \times D^2$, we have

$$\xi'_p = \operatorname{span}(\partial_r, \partial_\phi)$$

which is the tangent plane to the disk $D_0 = \{(\theta_0, (r, \phi)) \mid r \leq r_0, \phi \in S^1\}$. Thus D_0 is an overtwisted disk. □

A co-orientable contact structure ξ on M has trivial normal bundle, so $TM = \xi \oplus \underline{\mathbb{R}}$, and since M is parallelizable we have $w_2(\xi) = w_2(\xi) \oplus w_2(\underline{\mathbb{R}}) = w_2(TM) = 0$. Thus the Euler class of any contact structure is even.

Lemma 5. *A Lutz twist along the positively transverse knot $K \subset (M, \xi)$ changes the Euler class of ξ according to the formula*

$$e(\xi') - e(\xi) = -2PD(K).$$

Proof. Take a generic section s of ξ which equal to $r\partial_r = x\partial_x + y\partial_y$ in the model neighborhood of $S^1 \times D^2$, where

$$\xi = \ker(d\theta + r^2 d\phi).$$

Let $\psi : [0, 1] \to \mathbb{R}_{\geq 0}$ be a smooth, nondecreasing function which equals 0 for $r < \frac{1}{3}$ and 1 for $r > \frac{2}{3}$. Then

$$s' = \psi(r) \cdot r\partial_r + (1 - \psi(r)) \cdot r(g(r)\partial_\theta - f(r)\partial_\phi)$$

is a section of $\xi' = \ker(f(r)d\theta + g(r)d\phi)$ which is equal to $r\partial_r = s$ near $r = 1$, nonzero for $r \geq \frac{1}{3}$, and

$$g(r)\partial_\theta - f(r)\partial_\phi = -r^3\partial_\theta + r\partial_\phi = -r^3\partial_\theta + (-y\partial_x + x\partial_y)$$

for $r < \frac{1}{3}$. In particular, s and s' vanish to first order along the positive knots K in ξ and $-K$ in ξ', respectively, and they are equal away from $S^1 \times D^2$. Since the Euler class is Poincaré dual to the zero set of a generic section, we have $e(\xi) - PD(K) = e(\xi') + PD(K)$, as desired. □

Corollary 6. *Every even element $e \in H^2(M; \mathbb{Z})$ is the Euler class of a contact structure on M.*

Proof. We know that M admits some contact structure ξ, and $e(\xi)$ is even, so let c be a cohomology class satisfying $e(\xi) - e = 2c$. We can find an embedded link $L \subset M$ which is Poincaré dual to c – let L be the zero set of a generic section of the complex line bundle over M with first Chern class c – and then perform Lutz twists along each component of L to get a new contact structure ξ' with $e(\xi') = e(\xi) - 2c = e$. □

If we perform a Lutz twist along K, the new contact form on $S^1 \times D^2$ at $r = 0$ is given by

$$\alpha'|_{r=0} = -d\theta - r^2 d\phi = -\alpha|_{r=0}$$

and so K switches from positively transverse to negatively transverse (or vice versa). If we perform another Lutz twist along $-K$, then the result ξ'' is still overtwisted; we claim that it is homotopic to ξ as a plane field. We can describe the composition of these, a *full Lutz twist*, as replacing the contact form

$$\alpha = d\theta + r^2 d\phi$$

on $S^1 \times D^2$ with

$$\alpha'' = f(r)d\theta + g(r)d\phi,$$

where $(f, g) = (1, r^2)$ for $r \in [0, \epsilon]$ and $r \in [1-\epsilon, 1]$ and the graph of (f, g) travels once around the origin. Let $\chi(r) : [0, 1] \to \mathbb{R}_{\geq 0}$ be positive on $[\epsilon, 1-\epsilon]$ and supported on $[\frac{\epsilon}{2}, 1 - \frac{\epsilon}{2}]$. Then we define the family of 1-forms

$$\alpha_t = \chi(r)dr + (1-t)\alpha + t\alpha''.$$

Clearly $\alpha_t = \alpha = \alpha''$ for $r \in [0, \frac{\epsilon}{2}] \cup [\frac{\epsilon}{2}, 1]$, so $\ker(\alpha_t)$ is fixed near $\partial(S^1 \times D^2)$ and α_t is nonzero everywhere. Thus $\ker(\alpha_t)$ gives a homotopy of plane fields from ξ to ξ''. Since ξ'' is overtwisted just as in the case of a simple Lutz twist, we have proven:

Proposition 7. *Every contact structure is homotopic as a 2-plane field to an overtwisted contact structure.*

There is a natural inclusion

$$\{\text{overtwisted contact structures}\} \to \{2\text{−plane fields}\}$$

which, according to a celebrated theorem of Eliashberg, is a homotopy equivalence. We will eventually prove that it gives a bijection on π_0 of each space, i.e. that there is a unique overtwisted contact structure up to isotopy in each homotopy class of plane fields. The surjectivity part is due to Lutz; the injectivity is much harder and will be proven later. In order to prove surjectivity, we must first understand how to classify 2-plane fields on a closed 3-manifold M.

Since a co-oriented plane field can be uniquely determined by its normal vector at each point and $TM \cong M \times \mathbb{R}^3$, there is a natural one-to-one correspondence of homotopy classes

$$\{2\text{−plane fields}\} \leftrightarrow [M, S^2].$$

The Pontryagin-Thom construction puts $[M, S^2]$ in one-to-one correspondence with framed cobordism classes of framed links as follows: given a smooth map $f : M \to S^2$, we pick a regular value c of f and a fixed basis $\mathfrak{b}$ of $T_c S^2$ and associate to f the link $L_f = f^{-1}(c) \subset M$ with basis $f^*\mathfrak{b}$ for the normal bundle of L_f. Conversely, given a framed link $L \subset M$, we define the map $f_L : M \to S^2$ by projecting an open neighborhood $L \times \text{int}(D^2)$ onto $\text{int}(D^2) = S^2 \backslash \{p\}$ and then sending the complement of this neighborhood to p.

Lemma 8. *Let ξ_1 and ξ_2 be plane fields on M with associated framed links L_{ξ_1} and L_{ξ_2}, and define*

$$d^2(\xi_1, \xi_2) = PD(L_{\xi_1}) - PD(L_{\xi_2}) \in H^2(M; \mathbb{Z}).$$

Then ξ_1 and ξ_2 are homotopic on the complement of a 3-ball if and only if $d^2(\xi_1, \xi_2) = 0$.

Proof. If $d^2(\xi_1, \xi_2) = 0$ then L_{ξ_1} and L_{ξ_2} are homologous, so there is an unframed cobordism $W \subset M \times I$ from L_{ξ_1} to L_{ξ_2}. If we remove a small disk from W then we can extend the framing of each L_{ξ_i} to a trivialization of $N(W \backslash D^2)$, so W gives a *framed* cobordism from L_{ξ_1} to $L_{\xi_2} \sqcup U_n$, where U_n is the unknot with framing n. Since each L_{ξ_i} is a homotopy invariant, it follows that ξ_1 can be homotoped to agree with ξ_2 away from U_n, so the ξ_i are homotopic on the complement of a ball containing a neighborhood of U_n.

Conversely, if $\xi_1|_{M\backslash B^3} \simeq \xi_2|_{M\backslash B^3}$ then after a homotopy we may assume that $\xi_1 = \xi_2$ except on B^3, and so $L_{\xi_1} \sqcup L_1'$ is cobordant to $L_{\xi_2} \sqcup L_2'$ for some links $L_i' \subset B^3$. We may fill in Seifert surfaces for L_1' and L_2' to get a cobordism from L_{ξ_1} to L_{ξ_2}, so $[L_{\xi_1}] = [L_{\xi_2}]$ and $d^2(\xi_1, \xi_2) = 0$. □

If $d^2(\xi_1, \xi_2) = 0$, then we may assume after a homotopy that ξ_1 and ξ_2 coincide except on some ball B^3. We then construct a map

$$f : S^3 \to S^2$$

as follows: identify the upper hemisphere of S^3 with B^3 and let f be the normal vector to $\xi_1|_{B^3}$ there; then identify the lower hemisphere with $-B^3$ and let f be the normal vector to $\xi_2|_{B^3}$ there.

Definition 9. The obstruction class $d^3(\xi_1, \xi_2)$ is the Hopf invariant of the map $f : S^3 \to S^2$. It can be computed as the linking number of the preimages of two regular values of f, or via the isomorphism $\pi_3(S^2) \cong \mathbb{Z}$.

Note that both d^2 and d^3 are additive invariants, in the sense that

$$d^i(\xi_1, \xi_3) = d^i(\xi_1, \xi_2) + d^i(\xi_2, \xi_3).$$

It is also clear that $\xi_1 \simeq \xi_2$ if and only if $d^2(\xi_1, \xi_2) = 0$ and $d^3(\xi_1, \xi_2) = 0$.

Proposition 10. *If ξ' is obtained from ξ by a Lutz twist along K, then $d^2(\xi, \xi') = PD(K)$.*

Proof. Let us consider a contact structure ξ_0 such that $e(\xi_0) = 0$, in which case we can trivialize TM by oriented basis vectors $x_1, x_2 \in \xi_0$ and $x_3 \in \xi_0^\perp$. Construct ξ_1 by a Lutz twist along a transverse knot $K \subset (M, \xi_0)$; we claim that $d^2(\xi_0, \xi_1) = PD(K)$ as well, from which the proposition will follow using the additivity of d^2.

To see this, observe that the map $f_{\xi_0} : M \to S^2$ is constant with image x_3. Performing a homotopy to change the trivialization of TM on the $S^1 \times D^2$ neighborhood of K, which had contact form

$$d\theta + r^2 d\phi = d\theta + x dy - y dx$$

on ξ_0, so that $x_1 = \partial_\theta$, $x_2 = \partial_x$, and $x_3 = \partial_y$, it is still clear that $f_{\xi_0}(p) \neq -\partial_\theta$ everywhere. Since f_{ξ_0} misses the point $(-1,0,0)$, we have $L_{\xi_0} = \emptyset$ and $PD(L_{\xi_0}) = 0$. On the other hand, the Lutz twist replacing the contact form on $S^1 \times D^2$ with

$$f(r)d\theta + g(r)d\phi$$

turns K into a negatively transverse knot with $f_{\xi_1}^{-1}(-1,0,0) = -K$ (recall that the new contact form is $-d\theta$ at $r = 0$, i.e. along K), and so $L_{\xi_1} = -K$. Thus $d^2(\xi_0, \xi_1) = 0 - PD(L_{\xi_1}) = PD(K)$, as desired. □

In order to prove our main theorem, we first need a fact about transverse knots.

Definition 11. Let $T \subset (\mathbb{R}^3, \xi_{\mathrm{st}})$ be a positively transverse knot with Seifert surface Σ. Since ξ_{st} is trivial over Σ, we may choose a nonzero section s of $\xi_{\mathrm{st}}|_\Sigma$ and let T' be a push-off of T in that direction. Then the *self-linking number* of T is defined as

$$sl(T) = lk(T, T').$$

Proposition 12. *The self-linking number of T may be calculated as the writhe of its front ($xz-$) projection.*

Proof. Since $\xi_{\mathrm{st}} = \ker(dz - y dx)$, we may take ∂_y to be a section of ξ_{st}. Since T is positive, a smooth parametrization $\gamma(\theta) = (x(\theta), y(\theta), z(\theta))$ satisfies $z' > y \cdot x'$, and so the knot must be oriented upward at any vertical tangencies; furthermore, at a positive crossing we cannot have both strands pointing down, because the top strand has $x', z' < 0$ and hence $z' - yx' > 0$ becomes $y > \frac{z'}{x'} = \frac{dz}{dx} > 0$ while the bottom strand satisfies $z' < 0 < x'$ and hence $y < \frac{z'}{x'} = \frac{dz}{dx} < 0$, contradiction. One can check that any smooth diagram satisfying these conditions is the front projection of a transverse knot.

Now push T off in the ∂_y direction, achieved by lifting it slightly off the page on which its front projection is drawn. It is not hard to see that each crossing contributes its sign to $lk(T, T')$, and so $sl(T)$ is the writhe of the front projection. □

It is not hard to see that there are transverse knots with $sl(T) = \pm 1$: for $sl = -1$ we can take an unknot diagram with a single negative crossing, and for $sl = +1$ we can use a right-handed trefoil. In fact, for any odd $n \in \mathbb{Z}$ there is a transverse knot with $sl(T) = n$.

Theorem 13. *Let η be a 2-plane field on M. Then there is an overtwisted contact structure which is homotopic to η as a 2-plane field.*

Proof. We can find a positively transverse knot K with $PD(K) = d^2(\xi_0, \eta)$, and then Lutz twisting ξ_0 along K gives a contact structure ξ_1 with

$$d^2(\xi_0, \xi_1) = PD(K) = d^2(\xi_0, \eta).$$

But then $d^2(\xi_1, \eta) = 0$, so after a homotopy we may assume that ξ_1 and η coincide outside of a ball B^3. By making B^3 arbitrarily small, we can even assume that it is a Darboux ball.

Now let $T \subset (B^3, \xi_1)$ be a transverse knot with self-linking number n, and let ξ_2 be obtained from ξ_1 by Lutz twisting along T; since T is nullhomologous, we have

$$d^2(\xi_2, \eta) = d^2(\xi_2, \xi_1) + d^2(\xi_1, \eta) = 0.$$

We wish to compute $d^3(\xi_2, \xi_1)$ using the trivialization of TB^3 inherited from $\xi_0|_{B^3} = \xi_1|_{B^3}$, where (x_1, x_2) is a basis of ξ_0 and x_3 is the oriented normal. The map $f : S^3 \to S^2$ which determines $d^3(\xi_2, \xi_1)$ is then equal to $-x_3$ along the whole lower hemisphere, and in particular it avoids a neighborhood $U \subset S^2$ of $(-1, 0, 0)$. Now $d^3(\xi_2, \xi_1)$ is the linking number of $-T = f^{-1}(-1, 0, 0)$ with $f^{-1}(u)$ for any regular value $u \in U$, and $f^{-1}(u)$ is a transverse push-off of $-T$, so

$$d^3(\xi_2, \xi_1) = sl(T).$$

In particular, given transverse knots $T_\pm \subset (B^3, \xi_1)$ with $sl(T_\pm) = \pm 1$ we can perform Lutz twists along k unlinked copies of $T_\pm$ to get ξ_2 with

$$d^3(\xi_2, \eta) = d^3(\xi_2, \xi_1) + d^3(\xi_1, \eta) = \pm k + d^3(\xi_1, \eta).$$

Choosing the appropriate sign and value of k, we get a contact structure ξ_2 with the same d^2 and d^3 invariants as η, so $\xi_2 \simeq \eta$. $\square$

Next time we will begin to study embedded surfaces in contact manifolds. Among other things, this will show us that tightness is a much more restrictive condition: for example, only finitely many elements of $H^2(M; \mathbb{Z})$ can be the Euler class of a tight contact structure, whereas we have seen that any even element is the Euler class of an overtwisted one.