Math 273 Lecture 18

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We have already seen that contact (± 1)-surgery suffices to describe any closed contact 3-manifold. In this lecture we will first prove an interesting consequence related to symplectic filling and then study how more general $\frac{p}{q}$-surgeries relate to ± 1-surgeries.

Proposition 1. *Let (Y, ξ) be a contact structure with weak symplectic filling (X, ω). Then (X, ω) embeds symplectically into a closed symplectic manifold.*

Proof. We saw last time that there is a Legendrian link $L \subset (Y, \xi)$ such that Legendrian surgery on L results in a Stein fillable (Y', ξ'). Recall the construction of (Y', ξ'): we first express (Y, ξ) as contact (± 1)-surgery on a link $\mathbb{L}_- \cup \mathbb{L}_+$ in (S^3, ξ_{st}) and then identify L as a push-off of $\mathbb{L}_+$, so that (Y', ξ') is obtained from (S^3, ξ_{st}) as Legendrian surgery on $\mathbb{L}_-$.

We now add to $\mathbb{L}_- = L_1 \cup \ldots \cup L_n \subset (S^3, \xi_{\text{st}})$ a set of Legendrian right-handed trefoils $\mathbb{T} = T_1 \cup \ldots \cup T_n$, each with $tb = 1$, so that each T_i is linked once with L_i and not at all with any L_j, $i \neq j$. If we also do Legendrian surgery on $\mathbb{T}$, the result (Y'', ξ'') is still Stein fillable, but attaching the corresponding Weinstein handles to (X, ω) only gives a weak filling (X'', ω'') of (Y'', ξ''). We can improve our situation, however, by showing that (Y'', ξ'') is also an integral homology sphere. In that case, an argument of Eliashberg discussed earlier lets us deform ω'' near $\partial X''$ by gluing on a symplectic piece of the form $\partial X'' \times [1, C]$ so that it becomes a strong filling of (Y'', ξ''). In particular, we will have embedded (X, ω) into a strong symplectic filling of a Stein fillable contact structure.

Topologically, the group $H_1(Y''; \mathbb{Z})$ is generated by meridians of each component of $\mathbb{L}_- \cup \mathbb{T}$. However, the meridian μ_{L_i} of any L_i is homologous to the longitude λ_{T_i} of T_i, and Legendrian surgery on T_i is topologically a zero-surgery, so λ_{T_i} bounds a disk in the surgered manifold and thus $[\mu_{L_i}] = 0$. Furthermore, if we take a generic Seifert surface Σ_i for L_i and remove a small disk around each point where either T_i or some L_j intersects $\text{int}(\Sigma_i)$ transversely, then the result is a surface with boundary of the form $[\lambda_{L_i}] - [\mu_{T_i}] + \sum a_j [\mu_{L_j}]$ for some $a_j \in \mathbb{Z}$, so this sum is zero in homology and by the above argument we have $[\mu_{T_i}] = [\lambda_{L_i}]$ in $H_1(Y''; \mathbb{Z})$. On the other hand, the Legendrian surgery on L_i is a topological k-surgery for some integer k, so $[k\mu_{L_i} + \lambda_{L_i}] = 0$ and thus $[\mu_{T_i}] = -k[\mu_{L_i}] = 0$. We have now shown that every generator of $H_1(Y''; \mathbb{Z})$ vanishes, as desired.

Next, we note the theorem (originally due to Lisca and Matić [3] in a slightly stronger form) that any Stein domain embeds into a closed symplectic manifold. In particular, we apply this to the Stein domain $V = B^4 \cup \mathbb{H}$, where $\mathbb{H}$ is a set of handles corresponding to the Legendrian surgery on $\mathbb{L}_- \cup \mathbb{T}$, with contact type boundary (Y'', ξ''). This embeds into a closed symplectic manifold (X_0, ω_0), in which (Y'', ξ'') is now a separating hypersurface of contact type, so we can use a symplectic cut-and-paste operation along Y'' to form the closed manifold $(X'', \omega'') \cup (X_0 \backslash V, \omega_0|_{X_0 \backslash V})$. Since (X, ω) embeds symplectically into (X'', ω''), it embeds into this closed manifold as well. □

Proposition 2. *Let $K \subset (Y, \xi)$ be a Legendrian knot. Any contact $\frac{p}{q}$-surgery on K, $\frac{p}{q} < 0$, can be expressed as a sequence of Legendrian surgeries.*

Proof. We can assume that (Y, ξ) is actually a standard neighborhood $N = (S^1 \times D^2, \ker(\sin(\theta)dx + \cos(\theta)dy))$ of $K = S^1 \times \{0\}$. We note that any Legendrian surgeries performed inside N will preserve the tightness of ξ, because we can embed $N \subset (S^3, \xi_{\mathrm{st}})$ where Legendrian surgeries preserve fillability. Let μ_0 be a meridian $\{*\} \times \partial D^2$ and let λ_0 be a meridian $S^1 \times \{*\}$, so that the contact framing λ_{tb} is actually given by $-\mu_0 + \lambda_0$.

Let $K_1 \subset N$ be a Legendrian knot which is topologically isotopic to K with $tw(K_1) = r_1 + 1 \leq -1$; we have $-(r_1 + 1)$ ways to choose K_1 by picking different stabilizations of K. Then K_1 has a standard neighborhood $N_1 \subset N$ with contact framing $(r_1 + 1)\mu_0 + \lambda_0$ and meridian μ_0. We perform contact surgery by removing N_1 and gluing in a solid torus N_1', sending its meridian μ_1 to

$$\mu_0 - ((r_1 + 1)\mu_0 + \lambda_0) = -r_1\mu_0 - \lambda_0$$

and its longitude λ_1 (again, chosen so that $-\mu_1 + \lambda_1$ is the contact framing on N_1') to μ_0. In particular, the curve $-\mu_1 + \lambda_1$ is sent to $(r_1 + 1)\mu_0 + \lambda_0$, and there is a unique tight contact structure on N_1' with this contact framing, so this lets us define contact surgery on K_1. In matrix form, we have an identification

$$\begin{pmatrix} \mu_1 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} -r_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_0 \\ \lambda_0 \end{pmatrix}.$$

Now we can replace N with N_1' and repeat, using some knot K_2 with $tw(K_2) = r_2 + 1 \leq -1$, and so on; after n such surgeries we have

$$\begin{pmatrix} \mu_n \\ \lambda_n \end{pmatrix} = \begin{pmatrix} -r_n & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -r_2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -r_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_0 \\ \lambda_0 \end{pmatrix}.$$

In particular, if we let $(p_{-1}, q_{-1}) = (0, 1)$ and $(p_0, q_0) = (1, 0)$ then a quick induction shows that

$$\begin{pmatrix} \mu_i \\ \lambda_i \end{pmatrix} = \begin{pmatrix} p_i & q_i \\ p_{i-1} & q_{i-1} \end{pmatrix} \begin{pmatrix} \mu_0 \\ \lambda_0 \end{pmatrix}$$

where $p_i = -(p_{i-1}r_i + p_{i-2})$ and $q_i = -(q_{i-1}r_i + q_{i-2})$ for all $i \geq 1$. But then it is known that

$$\frac{p_n}{q_n} = r_1 - \cfrac{1}{r_2 - \cfrac{1}{\ddots - \frac{1}{r_n}}}$$

and this is the topological surgery coefficient, so if instead we take $\frac{p}{q} = [r_1 + 1, \ldots, r_n]$ then this procedure will perform a contact $\frac{p}{q}$-surgery. It is not hard to check that this allows us to perform all possible contact $\frac{p}{q}$-surgeries, since each choice of stabilizations along the way gives us a different contact structure and the total number of possibilities is exactly the number of tight contact structures on a solid torus. $\square$

We can make this procedure explicit: write $\frac{p}{q} = [r_1 + 1, \ldots, r_n]$ and let $K_0 = K$. For $i = 1, 2, \ldots, n$, then, we let K_i' be a Legendrian push-off of K_{i-1}, and let K_i be a stabilization of K_i' with $tw(K_i) = r_i + 1$. The contact $\frac{p}{q}$-surgery is then equivalent to Legendrian surgery on the link $K_1 \cup \ldots \cup K_n$. Furthermore, if $\frac{p}{q}$ is an integer $n < 0$ then we see that the surgery is equivalent to a single Legendrian surgery on a $(-n-1)$-fold stabilization of K.

Proposition 3. *Any contact $\frac{p}{q}$-surgery on a Legendrian knot K, $\frac{p}{q} > 0$, is equivalent to a contact $\frac{1}{k}$-surgery for some positive integer k followed by a contact r-surgery for some $r < 0$.*

Proof. Again, we restrict to a standard neighborhood N of K; let μ be a meridian of K and λ a longitude determined by the contact framing. For any p', q' with $\det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = 1$, we can define a $\frac{p}{q}$-surgery topologically by replacing a neighborhood $N(K) \subset N$ with another solid torus N' having meridian μ' and longitude λ' by the map $\mu' \mapsto p\mu + q\lambda$ and $\lambda' \mapsto p'\mu + q'\lambda$. The inverse of this map is $\begin{pmatrix} q' & -p' \\ -q & p \end{pmatrix}$, so it identifies $-p'\mu' + p\lambda'$ with λ and thus every tight solid torus with two dividing curves on its boundary of slope $-\frac{p}{p'}$ determines a contact $\frac{p}{q}$-surgery.

Now take $k > 0$ such that $\frac{p}{q} > \frac{1}{k}$. We begin by performing the uniquely defined $\frac{1}{k}$-contact surgery on K, replacing $N(K)$ with a standard neighborhood N_1 of a Legendrian knot K_1 with meridian μ_1 and contact framing λ_1 by the map

$$\begin{aligned} \mu_1 &\mapsto \mu + k\lambda \\ \lambda_1 &\mapsto \lambda. \end{aligned}$$

Next, we let $r = \frac{p}{q-kp} < 0$ and perform a contact r-surgery along K_1. We remove a neighborhood $N(K_1) \subset N_1$ and glue in a torus N_2 with meridian μ_2 and longitude λ_2, using the map

$$\begin{aligned} \mu_2 &\mapsto p\mu_1 + (q-kp)\lambda_1 \\ \lambda_2 &\mapsto p'\mu_1 + (q'-kp')\lambda_1 \end{aligned}$$

This also identifies μ_2 with $p(\mu + k\lambda) + (q - kp)\lambda = p\mu + q\lambda$, so the end result is a topological $\frac{p}{q}$-surgery with respect to the contact framing.

We need to check two things: first, that this procedure always results in a tight contact structure, and second, that every possible $\frac{p}{q}$-surgery can be performed by this procedure. We know that it is tight because we can embed the $\frac{1}{k}$-surgery on $K \subset N$ in the standard S^3, and then the r-surgery is equivalent to a series of Legendrian surgeries in S^3, so the result is tight. Furthermore, the curve $-p'\mu_2 + p\lambda_2$ is glued to λ_1, so again there is one r-surgery for every tight contact structure on a solid torus with boundary slope $-\frac{p}{p'}$; this is exactly the number of contact $\frac{p}{q}$-surgeries. □

We remark that the case of integral contact n-surgery on K has a simple description once again. If $n > 1$, we can perform a contact $(+1)$-surgery followed by a $\frac{n}{1-n}$-surgery, and $\frac{n}{1-n} < 0$, so the n-surgery is equivalent to a $(+1)$-surgery on K followed by a series of Legendrian surgeries. (Since $\frac{n}{1-n} = -1 - \frac{1}{n-1}$, this actually only requires two Legendrian surgeries.)

We have now described almost completely how to turn an arbitrary contact $\frac{p}{q}$-surgery into a series of (± 1)-surgeries; the only remaining detail is the case $\frac{p}{q} = \frac{1}{k} > 0$.

Lemma 4. *Let $n \geq 1$. The contact $\frac{1}{n}$-surgery on a Legendrian knot K is equivalent to contact $(+1)$-surgeries on each of n parallel push-offs of K.*

Proof. Again we restrict to a standard neighborhood N of K. By the argument of the previous lemma, contact $\frac{1}{n}$-surgery on K is topologically equivalent to a $\frac{1}{1}$-surgery and a $\frac{1}{n-1}$-surgery on parallel copies of K, so if we repeat this $n-1$ times then we can replace the original surgery with n contact $(+1)$-surgeries as desired. Now it remains to be seen that after all n surgeries we are left with something tight.

Suppose we start with the knot K, with standard neighborhood N embedded in (S^3, ξ_{st}), and take n push-offs $K'_1, \ldots, K'_n$. If we take N to be small enough that it is disjoint from all the other K'_i, and then locate n push-offs $K_1, \ldots, K_n$ of K inside N, then we can perform $(+1)$-surgery on each of the K_i and (-1)-surgery on each of the K'_i. The result is still contact isotopic to (S^3, ξ_{st}), since the (± 1)-surgeries cancel in pairs, so we have performed a $\frac{1}{k}$-surgery inside N and embedded the result in the tight (S^3, ξ_{st}) as desired. □

Corollary 5. *Contact $\frac{1}{k}$-surgery is inverse to contact $-\frac{1}{k}$-surgery for all $k \geq 1$.*

References

[1] Fan Ding and Hansjörg Geiges, *A Legendrian surgery presentation of contact 3-manifolds*, Math. Proc. Cambridge Philos. Soc. 136 (2004), no. 3, 583–598.

[2] Fan Ding, Hansjörg Geiges, and András I. Stipsicz, *Surgery diagrams for contact 3-manifolds*, Turkish J. Math. 28 (2004), no. 1, 41–74.

[3] P. Lisca and G. Matić, *Tight contact structures and Seiberg-Witten invariants*, Invent. Math. 129 (1997), no. 3, 509–525.