Math 273 Lecture 19

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Our goal today is to begin to prove the following theorem of Eliashberg [1], following a recent proof by Huang [2] which relies heavily on convex surface theory and bypasses.

Theorem 1. *Let Y be a closed, oriented 3-manifold. If overtwisted contact structures ξ and ξ' on Y are homotopic as plane fields, then they are isotopic.*

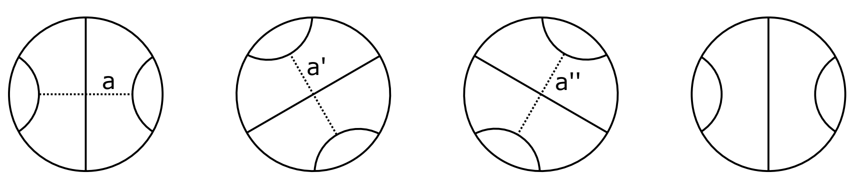
In this lecture, we will work to classify overtwisted contact structures on $S^2 \times I$ which have convex boundary and a tight neighborhood of $S^2 \times \partial I$, i.e. one dividing curve on each component $S^2 \times \{0,1\}$.

The following generalization of the Right-to-Life Principle shows that we have much more freedom to find bypasses in an overtwisted contact structure.

Lemma 2. *Let $\Sigma \subset (M, \xi)$ be a surface whose complement is overtwisted. If $\alpha \subset \Sigma$ is an admissible Legendrian arc for a bypass attachment, then there is a bypass in $M \backslash \Sigma$ attached to Σ along α.*

Proof. Let $D \subset M \backslash \Sigma$ be an overtwisted disk. Take a parallel copy α' of α with the same endpoints and with interior off of Σ; then $\alpha \cup \alpha'$ bounds a Legendrian disk and $tb(\alpha \cup \alpha') = -2$, so if we replace α' with the connected sum $\beta = \alpha' + \partial D$ then $tb(\alpha \cup \beta) = -1$ and $\alpha \cup \beta$ bounds a disk which is the desired bypass. □

Consider the effect of attaching *bypass triples*, which are triples of bypasses attached to some surface Σ in a neighborhood of an admissible arc, one after the other, as follows:



We will let σ_a, $\sigma_{a'}$, and $\sigma_{a''}$ denote the three bypasses and $\Delta_a = \sigma_a \circ \sigma_{a'} \circ \sigma_{a''}$ the union of all three. Note that Δ_a does not change Γ_Σ except by an isotopy supported near a.

Definition 3. A *minimal overtwisted ball* is an overtwisted (B^3, ξ) obtained from attaching a bypass triple to the standard (B^3, ξ_{st}).

We claim that this is well-defined. Since (B^3, ξ_{st}) is tight, its boundary S^2 has a single dividing curve, and up to isotopy there are only two choices a and b of attaching arcs for the bypass triple; one (say a) is a trivial bypass arc and the other is a forbidden arc. Then in $\Delta_a = \sigma_a \circ \sigma_{a'} \circ \sigma_{a''}$, attaching σ_a does not change the contact structure, and a' and a'' are isotopic to b and b' respectively, so $\Delta_a = \sigma_b \circ \sigma_{b'}$. At this point b'' is a trivial arc, so $\Delta_a = \sigma_b \circ \sigma_{b'} \circ \sigma_{b''} = \Delta_b$ as desired. We also see that these are overtwisted by Giroux's criterion, since after attaching σ_b the boundary S^2 has three dividing curves.

Proposition 4. *Let (Y, ξ) be a contact manifold with boundary, and let α and β be admissible arcs for bypasses on ∂Y. If ξ_a and ξ_b are the contact structures obtained by attaching bypass triangles Δ_a and Δ_b, respectively, then ξ_a is isotopic to ξ_b rel boundary.*

Proof. It suffices to assume that a and b are disjoint, since otherwise we can take an arc c disjoint from either one and apply the proposition to the pairs (a, c) and (c, b). We also describe the bypass triangles differently as follows: identify an I-invariant collar neighborhood $\partial Y \times [-1, 0]$, with $a = a \times \{0\}$, and if D_a is a small neighborhood of $a \subset \partial Y$ then we let N_a be the ball $D \times [-\frac{2}{3}, -\frac{1}{3}]$ after rounding corners, and note by Giroux's criterion that N_a is tight. If we cut out N_a and glue in a minimal overtwisted ball, then the result is isotopic to the result of attaching Δ_a instead.

Let γ be a Legendrian arc connecting the balls N_a and N_b with $\partial\gamma \subset \Gamma_{\partial N_a} \cup \Gamma_{\partial N_b}$. If we let $N = N_a \cup N(\gamma) \cup N_b$ and round corners, the result is a tight B^3, and so replacing exactly one of N_a and N_b with a minimal overtwisted ball turns N into a minimal overtwisted ball. Since this contact structure on B^3 is unique rel boundary, it follows that $\xi_a|_N$ and $\xi_b|_N$ are isotopic rel ∂N, and we can extend this isotopy trivially to the rest of Y. $\square$

We can therefore write Δ to denote a bypass triangle attachment along any admissible arc, since the result does not depend on the choice of arc.

Corollary 5. *Bypass attachment σ_a commutes with bypass triangle attachment Δ for any a.*

Proof. Attach Δ along some admissible arc disjoint from a, where it is clear that they commute. $\square$

Definition 6. Two contact structures ξ and ξ' on $S^2 \times [0, 1]$ are *stably isotopic* if $\xi \circ \Delta^n$ is isotopic to $\xi' \circ \Delta^n$ for some $n \geq 0$.

Definition 7. A contact structure ξ on $S^2 \times [0, 1]$ is *induced by an isotopy* if each $S^2 \times \{t\}$ is convex, $t \in [0, 1]$.

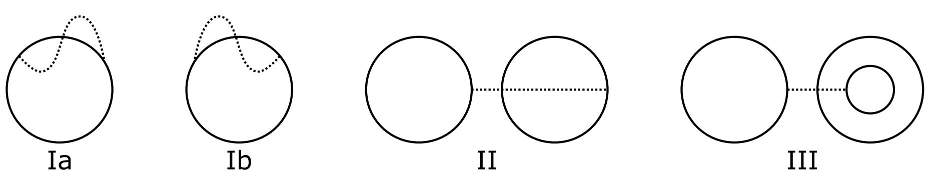
We make sense of this definition as follows: let $\phi_t : S^2 \to S^2$ be an isotopy for which $\phi_0 = \text{id}$ and $(\phi_t)_* \Gamma_{S^2 \times \{0\}} = \Gamma_{S^2 \times \{t\}}$. If ξ_0 is the I-invariant contact structure on $S^2 \times I$ with dividing set $\Gamma_{S^2 \times \{0\}}$ on each sphere, then the diffeomorphism $\Phi(x, t) = (\phi_t(x), t)$ of $S^2 \times I$ gives a contact structure $\Phi_*(\xi_0)$ for which each $S^2 \times \{t\}$ is convex and has the same dividing set with respect to both $\Phi_*(\xi_0)$ and ξ. It now follows that these contact structures are isotopic rel $S^2 \times \{0, 1\}$.

Lemma 8. *Let $(S^2 \times [0, \frac{1}{2}], \xi_\Phi)$ be induced by an isotopy ϕ_t, and let $(S^2 \times [\frac{1}{2}, 1], \sigma_a)$ be the contact structure obtained by attaching a bypass along an admissible arc $a \subset S^2 \times [\frac{1}{2}, 1]$. Then $a' = \phi_{1/2}^{-1}(a)$ is an admissible arc on $S^2 \times \{0\}$, and the contact structures $\xi_\Phi \circ \sigma_a$ and $\sigma_{a'} \circ \xi_\Phi$ on $S^2 \times I$ are isotopic rel boundary.*

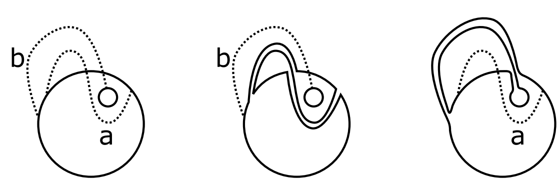
Proof. Let $D \subset S^2 \times [\frac{1}{2}, 1]$ be the bypass attached along a, and extend D to a disk $D' = D \cup \Phi(a' \times [0, \frac{1}{2}])$. Then $tb(\partial D') = tb(\partial D) = -1$, so D' is still a bypass. Now if we remove a neighborhood of $(S^2 \times \{0\}) \cup D'$ from $S^2 \times I$, then the rest can still be foliated by convex surfaces so that the contact structure is induced by Φ, so the resulting contact structure is indeed $\sigma_{a'} \circ \xi_\Phi$. □

Proposition 9. *Let ξ be an overtwisted contact structure on $S^2 \times [0, 1]$ for which the spheres $S^2 \times \{0, 1\}$ are convex with a single dividing curve. Then ξ is stably isotopic to some Δ^n.*

Proof. We know that up to isotopy, $S^2 \times \{0, 1\}$ have tight neighborhoods and that there are only finitely many times t_i at which $S^2 \times \{t\}$ fails to be convex. At each of those times we can describe ξ as a bypass attachment along some arc a_i, so that $\xi = \sigma_{a_1} \circ \sigma_{a_2} \circ \ldots \circ \sigma_{a_k}$. Define the *complexity* of this sequence as the maximum number of dividing curves on any convex $S^2 \times \{t\}$. Note also that in between each of the t_i, the contact structure is induced by an isotopy, so we can use the previous lemma to pull up all of the attaching arcs until they are in a neighborhood of $S^2 \times \{1\}$ and the rest of the contact structure on $S^2 \times [0, 1]$ is induced by an isotopy. We classify each arc a_i into one of four types depending on how it intersects the dividing set:

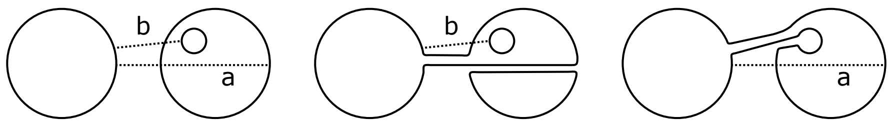


We claim that up to isotopy, we can assume that there are no arcs of type Ib. If we have a type Ib arc a which is not a trivial bypass arc, then the small half-disk cut off inside the dividing arc shown above must contain some dividing curves, and we can use one of them to find an admissible arc b as shown below.



The arc b becomes trivial after we attach the bypass along a as shown in the middle figure, so by the Right-to-Life Principle there was already a bypass along b of type II at the beginning, and we have $\sigma_a = \sigma_a \circ \sigma_b = \sigma_b \circ \sigma_a$. After we attach the bypass along b, we have the figure at right in which the arc a is of type II. Thus we can replace each nontrivial type Ib arc with a pair of type II arcs, and each trivial arc can be replaced by a contact structure induced by an isotopy and thus removed from the sequence.

Similarly, we can also assume there are no arcs of type II. Indeed, the upper half of the disk which is separated by a nontrivial type II arc a must contain some other dividing curves. We can use one of these other dividing curves to find an admissible arc b of type III as shown below on the left:

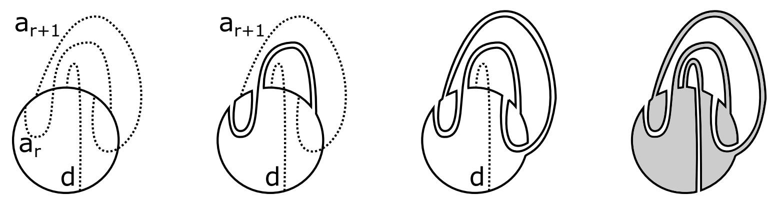


The middle picture shows the result of isotoping across the bypass along a. Now b becomes a trivial arc, so by the Right-to-Life Principle there is a bypass along b, and since b is disjoint from a this bypass was there all along. We have $\sigma_a = \sigma_a \circ \sigma_b = \sigma_b \circ \sigma_a$, and if we attach the bypass along b first then a becomes an arc of type Ia. Thus we have replaced the type II arc with a type III arc followed by a type Ia arc.

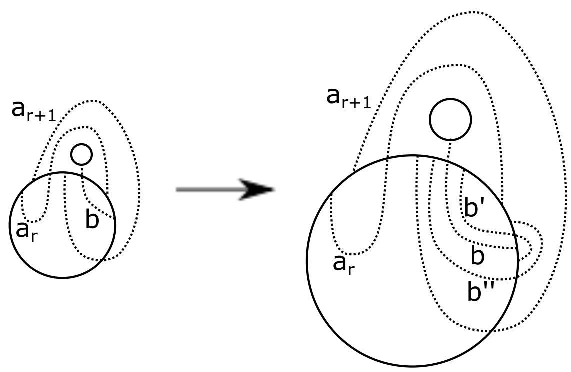
At this point we have described the contact structure by a sequence of arcs, all of which are either type Ia or type III. Note that type Ia arcs increase the number of dividing curves by 2, and type III arcs decrease it by 2. If we achieve the complexity c (i.e. the maximum number of dividing curves) in between times t_r and t_{r+1} and $c \geq 5$, then it follows that t_r must be type Ia and t_{r+1} must be type III. Suppose for now that a_r and a_{r+1} are disjoint, so that we can view them both as curves on the sphere $S^2 \times \{t_r - \epsilon\}$ and $\sigma_{a_r} \circ \sigma_{a_{r+1}} = \sigma_{a_{r+1}} \circ \sigma_{a_r}$, and let $\gamma \subset S^2$ be the dividing curve which intersects a_r. (We claim that this can be achieved by possibly adding in some bypass triangles.)

If $\gamma \cap a_{r+1}$ contains at most one point, then we may interchange $\sigma_{a_{r+1}}$ and σ_{a_r}. This preserves the types of both attaching arcs, but now we first decrease the number of dividing curves and then increase it again, so the number of dividing curves at each step from time $t_r - \epsilon$ to time $t_{r+1} + \epsilon$ is $c - 2 \stackrel{IV}{\to} c - 4 \stackrel{I}{\to} c - 2$. If instead $|\gamma \cap a_{r+1}| = 2$, then if we attach $\sigma_{a_{r+1}}$ first it must be a type II bypass by definition; afterward we can check the arc a_r becomes type III as well. Then $\sigma_{a_{r+1}} \circ \sigma_{a_r}$ is a pair of type II bypasses, which can each be turned into a type III followed by a type Ia, and so the number of dividing curves at each stage is $c - 2 \stackrel{III}{\to} c - 4 \stackrel{Ia}{\to} c - 2 \stackrel{III}{\to} c - 4 \stackrel{Ia}{\to} c - 2$. In either case we avoid having c dividing curves at any level by exchanging the bypasses and performing an isotopy.

Now suppose instead that $|\gamma \cap a_{r+1}| = 3$. Since each point of $\gamma \cap a_{r+1}$ must belong to a different dividing curve after attaching σ_{a_r}, there must be one point of $\gamma \cap a_{r+1}$ in each component of $\gamma \backslash a_r$. If the half-disk outside γ and cobounded by parts of γ and a_r does not contain any other dividing curves, then we may find a trivial arc d such that a_r, a_{r+1}, d form a bypass triple:



(We have shaded the inside of the disk on the right for clarity.) In particular, up to isotopy $\sigma_{a_r} \circ \sigma_{a_{r+1}} = \sigma_{a_r} \circ \sigma_{a_{r+1}} \circ d = \Delta$, and since Δ commutes with any bypass we can move it to the end of the sequence. Otherwise we can find a closed dividing curve inside that half-disk, take an admissible arc b which intersects that curve once and γ twice, and attach a bypass triangle $\Delta_b = \sigma_b \circ \sigma_{b'} \circ \sigma_{b''}$:



Now because the attaching arcs are all disjoint we have $\sigma_{a_r} \circ \sigma_{a_{r+1}} \circ \Delta_b = \sigma_b \circ \sigma_{a_r} \circ \sigma_{a_{r+1}} \circ \sigma_{b'} \circ \sigma_{b''}$, and one can show that all five of these bypasses are type II, so again we can replace them all with type III bypasses followed by type Ia bypasses and the number of dividing curves will never reach c in this sequence of moves.

By repeating this argument at any stage where we reach the maximum number $c \geq 5$ of dividing curves, we eventually show that $\xi \circ \Delta^l = \sigma_{a'_1} \circ \ldots \circ \sigma_{a'_{k'}} \circ \Delta^{l'}$ for some $l, l' \in \mathbb{N}$, where the complexity of $\sigma_{a'_1} \circ \ldots \circ \sigma_{a'_{k'}}$ is at most 3. If it is 1 then $\xi \circ \Delta^l = \Delta^{l'}$ and we are done. Otherwise it is exactly 3, and the bypasses appear in pairs $\sigma_a \circ \sigma_b$ where a and b are disjoint arcs of types Ia and III respectively. But then since $\#\Gamma_{S^2} = 1$ right before attaching σ_a, we must be in the above situation where there is a trivial arc d completing a bypass triple a, b, d, and so $\sigma_a \circ \sigma_b = \sigma_a \circ \sigma_b \circ \sigma_d = \Delta$. It follows that $\xi \circ \Delta^l$ is stably equivalent to some $\Delta^{l''}$, completing the proof. $\square$

The one claim we did not justify in the proof is that given consecutive bypasses $\sigma_{a_r} \circ \sigma_{a_{r+1}}$ of types Ia and III, we can assume up to stabilization that they are disjoint:

Lemma 10. *Let a be an attaching arc of type Ia on a convex (S^2, Γ) resulting in a new dividing set Γ', and let b be an attaching arc of type III on (S^2, Γ'). Then there is an attaching arc b' on Γ, disjoint from a, such that $\sigma_a \circ \sigma_b$ is stably isotopic to $\sigma_a \circ \sigma_{b'} \circ \Delta^l \circ \xi_\Phi$ for some integer l and isotopy Φ.*

See [2, Section 6] for the proof.

References

[1] Y. Eliashberg, *Classification of overtwisted contact structures on 3-manifolds,* Invent. Math. 98 (1989), no. 3, 623–637.

[2] Yang Huang, *A proof of the classification theorem of overtwisted contact structures via convex surface theory*, arXiv:1102.5398.