Math 273 Lecture 17

Steven Sivek

March 28, 2012

In this lecture we will begin a more in-depth look at contact surgery, starting with the case of (+1)-surgery. In order to do so, we will first examine the tight contact structure on $S^1 \times S^2$.

Lemma 1. *The unique tight contact structure ξ_0 on $S^1 \times S^2$ is strongly symplectically fillable.*

Proof. Give $S^1 \times \mathbb{R}^3$ the symplectic form $\omega = d\theta \wedge dx + dy \wedge dz$. It is not hard to check that the vector field

$$v = x\partial_x + \frac{y}{2}\partial_y + \frac{z}{2}\partial_z$$

is Liouville, i.e. that $\mathcal{L}_v\omega = d\iota_v\omega = \omega$, and that it points outward along

$$Y = \{(\theta, x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

so $Y \cong S^1 \times S^2$ is a hypersurface of contact type. In particular, $(Y, \ker(\iota_v\omega|_Y))$ is strongly fillable, and since it is then tight it must be isotopic to ξ_0. □

Remark 2. In fact, ξ_0 is Stein fillable, though we will not need to know this.

Proposition 3 ([2])**.** *Contact* (+1)*-surgery on a* $tb = -1$ *Legendrian unknot in* (S^3, ξ_{st}) *results in* $(S^1 \times S^2, \xi_0)$.

Proof. Topologically this is a 0-surgery on the unknot, which does produce $S^1 \times S^2$, so we need to check that the resulting contact structure ξ is tight. We will find an embedded convex torus $T \subset (S^1 \times S^2, \xi_0)$ which splits $S^1 \times S^2$ into a pair of solid tori, each of which has a unique contact structure: to define T, we let $f(\phi) = \epsilon \sin(\phi)$ for ϵ small and use the map

$$(\theta, \phi) \mapsto (\theta, f(\phi), \sqrt{1 - f(\phi)^2}\cos(\phi), \sqrt{1 - f(\phi)^2}\sin(\phi)).$$

The vectors ∂_θ and

$$v = -f'(\phi)\partial_x + \left(\frac{-ff'}{\sqrt{1-f^2}}\cos(\phi) - \sqrt{1-f^2}\sin(\phi)\right)\partial_y + \left(\frac{-ff'}{\sqrt{1-f^2}}\sin(\phi) + \sqrt{1-f^2}\cos(\phi)\right)\partial_z$$

span the tangent space of T, and if $\alpha = -xd\theta + \frac{1}{2}(ydz - zdy)$ is a contact form for ξ_0 then we have $\alpha(\partial_\theta) = -f$ and $\alpha(v) = \frac{1}{2}(1 - f^2)$. The characteristic foliation T_ξ is thus spanned by $\partial_\theta + \frac{2f}{1-f^2}v$, with closed orbits at $\phi \in \{0, \pi\}$, and we can see that T has dividing set $\phi \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. Since the curves $\phi = c$ all take the form $S^1 \times \{*\}$, the dividing curves on T are a pair of longitudes $\Gamma_T = S^1 \times (\pm 1, 0, 0)$.

Now let $K \subset (S^3, \xi_{\mathrm{st}})$ be the $tb = -1$ Legendrian unknot, and let N be a standard neighborhood of K with meridian μ and Seifert-framed longitude λ. The complement of N is a solid torus with meridian $\mu_N = \lambda$ and $\lambda_N = \mu$, and the contact framing determines a longitude $\lambda_{tb} = \lambda - \mu = \mu_N + \lambda_N$. In particular, we can identify $(S^3 \backslash N, \xi|_{S^3\backslash N})$ with one of the two components of $(S^1 \times S^2)\backslash T$ up to contact isotopy rel boundary. When we perform contact $(+1)$-surgery on K, we glue a solid torus N' to $S^3 \backslash N$ sending $\mu_{N'}$ to $-(\lambda_{tb} + \mu) = -\mu_N$ and $\lambda_{N'}$ to $\mu = \lambda_N$. In particular we glue $-\mu_{N'} - \lambda_{N'}$ to $\mu_N - \lambda_N = -\lambda_{tb}$, so the contact structure on N' has two dividing curves parallel to λ_{tb}. This tight contact structure is unique up to isotopy and agrees with ξ_0 on the other component of $(S^1 \times S^2)\backslash T$, so we conclude that the $(+1)$-surgery on K is isotopic to ξ_0, as desired. □

We can now prove that certain contact surgeries are "inverse" to each other.

Proposition 4. *Let $K \subset (Y, \xi)$ be a Legendrian knot with push-off K', and form (Y', ξ') by a contact (-1)-surgery along K and a contact $(+1)$-surgery along K'. Then (Y, ξ) is contact isotopic to (Y', ξ').*

Proof. This is actually true if we replace contact ± 1-surgery with contact $\pm\frac{1}{k}$-surgery; we will not prove this in full generality, but we will at least show that topologically a pair of contact $\pm\frac{1}{k}$ surgeries cancel each other out.

It suffices to restrict our attention to a standard neighborhood N of K, with $K' \subset N$, and prove that (N', ξ') is isotopic rel boundary to (N, ξ). Let μ_K and λ_K be a meridian and a curve representing the contact framing of K, respectively. Perform a contact $\frac{1}{k}$-surgery on K, by gluing in a contact torus $N_1 = D^2 \times S^1$ with Legendrian core L having meridian μ_L and contact framing λ_L so that μ_L is sent to $\mu_K + k\lambda_K$ and λ_L is sent to λ_K. If L is the Legendrian core of this torus, then a pushoff of L will be isotopic to λ_L in N_1, hence to λ_K in the surgered $(N \backslash N(K)) \cup N_1$. In other words, L is Legendrian isotopic to K', so it suffices to do a contact $-\frac{1}{k}$-surgery on L instead.

We perform contact $-\frac{1}{k}$-surgery on L by removing a neighborhood $N(L) \subset N_1$, whose complement is diffeomorphic to $N \backslash N(K)$, and gluing in another torus N_2, so that the result N' of the surgery on K and then on L is topologically identical to performing a single surgery on K. We take N_2 to be a standard neighborhood of a knot L' with meridian $\mu_{L'}$ and contact framing $\lambda_{L'}$, and the gluing map to send $\mu_{L'} \mapsto \mu_L - k\lambda_L$ and $\lambda_{L'} \mapsto \lambda_{L'}$. In N' we have $\mu_{L'} \mapsto (\mu_K + k\lambda_K) - k(\lambda_K) = \mu_K$ and $\lambda_{L'} \mapsto \lambda_K$, so this single surgery is a $\frac{1}{0}$-surgery, i.e. N' is diffeomorphic to N. Since there is a unique tight contact structure on N with the specified boundary conditions, we just need to see that (N', ξ') is tight as well.

In order to prove tightness in the case $k = 1$, we will embed (N', ξ') into a tight contact manifold. Let K be a Legendrian unknot in (S^3, ξ_{st}) with $tb = -1$, identify N with a standard neighborhood of K, and let $K' \subset N$ be its pushoff. Then the contact $(+1)$-surgery on K' produces the tight $S^1 \times S^2$, which we have shown to be strongly fillable, and since Legendrian surgery on K (now viewed in $S^1 \times S^2$) preserves fillability we see that the result of both contact surgeries is a strongly fillable contact structure on S^3 (in particular, it is ξ_{st}). It follows that N' is tight, hence isotopic rel boundary to N. □

Contact $(+1)$-surgery is in some sense not as nice as (-1)-surgery, however, because it often results in overtwisted contact structures even when performed on knots in the standard S^3. For example, there is a Legendrian right-handed trefoil with $tb = 1$, and contact $(+1)$-surgery on a stabilization of this trefoil gives the Poincaré homology sphere with reversed orientation, but we already know that there are no tight contact structures on this manifold. More generally:

Proposition 5. *Performing contact $(+1)$-surgery on any stabilized Legendrian knot in a contact manifold results in an overtwisted contact structure.*

Proof. Let K be a Legendrian knot and K' a pushoff, and suppose we are performing the surgery on a stabilization K'' of K'. Topologically, this surgery has framing $tb(K'')+1 = tb(K')$, which is also the linking number of K and K'', so the obvious annulus cobounded by K and K'' can be capped off inside the surgery torus by a disk. In other words, K bounds a disk inside the surgered manifold, and the surface framing of this disk agrees with the contact framing since the disk contains a pushoff of K, so K is the boundary of an overtwisted disk. □

Recall the definition of a Lutz twist along a transverse knot K, which has a tight model neighborhood $S^1_\theta \times D^2_{(r,\phi)}$ with contact form $\alpha = d\theta + r^2 d\phi$ and D^2 a disk of radius δ. (By rescaling in the r direction we can assume that $\delta > 1$.) We take functions $(f(r), g(r))$ which are equal to $(-1, -r^2)$ near $r = 0$ and $(1, r^2)$ for $r > 1-\epsilon$, such that when graphed parametrically (f, g) travels counterclockwise around the origin and avoids the positive y-axis. Then we replace $\xi = \ker(\alpha)$ with $\xi' = \ker(\alpha')$, where

$$\alpha' = f(r)d\theta + g(r)d\phi,$$

on $S^1 \times D^2$. We showed that by repeatedly Lutz twisting along knots, we can construct a contact structure in any homotopy class of plane field on a manifold.

Proposition 6 ([1])**.** *A Lutz twist can be performed by two contact $(+1)$-surgeries.*

Proof. There is a unique $r_0 \in (0, \delta)$ where $f(r_0) = -g(r_0) > 0$; let N be the solid torus $S^1 \times D^2_{r_0}$. On a torus $T_r \subset N$ of radius $r \leq r_0$, where $\alpha' = g(r)\partial_\theta - f(r)\partial_\phi$, the characteristic foliation is linear of slope $-\frac{g(r)}{f(r)}$, so as r increases from 0 to r_0 the slope decreases from 0 to $-\infty$ and then from ∞ down to 1. On the other hand, the characteristic foliation of a torus of radius r in $(S^1 \times \mathbb{R}^2, \ker(\alpha))$ is linear of slope $-r^2$, so if we apply several Dehn twists along a meridian of N

then its slopes are all negative and thus we can embed (N, ξ') in $(S^1 \times \mathbb{R}^2, \alpha)$. This implies that (N, ξ') is tight, and we can perturb ∂N to be convex with two parallel dividing curves of slope 1.

After the perturbation, (N, ξ') is now the unique tight contact structure determined by $\Gamma_{\partial N}$, and in particular it is a standard neighborhood of a Legendrian knot K with contact framing $\mu+\lambda$ where $\mu = \{*\} \times \partial D^2_{r_0}$ and $\lambda = S^1 \times \{*\}$. We perform a contact (-1)-surgery along K, removing a smaller standard neighborhood $N(K) \subset N$ and gluing in a solid torus $N' = S^1 \times D^2$ by a map sending $\mu_{N'} \mapsto \mu - (\mu + \lambda) = -\lambda$ and $\lambda_{N'} \mapsto \mu$. Let M denote the surgered manifold.

The torus T_1 of radius 1 in N has slope -1, so if we perturb it to be convex then it bounds a unique tight contact structure in N which we can identify as a standard neighborhood of a Legendrian knot. Then T_1 and the perturbation ∂N of T_{r_0} cobound a minimally twisting $T^2 \times I$ with boundary slopes decreasing from 1 to -1. These dividing curves are parallel to $\mu + \lambda$ and $\mu - \lambda$, which from N' are identified with $-\mu_{N'} + \lambda_{N'}$ and $\mu_{N'} + \lambda_{N'}$ respectively. Thus $N' \cup (T^2 \times [r_0, 1])$ is obtained by taking a tight solid torus with boundary slope -1 and gluing on a minimally twisting $T^2 \times I$ with boundary slopes -1 and 1, so that the result is the unique tight solid torus with boundary slope 1.

In summary, the Lutz twist and Legendrian surgery correspond to removing the neighborhood $(S^1 \times D^2_1, \xi)$ of a Legendrian knot K_0 with contact framing $\mu - \lambda$ and gluing in a tight solid torus N'' by a map sending $\mu_{N''}$ to $-\lambda = \mu - (\mu + \lambda)$. This is precisely a contact $(+1)$-surgery along K_0, and if we undo the Legendrian surgery along K by performing another $(+1)$-surgery on a push-off of K then we are left with the Lutz twist, so we conclude that the Lutz twist is equivalent to these two $(+1)$-surgeries. □

Using contact surgeries, it is easy to see that every closed 3-manifold Y admits a contact structure: we express Y in terms of integral surgeries on some link $L \subset S^3$, and then take a Legendrian representative of L in ξ_{st}. After stabilizing each component $L_i \subset L$ so that $tb(L_i)$ is less than the corresponding surgery coefficient c_i, we perform contact $(c_i - tb(L_i))$-surgery on each L_i to get a contact structure on Y. In general the contact surgery may not be uniquely defined, and the result will often be overtwisted, but it does provide a contact structure. We can refine this result as follows:

Theorem 7. *For any contact structure (Y, ξ), we can find a Legendrian link $\mathbb{L} = \mathbb{L}_- \cup \mathbb{L}_+$ in (S^3, ξ_{st}) such that ξ is the result of contact (-1)-surgery on each component of $\mathbb{L}_-$ and contact $(+1)$-surgery on each component of $\mathbb{L}_+$.*

Proof. Let U be a Legendrian unknot in (S^3, ξ_{st}) with $tb(U) = -2$, and let U' be a push-off of U. Let (S^3, ξ_{ot}) be the result of a contact $(+1)$-surgery on U', which is overtwisted by Proposition 5.

Now let $(Y, \xi') = (Y, \xi) \# (S^3, \xi_{\text{ot}})$. There is a link $L \subset Y$ for which some integral surgery on each component $L_i \subset L$ produces S^3. We can Legendrian realize L with respect to ξ' so that the contact framing on L_i is one less than the surgery framing: given an initial Legendrian realization, we can decrease the framing on any component by stabilizing it or increase the framing by taking the connected sum with the $tb = 0$ boundary of an overtwisted disk. This means that we have a Legendrian link $L \subset (Y, \xi')$ for which contact $(+1)$-surgery on every component gives some (possibly overtwisted) contact structure (S^3, ξ_1). Since we can get from (Y, ξ) to (Y, ξ') by a contact $(+1)$-surgery on $U \subset (S^3, \xi_{\text{st}}) \subset (Y, \xi)$, then, we can do contact $(+1)$-surgery on the link $L \cup U$ to get from (Y, ξ) to (S^3, ξ_1), hence Legendrian surgery on a push-off of $L \cup U$ inside (S^3, ξ_1) will produce (Y, ξ). If ξ_1 is actually ξ_{st} then we are done, and otherwise ξ_1 is overtwisted.

We have reduced the problem to the case of an overtwisted contact structure ξ_1 on S^3. We can perform a series of Lutz twists to turn ξ_1 into an overtwisted contact structure homotopic to ξ_{ot}, and by Eliashberg's classification of overtwisted contact structures (which we have not proved yet) this contact structure must actually be isotopic to ξ_{ot}. Now Proposition 6 gives us a Legendrian link $L \subset (S^3, \xi_1)$ for which contact $(+1)$-surgery on L results in ξ_{ot}, so equivalently we can do Legendrian surgery on a link $L' \subset (S^3, \xi_{\text{ot}})$ to construct (S^3, ξ_1). Letting $\mathbb{L}_- = L'$ and $\mathbb{L}_+ = U'$ finishes the proof. □

We have actually shown something stronger: we can assume that $\mathbb{L}_+$ has at most one component. (In fact, if $\mathbb{L}_+$ is empty, we can take some knot K and a push-off K' and then add K to $\mathbb{L}_-$ and K' to $\mathbb{L}_+$ so that $\mathbb{L}_+$ has exactly one component.)

Thus any contact structure on a closed 3-manifold can be described by a contact $(+1)$-surgery on some Legendrian knot in a Stein fillable contact manifold. Using the fact that Legendrian surgeries correspond to Weinstein 2-handle cobordisms, we immediately conclude as in [3]:

Corollary 8. *Given any contact manifold (Y, ξ), there is a symplectic cobordism from (Y, ξ) to a Stein fillable manifold.*

By a *symplectic cobordism* from (Y_0, ξ_0) to (Y_1, ξ_1) we mean a symplectic manifold (X, ω) with contact type boundary, where $\partial X = -Y_0 \sqcup Y_1$ and Y_0 and Y_1 are ω-concave and ω-convex respectively.

Proposition 9. *If (Y, ξ) is overtwisted and (Y', ξ') is arbitrary, there is a symplectic cobordism from (Y, ξ) to (Y', ξ').*

Proof. If (Y, ξ) is overtwisted then we can find a link $L \subset Y$ on which some integral surgeries produce Y', and then as before we can find Legendrian representatives of L for which the surgery framings are one less than the contact framings, so we can get a contact structure (Y', ξ'') by a Legendrian surgery in Y. We can also take L to avoid any overtwisted disks in Y, so that ξ'' is overtwisted as well. Then we can get from ξ' to ξ'' by a series of Lutz twists (again, this assumes the theorem that there is a unique overtwisted contact structure in each homotopy class of plane field), hence by contact $(+1)$-surgeries on some Legendrian link in (Y', ξ'), so equivalently some Legendrian surgeries on a link in (Y', ξ'') produce (Y', ξ') as desired. Thus we have a Legendrian link in (Y, ξ) for which Legendrian surgery results in (Y', ξ'). □

In both cases the symplectic cobordisms can actually be taken to be Stein cobordisms, since they are composed of Weinstein handles.

If (Y, ξ) is overtwisted, then we can find a symplectic (or Stein) cobordism W from it to (S^3, ξ_{st}). In particular, we can then identify a Darboux ball (B^4, ω) with contact-type boundary in any closed symplectic 4-manifold (X, ω) and glue W to $X \backslash B^4$. This gives us a symplectic manifold with ω-concave boundary (Y, ξ). On the other hand, we know that finding such a manifold with ω-convex boundary (Y, ξ) would be impossible because that would make (Y, ξ) strongly symplectically fillable, hence tight.

References

[1] Fan Ding and Hansjörg Geiges, *A Legendrian surgery presentation of contact 3-manifolds*, Math. Proc. Cambridge Philos. Soc. 136 (2004), no. 3, 583–598.

[2] Fan Ding, Hansjörg Geiges, and András I. Stipsicz, *Surgery diagrams for contact 3-manifolds*, Turkish J. Math. 28 (2004), no. 1, 41–74.

[3] John B. Etnyre and Ko Honda, *On symplectic cobordisms*, Math. Ann. 323 (2002), no. 1, 31–39.